Pure Mathematics 1

J. K. Backhouse
S. P. T. Houldsworth

This edition revised by
P. J. F. Horril

NEW EDITION

Pure Mathematics 1

By the same authors (with B. E. D. Cooper)

Pure Mathematics 2

(See page 579 of Book 1 for a list of contents)

Pure Mathematics Book 1

Fourth Edition

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Preface

I have personally used the previous editions of this book over many years and with pupils of a broad range of ability. I have always admired it for its common sense approach to the subject, for the large number and great variety of its examples and for the scope and grading of its exercises. In recent years, however, it has become increasingly clear that drastic revision was needed to take account of the many changes which have appeared in A-level syllabuses since the book was first written. At the time of writing the 'common core' syllabus is about to appear at A-level and this new edition, and its companion volume, have been prepared with this in mind.

Although many of the new chapters are concerned with so-called 'modern' topics, I make no apology for retaining the traditional style of the earlier editions. Indeed, I hope that readers will feel that the book conforms to the spirit of paragraph 582 of the Cockcroft Report, which says

> 'Syllabus changes during the last ten years have lessened the differences between the content of 'modern' and 'traditional' syllabuses and many feel it is no longer appropriate to distinguish between them .... We support the view that the distinction should no longer be maintained.'

This new edition of *Pure Mathematics, Books 1 and 2*, contains all the topics which are in the 'common core' syllabus and the symbols used throughout are those adopted by the Examining Boards. The pure mathematics content of most 'single subject' syllabuses is included and, while it cannot be guaranteed that *all* 'double subject' syllabuses are covered, the books will provide a sound course of study for most of them. These books are also suitable for the pure mathematics content of most AS level syllabuses.

The book has not been designed to be read straight through chapter by chapter in numerical order, and most readers will probably prefer to develop several branches simultaneously. Indeed, later sections of some chapters are better delayed for a second reading; where a natural break occurs this has been indicated in the text. New chapters and material have been incorporated without unduly disturbing the overall contents of previous editions, and, to make it easier to locate a particular topic, an index has now been provided. Another new feature of this edition is an appendix on Algebra revision which could be used at any stage in the course where the reader, or the teacher, feels that some revision of basic skills is necessary.

Chapter 1 introduces coordinates and the straight line. Chapter 2 introduces the idea of a function (this is a chapter to which the reader should return, when new functions are introduced at later stages in the course). Thereafter, the arrangement is

Chapters 3–8,	Calculus
Chapters 9–14,	Algebra (including an introduction to matrices)
Chapter 15,	Vectors
Chapters 16–19,	Trigonometry
Chapters 20–22,	Coordinate geometry

Finally there are chapters on variation, iterative methods, and an introduction to group theory.

Teachers who are familiar with the previous editions will find that *some* of the exercises have been pruned. Questions involving very heavy manipulation in algebra and trigonometry, which are now out of fashion with Examining Boards, have been replaced by more appropriate questions.

The individual reader has been kept in mind and he or she is advised to work through the questions marked **Qu.**; the class teacher will find that many of these questions are suitable for oral work. On some occasions proofs of important results have been left to the reader; when these appear in the exercises they are marked with an asterisk.

I would like to thank the previous authors for allowing me to tamper with their work, and for their detailed and constructive criticisms of my drafts.

My thanks are also due to Michael Spincer and Sue Justice of Longman Group for their help and encouragement. For the invaluable opportunity to give my undivided attention to this project, I am indebted to the Master and Fellows of Selwyn College, Cambridge, who kindly elected me a Fellow Commoner of the College for the Lent Term 1983, and to the Governors of Nottingham High School, for granting me the necessary leave of absence. I should also like to acknowledge the valuable help of my colleagues and pupils who tried the new material.

Lastly, but not least, I would like to thank my wife and family for their patience over the last few years, especially during my term in Cambridge, and to apologise for the many occasions when I have dodged the washing-up in order to 'work on my book'.

Nottingham
January 1984

Peter Horril

Note on degree of accuracy of answers

In order to avoid tedious repetition in the wording of questions the following conventions are observed throughout the book, unless there are specific instructions to the contrary:

(a) When possible an exact answer is given. To this end it is normally appropriate to retain surds and π in the answers where they occur. (The word *exact* is used here in the rather limited sense of being derived from the data without any intervening approximation.)

(b) When an answer is not exact, it is given correct to three significant figures, or, if it is an angle measured in degrees, to the nearest tenth of a degree.

Mathematical notation

The following notation is used in this book. It follows the conventions employed by most GCE Examining Boards.

1. Set notation

$\in$	is an element of.
$\notin$	is not an element of.
$\{a, b, c, \ldots\}$	the set with elements $a, b, c \ldots$
$\{x: \ldots\}$	the set of elements x, such that ...
$n(A)$	the number of elements in set A.
$\varnothing$	the empty set.
$\mathscr{E}$	the universal set.
A'	the complement of set A.
$\mathbb{N}$	the set of natural numbers (including zero) 0, 1, 2, 3 ...
$\mathbb{Z}$	the set of integers $0, \pm 1, \pm 2, \pm 3 \ldots$
$\mathbb{Z}^+$	the set of positive integers $+1, +2, +3 \ldots$
$\mathbb{Q}$	the set of rational numbers.
$\mathbb{R}$	the set of real numbers.
$\mathbb{C}$	the set of complex numbers.
$\subseteq$	is a subset of.
$\subset$	is a proper subset of.
$\cup$	union.
$\cap$	intersection.
$[a, b]$	the closed interval $\{x \in \mathbb{R}: a \leqslant x \leqslant b\}$.
(a, b)	the open interval $\{x \in \mathbb{R}: a < x < b\}$.

2. Miscellaneous symbols

$=$	is equal to.
$\neq$	is not equal to.
$>, <$	is greater than, is less than.
$\geqslant, \leqslant$	is greater than or equal to, is less than or equal to.
$\approx$	is approximately equal to.

3. Operations

$a+b$	a plus b.
$a-b$	a minus b.
$a \times b$, ab, $a.b$	a multiplied by b.
$a \div b$, $\frac{a}{b}$, a/b	a divided by b.
$\sum_{i=1}^{i=n} a_i$	$a_1 + a_2 + a_3 + \ldots + a_n$.

4. Functions

$f(x)$	the value of the function f at x.
$f\colon A \to B$	f is a function which maps each element of set A onto a member of set B.
$f\colon x \mapsto y$	f maps the element x onto an element y.
f^{-1}	the inverse of the function f.
$g \circ f$ or gf	the composite function $g(f(x))$.
$\lim_{x \to a} f(x)$	the limit of $f(x)$ as x tends to a.
δx	an increment of x.
$\frac{dy}{dx}$	the derivative of y with respect to x.
$\frac{d^n y}{dx^n}$	the nth derivative of y with respect to x.
$f'(x), f''(x), \ldots f^{(n)}(x)$	the first, second, ... nth derivatives of $f(x)$.
$\int y\, dx$	the indefinite integral of y with respect to x.
$\int_a^b y\, dx$	the definite integral, with limits a and b.
$[F(x)]_a^b$	$F(b) - F(a)$.

5. Exponential and logarithmic functions

e^x or $\exp x$	the exponential function.
$\log_a x$	logarithm of x in base a logarithms.
$\ln x$	$\log_e x$.
$\lg x$	$\log_{10} x$.

6. Circular and hyperbolic functions

$\sin x$, $\cos x$, $\tan x$	the circular functions sine, cosine, tangent.
$\operatorname{cosec} x$, $\sec x$, $\cot x$	the reciprocals of the above functions.
$\sin^{-1} x$ or $\arcsin x$	the inverse of the function $\sin x$ (with similar abbreviations for the inverses of the other circular functions).
$\sinh x$ etc.	the hyperbolic functions.

7. Other functions

$\sqrt{a}$	the positive square root of a.

$\lvert a \rvert$	the modulus of a.
$n!$	n factorial; $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$ $(0! = 1)$.
$\binom{n}{r}$	$\dfrac{n!}{r!(n-r)!}$ when $n, r \in \mathbb{N}$ and $0 \leqslant r \leqslant n$.
$\binom{n}{r}$	$\dfrac{n(n-1)\ldots(n-r+1)}{r!}$ when $n \in \mathbb{Q}$ and $r \in \mathbb{N}$.

8. Complex numbers

i	the square root of -1.
z or w	a typical complex number, e.g. $x + \mathrm{i}y$, where $x, y \in \mathbb{R}$.
$\mathrm{Re}(z)$	the real part of z; $\mathrm{Re}(x + \mathrm{i}y) = x$.
$\mathrm{Im}(z)$	the imaginary part of z; $\mathrm{Im}(x + \mathrm{i}y) = y$.
$\lvert z \rvert$	the modulus of z; $\lvert x + \mathrm{i}y \rvert = \sqrt{(x^2 + y^2)}$.
$\arg z$	the argument of z.
z^*	the complex conjugate of z.

9. Matrices

$\mathbf{M}$	a typical matrix $\mathbf{M}$.
$\mathbf{M}^{-1}$	the inverse of a matrix $\mathbf{M}$ (provided it exists).
$\mathbf{M}^{\mathrm{T}}$	the transpose of matrix $\mathbf{M}$.
$\det \mathbf{M}$	the determinant of a square matrix $\mathbf{M}$.
$\mathbf{I}$	the identity matrix.

10. Vectors

$\mathbf{a}$	the vector $\mathbf{a}$.
$\lvert \mathbf{a} \rvert$ or a	the magnitude of vector $\mathbf{a}$.
$\hat{\mathbf{a}}$	the unit vector with the same direction as $\mathbf{a}$.
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors parallel to the Cartesian coordinate axes.
$\overrightarrow{\mathrm{AB}}$	the vector represented by the line segment AB.
$\lvert \overrightarrow{\mathrm{AB}} \rvert$ or AB	the length of the vector $\overrightarrow{\mathrm{AB}}$.
$\mathbf{a}.\mathbf{b}$	the scalar product of $\mathbf{a}$ and $\mathbf{b}$.

Chapter 1

Coordinates and the straight line

Coordinates

1.1 The first thing that a reader new to this stage of mathematics will discover is that number, and the methods of algebra, may be brought to bear upon geometrical ideas to a much greater extent than before, and with great clarity and economy. To do this we must have a way of describing exactly and briefly the position of a point in a plane (i.e. a flat surface).

We may think for a moment of the pirate of old, who buried his treasure chest on a large flat featureless island, but was able to locate it when he returned. Starting at the most westerly point, he measured 400 paces due East, and then from there 100 paces due North. There, he knew, was the exact spot at which to dig.

This illustrates the method we shall use to fix the position of a point on a plane. Two straight lines cutting at right angles fix our directions, and we start our measurement from their point of intersection O (Fig. 1.1).

The point O is called the **origin**. The ***x*-axis** is drawn across the page, and the ***y*-axis** is drawn up the page; units of distance are marked off on them, positive in one direction, negative in the other. The plane containing these axes is called the **Cartesian** plane, after René Descartes (1588–1648) who did much to lay the foundations of the subject we now call Coordinate Geometry. When the axes are drawn in a vertical plane (for instance, when a teacher draws them on a board, fixed to a vertical wall), the x-axis is always drawn as a horizontal line and the y-axis as a vertical line; for this reason, they are often called the **horizontal axis** and the **vertical axis**, respectively (even though when they are drawn on the page of a book, lying on a horizontal table, both axes are horizontal!).

Consider the point A in Fig. 1.1. To reach A from O we travel 4 units in the direction of Ox, and then 1 unit in the direction of Oy.

The ***x*-coordinate** (or *abscissa*) of A is $+4$.
The ***y*-coordinate** (or *ordinate*) of A is $+1$.

We say that the **coordinates** of A are (4, 1), or that A is the point (4, 1). The x-coordinate is always given first, thus we distinguish between the points A(4, 1) and B(1, 4). By use of the sign of the coordinates we distinguish between the points A(4, 1) and C(−4, −1).

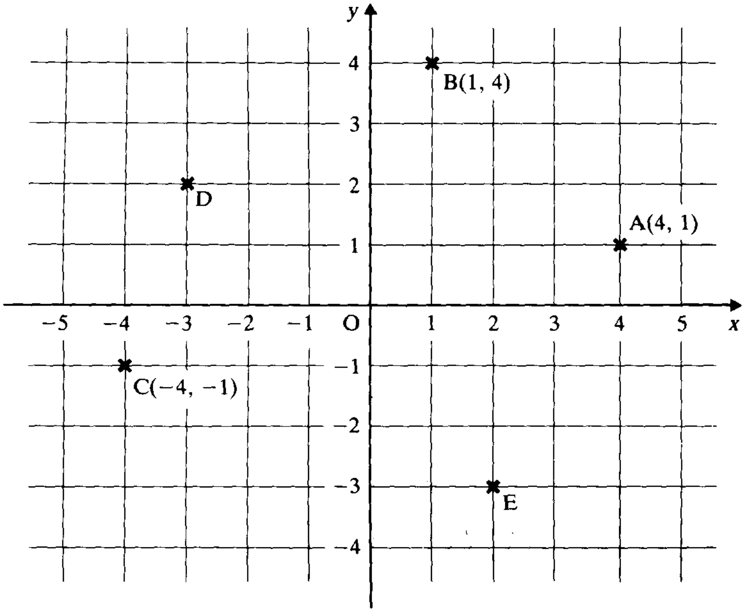

Figure 1.1

Qu. 1 Write down the coordinates of the points D, E, O in Fig. 1.1.
Qu. 2 Sketch your own axes and plot the points P(2, 4), Q(−5, 7), R(4, −2), S(0, 3), T(2, 0).

The length of a straight line

1.2 Example 1 *Find the length of the straight line joining* A(2, 1) *and* B(5, 5).

AC and CB are drawn parallel to the x-axis and y-axis respectively (Fig. 1.2). Applying Pythagoras' theorem to the right-angled triangle ABC,

$$\begin{aligned} AB^2 &= AC^2 + CB^2 \\ &= (5-2)^2 + (5-1)^2 \\ &= 9 + 16 \\ \therefore\ AB &= \sqrt{25} = 5 \end{aligned}$$

Notice that, if A had been the point (−2, 1) in the above example, the length of AC would still be the *difference* between the x-coordinates of A and B, since it would be $5-(-2) = 5+2 = 7$.

Qu. 3 Find the lengths of the straight lines joining the following pairs of points:
(a) A(3, 2) and B(8, 14), (b) C(−1, 3) and D(4, 7), (c) E(p, q) and F(r, s).

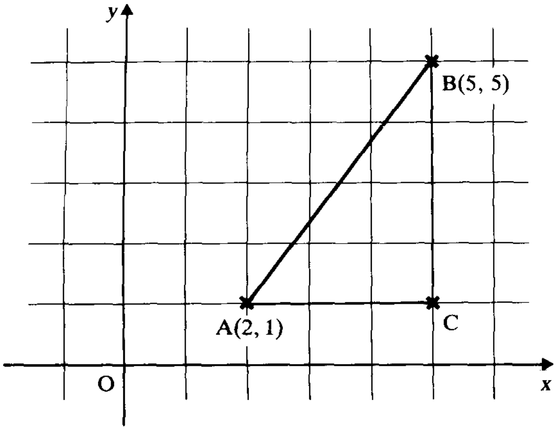

Figure 1.2

The mid-point of a straight line

1.3 Example 2 *Find the mid-point of the straight line joining* A(2, 1) *and* D(6, 5).

Let M, the mid-point of AD, have coordinates (p, q). FM and ED are drawn parallel to Oy; AFE is drawn parallel to Ox (Fig. 1.3).

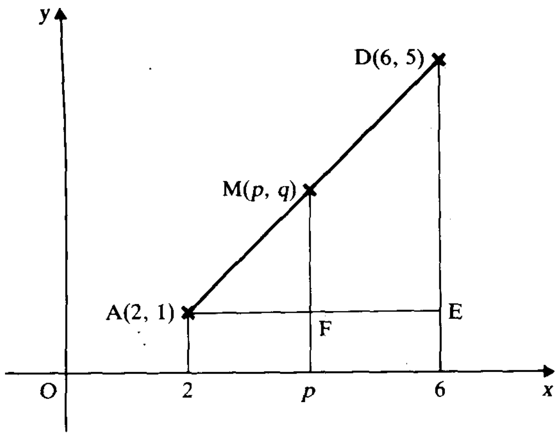

Figure 1.3

In the triangle ADE, applying the mid-point theorem, since M is the mid-point of AD, and MF is parallel to DE, F is the mid-point of AE. Thus

$$AF = FE$$

$$\therefore p - 2 = 6 - p$$

$$\therefore p = \frac{6+2}{2}$$

$$\therefore p = 4$$

The x-coordinate of M is seen to be the average of those of A and D. The y-coordinate of M may be found similarly.

$$q = \frac{5+1}{2}$$

$$\therefore q = 3$$

$\therefore$ the mid-point of AD is (4, 3).

In practice, of course, the working would be presented in shortened form thus:

the mid-point of AD is $\left(\frac{6+2}{2}, \frac{5+1}{2}\right)$, i.e. (4, 3)

Qu. 4 Find the coordinates of the mid-points of the straight lines joining the following pairs of points:
(a) A(4, 2) and B(6, 10), (b) C(−5, 6) and D(3, 2),
(c) E(−6, −1) and F(3, −4), (d) G(p, q) and H(r, s).

Exercise 1a

1 Find the lengths of the straight lines joining the following pairs of points:
(a) A(1, 2) and B(5, 2), (b) C(3, 4) and D(7, 1),
(c) E(−2, 3) and F(4, 3), (d) G(−6, 1) and H(6, 6),
(e) J(−4, −2) and K(3, −7), (f) L(−2, −4) and M(−10, −10).

2 Find the coordinates of the mid-points of the lines AB, CD, etc., in No. 1.

3 Find the distance of the point (−15, 8) from the origin.

4 P, Q, R are the points (5, −3), (−6, 1), (1, 8) respectively. Show that triangle PQR is isosceles, and find the coordinates of the mid-point of the base.

5 Repeat No. 4 for the points L(4, 4), M(−4, 1), N(1, −4).

6 A and B are the points (−1, −6) and (5, −8) respectively. Which of the following points lie on the perpendicular bisector of AB?
(a) P(3, −4), (b) Q(4, 0), (c) R(5, 2), (d) S(6, 5).

7 Three of the following four points lie on a circle whose centre is at the origin. Which are they, and what is the radius of the circle?

A(−1, 7), B(5, −5), C(−7, 5), D(7, −1).

8 A and B are the points (12, 0) and (0, −5) respectively. Find the length of AB, and the length of the median, through the origin O, of the triangle OAB.

The gradient of a straight line

1.4 Consider the straight line passing through A(1, 1) and B(7, 2) (Fig. 1.4). If we think of the x-axis as horizontal, and the line through A and B as a road, then someone walking from A to B would rise a vertical distance CB whilst at the same time he moves a horizontal distance AC.

The gradient of the road is CB/AC = (2 − 1)/(7 − 1) = 1/6. Instead of the two points A and B we might just as well have taken any other two points on the line, D and E; the gradient would then be expressed as FE/DF, which is the same as CB/AC, since the triangles ABC and DEF are similar.

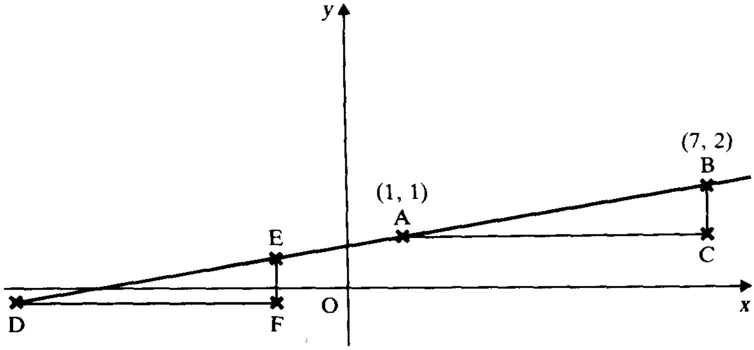

Figure 1.4

Definition

The gradient of a straight line is $\dfrac{\text{the increase in } y}{\text{the increase in } x}$ *in moving from one point on the line to another.*

In moving from A to B, since both x and y increase by positive amounts, the gradient is positive.

But now consider the gradient of PQ (Fig. 1.5). In moving from P to Q, the increase in x is $+2$, but since y decreases, we may say the increase in y is -3. Thus the gradient of PQ is $-\frac{3}{2}$.

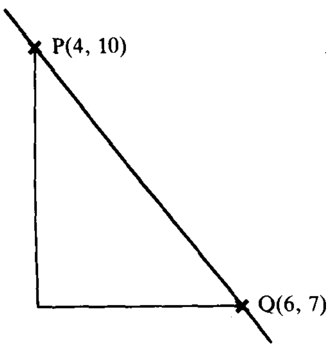

Figure 1.5

Until the reader is accustomed to the idea of positive and negative gradient it may help to think of it this way. In travelling along a line with x increasing (i.e.

moving from left to right across the page) if going *up*hill the gradient is *positive*: whereas if going *down*hill the gradient is *negative*. In calculating gradients a figure should not be necessary, but one similar to Fig. 1.5 will help in the first few examples.

Example 3 *Find the gradient of the line joining* R(4, 8) *and* S(5, −2).

$$\text{The gradient of RS} = \frac{y\text{-coord. of R} - y\text{-coord. of S}}{x\text{-coord. of R} - x\text{-coord. of S}}$$

$$= \frac{8-(-2)}{4-5}$$

$$= \frac{10}{-1} = -10$$

[Remember that the coordinates of R must appear first in the denominator *and* numerator (or second in both). In this case $\{8-(-2)\}/(4-5)$ and $(-2-8)/(5-4)$ both give the correct gradient.]

The gradient of the line joining A(2, 1) and B(2, 9) presents us with a problem. Proceeding as in Example 3, above, we might say that

$$\text{the gradient of AB} = \frac{y\text{-coord. of A} - y\text{-coord. of B}}{x\text{-coord. of A} - x\text{-coord. of B}}$$

$$= \frac{1-9}{0}$$

$$= \frac{-8}{0}$$

On the other hand we might also say that the gradient of AB

$$= \frac{y\text{-coord. of B} - y\text{-coord. of A}}{x\text{-coord. of B} - x\text{-coord. of A}}$$

$$= \frac{9-1}{0}$$

$$= \frac{+8}{0}$$

Now, what meaning should we attach to expressions like $-8/0$ and $+8/0$ and how can the line AB have two apparently different gradients? This illustrates just one of the difficulties which can arise when we attempt to divide by zero. Because it gives rise to many insuperable problems, division by zero is never allowed in mathematics; mathematicians say that an expression like 8/0 'does not exist'. So what are we to do about the gradient of the line AB? We have to accept that for a 'vertical line' such as AB, no numerical value can be given to its gradient; however, we can still say that 'AB is parallel to the y-axis'.

Qu. 5 Find the gradients of the lines joining the following pairs of points:
(a) (4, 3) and (8, 12), (b) $(-2, -3)$ and (4, 6),
(c) (5, 6) and (10, 2), (d) $(-3, 4)$ and $(8, -6)$,
(e) $(-5, 3)$ and (2, 3), (f) (p, q) and (r, s),
(g) $(0, a)$ and $(a, 0)$, (h) (0, 0) and (a, b).

Qu. 6 A and B are the points (3, 4) and (7, 1) respectively. Use Pythagoras' theorem to prove that OA is perpendicular to AB. Calculate the gradients of OA and AB, and find their product.

Qu. 7 Repeat Qu. 6 for the points A(5, 12) and B(17, 7).

Parallel and perpendicular lines

1.5 The gradient of a straight line was defined in §1.4; it may be proved that it is also the tangent of the angle between the line and the positive direction of the x-axis.

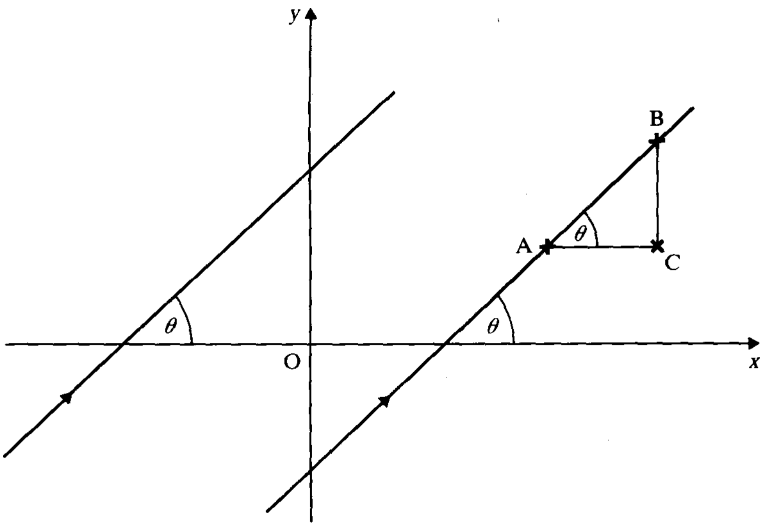

Figure 1.6

In Fig. 1.6 the gradient of AB is CB/AC, which is tan θ. The reader familiar with the tangent of an obtuse angle will appreciate that this covers negative gradient as well.

Since parallel lines make equal corresponding angles with the x-axis, *parallel lines have equal gradients.*

Qu. 6 and 7 of §1.4 will have led the reader to discover a useful property of the gradients of perpendicular lines. This we will now prove.

Consider the two straight lines AB and CD which cut at right angles at E. EF is drawn perpendicular to the x-axis (Fig. 1.7).

$$\alpha + \theta = 90^\circ$$
$$\alpha + \beta = 90^\circ$$
$$\therefore \theta = \beta$$

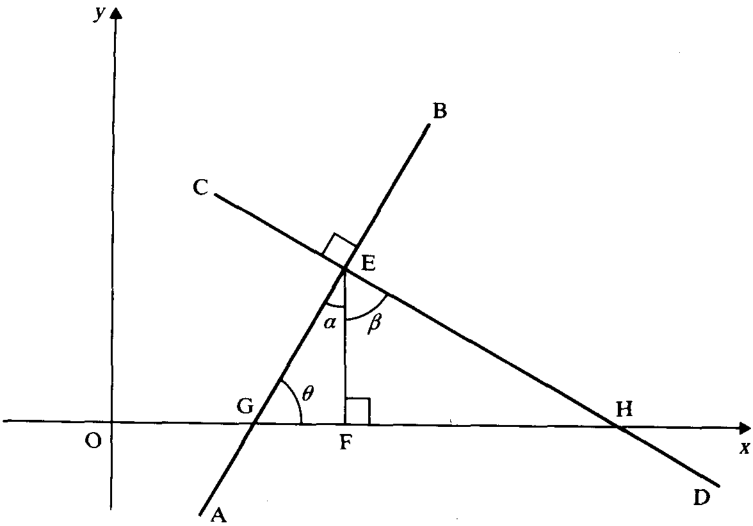

Figure 1.7

Let the gradient of AB be m, then

$$m = \frac{\text{FE}}{\text{GF}} = \tan\theta$$

$$\text{The gradient of CD} = -\frac{\text{FE}}{\text{FH}}$$
$$= -\frac{1}{\tan\beta}$$
$$= -\frac{1}{\tan\theta}$$
$$= -\frac{1}{m}$$

$\therefore$ the gradient of AB $\times$ the gradient of CD $= m \times \left(-\frac{1}{m}\right) = -1$

In general, *if two lines are perpendicular, the product of their gradients is* -1. Or in other words, if the gradient of one is m, the gradient of the other is $-1/m$.

Qu. 8 Write down the gradients of lines perpendicular to lines of gradient (a) 3, (b) $\frac{1}{4}$, (c) -6, (d) $-\frac{2}{3}$, (e) $2m$, (f) $-b/a$, (g) $-m/2$.

Qu. 9 Find if AB is parallel or perpendicular to PQ in the following cases:
(a) A(1, 4), B(6, 6), P(2, −1), Q(12, 3);
(b) A(−1, −1), B(0, 4), P(−4, 3), Q(6, 1);
(c) A(0, 3), B(7, 2), P(6, −1), Q(−1, −2).

The meaning of equations

1.6 The bare statement 'P is the point (x, y)' means that P can be anywhere in the plane. Previously, if we have been asked to find P, we have been given some data which enabled us to find one pair of numerical values for x and y, and so to fix the position of P.

Suppose however that the data is in the form of the equation $y = x^2 - 2x$. This does not give one pair of values for x and y, it gives as many as we like to find. But P is not now free to be anywhere in the plane, since for any chosen value of x there is only one corresponding value of y; P is now restricted to positions whose coordinates (x, y) satisfy the relationship $y = x^2 - 2x$.

The reader will be familiar with the process of making a table of values as shown below, in which certain suitable values of x are chosen, and the corresponding values of y calculated.

Table of values for $y = x^2 - 2x$

x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
x^2	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	$2\frac{1}{4}$	4	$6\frac{1}{4}$	9
$-2x$	2	1	0	-1	-2	-3	-4	-5	-6
y	3	$1\frac{1}{4}$	0	$-\frac{3}{4}$	-1	$-\frac{3}{4}$	0	$1\frac{1}{4}$	3

From this we find that the points $(-1, 3)$, $(-\frac{1}{2}, 1\frac{1}{4})$, $(0, 0)$, etc., have coordinates which satisfy the relationship $y = x^2 - 2x$, and by plotting these points and drawing a smooth curve through them (Fig. 1.8), we obtain all the possible

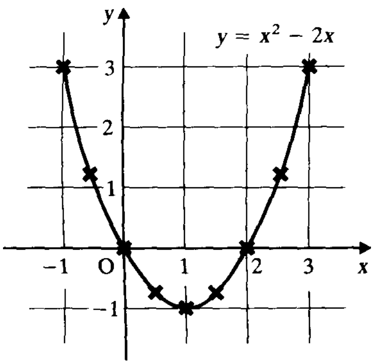

Figure 1.8

positions of P corresponding to the values of x from $x=-1$ to $x=3$.

Just as coordinates are used to name a point, so an equation is used to name a curve, and we refer to 'the curve $y=x^2-2x$'.

It must be stressed that the equation is the condition that the point (x, y) should lie on the curve.

Thus, only if $b=a^2-2a$ does the point (a, b) lie on the curve $y=x^2-2x$, and in that case we say that the coordinates of the point *satisfy* the equation.

If $q \neq p^2-2p$, the point (p, q) does *not* lie on the curve $y=x^2-2x$.

Example 4 *Do the points* $(-3, 9)$ *and* $(14, 186)$ *lie on the curve* $y=x^2$?

(a) The point $(-3, 9)$:
When $x=-3$, $y=x^2=(-3)^2=+9$,

$\therefore$ $(-3, 9)$ does lie on the curve $y=x^2$.

(b) The point $(14, 186)$:
When $x=14$, $y=x^2=14^2=196$,

$\therefore$ $(14, 186)$ does *not* lie on the curve $y=x^2$.

The next example illustrates another way of presenting this idea.

Example 5 *Does the point* $(-7, 6)$ *lie on the curve* $x^2-y^2=14$?

[We use L.H.S. as an abbreviation for 'the left-hand side' of the equation and R.H.S. for 'the right-hand side'.]

$$x^2-y^2=14$$

When $x=-7$ and $y=+6$,

$$\text{L.H.S.}=(-7)^2-6^2=49-36=13$$

$$\text{R.H.S.}=14$$

The coordinates of the point do not satisfy the equation. Therefore $(-7, 6)$ does *not* lie on the curve $x^2-y^2=14$.

Qu. 10 Find the y-coordinates of the points on the curve $y=2x^2-x-1$ for which $x=2, -3, 0$.

Qu. 11 Find the x-coordinates of the points on the curve $y=2x+3$ for which the y-coordinates are $7, 3, -2$.

Qu. 12 Find the points at which the curve in Qu. 10 cuts (a) the x-axis, and (b) the y-axis.

Qu. 13 Determine whether the following points lie on the given curves:

(a) $y=6x+7$, $(1, 13)$, (b) $y=2x+2$, $(13, 30)$,
(c) $3x+4y=1$, $(-1, \frac{1}{2})$, (d) $y=x^3-6$, $(2, -2)$,
(e) $xy=36$, $(-9, -4)$, (f) $x^2+y^2=25$, $(3, -4)$.

The relationship between a curve and its equation gives rise to two main groups of problems.

Firstly there are those problems in which we are given the equation, and from it we are required to find the curve. With this type the reader will already be familiar, in such work as the graphical solution of quadratic and other equations.

Secondly there are those problems in which we are given some purely geometrical facts about the curve, and from these we are required to discover the equation. It is this second type of problem with which we are now mainly concerned, but first we shall discuss a few more simple equations, to see what they represent.

$y = x$. This equation is satisfied by the coordinates of the points (0, 0), (1, 1), (2, 2), (3, 3), etc., and it is readily seen to represent a straight line through the origin. Its gradient is 1.

$x = 2$. Whatever the value of its y-coordinate, provided that its x-coordinate is 2, a point will lie on this curve. The points (2, 0), (2, 1), (2, 2), (2, 3), etc., lie on the curve, which is a straight line parallel to the y-axis, 2 units from it, on the side on which x is positive.

Qu. 14 Make a rough sketch of the lines represented by the following equations. Write down the gradient of each:
(a) $y = 3$, (b) $y = 2x$, (c) $y = 3x$, (d) $y = \frac{1}{2}x$, (e) $y = -x$.

The equation $y = mx + c$

1.7 We come now to the second type of problem mentioned above, in which from some geometrical facts about a curve we discover its equation. And the examples we do will, in turn, help us to interpret straight line equations more skilfully.

Example 6 *Find the equation of the straight line of gradient 4 which passes through the origin.*

If P(x, y) is any point on the line, other than O, the gradient of the line may be written y/x (Fig. 1.9).

$$\therefore \frac{y}{x} = 4$$

Hence $y = 4x$ is the required equation.

Qu. 15 Write down the equations of the straight lines through the origin having gradients (a) $\frac{1}{3}$, (b) -2, (c) m.

Qu. 16 Rearrange the following equations in the form $y = mx$, and hence write down the gradients of the lines they represent:
(a) $4y = x$, (b) $5x + 4y = 0$, (c) $3x = 2y$,
(d) $\frac{x}{4} = \frac{y}{7}$, (e) $\frac{x}{p} - \frac{y}{q} = 0$.

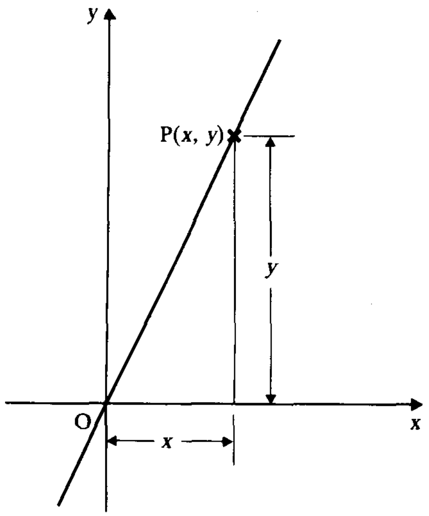

Figure 1.9

Example 7 *Find the equation of the straight line of gradient* 3 *which cuts the y-axis at* (0, 1).

Let P(x, y) be any point on the line other than (0, 1).
The gradient of the line may be written $(y-1)/x$ (Fig. 1.10).

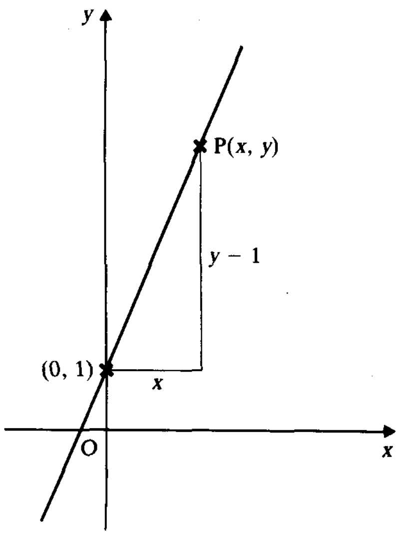

Figure 1.10

$$\therefore \frac{y-1}{x} = 3$$

Hence $y = 3x + 1$ is the required equation.

Qu. 17 By the method of Example 7, find the equations of the straight lines of given gradients cutting the y-axis at the named points:
(a) gradient 3, (0, 2), (b) gradient 3, (0, 4),
(c) gradient 3, (0, $-$1), (d) gradient $\frac{1}{5}$, (0, 2),
(e) gradient $\frac{1}{5}$, (0, 4).

If a straight line cuts the y-axis at the point $(0, c)$, the distance of this point from the origin is called the **intercept** on the y-axis.

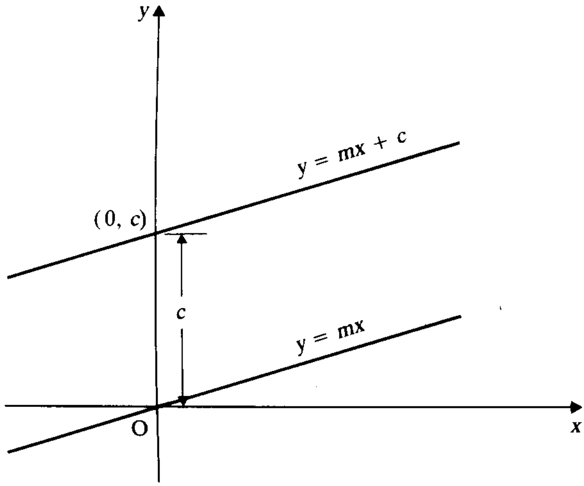

Figure 1.11

Then the equation of a straight line of gradient m, making an intercept c on the y-axis (Fig. 1.11) is

$$\frac{y-c}{x} = m$$

i.e. $\quad y = mx + c$

This line is parallel to $y = mx$, which passes through the origin, and it is m, the coefficient of x, which in each case determines the gradient. The effect of altering the value of the number c (c being the intercept on the y-axis) is to raise or lower the line, without altering its gradient; the sign of c determines whether the line cuts the y-axis above or below the origin.

We might be tempted to think at this stage that in $y = mx + c$ we have found the form in which all straight line equations may be written. But remember that on p. 6 we ran into trouble trying to find the gradient of a line parallel to the y-axis; for such a line it is impossible to find a numerical value for m, and the equation is $x = k$, where k is a constant.

The various straight line equations we have met are summarised below. It should be noted that only terms of the first degree in x and y and a constant term occur; this, in fact, is how we may recognise a straight line, or *linear*, equation.

$y = mx + c$ is a line of gradient m, passing through $(0, c)$.
$y = mx$ is a line of gradient m, passing through the origin.
$y = c$ is a line of zero gradient (i.e. parallel to the x-axis).
$x = k$ is a line parallel to the y-axis.

Example 8 *Find the gradient of the straight line* $7x + 4y + 2 = 0$, *and its intercepts on the axes.*

The equation may be written

$$4y = -7x - 2$$

or

$$y = -\tfrac{7}{4}x - \tfrac{1}{2}$$

This is now in the form $y = mx + c$, where $m = -\frac{7}{4}$, and $c = -\frac{1}{2}$, and we see that the gradient is $-\frac{7}{4}$, and that the intercept on the y-axis is $-\frac{1}{2}$. In fact, to find the intercepts on each axis it is better to go back to the original equation

$$7x + 4y + 2 = 0$$

To find the intercept on the y-axis:
putting $x = 0$, $4y + 2 = 0$, $\therefore\ y = -\frac{1}{2}$.
To find the intercept on the x-axis:
putting $y = 0$, $7x + 2 = 0$, $\therefore\ x = -\frac{2}{7}$.
The intercepts on the x-axis and y-axis are $-\frac{2}{7}$ and $-\frac{1}{2}$ respectively.

Qu. 18 Arrange the following equations in the form $y = mx + c$, hence write down the gradient of each line; also find the intercepts on the y-axis:
(a) $3y = 2x + 6$, (b) $x - 4y + 2 = 0$, (c) $3x + y + 6 = 0$,
(d) $7x = 3y + 5$, (e) $y + 4 = 0$, (f) $lx + my + n = 0$.

Qu. 19 Write down the equations of (a) the x-axis, (b) the y-axis, (c) a straight line parallel to the y-axis through $(4, 0)$, (d) a straight line parallel to the x-axis making an intercept of -7 on the y-axis.

Exercise 1b

1 Find the gradients of the straight lines joining the following pairs of points:
(a) $(4, 6)$ and $(9, 15)$, (b) $(5, -11)$ and $(-1, 3)$,
(c) $(-2\frac{1}{2}, -\frac{1}{2})$ and $(4\frac{1}{2}, -1)$, (d) $(7, 0)$ and $(-3, -2)$.

2 Show that the three given points are in each case collinear, i.e. they lie on the same straight line:
(a) $(0, 0)$, $(3, 5)$, $(21, 35)$, (b) $(-3, 1)$, $(1, 2)$, $(9, 4)$,
(c) $(-3, 4)$, $(1, 2)$, $(7, -1)$, (d) $(1, 2)$, $(0, -1)$, $(-2, -7)$.

3 Find the gradients of the straight lines which make the following angles with the x-axis, the angle in each case being measured anti-clockwise from the positive direction of the x-axis:
(a) $45°$, (b) $135°$, (c) $60°$, (d) $150°$.

4 Find if AB is parallel or perpendicular to PQ in the following cases:

(a) A(4, 3),	B(8, 4),	P(7, 1),	Q(6, 5);
(b) A(−2, 0),	B(1, 9),	P(2, 5),	Q(6, 17);
(c) A(8, −5),	B(11, −3),	P(1, 1),	Q(−3, 7);
(d) A(−6, −1),	B(−6, 3),	P(2, 0),	Q(2, −5);
(e) A(4, 3),	B(−7, 3),	P(5, 2),	Q(5, −1);
(f) A(3, 1),	B(7, 3),	P(−3, 2),	Q(1, 0).

5 Show that A(−3, 1), B(1, 2), C(0, −1), D(−4, −2) are the vertices of a parallelogram.

6 Show that P(1, 7), Q(7, 5), R(6, 2), S(0, 4) are the vertices of a rectangle. Calculate the lengths of the diagonals, and find their point of intersection.

7 Show that D(−2, 0), E($\frac{1}{2}$, $1\frac{1}{2}$), F($3\frac{1}{2}$, $-3\frac{1}{2}$) are the vertices of a right-angled triangle, and find the length of the shortest side, and the mid-point of the hypotenuse.

8 Find the y-coordinates of the points on the curve $y = x^2 + 1$ for which the x-coordinates are −3, 0, 1, 5. Find the coordinates of points on the curve whose y-coordinates are 5, and 17. Sketch the curve.

9 Find the coordinates of the points on the curve $y = x^3$ for which $x = -3$, −1, 1, 3; and also of the points for which $y = -8$, 0, +8. Sketch the curve.

10 Determine whether the following points lie on the given curve:
(a) $y = 3x - 5$, (−1, −8), (b) $5x - 2y + 7 = 0$, (1, −1),
(c) $y = x^3$, (−4, 64), (d) $x^2y = 1$, $(-2, \frac{1}{4})$.

11 Find the intercepts on the axes made by the straight line $3x - 2y + 10 = 0$. Hence find the area of the triangle enclosed by the axes and this line.

12 Find the coordinates of the points at which the following curves cut the axes:
(a) $y = x^2 - x - 12$, (b) $y = 6x^2 - 7x + 2$,
(c) $y = x^2 - 6x + 9$, (d) $y = x^3 - 9x^2$,
(e) $y = (x + 1)(x - 5)^2$, (f) $y = (x^2 - 1)(x^2 - 9)$.

13 Plot the following points on squared paper, and write down the equations of the straight lines passing through them, in the form $y = mx + c$:
(a) (−1, −1), (0, 0), (4, 4), (b) (−1, 1), (0, 0), (1, −1),
(c) (−4, −2), (0, 0), (8, 4), (d) (0, −4), (4, −2), (6, −1),
(e) (−5, 2), (−5, 0), (−5, −2), (f) (−3, 7), (3, 3), (6, 1).

14 Write down the equation of the straight line
(a) through (5, 11) parallel to the x-axis,
(b) which is the perpendicular bisector of the line joining (2, 0) and (6, 0),
(c) through (0, −10) parallel to $y = 6x + 3$,
(d) through (0, 2) parallel to $y + 8x = 0$,
(e) through (0, −1) perpendicular to $3x - 2y + 5 = 0$.

15 Find the equation of the straight line joining the origin to the mid-point of the line joining A(3, 2) and B(5, −1).

16 P(−2, −4), Q(−5, −2), R(2, 1), S are the vertices of a parallelogram. Find the coordinates of M, the point of intersection of the diagonals, and of S.

17 (a) Write down the gradient of the straight line joining (a, b) and (p, q). Write down the two conditions that these points should lie on the line $y = 7x - 3$. From these deduce the gradient of the line.

(b) Repeat for the line $3x + 2y - 1 = 0$, and check your result by writing the equation in the form $y = mx + c$.

The use of suffixes

1.8 When we wish to refer to points whose coordinates are not given, it is convenient to write them as

(x_1, y_1) read as 'x one, y one'
(x_2, y_2) read as 'x two, y two', etc.

It is important to write the number (the suffix) at the bottom of the letter, so as to avoid confusion between x_2 and x^2, x_3 and x^3, and so on. This is a suitable point at which to summarise some of the early results of this chapter, using this notation.

If A and B are the points (x_1, y_1) and (x_2, y_2) respectively,

the length of AB is $\sqrt{\{(x_1 - x_2)^2 + (y_1 - y_2)^2\}}$

the mid-point of AB is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

the gradient of AB is $\dfrac{y_1 - y_2}{x_1 - x_2}$ or $\dfrac{y_2 - y_1}{x_2 - x_1}$

the condition for A to lie on $ax + by + c = 0$ is

$$ax_1 + by_1 + c = 0$$

Finding the equation of a straight line

1.9 The method of Example 7 in §1.7 can of course be used to find the equation of any straight line provided (a) that we know one point through which the line passes, and (b) that we know, or can calculate, the gradient. Two examples will illustrate this.

Example 9 *Find the equation of the straight line of gradient* $-\frac{2}{3}$, *which passes through* $(-4, 1)$.

Let $P(x, y)$ be any point on the line other than $(-4, 1)$ (Fig. 1.12).
The gradient of the line may be written

$$\frac{y-1}{x-(-4)} = \frac{y-1}{x+4}$$

But the gradient is given as $-\frac{2}{3}$,

$$\therefore \frac{y-1}{x+4} = -\frac{2}{3}$$

$$\therefore 3(y-1) = -2(x+4)$$

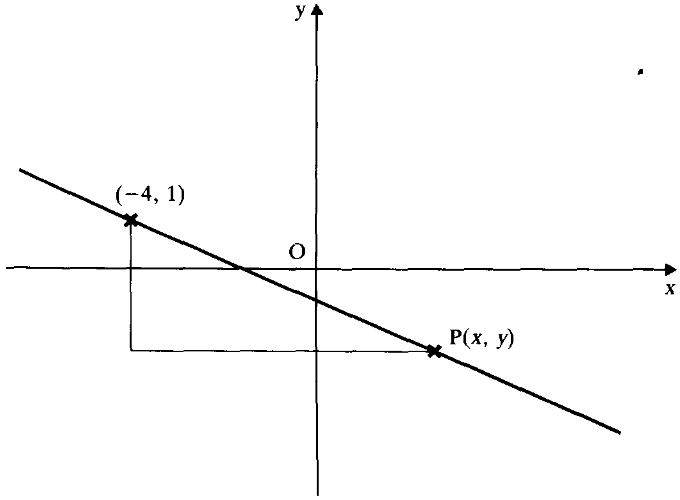

Figure 1.12

$$\therefore\ 3y - 3 = -2x - 8$$

Hence the required equation is $2x + 3y + 5 = 0$.

Example 10 *Find the equation of the straight line joining the points* $(-5, 2)$ *and* $(3, -4)$.

The gradient of the line $= \dfrac{2-(-4)}{-5-3} = \dfrac{6}{-8} = -\dfrac{3}{4}$.

If $P(x, y)$ is any point on the line other than $(3, -4)$, the gradient may be written

$$\frac{y-(-4)}{x-3} = \frac{y+4}{x-3}$$

$$\therefore\ \frac{y+4}{x-3} = -\frac{3}{4}$$

$$\therefore\ 4(y+4) = -3(x-3)$$

$$\therefore\ 4y + 16 = -3x + 9$$

Hence the required equation is $3x + 4y + 7 = 0$.

Examples 9 and 10 illustrate the most direct approach. The equation as first written is the direct statement of the condition that the point (x, y) should lie on the given line.

Another method is given below as an alternative solution to Example 9. We know that the equation $y = mx + c$ represents a straight line of gradient m; so the equation $y = -\frac{2}{3}x + c$ represents any line of gradient $-\frac{2}{3}$, according to the value of the constant c, and our problem is to find the appropriate value of c for the given line. To do this we use the fact that if the point (x_1, y_1) lies on the straight line $y = mx + c$, its coordinates satisfy the equation of the line, i.e. $y_1 = mx_1 + c$.

Example 9 (*Alternative solution*)

The equation is of the form $y = -\frac{2}{3}x + c$.
Since $(-4, 1)$ lies on this line,

$$1 = -\tfrac{2}{3}(-4) + c$$
$$\therefore\ c = 1 - \tfrac{8}{3} = -\tfrac{5}{3}$$

Hence the required equation is $y = -\frac{2}{3}x - \frac{5}{3}$, or $2x + 3y + 5 = 0$.

Qu. 20 Use the methods of Examples 9 (first solution) and 10 to find the equations of the straight lines
(a) through $(4, -3)$, of gradient $\frac{5}{2}$, (b) joining $(-3, 8)$ and $(1, -2)$.
Qu. 21 Using the method of Example 9 (alternative solution) find the equations of the straight lines
(a) through $(5, -2)$, of gradient $\frac{3}{2}$, (b) joining $(-2, 5)$ and $(3, -7)$.
Qu. 22 Write down the equation of the straight line through (x_1, y_1) of gradient m.

Points of intersection

1.10 If the two straight lines $x + y - 1 = 0$ and $2x - y - 8 = 0$ cut at the point $P(a, b)$ then the coordinates of P satisfy the equation of each line, and we may write

$$a + b - 1 = 0$$
$$2a - b - 8 = 0$$

The solution of these equations is $a = 3$, $b = -2$, which tells us that the given lines cut at $(3, -2)$. In practice we obtain the result by solving the equations simultaneously for x and y.

Qu. 23 Find the points of intersection of the following pairs of straight lines:
(a) $2x - 3y = 6$ and $4x + y = 19$, (b) $y = 3x + 2$ and $2x + 3y = 17$,
(c) $y = c$ and $y = mx + c$, (d) $x = -a$ and $y = mx + c$.
Qu. 24 Can you find the point of intersection of

$$3x - 2y - 10 = 0 \quad \text{and} \quad 4y = 6x - 7?$$

Qu. 25 Find the points of intersection of the curve $y = 12x^2 + x - 6$ and the x-axis.

Exercise 1c

1 Find the equations of the straight lines of given gradients, passing through the points named:
(a) $4, (1, 3)$, (b) $3, (-2, 5)$, (c) $\frac{1}{3}, (2, -5)$,

(d) $-\frac{3}{4}, (7, 5)$, (e) $\frac{1}{2}, (\frac{1}{3}, -\frac{1}{2})$, (f) $-\frac{5}{3}, (\frac{1}{4}, -3)$.

2 Find the equations of the straight lines joining the following pairs of points:
(a) $(1, 6)$ and $(5, 9)$, (b) $(3, 2)$ and $(7, -3)$,
(c) $(-3, 4)$ and $(8, 1)$, (d) $(-1, -4)$ and $(4, -3)$,
(e) $(\frac{1}{2}, 2)$ and $(3, \frac{1}{3})$, (f) $(-\frac{1}{2}, 0)$ and $(5, 11)$.

3 Find the points of intersection of the following pairs of straight lines:
(a) $x + y = 0$, $y = -7$,
(b) $y = 5x + 2$, $y = 3x - 1$,
(c) $3x + 2y - 1 = 0$, $4x + 5y + 3 = 0$,
(d) $5x + 7y + 29 = 0$, $11x - 3y - 65 = 0$.

4 Find the equation of the straight line
(a) through $(5, 4)$, parallel to $3x - 4y + 7 = 0$,
(b) through $(-2, 3)$, parallel to $5x - 2y - 1 = 0$,
(c) through $(4, 0)$, perpendicular to $x + 7y + 4 = 0$,
(d) through $(-2, -3)$, perpendicular to $4x + 3y - 5 = 0$.

5 Find the equation of the perpendicular bisector of AB, where A and B are the points $(-4, 8)$ and $(0, -2)$ respectively.

6 Repeat No. 5 for the points A(7, 3) and B(−6, 1).

7 Find the equation of the straight line joining A(10, 0) and B(0, −7). Also find the equation of the median through the origin, O, of the triangle OAB.

8 P, Q, R are the points $(3, 4)$, $(7, -2)$, $(-2, -1)$ respectively. Find the equation of the median through R of the triangle PQR.

9 Calculate the area of the triangle formed by the line $3x - 7y + 4 = 0$ and the axes.

10 Find the circumcentre of the triangle with vertices $(-3, 0)$, $(7, 0)$, $(9, -6)$. Show that the point $(1, 2)$ lies on the circumcircle.

11 Find the equation of the straight line through P(7, 5) perpendicular to the straight line AB whose equation is $3x + 4y - 16 = 0$. Calculate the length of the perpendicular from P to AB.

12 ABCD is a rhombus. A is the point $(2, -1)$, and C is the point $(4, 7)$. Find the equation of the diagonal BD.

13 L(−1, 0), M(3, 7), N(5, −2) are the mid-points of the sides BC, CA, AB respectively of the triangle ABC. Find the equation of AB.

14 Find the points of intersection of $x^2 = 4y$ and $y = 4x$.

15 The straight line $x - y - 6 = 0$ cuts the curve $y^2 = 8x$ at P and Q. Calculate the length of PQ.

Exercise 1d (Miscellaneous)

1 Find the equation of the line joining the points $(6, 3)$ and $(5, 8)$. Show also that these two points are equidistant from the point $(-2, 4)$.

2 What is the equation of the straight line joining the points A(7, 0) and B(0, 2)? Obtain the equation of the straight line AC such that the x-axis bisects the angle BAC.

3 Find the equations of the following straight lines, giving each in the form $ax + by + c = 0$:

(a) the line joining the points (2, 4) and (−3, 1),
(b) the line through (3, 1) parallel to the line $3x + 5y = 6$,
(c) the line through (3, −4) perpendicular to the line $5x - 2y = 3$.

4 Write down the condition that the straight lines

$$y = m_1x + c_1 \quad \text{and} \quad y = m_2x + c_2$$

should be at right angles. Find the equations of the straight lines through the point (3, −2) which are (a) parallel, and (b) perpendicular to the line $2y + 5x = 17$.

5 The points A, B, C have coordinates (7, 0), (3, −3), (−3, 3) respectively. Find the coordinates of D, E, F, the mid-points of BC, CA, AB respectively. Find the equations of the lines AD, BE, and the coordinates of K, their point of intersection. Prove that C, K, F are in a straight line.

6 Find the equation of the straight line
(a) joining the points (−3, 2) and (1, −4),
(b) through (−1, 3) parallel to the line $2x + 7y - 8 = 0$,
(c) through (2, −3) perpendicular to the line $5x - 2y - 11 = 0$.

7 Find the equations of the lines passing through the point (4, −2) and respectively (a) parallel, (b) perpendicular to the line $2x - 3y - 4 = 0$. Find also the coordinates of the foot of the perpendicular from (4, −2) to $2x - 3y - 4 = 0$.

8 A line is drawn through the point (2, 3) making an angle of 45° with the positive direction of the x-axis, and it meets the line $x = 6$ at P. Find the distance of P from the origin O, and the equation of the line through P perpendicular to OP.

9 Prove that the points (−5, 4), (−1, −2), (5, 2) lie at three of the corners of a square. Find the coordinates of the fourth corner, and the area of the square.

10 The vertices of a quadrilateral ABCD are A(4, 0), B(14, 11), C(0, 6), D(−10, −5). Prove that the diagonals AC, BD bisect each other at right angles, and that the length of BD is four times that of AC.

11 The coordinates of the vertices A, B, C of the triangle ABC are (−3, 7), (2, 19), (10, 7) respectively.
(a) Prove that the triangle is isosceles.
(b) Calculate the length of the perpendicular from B to AC, and use it to find the area of the triangle.

12 A triangle ABC has A at the point (7, 9), B at (3, 5), C at (5, 1). Find the equation of the line joining the mid-points of AB and AC; and find also the area of the triangle enclosed by the line and the axes.

13 One side of a rhombus is the line $y = 2x$, and two opposite vertices are the points (0, 0) and $(4\frac{1}{2}, 4\frac{1}{2})$. Find the equations of the diagonals, the co-ordinates of the other two vertices, and the length of the side.

14 Prove that the four points (4, 0), (7, −3), (−2, −2), (−5, 1) are the vertices of a parallelogram and find the equations of its diagonals.

15 Find the equation of the line which is parallel to the line $x + 4y - 1 = 0$, and which passes through the point of intersection of the lines $y = 2x$ and $x + y - 3 = 0$.

16 Find the equations of the lines which pass through the point of intersection of the lines $x - 3y = 4$ and $3x + y = 2$, and are respectively parallel and perpendicular to the line $3x + 4y = 0$.

17 The three straight lines $y = x$, $2y = 7x$, and $x + 4y - 60 = 0$ form a triangle. Find the equations of the three medians, and calculate the coordinates of their point of intersection.

18 The points D(2, -3), E(-1, 7), F(3, 5) are the mid-points of the sides BC, CA, AB respectively of a triangle. Find the equations of its sides.

19 Prove that the points (1, -1), (-1, 1), ($\sqrt{3}$, $\sqrt{3}$) are the vertices of an equilateral triangle. Find the coordinates of the point of intersection of the medians of this triangle.

20 The points A(-7, -7), B(8, -1), C(4, 9), D are the vertices of the parallelogram ABCD. Find the coordinates of D. Prove that ABCD is a rectangle and find its area.

21 Find the equation of the line which is parallel to the line $3x + 4y = 12$ and which makes an intercept of 5 units on the x-axis. Find also the equation of the line which is perpendicular to the given line and which passes through the point (4, 5).

22 A, B, C are the points (1, 6), (-5, 2), (3, 4) respectively. Find the equations of the perpendicular bisectors of AB and BC. Hence find the coordinates of the circumcentre of the triangle ABC.

23 Find the equation of the straight line joining the feet of the perpendiculars drawn from the point (1, 1) to the lines $3x - 3y - 4 = 0$ and $3x + y - 6 = 0$.

24 Through the point A(1, 5) is drawn a line parallel to the x-axis to meet at B the line PQ whose equation is $3y = 2x - 5$. Find the length of AB and the sine of the angle between PQ and AB; hence show that the length of the perpendicular from A to PQ is $18 \div \sqrt{13}$. Calculate the area of the triangle formed by PQ and the axes.

Chapter 2

Functions

Real numbers

2.1 Any student of mathematics who has progressed this far will be thoroughly familiar with the **real numbers**; they are the bricks and mortar of arithmetic. All the weighing, measuring and calculating that are used in commerce and science require the use of the real numbers. To the mathematician, they are the numbers, both positive and negative, which can be represented by points on the 'real number line'. Some of them are illustrated in Fig. 2.1.

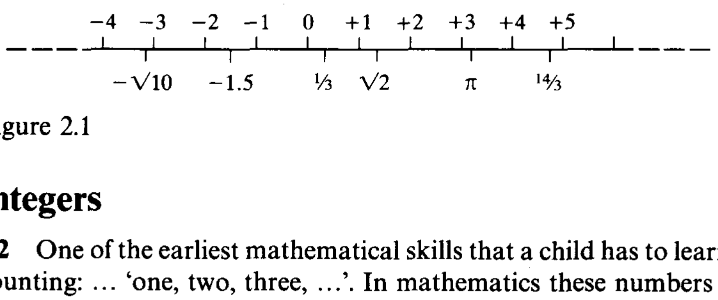

Figure 2.1

Integers

2.2 One of the earliest mathematical skills that a child has to learn is the skill of counting: ... 'one, two, three, ...'. In mathematics these numbers are called the **counting** or **natural** numbers. However, in order to develop mathematical ideas beyond very elementary arithmetic, it is necessary to extend the concept of natural numbers in two important directions. One of these is the extension to negative, as well as positive, numbers. Mathematicians refer to the positive and negative numbers, together with zero, as **integers**. An integer then is any number of the form $\ldots, -4, -3, -2, -1, 0, +1, +2, +3, \ldots$.

Rational numbers

2.3 The other important extension of the natural numbers is the idea of fractions, e.g. $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \ldots$. In mathematics we extend this idea still further to include fractions which are bigger than one, e.g. 7/5, 22/7 (these are often called improper fractions), and we also allow them to be positive or negative. The collective name for all such numbers is **rational numbers** (rational is the adjective derived from the noun 'ratio').

Unfortunately, but for mathematicians rather interestingly, that is not the end of the story; the rational numbers do not 'fill' the number line. There are points on the number line, that representing $\sqrt{2}$ for example, which do not represent rational numbers. In other words, some real numbers are not rational numbers. In the next section we shall prove that $\sqrt{2}$ is not rational.

Before we can do this, we must state clearly and unambiguously what we mean by a rational number. *A rational number is a number of the form a/b, in which a and b are integers with no common factor.* (If there is a common factor, it should be cancelled, e.g. 12/15 should be simplified to 4/5.) *The number b must not be zero.* Notice however that b can be 1; this enables us to regard any integer, including zero itself, as a rational number. An integer is simply a rational number whose denominator b is equal to 1. Notice that a can be larger than b; 5/3 is a perfectly acceptable rational number.

The irrationality of $\sqrt{2}$

2.4 The Greek mathematicians of the 4th century BC knew all about the theorem of Pythagoras so they knew that the hypotenuse of a right-angled triangle, whose other two sides have a length of 1 unit, would have a length of $\sqrt{2}$ units. They discovered the proof that $\sqrt{2}$ is irrational, which is expressed in modern terms below.

Firstly we assume that $\sqrt{2}$ *can* be expressed as a rational number. That is, we assume that two integers, a and b, with no common factor, can be found such that

$$\sqrt{2} = \frac{a}{b}$$

Multiplying both sides by b gives

$$\sqrt{2}b = a$$

and squaring both sides we have

$$2b^2 = a^2$$

This equation tells us that a^2 is a multiple of 2, that is, it is an even number. Now, the squares of even numbers are even and the squares of odd numbers are odd, so we can deduce that a itself is an even number. Consequently it can be written as $2c$, where c is a natural number. Substituting $2c$ for a in the last equation, we have

$$2b^2 = (2c)^2 = 4c^2$$

and dividing through by 2 gives

$$b^2 = 2c^2$$

As before we can now deduce that b^2, and hence b itself, is an even number.

Thus the initial assumption that $\sqrt{2}$ is a rational number has led us to the conclusion that both a and b are even numbers, that is, they have a common factor of 2. But a and b have *no* common factor, so we have contradicted ourselves. Now there are only two possible ways out of this *impasse*; either the argument is faulty (the reader should go through it again to satisfy himself that this is not the case) or the original assumption is false. Hence $\sqrt{2}$ is not a rational number.

This proof is an example of a very important type of argument called *reductio ad absurdum.*

With only minor amendments it can be adapted to prove that the square root of any prime number is irrational. If such a square root is multiplied by a rational number, the result is also irrational. Numbers such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{6}$ are often called **surds**; we shall return to these in Chapter 9.

There are other irrational numbers, π for example, but we shall not go into the details here. Readers who wish to know more should consult a more advanced mathematics book. In particular, they should look for the names Cantor (1845–1918) and Dedekind (1831–1916) who were largely responsible for investigating irrational numbers.

Qu. 1 Are the following statements true or false? If you think they are false explain clearly why you have come to this conclusion.
(a) All prime numbers are odd numbers.
(b) Any natural number can be expressed as a rational number.
(c) The square root of a natural number is an irrational number.
(d) $\pi = 22/7$, so π is a rational number.

Infinity

2.5 If you have a calculator, work out the value of $1/n$ for $n = 0.1, 0.001, 0.0001, 0.000\,0001$. (Even if you do not have a calculator it is easy to find the answers!) You should find that $1/n$ gets bigger and bigger as n gets smaller and smaller; we say that $1/n$ 'tends to infinity as n tends to zero'. The symbol ∞ is normally used for infinity. However the idea of 'infinity' is a very risky one for the unwary. Consider, for example the two lists

$$\begin{array}{cccccccc} 1, & 2, & 3, & 4, & 5, & 6, & \ldots \\ 1, & 4, & 9, & 16, & 25, & 36, & \ldots \end{array}$$

How many terms are there in each of these lists? One might say 'infinity', but look carefully; since each number in the second list is the square of the corresponding number in the first, one could claim that each list contains the same number of terms. Yet the second list clearly omits many of the terms which are in the first, so one could also claim that the second list contains fewer terms than the first. 'Infinity' then is a dangerous concept and should be handled with great care. Mathematicians, unless they are very brave or very foolish, usually try to dodge it. In particular they never divide by zero; instead they usually say that an expression like $1/0$ does not exist. Infinity itself is not a number.

Example 1 *Find the values of x for which the expression* $\dfrac{2x+5}{x^2-x-6}$ *does not exist.*

The expression does not exist if

$$x^2 - x - 6 = 0$$
$$(x-3)(x+2) = 0$$

i.e. either $x - 3 = 0$ or $x + 2 = 0$

The expression does not exist when $x = 3$ or -2.

Qu. 2 Find the values of x for which the following expressions do not exist:

(a) $\dfrac{x}{2x+5}$, (b) $\dfrac{1}{x^2+8x+15}$, (c) $\dfrac{x}{x^2-25}$, (d) $\dfrac{10}{x^2-3}$.

Sets

2.6 In the previous sections we have already encountered the need to refer to particular collections, or **sets**, of numbers. For the benefit of any reader who has not met the idea of a set in mathematics before, a set is any clearly defined collection of objects (in this chapter the objects will always be numbers, but in later chapters you will meet sets of other mathematical objects or **elements**). The members of a set may be defined by listing them, or by describing them carefully in words. It is usual to enclose the list of members of a set in curly brackets, e.g.

$\{2, 4, 6, 8\}$	is the set of even numbers less than ten
$\{2, 3, 5, 7\}$	is the set of prime numbers less than ten
$\{3, 6, 9, \ldots, 99\}$	is the set of multiples of three, less than a hundred

Notice that when the pattern has been clearly established, as in the last case, the three dots indicate that the pattern continues until the last term is reached. In some cases there may be no last term, for example the set of square numbers,

$$\{1, 4, 9, 16, 25, 36, \ldots\}$$

When listing the members of a set, an individual member is never repeated. Thus the set of prime factors of 1200 is $\{2, 3, 5\}$.

When we wish to indicate that a particular number belongs to a certain set, the symbol $\in$ is used. Thus if P is the set of prime numbers we may write

$$37 \in P$$

and this means '37 is a member of the set of prime numbers'. In contrast,

$$36 \notin P$$

means '36 is not a member of the set of prime numbers'.

The symbol : is often used in this context to mean 'such that'. Thus if we use $\mathbb{N}$ to indicate the set of natural numbers, the mathematical statement

$$A = \{x^3 : x \in \mathbb{N}\}$$

means 'A is the set whose members have the form x^3, where x is such that it belongs to the set of natural numbers'. Thus $A = \{1, 8, 27, 64, \ldots\}$. Or again,

$$B = \{3n^2 : n \in \mathbb{N}\}$$

means 'B is the set whose members have the form $3n^2$, where n is a member of the set of natural numbers', i.e. $B = \{3, 12, 27, 48, 75, \ldots\}$.

$$C = \{x : -3 \leqslant x \leqslant +3\}$$

means that C is the set which contains any real number x between -3 and $+3$, inclusive.

Some very important sets have standard symbols:

$\mathbb{N}$ is used for the set of natural numbers, $\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$,
$\mathbb{Z}$ is the set of integers, positive or negative, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, +1, +2, +3, \ldots\}$,
$\mathbb{Z}^+$ is the set of positive integers, $\mathbb{Z}^+ = \{+1, +2, +3, +4, +5, +6, \ldots\}$,
$\mathbb{Q}$ is the set of rational numbers, (see §2.3),
$\mathbb{R}$ is the set of real numbers.

In a later chapter you will meet $\mathbb{C}$, the set of complex numbers.

The algebra of sets

2.7 (Readers who have studied this in an elementary course may wish to omit this section; on the other hand readers who have not met it before may need to supplement the section with further exercises from a more elementary textbook.)

Given two sets A and B, the set consisting of all those elements which belong both to A and B is called the **intersection of** A **and** B. The symbol for it is $A \cap B$. Thus if

$$A = \{2, 4, 6, 8, 10, 12\} \quad \text{and} \quad B = \{3, 6, 9, 12\}$$

the intersection of A and B is the set $\{6, 12\}$ and we write

$$A \cap B = \{6, 12\}$$

The set consisting of those elements which belong to A or B, *or both*, is called the **union of** A **and** B and the symbol for it is $A \cup B$. (The symbol $\cup$ can be remembered as the initial letter of union.) It is important to remember that when we list the members of a set we never repeat any individual element. Thus in the case of the sets A and B in the previous paragraph,

$$A \cup B = \{2, 3, 4, 6, 8, 9, 10, 12\}$$

Example 2 *Given that A is the set of odd numbers less than 20, and B is the set of prime numbers less than 20, list the members of A, B, $A \cap B$, $A \cup B$.*

$$A = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$$
$$B = \{2, 3, 5, 7, 11, 13, 17, 19\}$$
$$A \cap B = \{3, 5, 7, 11, 13, 17, 19\}$$
$$A \cup B = \{1, 2, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$$

Notice that if P is the set of odd numbers and Q is the set of multiples of 2 then there would be no number which belongs to $P \cap Q$. Such a set, that is a set with no members, is called an **empty set**; the symbol for it is $\varnothing$. Thus in the example above we write $P \cap Q = \varnothing$. ($\varnothing$ is pronounced 'ur', as in hurt.)

Sometimes it is convenient to have a special symbol for *all* the elements which are involved in a particular topic, or in a particular question. The normal symbol used for this is $\mathscr{E}$; it is called the **universal set**. In this context, it is also frequently useful to have a symbol for all the elements of the universal set $\mathscr{E}$ which are not in a given set A. The symbol used for this A' and this set is called the **complement of set** A. For example, given that

$$\mathscr{E} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \quad \text{and that} \quad X = \{4, 8\}$$

the complement of X is the set

$$X' = \{1, 2, 3, 5, 6, 7, 9, 10\}$$

Notice that for any set P,

$$P \cap P' = \varnothing \quad \text{and} \quad P \cup P' = \mathscr{E}$$

If every member of a certain set H is also a member of a set K, then H is called a **subset** of K. For example, $\{2, 4, 6, 8\}$ is a subset of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and the symbol used for this purpose is $\subset$. Thus $H \subset K$ reads 'H is a subset of K'.

Finally, the notation $n(A)$ is used to denote 'the number of elements in set A'. Thus in Example 2 above, $n(A) = 10$, $n(B) = 8$, $n(A \cap B) = 7$ and $n(A \cup B) = 11$. Notice that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

The reader should think carefully about this equation and should be able to see that it is true for *any* sets A and B.

Exercise 2a

1 Given that $A = \{1, 2, 3, 4, 5\}$, list the members of the following sets:
(a) $\{x^2 : x \in A\}$, (b) $\{1/x : x \in A\}$,
(c) $\{2x : x \in A\}$, (d) $\{4x + 1 : x \in A\}$.

2 Given that $A = \{-3, -2, -1, 0, +1, +2, +3\}$ list the members of the following sets:
(a) $\{x^2 : x \in A\}$, (b) $\{x^3 - x : x \in A\}$,
(c) $\{x^4 : x \in A\}$, (d) $\{1/(x+5) : x \in A\}$.

3 In this question, $x \in \mathbb{Z}^+$. List the members of the following sets:
(a) $\{x^2 : x < 10\}$, (b) $\{10x - x^2 : x < 10\}$,
(c) $\{10 - x : x < 10\}$, (d) $\{x/2 : x < 10\}$.

4 Are the following statements true or false? If you think a statement is false, give a clear reason for your conclusion.
(a) All factors of an even integer are even.
(b) All the factors of an odd integer are odd.

(c) $\mathbb{Z} \subset \mathbb{Q}$.
(d) Any odd square number can be expressed in the form $4m+1$, where $m \in \mathbb{Z}^+$.

5 List the members of the set of real numbers for which the expression $\dfrac{1}{(x-1)(x-2)(x-3)}$ does not exist.

6 In this question, $\mathscr{E}$ is the set of positive integers less than 100 and the sets A and B are subsets of $\mathscr{E}$. A is the set of multiples of 5, and B is the set of multiples of 7.
(a) List the members of A, B, $A \cap B$, $A \cup B$.
(b) Describe in words the members of set $A \cap B$.
(c) Write down the values of $n(A)$, $n(B)$, $n(A \cap B)$ and $n(A \cup B)$. Verify that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

7 Given that $\mathscr{E}$ is the set of natural numbers less than or equal to 20, list the members of the following subsets of $\mathscr{E}$:
(a) A, the multiples of 3, (b) B, the multiples of 4,
(c) A', the complement of A, (d) B',
(e) $(A \cup B)'$, (f) $A' \cap B'$.
Comment on your answers.

8 Express as recurring decimals the rational numbers (a) 1/3, (b) 2/7, (c) 3/11.

9 Express the recurring decimal $0.\dot{7}$ as a rational number. (Hint: let $x = 0.7$ and consider $10x$.)

10 Express the following recurring decimals as rational numbers:
(a) $0.\dot{1}\dot{2}$, (b) $0.\dot{6}5\dot{7}$, (c) $0.\dot{4}2857\dot{1}$.

Functions

2.8 Consider the two exercises (1) and (2) below.

(1) *A stone is projected vertically upwards. Its height, h metres, after t seconds, is given approximately by the formula* $h = 20t - 5t^2$. *Use the formula to calculate its height after* 0, 1, 2, 3, 4 *seconds.*

The answers to this exercise are shown in the table below:

t	0	1	2	3	4
h	0	15	20	15	0

(2) *Given that* $x \in \{1, 2, 3, 4, 5\}$ *find the corresponding set of values of y, where y is given by the rule:*

(a) $y = x^2$, (b) $y = 1/x$, (c) $y = \sqrt{(5-x)}$.

The three answers to this exercise are

(a) $\{1, 4, 9, 16, 25\}$, (b) $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$, (c) $\{2, \sqrt{3}, \sqrt{2}, 1, 0\}$.

All exercises like these have certain features in common. In each case, a set of values is given for one of the variables. Then a rule is given and this is applied to the given set of numbers, to produce a set of values of the other variable. In mathematics there are standard terms which are used to describe these features. The variable for which the values are given (t in exercise (1), x in (2)) is called the **independent variable** and the set of values of the independent variable is called the **domain.** The rule which is applied to the independent variable is called the **function** and the variable which is produced by the rule is called the **dependent variable.** (In (1) h is the dependent variable and in (2) y is the dependent variable.) The set of values of the dependent variable is called the **range** of the function. In exercise (2) part (a), the range is the set $\{1, 4, 9, 16, 25\}$.

When these standard terms are used there are some important restrictions which must be observed in order to avoid certain difficulties and possible misunderstandings. In many instances the domain will be $\mathbb{R}$, the set of real numbers. However, it may be necessary to restrict $\mathbb{R}$, to exclude numbers to which the rule cannot be applied. For instance in exercise (2) part (b), x must not be zero, and in part (c) x must not be greater than 5 since this would require us to find the square root of a negative number. The other restriction, which must be observed, is that the rule must provide one and *only one* value of the dependent variable. There is no difficulty over this point in the exercises above, but suppose that the rule is 'y is the angle whose sine is x'. In this case, if $x = 0.5$, then y could be $30°$, $150°$, $390°$, ...; in fact this particular rule would produce infinitely many values of y for a given value of x. This difficulty has to be faced by the manufacturer of a pocket calculator. Since a calculator has a single display for showing numbers, it is only possible for a calculator to show *one* answer to a given calculation. In the case of finding an angle whose sine is given, the designer must use a standard convention for deciding which answer should appear in the display. Similarly, when we define a function, we must define it carefully so that it produces just one value of the dependent variable. Another example in which this difficulty could arise would be the rule 'y is the square root of x', because any positive value of x would yield *two* values of y (both $+5$ and -5 are square roots of 25). It should be noted however that there is a convention in mathematics that the square root sign $\surd$ is reserved for the positive square root only, that is $\sqrt{25} = +5$ (not -5). With this convention it is perfectly in order to regard $y = \sqrt{x}$ as a function.

The member of the range which corresponds to a certain member of the domain is usually called the **image** of that member, e.g. in (2) (a) above, 25 is the image of 5. Notice that there is no objection to having two distinct members of the domain with the same image, *vide* (1) above, in which both $t = 1$ and $t = 3$ have the image 15. The converse however is not allowed; a member of the domain must not have more than one image. When each member of the range has exactly one corresponding member of the domain the function is called a 'one-to-one function'. Thus if the domain is $\mathbb{R}$, the set of all real numbers, $y = x^3$ represents a one-to-one function, but $y = x^2$ does not. A function which is not one-to-one is said to be 'many-to-one'.

Qu. 3 For each of the rules below, state carefully the largest possible subset of $\mathbb{R}$ which would be a suitable domain. In each case describe the corresponding range.
(a) $y = 1/(x-3)$, (b) $y = \sqrt{(10-x)}$, (c) $y = \sqrt{(25-x^2)}$,
(d) $y = 1/(25-x^2)$, (e) $y = 1/(25+x^2)$.

Qu. 4 Which of the rules below represent functions (distinguish between one-to-one functions and many-to-one functions)? In each part, the domain is $\mathbb{R}$.
(a) $y = x^4$, (b) $y = x^5$, (c) $y^2 + x^2 = 25$, (d) $y = x^3 - x$.

The function notation

2.9 Sometimes we need to discuss several functions simultaneously and consequently a notation which enables us to distinguish between them can be very convenient. Suppose we have two functions, both having $\mathbb{R}$ as the domain, and suppose one of them squares each member of the domain and the other doubles each member of the domain. We write $\mathrm{f}(x)$ to represent the image of x under the function f; our first function would be represented by $\mathrm{f}(x) = x^2$ and the second by $\mathrm{g}(x) = 2x$. The usual letters to use for this purpose are f, g, h and their corresponding capital letters, but other letters may be used if desired. In the illustration above, $\mathrm{f}(5) = 25$ and $\mathrm{g}(5) = 10$. We can also write $\mathrm{f}(a) = a^2$, $\mathrm{f}(a+h) = (a+h)^2$, $\mathrm{g}(k) = 2k$, and so on.

Example 3 *Given that* $\mathrm{h}(x) = x^2 - x$, *find the values of* $\mathrm{h}(10)$, $\mathrm{h}(-3)$, $\mathrm{h}(\frac{1}{2})$, $\mathrm{h}(t+1)$, $\mathrm{h}(5k)$.

$$h(10) = 10^2 - 10 = 100 - 10 = 90$$
$$h(-3) = (-3)^2 - (-3) = 9 + 3 = 12$$
$$h(\tfrac{1}{2}) = (\tfrac{1}{2})^2 - (\tfrac{1}{2}) = \tfrac{1}{4} - \tfrac{1}{2} = -\tfrac{1}{4}$$
$$h(t+1) = (t+1)^2 - (t+1) = t^2 + 2t + 1 - t - 1 = t^2 + t$$
$$h(5k) = (5k)^2 - 5k = 25k^2 - 5k$$

There is an alternative to this notation, which can also be quite useful. In this notation the function $\mathrm{f}(x) = x^3$ is written

$$\mathrm{f}: x \mapsto x^3$$

This statement should be read 'f is a function which **maps** x onto x^3'. The function $\mathrm{g}(x) = 2x$ now becomes $\mathrm{g}: x \mapsto 2x$. When $x = 5$, we write $\mathrm{f}: 5 \mapsto 125$ and $\mathrm{g}: 5 \mapsto 10$.

Composite functions

2.10 In this section f and g will be used to represent the functions $\mathrm{f}(x) = x^2$ and $\mathrm{g}(x) = x + 5$. The domain of both functions will be $\mathbb{R}$.

Notice that $\mathrm{f}(3) = 9$ and $\mathrm{g}(9) = 14$. Thus if we start with $x = 3$ and apply to it first function f and then function g, we shall obtain the number 14. This could be written $\mathrm{g}(\mathrm{f}(3)) = \mathrm{g}(9) = 14$, but it is usually abbreviated to $\mathrm{gf}(3) = 14$ (alternatively, the notation $\mathrm{g} \circ \mathrm{f}(3) = 14$ may be used). Similarly $\mathrm{gf}(10) = 100 + 5 = 105$. In general

$$gf(x) = x^2 + 5$$

The function gf(x) is called a **composite function**. Notice that the order of the functions which make up a composite function is very important. With f and g defined as above,

$$fg(x) = (x+5)^2 = x^2 + 10x + 25$$

Remember that when a composite function is written down, the individual functions must be read from right to left.

Example 4 *Given that* F: $x \mapsto (10+x)$, G: $x \mapsto x^3$ *and* H: $x \mapsto x/2$, *write down the functions* (a) FG, (b) GF, (c) FGH.

(a) FG: $x \mapsto (10 + x^3)$
(b) GF: $x \mapsto (10 + x)^3$
(c) H: $x \mapsto (x/2)$
GH: $x \mapsto (x/2)^3$, hence GH: $x \mapsto x^3/8$
FGH: $x \mapsto 10 + x^3/8$

Example 5 *Given that* $f(x) = 25 - x^2$ *and that* $g(x) = \sqrt{x}$, *find, where possible, the values of* (a) gf(0), (b) gf(4), (c) gf(13).

(a) $f(0) = 25$, $gf(0) = g(25) = 5$,
(b) $f(4) = 9$, $gf(4) = g(9) = 3$,
(c) $f(13) = 25 - 169 = -144$, but we cannot evaluate $g(-144)$ because a negative number does not have a real square root.

Example 5(c) illustrates a difficulty which can arise when forming a composite function. If the domain of the function f(x), above, is $\mathbb{R}$ then its range is $\{y: y \in \mathbb{R}, y \leqslant 25\}$ and this includes negative numbers, which are not in the domain of the function $g(x) = \sqrt{x}$. This can only be avoided if we restrict the domain of f(x) to $\{x: x \in \mathbb{R}, -5 \leqslant x \leqslant +5\}$. In general, when a composite function gf(x) is formed, the range of the function f(x) must be a subset of the domain of the function g(x).

Some mathematicians insist that whenever a function is defined its domain should be explicitly stated and, strictly speaking, they are correct. However this soon becomes rather tedious and most people adopt the less rigid convention that, unless the domain has some special features that need comment, it may be assumed that the domain is intended to be $\mathbb{R}$; the reader is normally expected to use common-sense to exclude any members of the domain which give rise to obvious difficulties (e.g. square roots of negative numbers, fractions with a zero denominator). This is the convention which will generally be employed in this book, although in *this* chapter, the domain will be described in full.

The term **co-domain** is sometimes used for any set which contains the range. For example, the function $f(x) = x^2$ maps real numbers onto real numbers and so one can say the domain is $\mathbb{R}$ and the co-domain is $\mathbb{R}$, but since all the images are positive (or, to be precise, non-negative) the range is the set of non-negative real numbers.

Exercise 2b

1 Given that $g(x) = x^3 + 1$, find the values of
(a) $g(0)$, (b) $g(5)$, (c) $g(\frac{3}{4})$, (d) $g(-2)$.
2 The domain of the function $g(x) = 5x + 1$ is $\{0, 1, 2, 3, 4, 5\}$. Find its range.
3 The domain of the function $f(x) = x^2 + 1$ is $\mathbb{R}$. Find its range.
4 The domain of the function $f(x) = 1/(1 + x^2)$ is $\mathbb{R}$. Find its range.
5 The domain of the function $f(x) = 1/\sqrt{(25 - x)}$ is a subset of $\mathbb{R}$. Write down the largest possible set which is a suitable domain.

In Nos. 6–10 the domain is $\mathbb{R}$.

6 Given that $f: x \mapsto 5x + 1$ and that $g: x \mapsto x^2$, express the composite functions fg and gf in their simplest possible forms.
7 Given that $f(x) = x^2$, express as simply as possible
(a) $f(5 + h)$, (b) $\dfrac{f(5 + h) - f(5)}{h}$, $(h \neq 0)$.
8 If $f(x) = x^2$ express as simply as possible

$$\frac{f(a + h) - f(a)}{h}, \quad (h \neq 0).$$

9 Given that $f(x) = x^3$ find
(a) $f(2)$, (b) $f(-10)$, (c) $f(\frac{1}{2})$,
(d) $f(5a)$, (e) $f(a/3)$, (f) $f(a + h)$,
(g) $f(a + h) - f(a - h)$, (h) $\dfrac{f(a + h) - f(a - h)}{2h}$, $(h \neq 0)$.
10 If $f(x) = 7x$ and $g(x) = x + 3$ and $fg: x \mapsto y$, express as simply as possible the rule which maps x onto y. Find the values of p, q, r such that
(a) $fg: 5 \mapsto p$, (b) $fg: 10 \mapsto q$, (c) $fg: r \mapsto 35$.
Find also the function, F, which reverses the function fg, that is, it maps y onto x.

Graphs of functions

2.11 When the domain is the set of real numbers $\mathbb{R}$, it is always represented by the horizontal axis, and the corresponding values of the dependent variable are represented by points on the vertical axis. When x and y are used to represent typical members of the domain and the co-domain, these axes are called the x-axis and the y-axis respectively. Fig. 2.2 shows the graph of a function $y = f(x)$. A typical member a of the domain and its image $f(a)$ are shown.

Bearing in mind that each member of the domain has exactly one image in the co-domain, a graph like the one shown in Fig. 2.3 does not represent the graph of a function. In this diagram, a has three possible images in the co-domain.

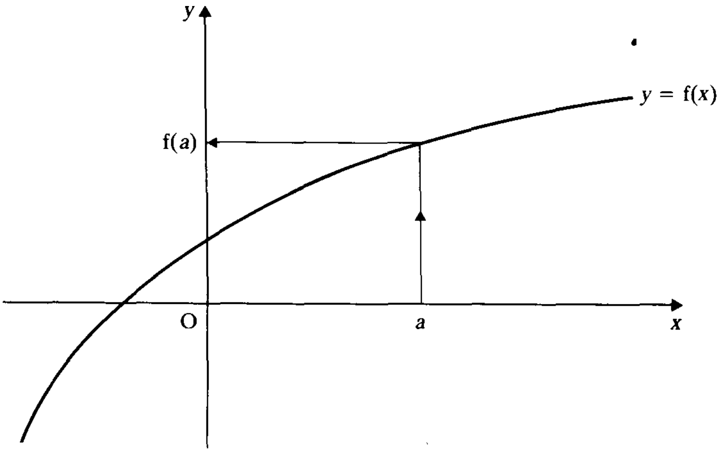

Figure 2.2

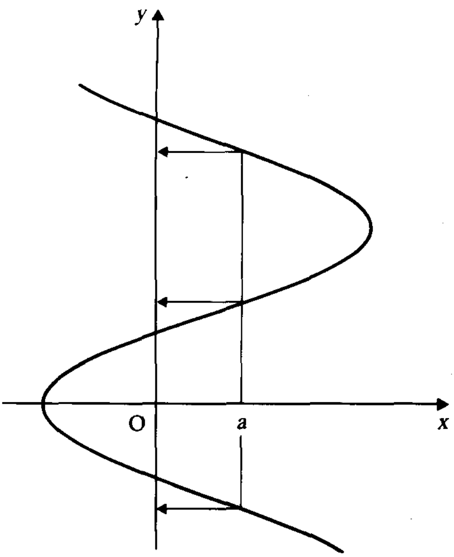

Figure 2.3

Although this is a purely artificial example, made up to illustrate the point, consider the circle, centre (0, 0), radius 10. The coordinates of any point P on the circle satisfy the equation $x^2 + y^2 = 100$, so a relation exists between the values of x and y at each point, and a graph can be drawn, but this is not the graph of a function because there are values of x for which there are two possible values of y, e.g. when $x = 6$, $y = +8$ or $y = -8$ (see Fig. 2.4).

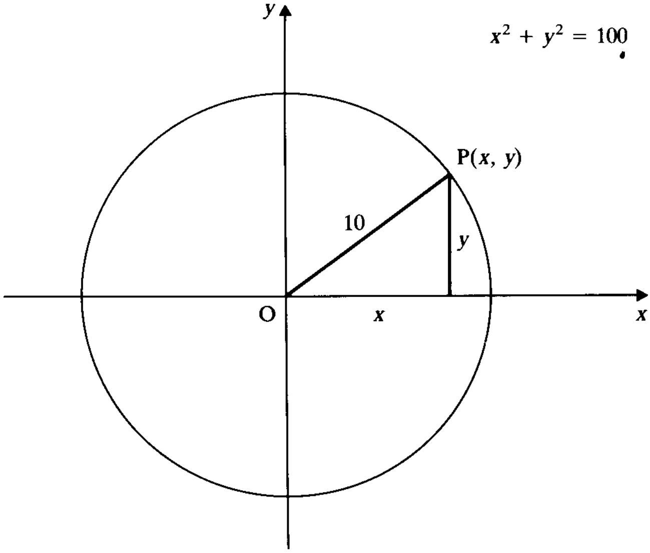

Figure 2.4

Graphs of some common functions

2.12 Fig. 2.5 illustrates the graphs of $y = x^2$, $y = x^3$ and $y = x^4$; any reader who is not familiar with these already is advised to draw and save graphs of these functions. (Start by plotting the values of y corresponding to values of x from $x = -2$ to $+2$, at intervals of $\frac{1}{4}$ of a unit.) Note that all of these graphs pass through the point (1, 1); $y = x^2$ and $y = x^4$ also pass through $(-1, 1)$, while $y = x^3$ passes through $(-1, -1)$. Notice also that the graphs 'flatten out' between $x = -1$ and $x = +1$, as the power increases. (Try plotting $y = x^{10}$: a calculator may be needed for some of the calculations.)

Fig. 2.6 shows the graphs of $y = 1/x$, $(x \neq 0)$ and $y = \sqrt{x}$, $(x \geqslant 0)$. (Remember that the square-root sign means the positive square root.)

Any reader who is not familiar with these graphs is advised to make careful copies, using a calculator where necessary, and to save the graphs for future reference. Notice also that if the functions are changed to $y = 1/(x - 2)$, $(x \neq 2)$, and $y = \sqrt{(x - 2)}$, $(x \geqslant 2)$, then the shape of the graphs is unaltered but the graph is translated 2 units to the right. In general, the graph of $y = f(x - a)$ will have the same shape as $y = f(x)$ but it will be translated a units to the right.

The modulus of x, written $|x|$, is probably new to many readers; *the modulus of x is the magnitude of x*, thus $|+5| = +5$ and $|-7| = +7$. A table of values of $|x|$ for $x = -4$ to $x = +4$ is shown below:

x	-4	-3	-2	-1	0	$+1$	$+2$	$+3$	$+4$
$\lvert x\rvert$	$+4$	$+3$	$+2$	$+1$	0	$+1$	$+2$	$+3$	$+4$

and the graph of $y = |x|$ is shown in Fig. 2.7.

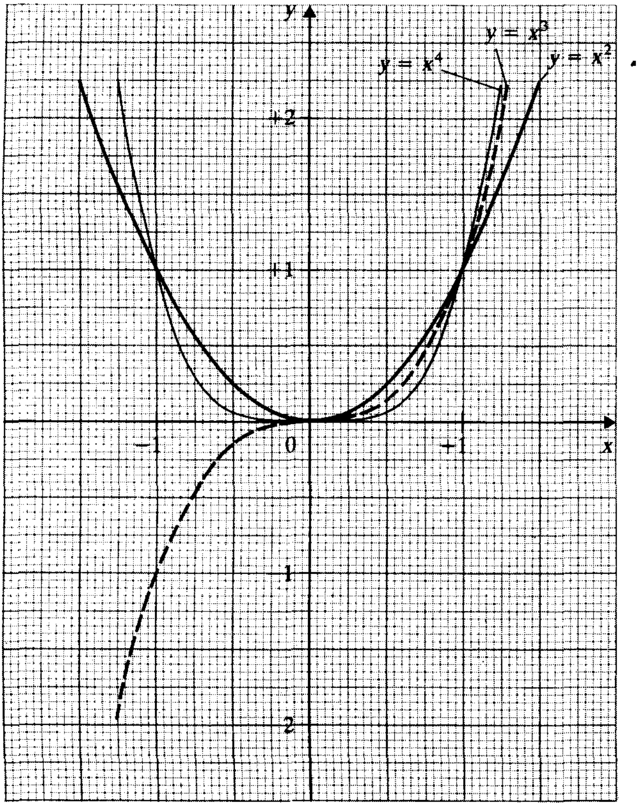

Figure 2.5

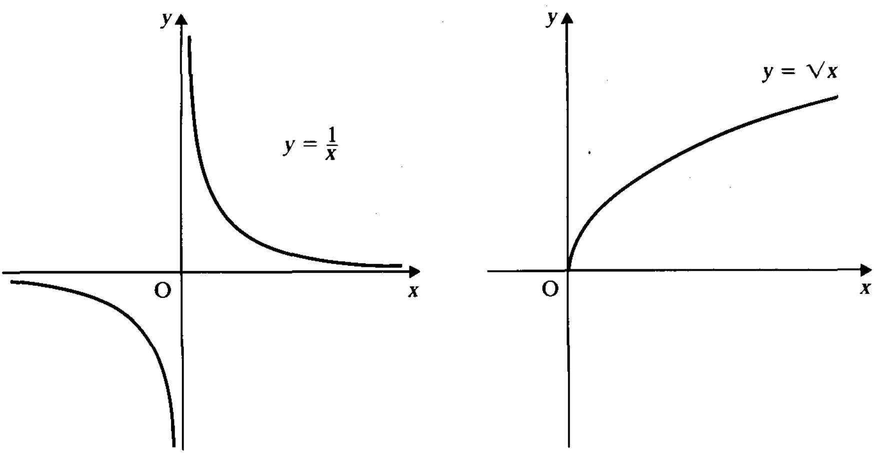

Figure 2.6

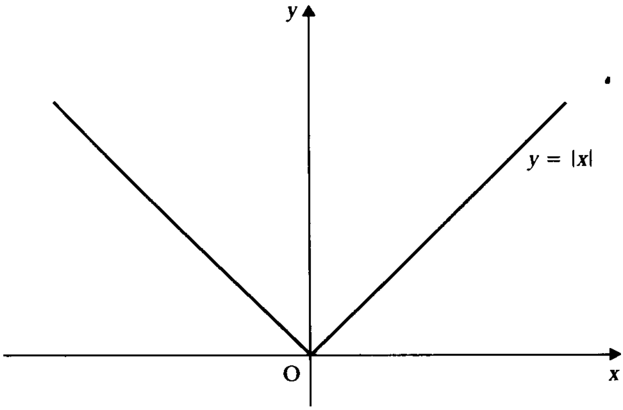

Figure 2.7

The instructions 'plot the graph of ...' and 'sketch the graph of ...' have very definite, but distinct, meanings in mathematics. The instruction 'plot the graph of $y = x^2$, for $x = 0$ to 5' means that the necessary values of y should be calculated, the points should be accurately plotted on graph paper and the points should be joined with a neat smooth curve. In contrast, a *sketch* of a curve should not be done on graph paper; plain, or ordinary lined paper should be used. Only a few points need to be plotted, but points which have special importance should be marked. The sketch should not be limited to a small part of the domain. Instead, every effort should be made to convey the overall appearance of the graph throughout its domain.

Example 6 *Sketch the graph of* $y = \dfrac{1}{x-3} + 2, \qquad (x \neq 3).$

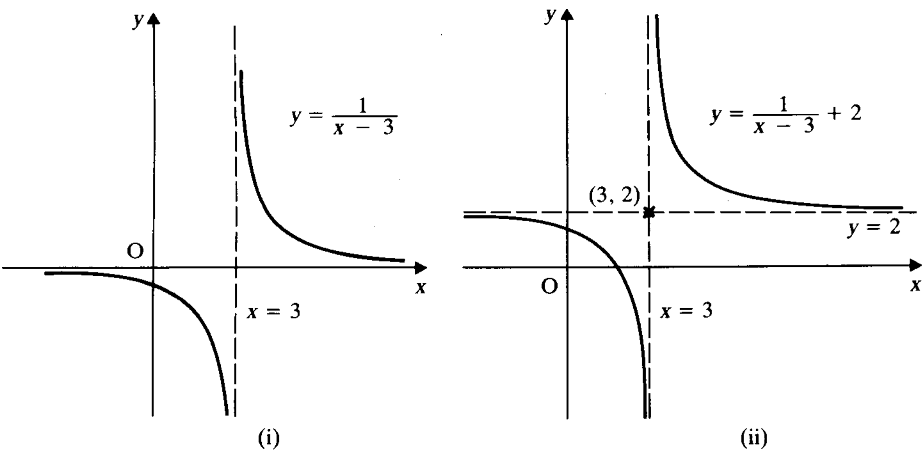

Figure 2.8

The graph of $y = 1/x$ is one of the standard graphs; a sketch of it is shown in Fig. 2.6. Replacing x by $x - 3$ translates the graph 3 units to the right, and so a sketch of $y = 1/(x - 3)$ would be like Fig. 2.8(i). When the final 2 is added to $1/(x - 3)$ the graph is translated 2 units vertically upwards. Hence the sketch graph of $y = 1/(x - 3) + 2$ should look like Fig. 2.8(ii).

Exercise 2c

Sketch (detailed plotting is not required) the graphs of the following functions. Where possible, the sketch should be obtained by modifying one of the standard graphs in the preceding section.

1 $y = 2x + 1$.
2 $y = (x + 2)^3$.
3 $y = x^2 + 5$.
4 $y = 1/(x + 4)$, $(x \neq -4)$.
5 $y = -x^2$.
6 $y = 5x^2$.
7 $y = \sqrt{(10 - x)}$, $(x \leqslant 10)$.
8 $y = 1/x^2$, $(x \neq 0)$.

So far x has always been used for the independent variable and y for the dependent variable, but x and y are not the only letters which may be used. In Nos. 9–15 t is used for the independent variable, hence the t-axis is horizontal, and z is used for the dependent variable.

9 $z = (t - 4)^3$.
10 $z = 100 - t^2$, $(-10 \leqslant t \leqslant +10)$.
11 $z = |t - 3|$.
12 $z = |(t + 4)(t - 4)|$.
13 $z = |1/(1 + t)|$.
14 $z = 1/(1 + |t|)$.
15 $z = |t| - |t + 1|$.
16 Fig. 2.9 shows the graph of an unspecified function $y = f(x)$. Trace the

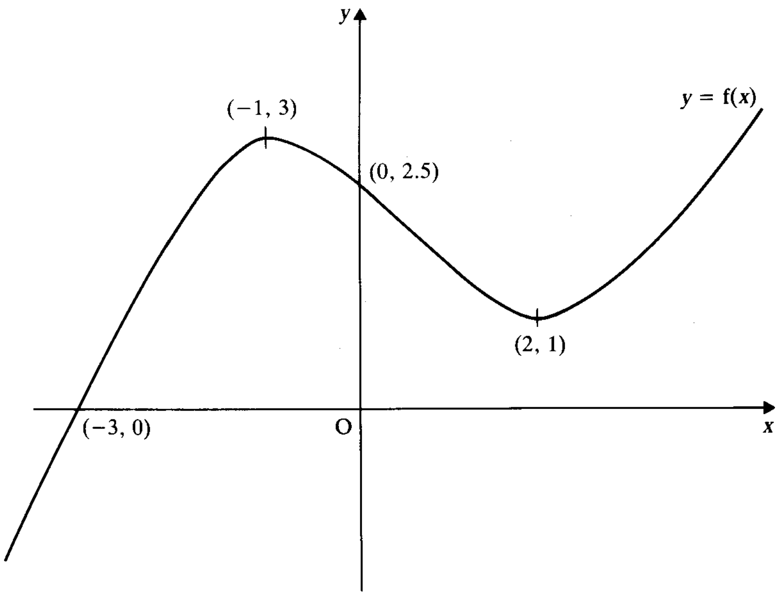

Figure 2.9

diagram and use the tracing to show, on a single diagram, sketch graphs of
(a) $y = f(x-6)$, (b) $y = f(x+3)$, (c) $y = f(x) + 2$.

17 Use the tracing from No. 16 to draw the graph of $y = f(x)$ and superimpose on it sketches of the following graphs, showing clearly their relationship to the graph of $y = f(x)$:
(a) $y = 5f(x)$, (b) $y = f(5x)$, (c) $y = -f(x)$, (d) $y = f(-x)$.

***18** Describe, in words, the appearance of the following graphs, relative to the graph of $y = f(x)$:
(a) $y = f(x-a)$, (b) $y = f(x) + a$, (c) $y = k \times f(x)$,
(d) $y = -f(x)$, (e) $y = f(-x)$.

Further functions

2.13 Example 7 *In* 1981 *the cost of posting a parcel, weighing not more than* 10 kg, *was given by the table below. Explain why this table expresses the cost of the parcel as a function of its weight and draw a graph of the function.*

Not over	Cost	Not over	Cost
1 kg	£1.10	6 kg	£2.20
2 kg	£1.43	7 kg	£2.35
3 kg	£1.73	8 kg	£2.45
4 kg	£1.90	9 kg	£2.55
5 kg	£2.05	10 kg	£2.65

The table expresses the cost as a function of the weight because if the weight is known, the table indicates the cost of postage. A function is any rule which enables the dependent variable to be found, when the independent variable is known. It is not necessary to express the rule as a formula. The graph is shown in Fig. 2.10.

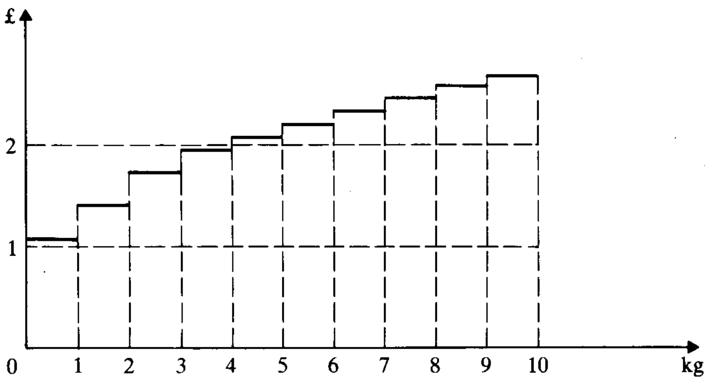

Figure 2.10

The function in Example 7 differs from the functions discussed earlier in the chapter, in that different rules apply to different parts of the domain. Many of

The graph of $y = 1/x$ is one of the standard graphs; a sketch of it is shown in Fig. 2.6. Replacing x by $x - 3$ translates the graph 3 units to the right, and so a sketch of $y = 1/(x - 3)$ would be like Fig. 2.8(i). When the final 2 is added to $1/(x - 3)$ the graph is translated 2 units vertically upwards. Hence the sketch graph of $y = 1/(x - 3) + 2$ should look like Fig. 2.8(ii).

Exercise 2c

Sketch (detailed plotting is not required) the graphs of the following functions. Where possible, the sketch should be obtained by modifying one of the standard graphs in the preceding section.

1 $y = 2x + 1$.
2 $y = (x + 2)^3$.
3 $y = x^2 + 5$.
4 $y = 1/(x + 4)$, $(x \neq -4)$.
5 $y = -x^2$.
6 $y = 5x^2$.
7 $y = \sqrt{(10 - x)}$, $(x \leqslant 10)$.
8 $y = 1/x^2$, $(x \neq 0)$.

So far x has always been used for the independent variable and y for the dependent variable, but x and y are not the only letters which may be used. In Nos. 9–15 t is used for the independent variable, hence the t-axis is horizontal, and z is used for the dependent variable.

9 $z = (t - 4)^3$.
10 $z = 100 - t^2$, $(-10 \leqslant t \leqslant +10)$.
11 $z = |t - 3|$.
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13 $z = |1/(1 + t)|$.
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15 $z = |t| - |t + 1|$.
16 Fig. 2.9 shows the graph of an unspecified function $y = f(x)$. Trace the

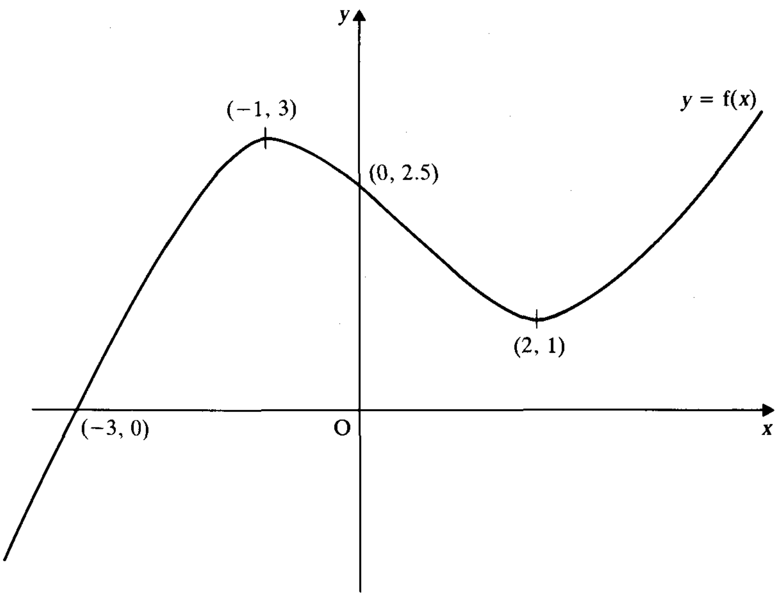


Figure 2.9

the functions which arise from real life problems are like this. When a multi-stage rocket is fired, the function which expresses its velocity in terms of time will have different mathematical formulae corresponding to each stage of the rocket motor. Examples 8 and 9 illustrate further functions which display this characteristic.

Example 8 *The domain of function* f *is* $\mathbb{R}$.

$$f: x \mapsto 1 \quad \text{when } x < 0, \qquad \text{and}$$
$$f: x \mapsto x^2 + 1 \quad \text{when } x \geqslant 0.$$

Sketch the graph of this function. (See Fig. 2.11.)

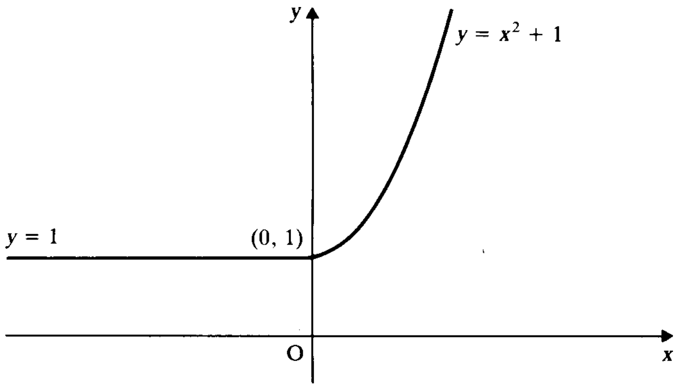


Figure 2.11

Example 9 *The domain of the function* f *is* $\mathbb{R}$.

$$f: x \mapsto 1, \quad \text{if } x \in \mathbb{Z}, \qquad \text{and}$$
$$f: x \mapsto 2, \quad \text{if } x \notin \mathbb{Z}.$$

Write down $f(+5)$, $f(-1)$, $f(0)$, $f(3.4)$, $f(\sqrt{2})$ *and* $f(\pi)$.
Sketch the graph of $y = f(x)$.

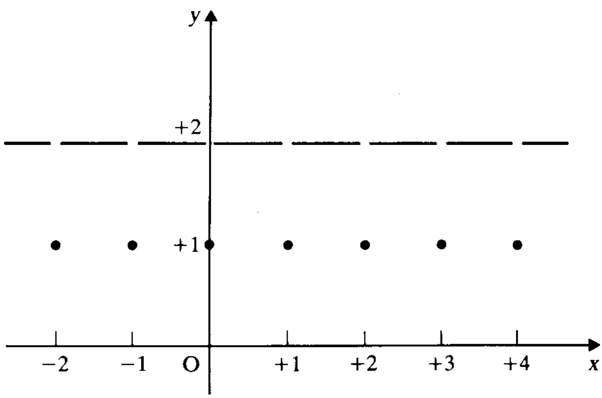


Figure 2.12

Functions of the form $f(x)=x^n$, where n is an odd number, will be odd functions. Another important odd function is $f(x)=\sin x$ (see Chapter 16).

Example 10 *Prove that the sum of two even functions is an even function and that the sum of two odd functions is an odd function.*

Let $f(x)$ and $g(x)$ be two even functions. Then $f(x)$ and $g(x)$ have the property $f(-a)=f(a)$ and $g(-a)=g(a)$, for any member a of the domain.

Let $F(x)$ be the sum of $f(x)$ and $g(x)$, that is $F(x)=f(x)+g(x)$. Then if a is any member of the domain

$$\begin{aligned} F(-a) &= f(-a)+g(-a) \\ &= f(a)+g(a) \\ &= F(a) \end{aligned}$$

hence $F(x)$ is an even function.

Similarly if $f(x)$ and $g(x)$ are odd functions, then

$$\begin{aligned} F(-a) &= f(-a)+g(-a) \\ &= -f(a)-g(a) \\ &= -F(a) \end{aligned}$$

hence $F(x)$ is an odd function.

Qu. 5 Prove that the product of two even functions is an even function.
Qu. 6 Prove that the product of two odd functions is an even function.
Qu. 7 Is the product of an even function and an odd function odd or even?

Periodic functions

2.15 A function whose graph repeats itself at regular intervals is called a **periodic function** (see Fig. 2.15). Such functions are especially important in science. The sound wave of a note of constant pitch, for example, is periodic.

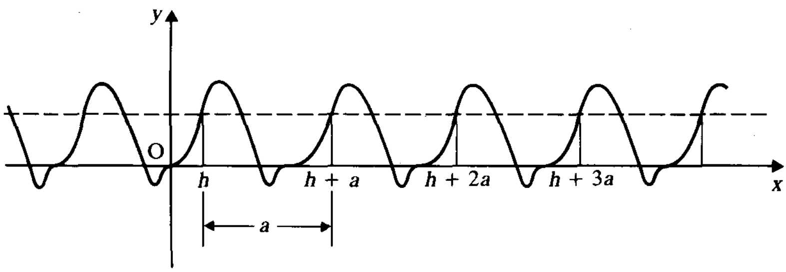


Figure 2.15

The length of the interval between repeats is called the **period** of the function. If the period is a, then for any value of h in the domain of the function,

$$f(h+a)=f(h)$$

The most common periodic functions are the trigonometric functions (see Chapter 16) sin x and cos x; they have a period of 360°, because

$$\sin(x+360)^\circ = \sin x^\circ \quad \text{and} \quad \cos(x+360)^\circ = \cos x^\circ.$$

Example 11 *Sketch the graph of the periodic function such that* $f(x)=x$, *for* $-1<x\leqslant +1$, *where the period of* $f(x)$ *is* 2.

Between $x=-1$ and $x=+1$, the graph is the ordinary straight line $y=x$. Outside this interval, the graph repeats itself every 2 units (Fig. 2.16).

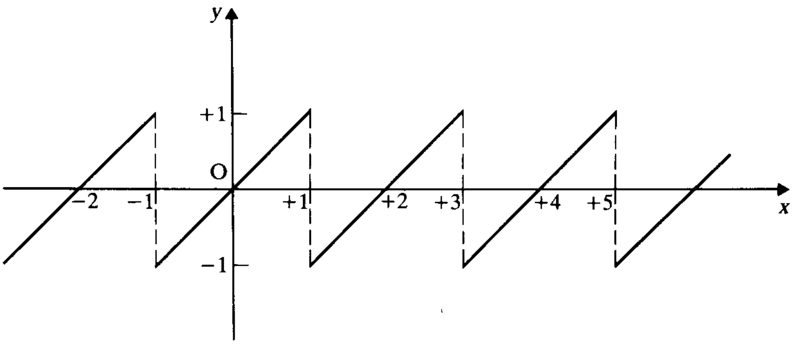


Figure 2.16

Example 12 *Sketch the graph of* $y=f(x)$ *where* $f(x)=\sqrt{(1-x^2)}$, *when* $0\leqslant x\leqslant 1$, *and* $f(x)$ *is an even function with a period of* 2.

The equation $y=\sqrt{(1-x^2)}$ produces an arc of a circle between $x=0$ and $x=+1$. Because the function is even, the graph is symmetrical about the vertical axis. Thus between $x=-1$ and $x=+1$, the graph is a semi-circle, and this is then repeated at regular intervals of 2 units (Fig. 2.17).

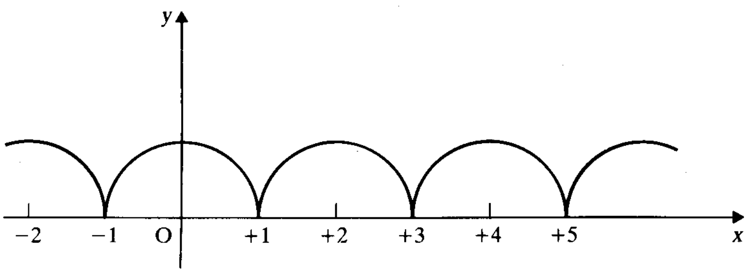


Figure 2.17

The inverse of a function

2.16 Consider the function $y=f(x)$, where $f(x)=\frac{1}{8}x^3+1$. A sketch of its graph is shown in Fig. 2.18.

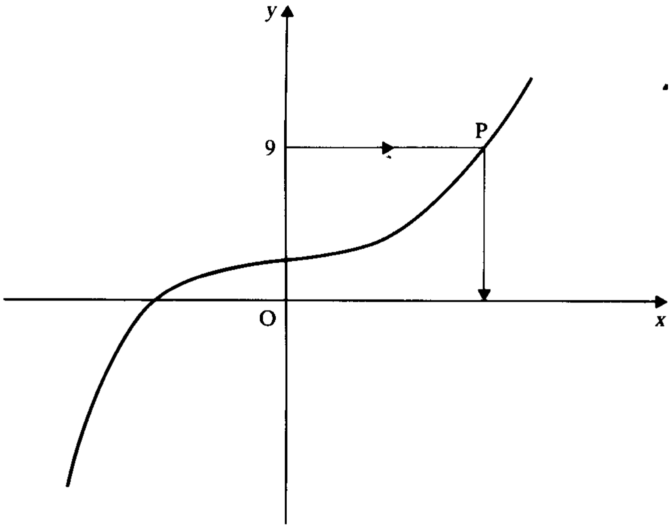


Figure 2.18

If we are given a member of the range, say $y = 9$, is it possible to find the corresponding member of the domain? On the graph this would mean starting from $y = 9$ on the vertical axis, drawing a line horizontally to the point P on the curve and then drawing a vertical line down to the x-axis. The point where the line meets the axis gives the value of x which is required. In this particular example it is fairly easy to solve the problem algebraically. The value of x required is found by solving the equation

$$\tfrac{1}{8}x^3 + 1 = 9$$

$$\tfrac{1}{8}x^3 = 8$$

$$x^3 = 64$$

$$\therefore\ x = 4$$

Indeed it is quite simple to generalise this. Starting with the given value from the range of function f, we first subtract 1, then we multiply by 8 and finally we find the cube root. The whole operation is called the inverse of function f and it is written f^{-1}. Following the usual convention of writing x for a typical member of the domain of function f^{-1}, we can write our inverse function as follows:

$$f^{-1}(x) = \sqrt[3]{\{8(x-1)\}}$$

Thus

$$f^{-1}(9) = \sqrt[3]{\{8(9-1)\}} = \sqrt[3]{(8 \times 8)} = \sqrt[3]{64} = 4$$

There is however one problem; when we draw the horizontal line from the given number to the graph of $y = f(x)$, this line must meet the curve *once only.* Otherwise there will be more than one possible answer and we are not allowed to use the word function to describe such a situation. For example, $f(x) = x^2$ is a perfectly acceptable function, but it maps both $+5$ and -5 onto the same image, namely 25. There is no objection to this, we simply agree to call it a many-to-one function. However if we attempt to find $f^{-1}(25)$, there are two possible answers, namely $+5$ and -5. So $f(x)$ does not have an inverse function. This difficulty can be by-passed if we agree in advance to limit the domain of $f(x) = x^2$ to the non-negative real numbers; in that case we shall not be applying it to -5 and the difficulty of having two possible answers will not arise.

To sum up then, we can only have an inverse function if the original function is a one-to-one function. (However the fact that an inverse function exists does not necessarily mean we shall be able to write down the rule which gives the inverse.)

In general, if (a, b) is a point on the graph of $y = f(x)$, then (b, a) will be a point on the graph of $y = f^{-1}(x)$, and consequently the graph of $y = f^{-1}(x)$ will be the reflection of the graph of $y = f(x)$ in the line $y = x$ (see Fig. 2.19).

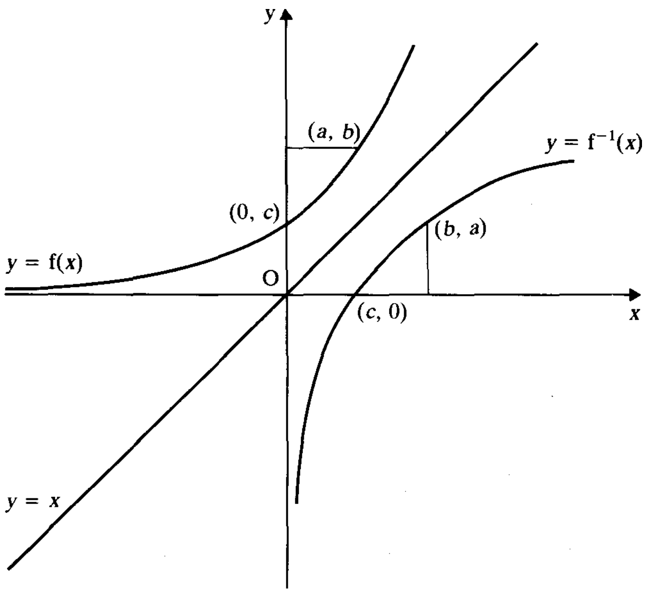


Figure 2.19

Here are some examples of some common functions and their inverses:

(a) $f(x) = x + a$ $\quad f^{-1}(x) = x - a$,
(b) $f(x) = kx$ $\quad f^{-1}(x) = x/k$,
(c) $f(x) = x^2$, $(x \geqslant 0)$ $\quad f^{-1}(x) = \sqrt{x}$,
(d) $f(x) = a - x$ $\quad f^{-1}(x) = a - x$,
(e) $f(x) = 1/x$ $\quad f^{-1}(x) = 1/x$.

Functions, like (d) and (e), which are the same as their inverses are called **self-inverse functions.**

If a function f is applied to a number a, and then f^{-1} is applied, the final result will be the original number a. For example using function (c) above, $f(3)=9$ and $f^{-1}(9)=3$. (This can be clearly observed on a pocket calculator. First key in any number a, then press a function button, say x^2, and then the button of the inverse function $\sqrt{x}$, and the original number a should be displayed. Although the following functions are as yet unknown to you, you can observe the same phenomenon by pressing the buttons representing the following pairs of functions and their inverses: $\log x$, 10^x; $\ln x$, e^x.)

We have already seen that $fg(x)$ is the composite function, in which the function g is applied first and then function f is applied to the result. The inverse of this composite function is $g^{-1}f^{-1}(x)$. (This is rather like packing and unpacking a parcel. Suppose you wrap the parcel in paper and then tie it up with string. When the parcel is unpacked, first the string must be untied and, after that, the paper removed.)

Example 13 *Given that* $f(x)=10x$ *and* $g(x)=x+3$, *find* $fg(x)$ *and* $(fg)^{-1}(x)$. *Verify that if* $b=fg(a)$, *then* $(fg)^{-1}(b)=a$.

$$g(x)=x+3$$
$$fg(x)=10x\,(x+3)$$

The inverses of g and f are $g^{-1}(x)=x-3$ and $f^{-1}(x)=x/10$. Hence

$$(fg)^{-1}(x)=g^{-1}f^{-1}(x)$$
$$=g^{-1}\left(\frac{x}{10}\right)$$
$$=\frac{x}{10}-3$$

In the general case, we are given $b=fg(a)$,

$$\therefore\ b=10(a+3)$$

and hence,

$$(fg)^{-1}(b)=\frac{10(a+3)}{10}-3$$
$$=a+3-3$$
$$=a$$

In some cases the inverse function can be found by regarding $y=f(x)$ as an equation in which y is *known*, and solving the equation for the *unknown* x. For instance, if

$$y=\frac{5x+7}{3x+2}$$

then

$$y(3x+2)=5x+7$$

$$3xy + 2y = 5x + 7$$

$$3xy - 5x = 7 - 2y$$

$$x(3y - 5) = 7 - 2y$$

$$x = \frac{7 - 2y}{3y - 5}$$

So the inverse of $f(x) = (5x + 7)/(3x + 2)$ is $g(y) = (7 - 2y)/(3y - 5)$. However, since we need to emphasise that g is the inverse of f and since the letter x is normally used to represent the independent variable, we express this result as

$$f^{-1}(x) = \frac{7 - 2x}{3x - 5}$$

A result such as this can be checked by verifying that $f^{-1}(f(x)) = x$. In this case,

$$f^{-1}(f(x)) = \frac{7 - 2(5x + 7)/(3x + 2)}{3(5x + 7)/(3x + 2) - 5}$$

$$= \frac{7(3x + 2) - 2(5x + 7)}{3(5x + 7) - 5(3x + 2)}$$

$$= \frac{21x + 14 - 10x - 14}{15x + 21 - 15x - 10}$$

$$= \frac{11x}{11}$$

$$= x$$

Exercise 2d

1 Given that $f(x) = 5x + 1$, find the values of
(a) $f^{-1}(36)$, (b) $f^{-1}(-14)$, (c) $f^{-1}(0)$, (d) $f^{-1}(a)$.

2 Given that $g(t) = 1/(t - 5)$, $(t \neq 5)$, find the values of
(a) $g^{-1}(\frac{1}{2})$, (b) $g^{-1}(2)$, (c) $g^{-1}(-1)$, (d) $g^{-1}(a)$.

3 Find the inverses of the following functions:
(a) $f(x) = 12 - \frac{1}{2}x$, (b) $f(x) = \frac{1}{2}(x - 3)$,
(c) $f(x) = (2x + 1)/5$, (d) $f(x) = (7 - 3x)/10$.

4 Find the inverses of the following functions:
(a) $f: x \mapsto \frac{5}{9}(x - 32)$, (b) $f: x \mapsto 180(x - 2)$,
(c) $f: x \mapsto 2\pi x$, (d) $f: x \mapsto 5(x + 7)/3 - 9$.

5 Find the inverses of the following functions:
(a) $F: t \mapsto t^2 + 5$, $(t \geqslant 0)$, (b) $F: t \mapsto 5\sqrt{t}$, $(t \geqslant 0)$,
(c) $F: t \mapsto (t - 5)^3$, (d) $F: t \mapsto \sqrt[3]{(t + 1)}$.

6 Find the inverses of the following functions:

(a) $g: x \mapsto \dfrac{1}{x - 3}$, $(x \neq 3)$, (b) $g: x \mapsto \dfrac{1}{2x + 1}$, $(x \neq -\frac{1}{2})$,

(c) $g: x \mapsto \dfrac{3}{4-x}$, $(x \neq 4)$, (d) $g: x \mapsto \dfrac{2x}{1+x}$, $(x \neq -1)$.

7 Show that the function $f(x) = 1/(1-x)$, $(x \neq 1)$, is the inverse of the function $g(x) = (x-1)/x$, $(x \neq 0)$.

8 Show that the function $H(x) = x/(x-1)$ is a self-inverse function.

9 Sketch the graph of the function $y = f(x)$, where $f(x) = x^3 + 1$ and, on the same diagram, sketch the graph of the inverse function $y = f^{-1}(x)$.

10 Fig. 2.20 shows the graph of a function $y = f(x)$. Copy the diagram carefully, using tracing paper if necessary, and on the same diagram, sketch the graph of the inverse function.

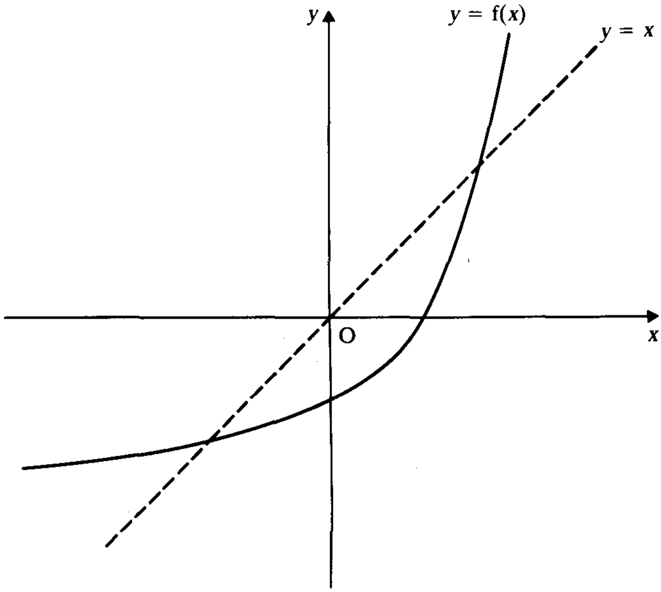


Figure 2.20

Investigating limits, using a calculator

2.17 In this section we shall investigate the limits of some functions using a calculator. It is important to understand that our investigations will only tell us the value of the function at the points we examine. To prove that the limits are what we think they are, we must turn to algebra, which we shall do in the next section. Nevertheless, the calculator can give us some very strong clues to the behaviour of certain functions.

The phrase 'x tends to zero', which is written '$x \to 0$', means that x can be made as small as we please. If any prearranged small number is chosen, then it must be possible to make x smaller than that number.

Example 14 *Investigate the function* $f(x) = x/\sin x$, *as* $x \to 0$, *using your calculator in degree mode.*

(Notice that this function does not exist at $x = 0$, since when $x = 0$, the function would give 0/0.)

x	1.0	0.5	0.1	0.01
$f(x)$	57.299	57.297	57.296	57.296

This function seems pretty determined to approach 57.296 (to five significant figures) as x tends to zero.

When we say 'x tends to a', where a is a fixed real number, we mean that x can be made as close to a as we please; or, to put it another way, $|x - a| \to 0$. In the following example, x tends to 2.

Example 15 *Investigate the function* $f(x) = \dfrac{x^3 - 8}{x - 2}$, *as* $x \to 2$.

(First it should be noticed that $f(x)$ does not exist when $x = 2$; with this value of x the function gives 0/0.)

Set out below are two tables; the first shows the values of $f(x)$ when x approaches 2 from below, and the second shows the values of $f(x)$ when x approaches 2 from above.

x	1.9	1.99	1.999	1.9999
$f(x)$	11.41	11.940	11.994	11.999

x	2.1	2.01	2.001	2.0001
$f(x)$	12.61	12.060	12.006	12.000

This suggests that $f(x)$ approaches 12, as x tends to 2.

A function $f(x)$ is said to tend to a **limit** L, if $|f(x) - L| \to 0$, as $x \to a$. The *same* number L must be reached whether x approaches the fixed number a from above or below. The function itself may, in some cases, be undefined at $x = a$. (In Example 15, above, we say that the limit of $f(x)$ is 12, as $x \to 2$.)

The phrase 'x tends to infinity', means that x gets bigger and bigger, without any limit on its size. If we choose a large number N, then it must be possible for x to exceed N. (Infinity itself is not a real number; see §2.5.) Thus we can say that $1/n$ tends to zero as n tends to infinity. In other words $1/n$ gets smaller and smaller as n gets bigger and bigger. If we choose a very small number, say 10^{-6}, and ask whether we can make $1/n$ smaller than this number, the answer is 'yes'; all we have to do is to make n bigger than 10^{+6}. In writing, this statement is abbreviated to '$1/n \to 0$, as $n \to \infty$'.

Example 16 *Investigate the function* $f(x) = \dfrac{2x}{1+x}$, *as* $x \to \infty$.

The table below shows some values of f(x) for some increasingly large values of x. (The values of f(x) are given to five significant figures.)

x	10	100	1000	10 000
f(x)	1.8182	1.9802	1.9980	1.9998

From this table it seems reasonable to suppose that $f(x) \to 2$, as $x \to \infty$.

Example 17 *Investigate* $f(n) = (1 + 1/n)^n$, *as* $n \to \infty$.

The table below shows the values of f(n), for some increasingly large values of n. (The values of f(n) have been corrected to four significant figures.)

n	1	5	10	100	1000	1 000 000
f(n)	2	2.488	2.594	2.705	2.717	2.718

The table suggests that the limit of this function is 2.718. (It is difficult to investigate the limit of this function rigorously, but it can be shown that it is a number called e. We will meet e again in Book 2; like π, it plays a very important role in higher mathematics.)

In Qu. 8–12, use your calculator to investigate each function, as x tends to the number stated.

Qu. 8 $\dfrac{2x-7}{x-4}$, $x \to \infty$. **Qu. 9** $\dfrac{x^2+5x-14}{x-2}$, $x \to 2$.

Qu. 10 $\left(1 + \dfrac{2}{x}\right)^x$, $x \to \infty$.

Qu. 11 $\dfrac{x}{\sin x}$, $x \to 0$, using your calculator in radian mode.

Qu. 12 $\dfrac{1-\cos x}{x^2}$, $x \to 0$, using your calculator in radian mode.

Finding limits algebraically

2.18 Some of the functions which you have investigated in the preceding sections can be examined more rigorously using algebra.

In Example 16, above, if we divide the numerator and the denominator by x, the function can be written

$$f(x) = \frac{2}{1/x + 1}$$

If now we let $x \to \infty$, the term $1/x$ will tend to zero and we can see that $f(x)$ will tend to 2. Notice that, since x is positive, the denominator will always be slightly bigger than 1, so $f(x)$ will always be slightly less than 2. We say that $f(x)$ tends to 2 from below. On the other hand, when $x \to -\infty$, the denominator will be slightly less than 1 and so $f(x)$ will approach 2 from above.

In Example 15, put $x = 2 + h$, where h is small (in due course, we shall let h tend to zero).

$$x^3 = (2+h)^3 = 8 + 12h + 6h^2 + h^3$$

hence

$$\frac{x^3 - 8}{x - 2} = \frac{12h + 6h^2 + h^3}{h}$$

$$= 12 + 6h + h^2 \qquad (h \neq 0)$$

Although we must not put h *equal* to zero, we can let h *tend* to zero, that is, we can let it get smaller and smaller. As it does so, the terms $6h$ and h^2 tend to zero and we see that the function tends to 12. This confirms the result of our investigation by calculator.

If $f(x)$ tends to L as x tends to a, we frequently say that the limit of $f(x)$, as x tends to a, is L. This is usually abbreviated to

$$\lim_{x \to a} f(x) = L$$

Thus the outcome of Example 15 could be written

$$\lim_{x \to 2} \left(\frac{x^3 - 8}{x - 2} \right) = 12$$

Continuity

2.19 Looking back at Examples 7 and 8 (§2.13), the reader will notice that there is an important difference between them. The graph of Example 8 could, at least in our imagination, be drawn with a single sweep of the pencil, whereas in Example 7 the pencil must be lifted off the page at each integer point of the domain. We say that the function in Example 8 is **continuous**, but the function in Example 7 is **discontinuous** at 1, 2, 3, ...

Fig. 2.21 shows sketches of the graphs of $y = x^2$, $y = 1/x$ and $y = 1/x^2$.

$f(x) = x^2$ is plainly a continuous function, but the other two are discontinuous at $x = 0$ (they are, of course, both undefined at this point).

The function given by

$$f(x) = +1, \quad \text{when } x \geqslant 0$$
$$f(x) = -1, \quad \text{when } x < 0$$

is defined at every point of $\mathbb{R}$, but it is discontinuous at $x = 0$. A sketch of its graph is shown in Fig. 2.22.

In all these cases the break in the graph has been pretty obvious, but a

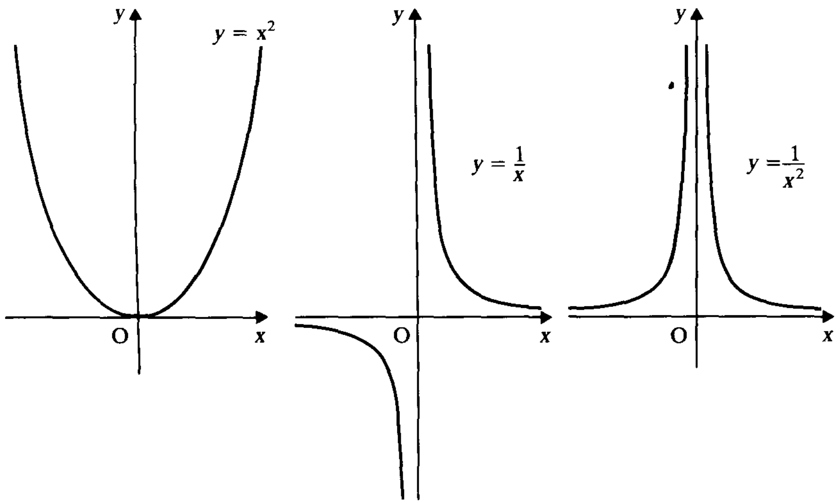


Figure 2.21

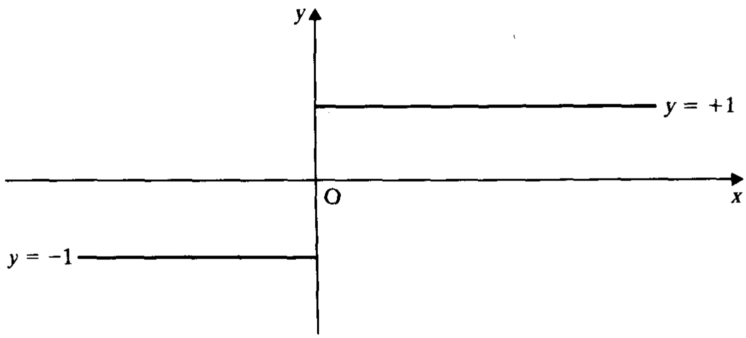


Figure 2.22

discontinuity can be more subtle than this. Consider for example the function

$$F(x) = \frac{x^2 - 4}{x - 2} \qquad (x \neq 2)$$

For all values of x, except $x = 2$, this function is equal to $(x + 2)$, and consequently its graph is the straight line $y = x + 2$, with a 'hole' in it at $x = 2$ (Fig. 2.23).

It is perfectly legitimate to say that $\lim_{x \to 2} F(x) = 4$, but we must not actually put x *equal* to 2. At the moment the graph is undefined at this point. Now, if we wish, we can 'plug the gap' by *defining* F(2) as 4. In doing so we shall have made F(x) continuous at $x = 2$. But, if we wish to be difficult, we could choose to define F(2) as something else, say F(2) = 0; in this case F(x) is discontinuous at $x = 2$.

Notice that in the case of $f(x) = 1/x$ and $f(x) = 1/x^2$, we can decide to define

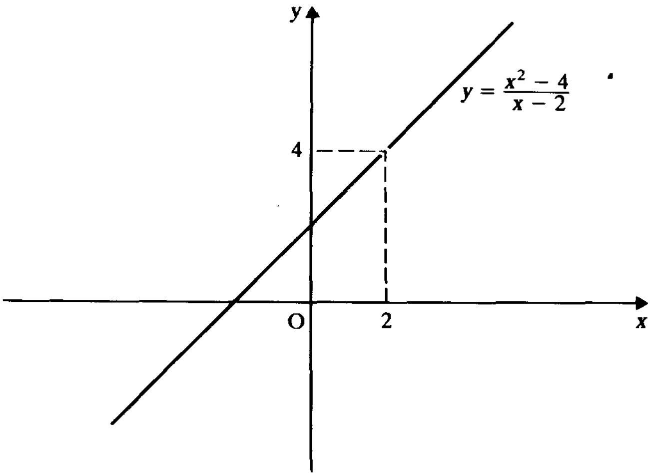


Figure 2.23

the function at $x = 0$, if we wish, but there is no number which we could assign to it which would make these functions continuous at $x = 0$.

We can express this more formally by saying that, if

$$\lim_{x \to a} f(x) \neq f(a)$$

then $f(x)$ is discontinuous at $x = a$. But if

$$\lim_{x \to a} f(x) = f(a)$$

then the function is continuous at $x = a$. A function which is continuous at every point in its domain is called a continuous function.

Exercise 2e

1 Find the limits of the following expressions as $x \to \infty$:

(a) $\dfrac{5x+1}{10+2x}$, (b) $\dfrac{x+1}{x^2}$, (c) $\dfrac{x^2+1}{x}$, (d) $\dfrac{5}{1+x}$.

2 Find the limits of the following expressions as $x \to 5$:

(a) $\dfrac{x^2-4x-5}{x-5}$, (b) $\dfrac{x^2-25}{x-5}$, (c) $\dfrac{x^3-125}{x-5}$, (d) $\dfrac{x^2-25}{(x-5)^2}$.

3 The following functions are not defined at $x = 0$. Define them, if possible, so that each function is continuous at $x = 0$.

(a) $f(x) = \dfrac{x^2+x}{x}$, (b) $f(x) = x^2 + \dfrac{5}{x}$,

(c) $f(x) = \dfrac{|x|}{x}$, (d) $f(x) = \dfrac{10+6/x}{5+2/x}$.

4 Which of the following functions are continuous at $x = 0$? Sketch the graph in each case.

(a) $f(x) = x$, when $x \geqslant 0$,
$= 0$, when $x < 0$.

(b) $f(x) = x$, when $x \geqslant 0$,
$= 1$, when $x < 0$.

(c) $f(x) = x + 1$, when $x \geqslant 0$,
$= 0$, when $x < 0$.

(d) $f(x) = 2^x$, when $x \geqslant 0$,
$= 1$, when $x < 0$.

5 The function $f(x) = \dfrac{x^3 + x^2 - 9x - 9}{x^2 - 9}$ does not exist for two members of $\mathbb{R}$.

Find these two members of $\mathbb{R}$ and define $f(x)$ at each of these points, so that it becomes a continuous function.

Exercise 2f (Miscellaneous)

1 The domain of the function $f(x)$ is $\{1, 2, 3, 4, 5\}$. Find the range if
(a) $f(x) = 5x^2 + 3$, (b) $f(x) = x/(x + 1)$.

2 Given that $F(t) = 30/(t + 2)$, find
(a) $F(3)$, (b) $F(\frac{1}{2})$, (c) $F(-1)$, (d) $F(-2.5)$.

3 Given that $g(x) = 5 + x/2$, find the values of
(a) $g^{-1}(6)$, (b) $g^{-1}(0)$, (c) $g^{-1}(-1)$, (d) $g^{-1}(a)$.

4 The domain of the function $h(t) = |t| - t$ is $\mathbb{Z}$. Describe its range. Describe in words the set of numbers $\{a\colon h(a) = 0\}$.

5 The domain of the functions $f(x) = 5x$, $g(x) = x^2$, and $h(x) = x + 1$, is $\mathbb{R}$. Write down as simply as possible, the composite functions
(a) $fgh(x)$, (b) $hgf(x)$.

6 The domain of the functions $f(x) = x/5$ and $g(x) = 7 - x$ is $\mathbb{R}$. Write down, as simply as possible,
(a) $f^{-1}(x)$, (b) $g^{-1}(x)$, (c) $fg(x)$, (d) $(fg)^{-1}(x)$.

7 The domain of the function $f(x) = 1/(1 + x^2)$ is $\mathbb{R}$. Explain why the denominator is never zero. Find the range of the function. Sketch the graph of $y = f(x)$.

8 Given that $f(x) = x + 2 - 15/x$ and that $g(x) = 1/x$, $(x \neq 0)$, write down the composite function $gf(x)$, in its simplest form, stating clearly any restrictions on the domain which are necessary.

9 State, with reasons, whether the following functions are one-to-one or many-to-one:
(a) $f\colon \mapsto 10x + 2$, $x \in \mathbb{R}$,
(b) $g\colon \mapsto 1/(x + 4)$, $x \in \mathbb{R}, x \neq -4$,
(c) $h\colon \mapsto x^2 + 1$, $x \in \mathbb{R}$.
Find the composite function $fgh(x)$ in its simplest possible form. Is $\mathbb{R}$ a suitable domain for $fgh(x)$? Find the range of the function $fgh(x)$ and sketch the graph of $y = fgh(x)$.

10 Show that the function $g(x) = (2x - 1)/(x - 3)$ can be expressed in the form

$$g(x) = \frac{a}{x - 3} + b$$

where a and b are real numbers and $x \neq 3$. Hence, or otherwise find $\lim\limits_{x \to \infty} g(x)$.

Show, also, that the graph of $g(x)$ can be obtained from the graph of $y = a/x$, by suitably chosen translations parallel to the axes. Sketch the graph of $y = g(x)$, showing clearly how it can be obtained by translating the graph of $y = a/x$.

11 Functions f and g, whose domain is the set of real numbers, are defined as follows:

$$f: x \mapsto x^3 + 2 \qquad g: x \mapsto x - 3$$

Find (a) $gf(2)$, (b) $(fg)^{-1}(-6)$. (O & C: SMP)

12 (a) Sketch the graph of the function $f: x \mapsto \big||x+2| - |x|\big|$, where x is real. State the range of f.

(b) The function g is defined by $g: x \mapsto 4/(1 + x^2)$. Give a suitable domain for x so that g is a one-to-one function and state the range of g for this domain. Define an inverse function g^{-1} stating its domain and the corresponding range. (C)

13 The real function f, defined for all $x \in \mathbb{R}$, is said to be multiplicative if, for all $x \in \mathbb{R}$, $y \in \mathbb{R}$,

$$f(xy) = f(x)f(y)$$

Prove that if f is a multiplicative function then
(a) either $f(0) = 0$ or $f(x) = 1$,
(b) either $f(1) = 1$ or $f(x) = 0$,
(c) $f(x^n) = \{f(x)\}^n$ for all positive integers n.
Give an example of a non-constant multiplicative function. (C)

14 Functions f, g and h, with domains and co-domains

$$\mathbb{R}^+ = \{x\colon x \text{ real}, x > 0\}$$

are defined as follows:

$$f: x \mapsto 3x^2, \qquad g: x \mapsto \frac{1}{\sqrt{(1+x)}}, \qquad h: x \mapsto (1+x)/x.$$

Prove that the composite function L defined on $\mathbb{R}^+$ by $L = hgf$ is given by

$$L: x \mapsto 1 + \sqrt{(1 + 3x^2)}$$ (L)

Chapter 3

The gradient of a curve*

The gradient of a curve

3.1 So far we have only discussed the gradient of a straight line. A man walking up the ramp AB (Fig. 3.1) is climbing a gradient of $\frac{2}{7}$.

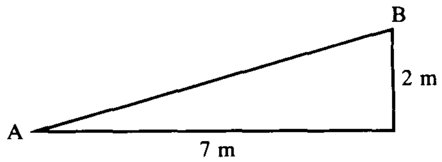


Figure 3.1

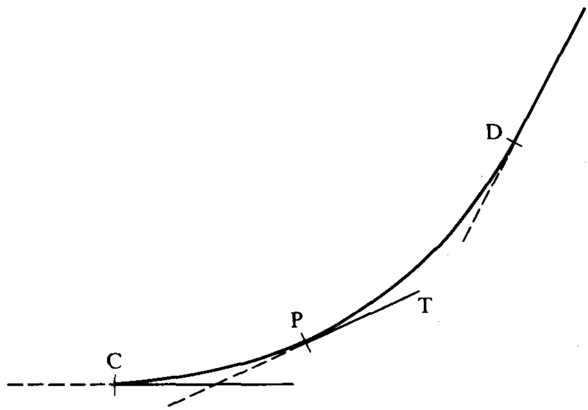


Figure 3.2

Let us now consider a man walking up the slope represented by the curve CPD (Fig. 3.2). Between C and D the gradient is steadily increasing. If, when he had reached the point P, the gradient had stopped increasing, and had remained

*Note. Most of the questions in the text in this chapter should be worked by the pupils themselves.

constant from then on, he would have climbed up the slope represented by the straight line PT, the tangent to the curve at P. Thus in walking up the slope CD, when the man is at the point P (and only at that instant) he is climbing a gradient represented by the gradient of PT.

Definition

The gradient of a curve at any point is the gradient of the tangent to the curve at that point.

The gradient at a point

3.2 If we wish to find approximately the gradient of a curve at a certain point, we could draw the curve, draw the tangent at that point by eye, and measure its gradient. But to develop our study of curves and their equations, it is important that we should discover a method of calculating exactly the gradient of a curve at any point; to do this we shall think of a tangent to a curve in the following way.

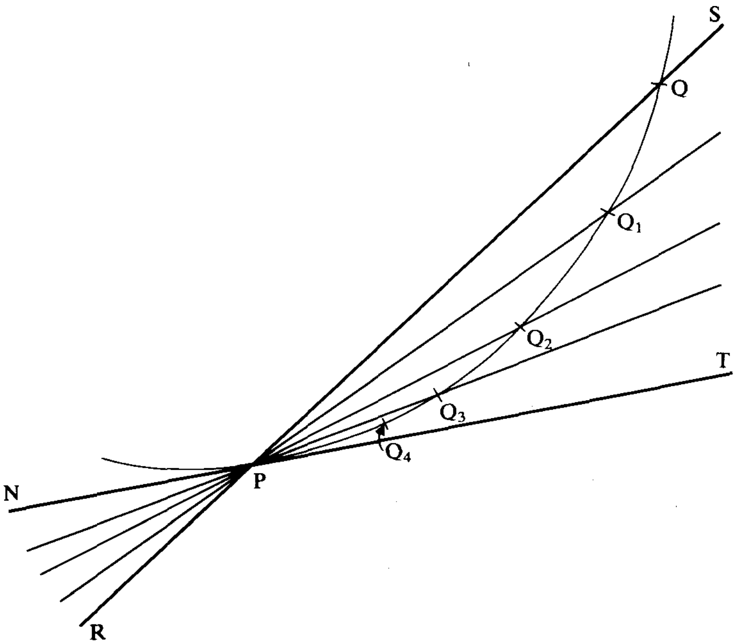


Figure 3.3

First we start with two distinct points on a curve, P and Q (Fig. 3.3), and the chord PQ is drawn and produced in both directions. Now consider RPQS as a straight rod hinged at P, which is rotated clockwise about P to take up successive positions shown by PQ_1, PQ_2, PQ_3, etc. Notice that the points at which it cuts the curve, Q_1, Q_2, Q_3, are successively nearer the fixed point P. The nearer this second point of intersection approaches P, the nearer does the gradient of the chord approach the gradient of the tangent NPT. *By taking* Q *sufficiently close to* P, *we can make the gradient of the chord* PQ *as near as we please to the gradient of the tangent at* P.

To see precisely how this happens, place the edge of a ruler along RPQS and then rotate it clockwise about P. You will see the second point of intersection approach P along the curve, until it actually coincides with P when the ruler lies along the tangent NPT. Using an arrow to denote 'tends to' or 'approaches' we may write:

as Q → P along the curve,
the gradient of the chord PQ → the gradient of the tangent at P,
the tangent at P is called the **limit** of the chord PQ (or more exactly of the secant RPQS), and
the gradient of the curve at P is the limit of the gradient of the chord PQ.

Qu. 1 A regular polygon of n sides is inscribed in a circle. What is the limit of the polygon as $n \to \infty$?
Qu. 2 OP is a radius of a circle centre O. A straight line PQR cuts the circumference at Q. What is the limit of the angle QPR as Q approaches P along the circumference?
Qu. 3 P is a point on the straight line $y = \frac{1}{3}x$. Q is the foot of the perpendicular from P to the x-axis. As P approaches O, the origin, what happens to PQ and QO? What can you say about the value of PQ/QO?

The gradient of $y = x^2$ at (2, 4)

3.3 We shall now use this idea of a tangent being the limit of a chord, to find the gradient of the curve $y = x^2$ at a particular point, namely (2, 4).

P is the point (2, 4) on the curve $y = x^2$ (Fig. 3.4). Q is another point on the curve, which we take first as (3, 9). Then, as the chord PQ rotates clockwise about P, Q moves along the curve to Q_1, and then nearer and nearer to P. By studying the behaviour of the gradient of PQ as this is happening we hope to be able to deduce the gradient of the tangent at P.

$$\text{The gradient of PQ} = \frac{\text{RQ}}{\text{PR}}$$

$$= \frac{\text{RQ}}{\text{MN}}$$

$$= \frac{\text{NQ} - \text{NR}}{\text{ON} - \text{OM}}$$

$$= \frac{9-4}{3-2}$$

$$= 5$$

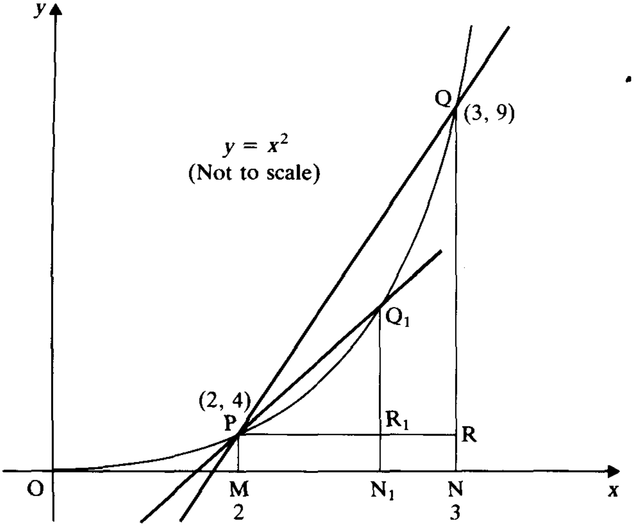


Figure 3.4

If Q now moves to the position Q_1, whose coordinates are $(2\frac{1}{2}, 6\frac{1}{4})$,

$$\text{the gradient of } PQ_1 = \frac{N_1Q_1 - N_1R_1}{ON_1 - OM}$$

$$= \frac{6\frac{1}{4} - 4}{2\frac{1}{2} - 2}$$

$$= \frac{2\frac{1}{4}}{\frac{1}{2}}$$

$$= 4\frac{1}{2}$$

We now let Q approach yet closer to P along the curve, and the table opposite gives the gradient of the chord PQ as it approaches the gradient of the tangent at P.

Comparing the first and last columns of this table, we see that for each position of Q, the gradient of PQ exceeds 4 by the same amount as the x-coordinate of Q exceeds 2. The actual equality is not important; what is important is that these values we have taken so far suggest that by taking Q sufficiently near P (i.e. by taking the x-coordinate of Q sufficiently near 2) we can make the gradient of PQ as near 4 as we please (see §2.17). This suggests that the limit of the gradient of PQ is 4, and that the gradient of the tangent at P is 4.

Qu. 4 Draw a figure similar to Fig. 3.4, taking P as the point (1, 1). Taking the x-coordinate of Q successively as 2, $1\frac{1}{2}$, 1.1, 1.01, make out a table similar to the one opposite. What appears to be the limit of the gradient of PQ in this case?
Qu. 5 Add a last line to your table for Qu. 4 by taking the x-coordinate of Q to

be $1 + h$. What happens to Q as $h \to 0$? What happens to the gradient of PQ as $h \to 0$? Deduce the gradient of $y = x^2$ at (1, 1).

Qu. 6 Add a last line to the table in the book, taking the x-coordinate of Q as $(2 + h)$. Deduce the gradient of $y = x^2$ at (2, 4).

ON (x-coord. of Q)	NQ (y-coord. of Q)	PR (ON − 2)	RQ (NQ − 4)	$\frac{RQ}{PR}$ Gradient of PQ
3	9	1	5	5
$2\frac{1}{2}$	$6\frac{1}{4}$	$\frac{1}{2}$	$2\frac{1}{4}$	$\frac{2\frac{1}{4}}{\frac{1}{2}} = 4\frac{1}{2}$
2.1	4.41	0.1	0.41	$\frac{0.41}{0.1} = 4.1$
2.01	4.0401	0.01	0.0401	$\frac{0.0401}{0.01} = 4.01$
2.001	4.004 001	0.001	0.004 001	$\frac{0.004\,001}{0.001} = 4.001$

The gradient function of $y = x^2$

3.4 We now use the method suggested in Qu. 5 to find the gradient of $y = x^2$ at *any* point.

P is the point (a, a^2), and Q is another point on the curve whose x-coordinate is $a + h$ (Fig. 3.5).

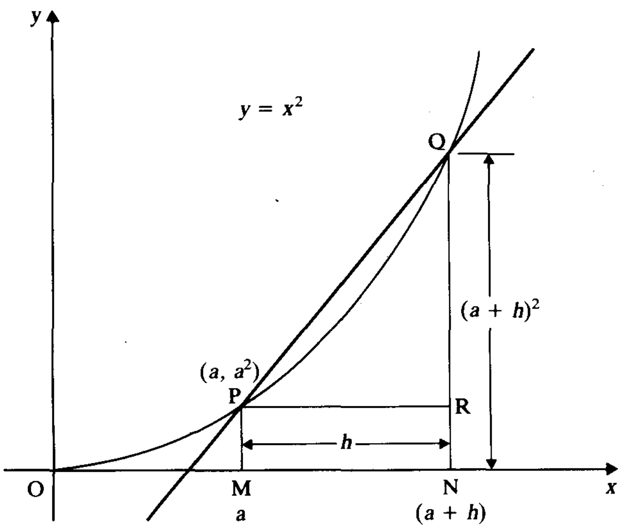


Figure 3.5

$$\begin{aligned}\text{RQ} &= \text{NQ} - \text{NR}\\ &= (a+h)^2 - a^2\\ &= 2ah + h^2\end{aligned}$$

and $\text{PR} = h$

The gradient of the chord PQ is

$$\begin{aligned}\frac{\text{RQ}}{\text{PR}} &= \frac{2ah+h^2}{h}\\ &= 2a + h\end{aligned}$$

As we let the chord rotate clockwise about P, Q approaches P along the curve, and the gradient of the chord PQ $\to$ the gradient of the tangent at P, and $h \to 0$.

But as $h \to 0$, the gradient of the chord PQ, $(2a+h) \to 2a$.

It follows that the gradient of the tangent at P is $2a$.

Thus the gradient of $y = x^2$ at (a, a^2) is $2a$, and since a is the x-coordinate of the point (a, a^2), the gradient of $y = x^2$ at (x, x^2) is $2x$.

Just as x^2 is the expression in which we substitute a value of x to find the corresponding y-coordinate and plot a point on the curve $y = x^2$, so we now have another expression, $2x$, in which we can substitute the value of x to find the gradient at that point.

$2x$ is called the **gradient function** of the curve $y = x^2$.

Example 1 *Find the coordinates of the points on the curve $y = x^2$, given by $x = 4$ and -10, and find the gradient of the curve at these points.*

$$y = x^2$$

When $x = 4$, $y = 4^2 = 16$.

The gradient function $= 2x$

$\therefore$ the gradient $= 8$, when $x = 4$

$\therefore$ the point is $(4, 16)$, and the gradient is 8.

When $x = -10$, $y = x^2 = +100$.

The gradient function $= 2x = -20$

$\therefore$ the point is $(-10, 100)$, and the gradient is -20.

Qu. 7 Calculate the gradients of the tangents to $y = x^2$ at the points given by $x = -1\frac{1}{2}, -1, +\frac{1}{2}, +2$.

Qu. 8 Use the method of §3.4 to find the gradient functions of the following curves, making a sketch in each case, and compare each result with the gradient function of $y = x^2$: (a) $y = 3x^2$, (b) $y = 5x^2$, (c) $y = \frac{1}{2}x^2$, (d) $y = cx^2$, where c is a constant, (e) $y = x^2 + 3$, (f) $y = x^2 + k$, where k is a constant.

Clearly we need an abbreviation for the statement 'the gradient function of

$y = x^2$ is $2x$'. A convenient way of writing this is

'if $y = x^2$
grad $y = 2x$'

The process of finding the gradient function of a curve is known as **differentiation**, and it is useful if we understand 'grad' also to be an instruction to differentiate. Thus,

grad $(x^2) = 2x$

The differentiation of x^3

3.5 P is any point (a, a^3) on the curve $y = x^3$. Q is another point on the curve with x-coordinate $(a + h)$ (Fig. 3.6).

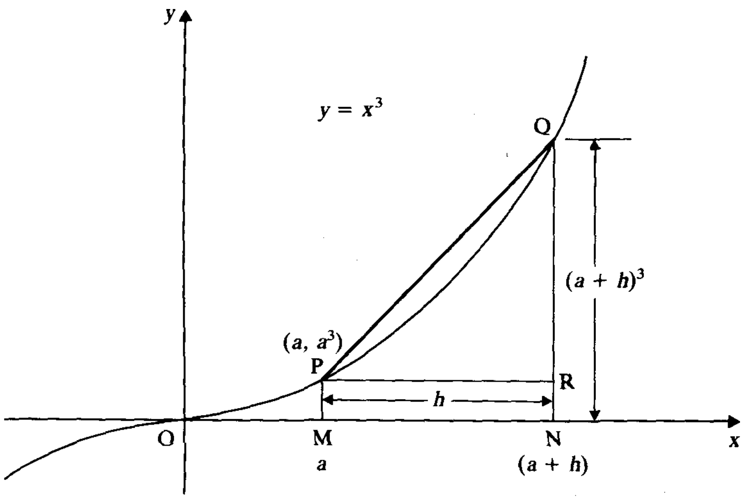


Figure 3.6

$$\begin{aligned} RQ &= NQ - NR \\ &= (a+h)^3 - a^3 \\ &= a^3 + 3a^2h + 3ah^2 + h^3 - a^3 \\ &= 3a^2h + 3ah^2 + h^3 \end{aligned}$$

$$PR = h$$

$$\text{The gradient of PQ} = \frac{RQ}{PR}$$

$$= \frac{3a^2h + 3ah^2 + h^3}{h}$$

$$= 3a^2 + 3ah + h^2$$

As Q approaches P along the curve, $h \to 0$, and the terms $3ah$ and h^2 each tend to zero; therefore the gradient of PQ $\to 3a^2$.

It follows that the gradient of $y = x^3$ at (a, a^3) is $3a^2$, or

$$\text{grad } x^3 = 3x^2$$

Qu. 9 Use the method of §3.5 to find grad x^4.
[Hint: $(a+h)^4 = a^4 + 4a^3h + 6a^2h^2 + 4ah^3 + h^4$.]
Qu. 10 Differentiate $2x^3$ by the same method.

Summary of results

3.6 We have now confirmed the following:

$$\text{grad } x^2 = 2x$$
$$\text{grad } x^3 = 3x^2$$
$$\text{grad } x^4 = 4x^3$$

The form of these results suggests that the rule for differentiating a power of x is *multiply by the index, and reduce the index by* 1; this means that grad x^5 would be $5x^4$, grad x^6 would be $6x^5$, and so on.

At this stage we must dispense with a formal proof of the validity of this process in general, and we shall assume that

$$\textbf{grad } \boldsymbol{x^n = nx^{n-1}}$$

when $n \in \mathbb{Z}^+$.

It is now time to link up these ideas with our earlier work on a straight line, and to extend them further.

y = c

Straight lines of this form, such as $y = 4$ and $y = -2$, are parallel to the x-axis, and have zero gradient. It follows that grad $4 = 0$ and grad $-2 = 0$. Thus, *if we differentiate a constant we get* 0.

[Note that this does agree with the general result, grad $x^n = nx^{n-1}$. Since $x^0 = 1$ (see §9.4), we may write grad $4 = \text{grad } 4x^0 = 0 \times 4x^{-1} = 0$.]

y = kx, y = kx^n

We know that the straight line $y = mx + c$ has gradient m, e.g. $y = x$ has gradient 1, and $y = 3x$ has gradient 3. Thus

$$\text{grad } x = 1$$

[Again, this agrees with the general result, since grad $x^1 = 1 \times x^0 = 1$.] Also,

$$\text{grad } 3x = 3 \times \text{grad } x = 3 \times 1 = 3$$

and as Qu. 8 showed,

$$\text{grad } 3x^2 = 3 \times \text{grad } x^2 = 3 \times 2x = 6x$$

This illustrates the general property that *if a function has a constant factor, that constant remains unchanged as a factor of the gradient function* (Fig. 3.7).

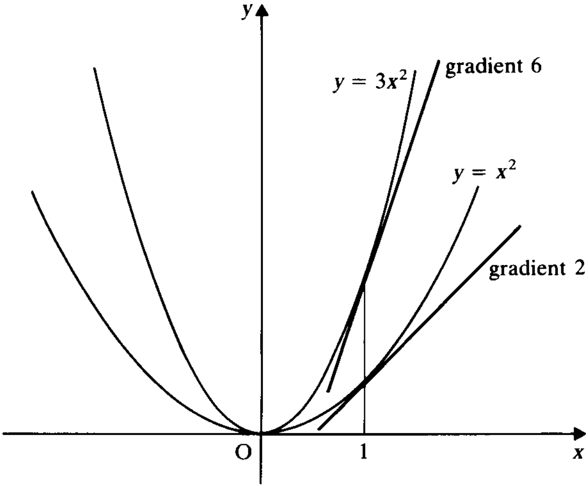


Figure 3.7

Qu. 11 Differentiate:
(a) $4x^3$, (b) $5x^4$, (c) ax^2, (d) $4x^n$, (e) Kx^{n+1}.

The differentiation of a polynomial

3.7 So far we have differentiated functions of one term only. What happens if there are two or more terms?

$y = mx + c$

The straight lines $y = 3x$, $y = 3x + 4$, and $y = 3x - 2$ all have gradient 3. Thus

$$\begin{aligned} \text{grad } 3x &= 3 \\ \text{grad } (3x + 4) &= 3 \\ \text{grad } (3x - 2) &= 3 \end{aligned}$$

The above lines are parallel, and as we discovered in §1.7, the effect of giving the different values $c = 0$, $+4$ and -2, is to raise or lower the line, but not to alter its gradient.

Clearly the same applies to the curves $y = x^2$, $y = x^2 + 4$ and $y = x^2 - 2$ (Fig. 3.8). At the point on each curve for which $x = a$, the tangents are parallel, each having gradient $2a$.

$$\begin{aligned} \text{grad } x^2 &= 2x \\ \text{grad } (x^2 + 4) &= 2x \\ \text{grad } (x^2 - 2) &= 2x \end{aligned}$$

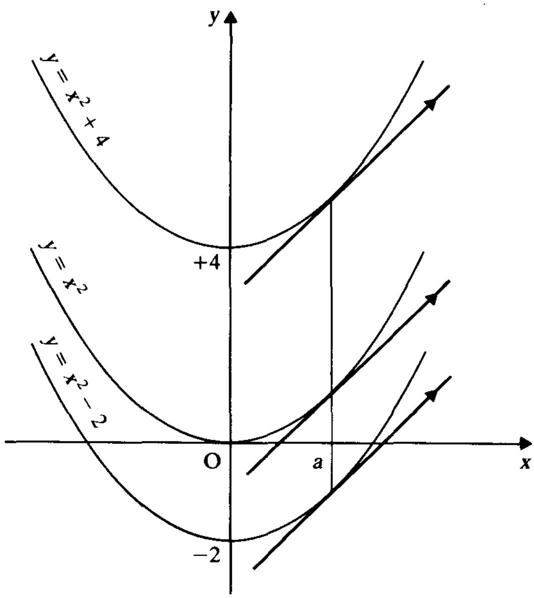


Figure 3.8

In the above cases where the function consists of two terms, we should get the same result by differentiating each term separately. Thus,

$$\begin{aligned} \text{grad}\,(x^2+4) &= \text{grad}\,x^2 + \text{grad}\,4 \\ &= 2x + 0 \\ &= 2x \end{aligned}$$

This leads us to investigate whether this method is valid in general.

$y = x^2 + 3x - 2$

To find the gradient function of this curve, let P be any point $(a, a^2 + 3a - 2)$ on it. Q is another point on the curve with x-coordinate $(a + h)$ (Fig. 3.9).

$$\begin{aligned} \text{RQ} &= \text{NQ} - \text{NR} \\ &= \{(a+h)^2 + 3(a+h) - 2\} - \{a^2 + 3a - 2\} \\ &= a^2 + 2ah + h^2 + 3a + 3h - 2 - a^2 - 3a + 2 \\ &= 2ah + h^2 + 3h \end{aligned}$$

$$\text{PR} = h$$

$$\begin{aligned} \text{The gradient of PQ} &= \frac{\text{RQ}}{\text{PR}} \\ &= \frac{2ah + h^2 + 3h}{h} \\ &= 2a + h + 3 \end{aligned}$$

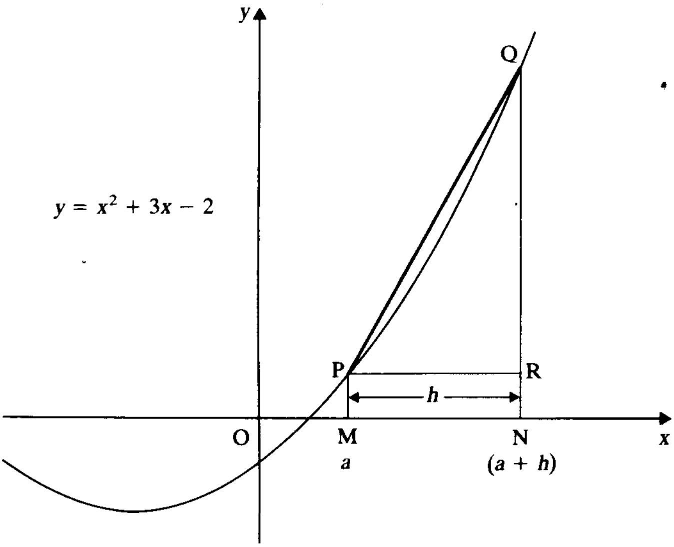


Figure 3.9

As Q approaches P along the curve, $h \to 0$ and the gradient of PQ $\to 2a + 3$. It follows that the gradient of $y = x^2 + 3x - 2$ at $(a, a^2 + 3a - 2)$ is $2a + 3$, or

$$\text{grad } (x^2 + 3x - 2) = 2x + 3$$

Now, if we try differentiating each term separately,

$$\begin{aligned}\text{grad } (x^2 + 3x - 2) &= \text{grad } x^2 + \text{grad } 3x + \text{grad } -2\\ &= 2x + 3 + 0\\ &= 2x + 3\end{aligned}$$

This illustrates the general property that *the gradient function of the sum of a number of terms is obtained by differentiating each term separately.*

Qu. 12 Differentiate:
(a) $x^3 + 2x^2 + 3x$, (b) $4x^4 - 3x^2 + 5$, (c) $ax^2 + bx + c$.

A special method of dealing with products and quotients will be met later, but for the present we must reduce a function in this form to the sum of a number of terms before differentiating. (The reader may check that to differentiate each factor separately in the following examples does *not* lead to the correct result.)

$$\text{grad } \{x^2(2x + 3)\} = \text{grad } (2x^3 + 3x^2) = 6x^2 + 6x$$

$$\text{grad } \left\{\frac{x^3 + 4x^2}{x}\right\} = \text{grad } (x^2 + 4x) = 2x + 4$$

Qu. 13 Differentiate:

(a) $x^2(4x - 2)$, (b) $(x + 3)(x - 4)$, (c) $\dfrac{5x^3 + 3x^2}{x^2}$.

Differentiation and the function notation

3.8 In the preceding sections we have considered a variety of functions and we have found their corresponding gradient functions. The gradient function is often called the **derived function**, or **derivative**.

If we have a given function f(x) it is very convenient to have a standard notation for its corresponding gradient function; the normal way of doing this is to write f′(x). Thus if $f(x) = x^3 + 5x^2 + 3x - 7$ then we write its derivative $f'(x) = 3x^2 + 10x + 3$. Alternatively

$$f: x \mapsto x^3 + 5x^2 + 3x - 7$$
$$f': x \mapsto 3x^2 + 10x + 3$$

The process of finding the derived functions in the case of $f(x) = x^2$ and $f(x) = x^3$, has been written out in full in §3.4 and §3.5 respectively. The general case is set out below.

Fig. 3.10 shows the graph of a general function $y = f(x)$; M and N are the points $(a, 0)$ and $(a + h, 0)$ respectively. P and Q are the points on the curve given by $x = a$ and $x = a + h$. So MP = f(a) and NQ = f($a + h$).

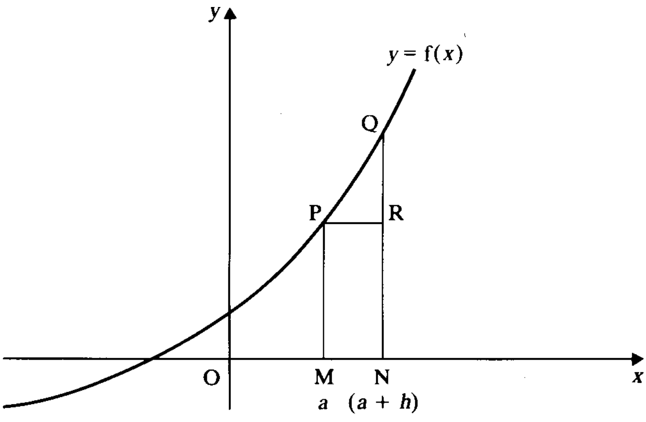


Figure 3.10

$$\begin{aligned} RQ &= NQ - NR \\ &= NQ - MP \\ &= f(a+h) - f(a) \end{aligned}$$

The gradient of PQ

$$= \frac{RQ}{PR}$$

$$= \frac{f(a+h) - f(a)}{h}$$

Hence the gradient of the tangent at P $= \lim_{h \to 0} [f(a+h) - f(a)]/h$, and hence the derived function $f'(x)$ is given by

$$\mathbf{f}'(x) = \lim_{h \to 0} \frac{\mathbf{f}(x+h) - \mathbf{f}(x)}{h} \tag{1}$$

In saying this, we are assuming that this limit exists and that it is the same whether h tends to zero from above or from below (see §2.17).

If you are ever required to differentiate a given function from first principles, you should start the proof by quoting the formula marked (1).

Example 2 *Find, from first principles, the derivative of the function* $f(t) = kt^4$, *where k is a constant.*

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

$$\begin{aligned} f(t+h) &= k(t+h)^4 \\ &= k(t^4 + 4t^3h + 6t^2h^2 + 4th^3 + h^4) \end{aligned}$$

$$\begin{aligned} \therefore f(t+h) - f(t) &= kt^4 + 4kt^3h + 6kt^2h^2 + 4kth^3 + kh^4 - kt^4 \\ &= 4kt^3h + 6kt^2h^2 + 4kth^3 + kh^4 \end{aligned}$$

$$\frac{f(t+h) - f(t)}{h} = 4kt^3 + 6kt^2h + 4kth^2 + kh^3$$

and hence

$$\begin{aligned} f'(t) &= \lim_{h \to 0} (4kt^3 + 6kt^2h + 4kth^2 + kh^3) \\ &= 4kt^3 \end{aligned}$$

Example 3 *Find from first principles,* $f'(x)$ *when* $f(x) = |x|$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{|x+h| - |x|}{h} \end{aligned}$$

Now if x and $x+h$ are both positive, then $|x+h| = x+h$ and $|x| = x$. Consequently in this case

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{x+h-x}{h} \\ &= \lim_{h \to 0} \left(\frac{h}{h}\right) \\ &= +1 \end{aligned}$$

But, if x and $x+h$ are both negative, $|x+h| = -(x+h)$ and $|x| = -x$. In this case

$$f'(x) = \lim_{h \to 0} \left(\frac{-x-h-(-x)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{-h}{h} \right)$$

$$= -1$$

The remaining case, namely f′(0), is rather tricky!

$$f'(0) = \lim_{h \to 0} \frac{|0+h|-0}{h}$$

$$= \lim_{h \to 0} \left(\frac{|h|}{h} \right)$$

But $|h|/h = +1$ if $h > 0$, or -1 if $h < 0$. Consequently, the limit as $h \to 0$ from above is $+1$, but it is -1 when h tends to 0 from below. Hence f′(0) cannot be found. This may seem rather strange, but it makes sense if we consider the graph of $y = |x|$ (Fig. 3.11).

It is clear from the graph that when $x > 0$, the gradient is $+1$, while if $x < 0$ the gradient is -1. At $x = 0$, however, the graph comes to a point and its gradient here does not exist.

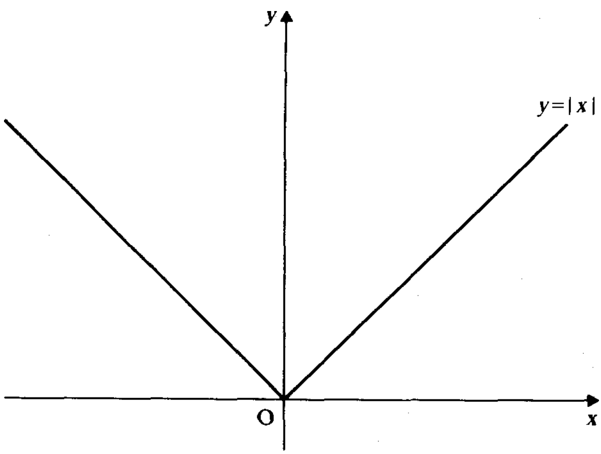


Figure 3.11

Exercise 3a

Write down the gradient functions of the following curves:

1 $y = x^{12}$. **2** $y = 3x^7$. **3** $y = 5x$.
4 $y = 5x + 3$. **5** $y = 3$. **6** $y = 5x^2 - 3x$.

Write down the derived function f′(x), for each of the following functions:

7 $f(x) = 3x^4 - 2x^3 + x^2 - x + 10.$ **8** $f(x) = 2x^4 + \frac{1}{3}x^3 - \frac{1}{4}x^2 + 2.$

9 $f(x) = ax^3 + bx^2 + cx.$ **10** $f(x) = 2x(3x^2 - 4).$

11 $f(x) = \dfrac{10x^5 + 3x^4}{2x^2}.$

Differentiate the following functions:

12 $-x.$ **13** $+10.$ **14** $4x^3 - 3x + 2.$

15 $\frac{1}{2}ax^2 - 2bx + c.$ **16** $2(x^2 + x).$ **17** $3x(x - 1).$

18 $\frac{1}{3}(x^3 - 3x + 6).$ **19** $(x + 1)(x - 2).$

Find the derivatives of the following functions:

20 $f: x \mapsto 3(x + 1)(x - 1).$ **21** $f: x \mapsto \dfrac{(x + 3)(2x + 1)}{4}.$

22 $f: x \mapsto \dfrac{2x^3 - x^2}{3x}.$ **23** $f: x \mapsto \dfrac{x^4 + 3x^2}{2x^2}.$

Find the y-coordinate, and the gradient, at the points on the following curves corresponding to the given values of x:

24 $y = x^2 - 2x + 1, x = 2.$ **25** $y = x^2 + x + 1, x = 0.$

26 $y = x^2 - 2x, x = -1.$ **27** $y = (x + 2)(x - 4), x = 3.$

28 $y = 3x^2 - 2x^3, x = -2.$ **29** $y = (4x - 5)^2, x = \frac{1}{2}.$

Find the coordinates of the points on the following curves at which the gradient has the given values:

30 $y = x^2$; 8. **31** $y = x^3$; 12.

32 $y = x(2 - x)$; 2. **33** $y = x^2 - 3x + 1$; 0.

34 $y = x^3 - 2x + 7$; 1. **35** $y = x^3 - 6x^2 + 4$; -12.

36 $y = x^4 - 2x^3 + 1$; 0. **37** $y = x^2 - x^3$; -1.

38 $y = x(x - 3)^2$; 0.

Tangents and normals

3.9 Definition

A normal to a curve at a point is the straight line through the point at right angles to the tangent at the point (Fig. 3.12).

We are now able to find the equations of tangents and normals.

Example 4 *Find the equation of the tangent to the curve $y = x^3$ at the point* (2, 8).

$$y = x^3$$
$$\therefore \text{grad } y = 3x^2$$

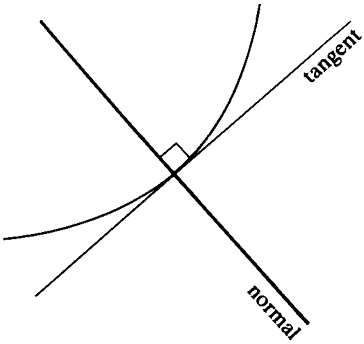


Figure 3.12

When $x = 2$,

$$\text{grad } y = 12$$

Thus the gradient of the tangent at (2, 8) is $+12$. Its equation is

$$\frac{y-8}{x-2} = 12$$

$$\therefore\ y - 8 = 12x - 24$$

$\therefore$ the equation of the tangent is $12x - y - 16 = 0$.

We can generalise Example 4 as follows.

Example 5 *Find the equation of the tangent to the curve $y = \mathrm{f}(x)$ at the point (a, b).*

Putting $x = a$ in the equation gives

$$b = \mathrm{f}(a)$$

The gradient at the given point is obtained by differentiating and putting $x = a$. Hence the gradient required is $\mathrm{f}'(a)$.

The equation of the tangent has the form

$$\frac{y-b}{x-a} = m$$

where m is the gradient. Hence the equation of the tangent is

$$y - \mathrm{f}(a) = \mathrm{f}'(a)(x - a)$$

Example 6 *Find the equation of the normal to the curve $y = (x^2 + x + 1)(x - 3)$ at the point where it cuts the x-axis.*

$$y = (x^2 + x + 1)(x - 3)$$

When $y = 0$,

$$(x^2 + x + 1)(x - 3) = 0$$

But $x^2 + x + 1 = 0$ has no real roots,

$\therefore x = +3$

$\therefore$ the curve cuts the x-axis at (3, 0)

$$y = x^3 - 2x^2 - 2x - 3$$

$\therefore$ grad $y = 3x^2 - 4x - 2$

When $x = 3$,

grad $y = 27 - 12 - 2 = 13$

The gradient of the tangent at (3, 0) is $+13$, therefore the gradient of the normal at (3, 0) is $-\frac{1}{13}$ (see §1.5) and its equation is

$$\frac{y - 0}{x - 3} = -\frac{1}{13}$$

$\therefore 13y = -x + 3$

$\therefore$ the equation of the normal is $x + 13y - 3 = 0$.

Exercise 3b

1 Find the equations of the tangents to the following curves at the points corresponding to the given values of x:
(a) $y = x^2$, $x = 2$;
(b) $y = 3x^2 + 2$, $x = 4$;
(c) $y = 3x^2 - x + 1$, $x = 0$;
(d) $y = 3 - 4x - 2x^2$, $x = 1$;
(e) $y = 9x - x^3$, $x = -3$.

2 Find the equations of the normals to the curves in No. 1 at the given points.

3 Find the equation of the tangent and the normal to the curve $y = x^2(x - 3)$ at the point where it cuts the x-axis. Sketch the curve.

4 Repeat No. 3 for the curve $y = x(x - 4)^2$.

5 Find the equation of the tangent to the curve $y = 3x^3 - 4x^2 + 2x - 10$ at the point of intersection with the y-axis.

6 Repeat No. 5 for the curve $y = x^2 - 4x + 3$.

7 Find the values of x for which the gradient function of the curve

$$y = 2x^3 + 3x^2 - 12x + 3$$

is zero. Hence find the equations of the tangents to the curve which are parallel to the x-axis.

8 Repeat No. 7 for the curve

$$y = 2x^3 - 9x^2 + 10.$$

Exercise 3c (Miscellaneous)

1 Find the gradient of the curve $y = 9x - x^3$ at the point where $x = 1$. Find the equation of the tangent to the curve at this point. Where does this tangent meet the line $y = x$?

2 Find the equation of the tangent at the point (2, 4) to the curve $y = x^3 - 2x$. Also find the coordinates of the point where the tangent meets the curve again.

3 Find the equation of the tangent to the curve $y = x^3 - 9x^2 + 20x - 8$ at the point (1, 4). At what points of the curve is the tangent parallel to the line $4x + y - 3 = 0$?

4 Find the equation of the tangent to the curve $y = x^3 + \frac{1}{2}x^2 + 1$ at the point $(-1, \frac{1}{2})$. Find the coordinates of another point on the curve where the tangent is parallel to that at the point $(-1, \frac{1}{2})$.

5 Find the points of intersection with the x-axis of the curve $y = x^3 - 3x^2 + 2x$, and find the equation of the tangent to the curve at each of these points.

6 Find the equations of the normals to the parabola $4y = x^2$ at the points $(-2, 1)$ and $(-4, 4)$. Show that the point of intersection of these two normals lies on the parabola.

7 Find the equation of the tangent at the point $(1, -1)$ to the curve

$$y = 2 - 4x^2 + x^3$$

What are the coordinates of the point where the tangent meets the curve again? Find the equation of the tangent at this point.

8 Find the coordinates of the point P on the curve $8y = 4 - x^2$ at which the gradient is $\frac{1}{2}$. Write down the equation of the tangent to the curve at P. Find also the equation of the tangent to the curve whose gradient is $-\frac{1}{2}$, and the coordinates of its point of intersection with the tangent at P.

9 Find the equations of the tangents to the curve $y = x^3 - 6x^2 + 12x + 2$ which are parallel to the line $y = 3x$.

10 Find the coordinates of the points of intersection of the line $x - 3y = 0$ with the curve $y = x(1 - x^2)$. If these points are in order P, O, Q, prove that the tangents to the curve at P and Q are parallel, and that the tangent at O is perpendicular to them.

11 Find the equations of the tangent and normal to the parabola $x^2 = 4y$ at the point (6, 9). Also find the distance between the points where the tangent and normal meet the y-axis.

12 The curve $y = (x - 2)(x - 3)(x - 4)$ cuts the x-axis at the points P(2, 0), Q(3, 0), R(4, 0). Prove that the tangents at P and R are parallel. At what point does the normal to the curve at Q cut the y-axis?

13 Find the equation of the tangent at the point P(3, 9) to the curve

$$y = x^3 - 6x^2 + 15x - 9$$

If O is the origin, and N is the foot of the perpendicular from P to the x-axis, prove that the tangent at P passes through the mid-point of ON. Find the coordinates of another point on the curve, the tangent at which is parallel to the tangent at the point (3, 9).

14 A tangent to the parabola $x^2 = 16y$ is perpendicular to the line

$$x - 2y - 3 = 0$$

Find the equation of this tangent and the coordinates of its point of contact.

15 Find the equation of the tangent to $y = x^2$ at the point (1, 1) and of the tangent to $y = \frac{1}{6}x^3$ at the point $(2, \frac{4}{3})$. Show that these tangents are parallel, and find the distance between them.

16 The point (h, k) lies on the curve $y = 2x^2 + 18$. Find the gradient at this point and the equation of the tangent there. Hence find the equations of the two tangents to the curve which pass through the origin.

17 For the curve $y = x^2 + 3$ show that $y = 2ax - a^2 + 3$ is the equation of the tangent at the point whose x-coordinate is a. Hence find the coordinates of the two points on the curve, the tangents at which pass through the point (2, 6).

18 Functions f and g are given by

$$\text{f}: x \mapsto 3x + 4 \qquad \text{and} \qquad \text{g}: x \mapsto x^2$$

(a) Find the functions f′ and g′.
(b) Calculate the values of f′(2) and g′(10).
(c) If h = gf, find h(x) and h′(x).
(d) Verify that h′(2) = f′(2)g′(10). (O & C: SMP)

Chapter 4

Velocity and acceleration

Gradient and velocity

4.1 The reader will have met 'travel graphs' in his study of mathematics. One such graph is shown in Fig. 4.1, representing a man walking to see a friend who lives 5 km away, staying 2 hours, and then returning home. On his outward journey represented by OA, he travels 5 km in 2 hours, and his velocity, $\frac{5}{2}$ km/h, is represented by DA/OD, the gradient of OA.

Whilst with his friend his velocity is zero; this is represented by the gradient of AB.

On his return journey, the gradient of BC gives his velocity as $-\frac{5}{3}$ km/h. The negative sign denotes that he is now travelling in the opposite direction; he is *decreasing* the distance from home.

This type of graph in which the distance, s, is plotted against the time, t, is called a **space-time graph**.

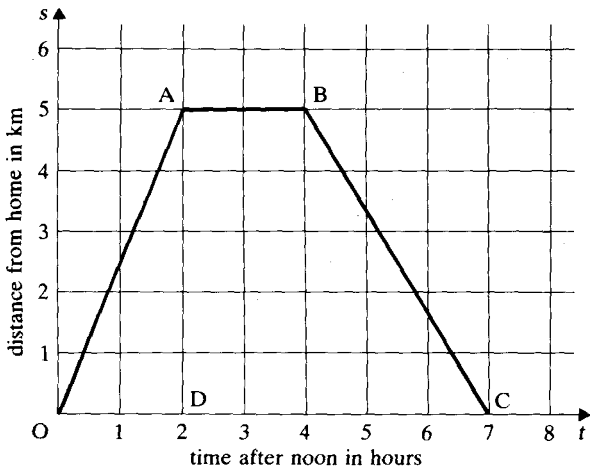


Figure 4.1

Variable velocity

4.2 When velocity is variable, as in a car journey, we may be concerned with the average velocity, which we need to define.

Definition

Average velocity is $\dfrac{\textit{total distance travelled}}{\textit{total time taken}}$ *or* $\dfrac{\textit{increase in } s}{\textit{increase in } t}$.

When the speed of a car changes, the speedometer moves, indicating the speed *at any instant.* We must now deal with the idea of *the velocity at an instant.*

Suppose that a car, starting from rest, increases its velocity steadily up to 80 km/h. Then the space-time graph is similar to the curve OPQ in Fig. 4.2. The point P we shall take to correspond to the instant at which the speedometer needle reaches the 60 km/h mark. If from that instant onward the velocity had instead been kept constant at 60 km/h, then the space-time graph would have consisted of the curve OP and the straight line PT of gradient 60.

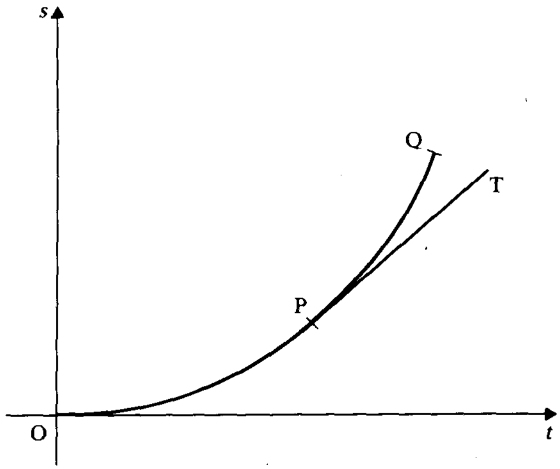


Figure 4.2

It would appear that PT is the tangent at P to the original space-time curve OPQ (like cotton under tension leading off a reel), and in that case its gradient would be the same as the gradient of the curve OPQ at P. This suggests that, when the velocity is variable, we mean, by the velocity at an instant, the velocity represented by the gradient of the space-time curve at the corresponding point. However, we must proceed to find a precise definition.

Velocity at an instant

4.3 We consider a stone falling from rest, its velocity steadily increasing. It can be verified by experiment that under certain conditions, it will be s m below its starting point t seconds after the start, where s is given by the formula $s = 4.9t^2$. From this we may make a table of values giving the position of the stone at different times.*

Value of t	0	0.5	1.0	1.5	2.0	2.5	3.0
Value of s	0	1.2	4.9	11.0	19.6	30.6	44.1

Part of the space-time graph is given in Fig. 4.3.

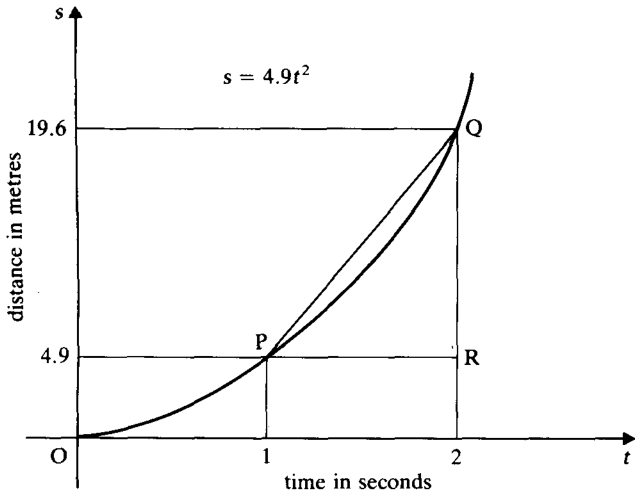


Figure 4.3

From $t = 1$ to $t = 2$, the *average velocity* is represented by the gradient of the chord PQ.

$$\frac{RQ}{PR} = \frac{19.6 - 4.9}{2 - 1} = 14.7$$

$\therefore$ the average velocity is 14.7 m/s.

Qu. 1 How far does the stone move in the interval $t = 1$ to $t = 1.5$? What is the average velocity during this interval?

Qu. 2 Repeat Qu. 1 for the intervals (a) $t = 1$ to $t = 1.1$, and (b) $t = 1$ to $t = 1 + h$.

The smaller we make the time interval (letting $Q \to P$ along the curve), the nearer the average velocity (the gradient of PQ) approaches the velocity given by the gradient of the curve at P.

Now we have seen that the gradient of the curve at P is the limit of the gradient of PQ as $Q \to P$ (§3.2); this leads to the following definition.

*Throughout §4.3, including Qu. 1 to 5, we work to one decimal place.

Definition

The velocity at an instant is the limit of the average velocity for an interval following that instant, as the interval tends to zero.

Qu. 3 From your answer to Qu. 2 (b) determine the actual velocity at the instant when $t = 1$.

Qu. 4 Calculate the distance moved, and the average velocity during the following intervals:
(a) $t = 2$ to $t = 3$, (b) $t = 2$ to $t = 2.5$,
(c) $t = 2$ to $t = 2.1$, (d) $t = 2$ to $t = 2 + h$.
Deduce the velocity when $t = 2$.

The definition given above identifies the velocity at an instant with the gradient of the space-time graph for the corresponding value of t. If we are given s in terms of t we can therefore find an expression for the velocity of the stone at any instant by differentiation, that is, if $s = \mathrm{f}(t)$, then the velocity v is given by

$$v = \mathrm{f}'(t)$$

In the case we considered above, $\mathrm{f}(t) = 4.9t^2$ and so the velocity, v m/s, is given by

$$v = \mathrm{f}'(t) = 9.8t$$

Thus when $t = 0$, $v = 0$,
when $t = 1$, $v = 9.8$,
when $t = 2$, $v = 19.6$, etc.

Qu. 5 A stone is thrown vertically downwards from the top of a cliff, and the depth below the top, s m, after t s, is given by the formula $s = 2t + 4.9t^2$.
(a) Where is the stone after 1, 2, 3, 4 s?
(b) What is its velocity at these times?
(c) What is its average velocity during the 3rd second (from $t = 2$ to $t = 3$)?

The symbols δs and δt

4.4 The idea of gradient helped us to arrive at the definition of velocity at an instant. It is instructive to take the definition as our starting point; and now, without reference to graphical ideas, we shall again demonstrate that velocity is found by differentiating the expression for s in terms of t. To do this it is convenient to introduce some new symbols, which will be of great use from now onwards.

Again we deal with the stone which falls s metres from rest in t seconds. Suppose that it falls a further small distance δs metres in the additional small interval of time δt seconds.

[The symbol δt, read as 'delta t', is used to denote a small increase, or *increment*, in time. Note that δt is a single symbol; it does not mean δ multiplied by t. Similarly δs is the corresponding *increment* in distance.]

The average velocity for the time interval δt (i.e. from t to $t + \delta t$) is $\delta s/\delta t$ m/s, and we now obtain an expression for this in terms of t.

Since the stone falls $(s + \delta s)$ metres in $(t + \delta t)$ seconds

$$s + \delta s = 4.9(t + \delta t)^2$$

i.e. $s + \delta s = 4.9t^2 + 9.8t \times \delta t + 4.9 \times (\delta t)^2$

But $s = 4.9t^2$

and subtracting,

$$\delta s = 9.8t \times \delta t + 4.9 \times (\delta t)^2$$

To find the average velocity between time t and time $(t + \delta t)$ we divide each side by δt, giving

$$\frac{\delta s}{\delta t} = 9.8t + 4.9 \times \delta t$$

As $\delta t \to 0$ the R.H.S. $\to 9.8t$.

By the definition of velocity at an instant, the velocity, v m/s, at time t, is the limit of $\delta s/\delta t$ as $\delta t \to 0$, hence

$$v = 9.8t$$

The fact that this process is identical with that of finding the gradient function of $s = 4.9t^2$ is readily seen from Fig. 4.4.

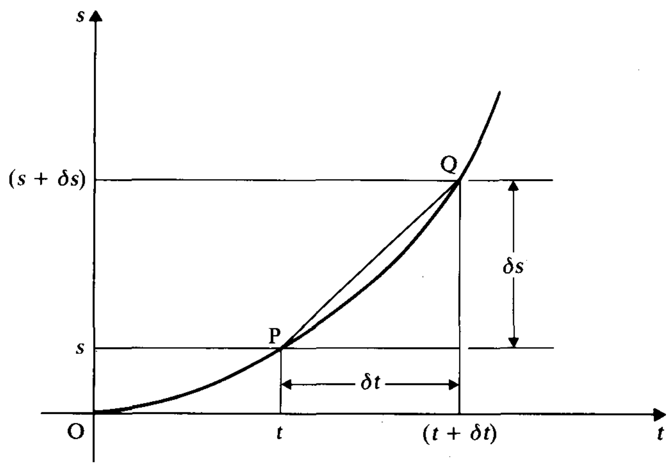


Figure 4.4

Exercise 4a

1 A stone is thrown vertically upwards at 35 m/s. It is s m above the point of projection t s later, where $s = 35t - 4.9t^2$.

(a) What is the distance moved, and the average velocity during the 3rd second (from $t = 2$ to $t = 3$)?
(b) Find the average velocities for the intervals $t = 2$ to $t = 2.5$, $t = 2$ to $t = 2.1$, $t = 2$ to $t = 2 + h$.
(c) Deduce the actual velocity at the end of the 2nd second.

2 A stone is thrown vertically upwards at 24.5 m/s from a point on the level with but just beyond a cliff ledge. Its height above the ledge t s later is $4.9t(5 - t)$ m. If its velocity is v m/s, differentiate to find v in terms of t.
(a) When is the stone at the ledge level?
(b) Find its height and velocity after 1, 2, 3, and 6 s.
(c) What meaning is attached to a negative value of s? A negative value of v?
(d) When is the stone momentarily at rest? What is its greatest height?
(e) Find the total distance moved during the 3rd second.

3 A particle moves along a straight line so that it is s m from a fixed point O on the line t s after a given instant, where $s = 3t + t^2$. After $(t + \delta t)$ s it is $(s + \delta s)$ m from O. Find the average velocity during the time interval t to $(t + \delta t)$ as was done in §4.4, and deduce an expression for the velocity v m/s, at time t. Check by differentiation.
(a) Where is the particle and what is its velocity at the instant from which time is measured (i.e. when $t = 0$)?
(b) When is the particle at O?
(c) When is the particle momentarily at rest? Where is it then?
(d) What is the velocity the first time the particle is at O?

4 A particle moves along a straight line OA in such a way that it is s m from O t s after the instant from which time is measured, where $s = 6t - t^2$. A is to be taken as being on the positive side of O.
(a) Where is the particle when $t = 0, 2, 3, 4, 6, 7$? What is the meaning of a negative value of s?
(b) Differentiate the given expression to find the velocity, v m/s, in terms of t. Find the value of v when $t = 0, 2, 4, 6$. What is the meaning of a negative value of v?
(c) When and where does the particle change its direction of motion?

5 A slow train which stops at every station passes a certain signal box at noon. Its motion between the two adjacent stations is such that it is s km past the signal box t min past noon, where $s = \frac{1}{3}t + \frac{1}{9}t^2 - \frac{1}{27}t^3$. Find
(a) the time of departure from the first station, and the time of arrival at the second,
(b) the distance of each station from the signal box,
(c) the average velocity between the stations,
(d) the velocity with which the train passes the signal box.

6 Repeat No. 5 in the case where $s = \frac{1}{72}t(36 - 3t - 2t^2)$.

7 A stone is thrown vertically downwards at 19.6 m/s from the top of a cliff 24.5 m high. It is s m below the top after t s, where $s = 19.6t + 4.9t^2$. Calculate the velocity with which it strikes the beach below.

Constant acceleration

4.5 Earlier in this chapter we used the formula $s = 4.9t^2$ for a stone falling from rest. On differentiation $v = \text{grad } s = 9.8t$. The stone's velocity is 9.8, 19.6, 29.4, 39.2 ... m/s at the end of successive seconds, and it is steadily increasing by 9.8 m/s in each second. This *rate* at which the stone's velocity increases is called its **acceleration.** This particular formula is based on the assumption that gravity is producing a *constant* acceleration of 9.8 m per second per second, written usually as 9.8 m/s^2 or 9.8 m s^{-2}.

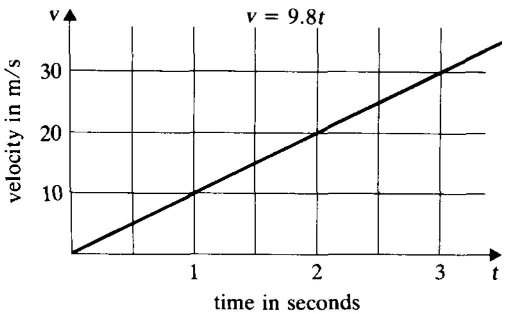


Figure 4.5

Fig. 4.5 shows the corresponding *velocity-time graph.* The equation $v = 9.8t$ (being of the form $y = mx$) represents graphically a straight line through the origin of gradient 9.8. In this case then, the acceleration is represented by the gradient of the velocity-time graph.

Qu. 6 A stone is thrown vertically downwards with a velocity of 10 m/s, and gravity produces on it an acceleration of 9.8 m/s^2.
(a) What is the velocity after 1, 2, 3, t s?
(b) Sketch the velocity-time graph.

If a particle has an initial velocity u m/s and a constant acceleration a m/s^2, then its velocity after t s is $(u + at)$ m/s and the equation $v = u + at$ (being of the form $y = mx + c$) represents a straight line of gradient a.

Thus when acceleration is constant, it is represented by the *gradient* of the straight-line *velocity-time graph.*

Exercise 4b

In this exercise, acceleration is constant.

1 At the start and end of a two-second interval, a particle's velocity is observed to be 5, 10 m/s. What is its acceleration?

2 A body starts with velocity 15 m/s, and at the end of the 11th second its velocity is 48 m/s. What is its acceleration?

3 Express an acceleration of 5 m/s^2 in (a) km/h per s, (b) km/h^2.
4 A car accelerates from 5 km/h to 41 km/h in 10 s. Express this acceleration in (a) km/h per s, (b) m/s^2, (c) km/h^2.
5 A car can accelerate at 4 m/s^2. How long will it take to reach 90 km/h from rest?
6 Sketch the velocity-time curve for a cyclist who, starting from rest, reaches 3 m/s in 5 s, travels at that speed for 20 s, and then comes to rest in a further 2 s. What is his acceleration when braking? What is the gradient of the corresponding part of the graph?
7 An express train reducing its velocity to 40 km/h, has to apply the brakes for 50 s. If the retardation produced is 0.5 m/s^2, find its initial velocity in km/h.

Variable acceleration

4.6 A car starts from rest and moves a distance s m in t seconds, where $s=\frac{1}{6}t^3+\frac{1}{4}t^2$. If its velocity after t s is v m/s, then $v=\text{grad } s=\frac{1}{2}t^2+\frac{1}{2}t$. The following table gives some corresponding values of v and t:

t	0	1	2	3	4
v	0	1	3	6	10

The increases in velocity during the first four seconds are 1 m/s, 2 m/s, 3 m/s, 4 m/s respectively. Since the rate of increase of the velocity is not constant in this case, we shall first investigate the average rate of increase over a given time interval.

Definition

Average acceleration is $\dfrac{\textit{increase in } v}{\textit{increase in } t}$.

Thus from $t=0$ to $t=2$, the average acceleration $=\dfrac{3-0}{2}=1\frac{1}{2}$ m/s^2 and from $t=2$ to $t=4$, the average acceleration $=\dfrac{10-3}{2}=3\frac{1}{2}$ m/s^2.

Clearly the acceleration itself is increasing with the time, and the next step is to define what is meant by the acceleration at an instant.

Definition

The acceleration at an instant is the limit of the average acceleration for an interval following that instant, as the interval tends to zero.

Using the notation of §4.4, if δv is the small increase in velocity which occurs in time δt, then the average acceleration for that interval is $\delta v/\delta t$, and the acceleration at time t is the limit of this as $\delta t \to 0$.

Reference to the velocity-time graph given in Fig. 4.6 shows that the average acceleration $\delta v/\delta t$ is the gradient of the chord PQ, and the limit is the gradient of the graph at P.

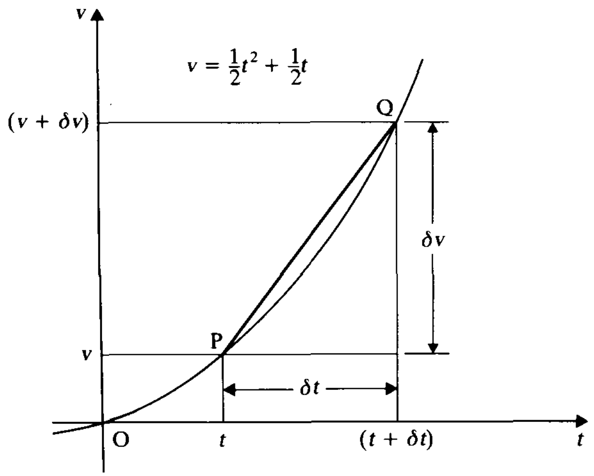


Figure 4.6

Thus an expression for the acceleration at time t may be found by differentiating the expression for v, that is, if $v = g(t)$, then a the acceleration is given by $a = g'(t)$.

Notice that if we start with the distance given by $s = f(t)$, then we differentiate once to obtain the velocity v and we differentiate again to find the acceleration a. We are already familiar with the symbol $f'(t)$ for the derivative of $f(t)$; when this in turn is differentiated we write $f''(t)$. Thus we can sum up the preceding statement as follows:

$$s = f(t)$$
$$v = f'(t)$$
$$a = f''(t)$$

Example 1 *A car starts from rest and moves a distance* s m *in* t s, *where* $s = \frac{1}{6}t^3 + \frac{1}{4}t^2$. *What is the initial acceleration, and the acceleration at the end of the 2nd second?*

$$s = f(t) = \tfrac{1}{6}t^3 + \tfrac{1}{4}t^2$$
$$v = f'(t) = \tfrac{1}{2}t^2 + \tfrac{1}{2}t$$
$$a = f''(t) = t + \tfrac{1}{2}$$

When $t = 0$, $a = \frac{1}{2}$ and when $t = 2$, $a = 2\frac{1}{2}$.

Hence the required accelerations are $\frac{1}{2}$ m/s^2, and $2\frac{1}{2}$ m/s^2.

Before reading Example 2 the reader should refer once again to the definitions of *average velocity* and *average acceleration*. In particular it should be noted that

(a) average velocity is not the same as the average of the initial and final velocities (unless the acceleration is constant); and (b) average acceleration is not necessarily the same as the average of the initial and final accelerations.

Example 2 *A particle moves along a straight line in such a way that its distance from a fixed point O on the line after t s is s m, where* $s = \frac{1}{6}t^4$. *Find* (a) *its velocity after* 3 s, *and after* 4 s, (b) *its average velocity during the* 4*th second,* (c) *its acceleration after* 2 s, *and after* 4 s, *and* (d) *its average acceleration from* $t = 2$ *to* $t = 4$.

$$s = f(t) = \tfrac{1}{6}t^4$$
$$v = f'(t) = \tfrac{2}{3}t^3$$
$$a = f''(t) = 2t^2$$

(a) When $t = 3$, $v = \frac{2}{3} \times 3^3 = 18$ m/s and when $t = 4$, $v = \frac{2}{3} \times 4^3 = 42\frac{2}{3}$ m/s.

Hence after 3 s and 4 s, the velocity is 18 m/s and $42\frac{2}{3}$ m/s respectively.

(b) When $t = 3$, $s = \frac{81}{6} = 13\frac{1}{2}$ m and when $t = 4$, $s = \frac{256}{6} = 42\frac{2}{3}$ m.

∴ the average velocity during the 4th second is

$$\frac{42\frac{2}{3} - 13\frac{1}{2}}{1} = 29\tfrac{1}{6} \text{ m/s}$$

(c) When $t = 2$, $a = 2 \times 2^2 = 8$ m/s^2 and when $t = 4$, $a = 2 \times 4^2 = 32$ m/s^2.

(d) When $t = 2$, $v = \frac{2}{3} \times 2^3 = 5\frac{1}{3}$ m/s and when $t = 4$, $v = \frac{2}{3} \times 4^3 = 42\frac{2}{3}$ m/s.

The change in velocity $= 37\frac{1}{3}$ m/s.

∴ the average acceleration from $t = 2$ to $t = 4$ is

$$\frac{37\frac{1}{3}}{2} \text{ m/s}^2 = 18\tfrac{2}{3} \text{ m/s}^2$$

Exercise 4c

1 A stone is thrown vertically upwards, and after t s its height is h m, where $h = 10.5t - 4.9t^2$. Determine, with particular attention to the signs, the height, velocity and acceleration of the stone (a) when $t = 1$, (b) when $t = 2$, and (c) when $t = 3$. Also state clearly in each case whether the stone is going up or down, and whether its speed is increasing or decreasing.

2 A stone is thrown downwards from the top of a cliff, and after t s it is s m below the top, where $s = 20t + 4.9t^2$. Find how far it has fallen, its velocity, and its acceleration at the end of the first second.

3 A ball is thrown vertically upwards and its height after t s is s m where $s = 25.2t - 4.9t^2$. Find
(a) its height and velocity after 3 s,
(b) when it is momentarily at rest,
(c) the greatest height reached,

(d) the distance moved in the 3rd second,
(e) the acceleration when $t = 2\frac{4}{7}$.

4 A particle moves in a straight line so that after t s it is s m from a fixed point O on the line, where $s = t^4 + 3t^2$. Find
(a) the acceleration when $t = 1$, $t = 2$, and $t = 3$,
(b) the average acceleration between $t = 1$ and $t = 3$.

5 At the instant from which time is measured a particle is passing through O and travelling towards A, along the straight line OA. It is s m from O after t s where $s = t(t-2)^2$.
(a) When is it again at O?
(b) When and where is it momentarily at rest?
(c) What is the particle's greatest displacement from O, and how far does it move, during the first 2 s?
(d) What is the average velocity during the 3rd second?
(e) At the end of the 1st second where is the particle, which way is it going, and is its speed increasing or decreasing?

6 Repeat No. 5(e) for the instant when $t = -1$.

7 A particle moves along a straight line so that after t s, its distance from O a fixed point on the line is s m where $s = t^3 - 3t^2 + 2t$.
(a) When is the particle at O?
(b) What is its velocity and acceleration at these times?
(c) What is its average velocity during the 1st second?
(d) What is its average acceleration between $t = 0$ and $t = 2$?

Exercise 4d (Miscellaneous)

1 The distance of a moving point from a fixed point in its straight line of motion is s m, at a time t s after the start. If $s = \frac{1}{10}t^2$, find the distances travelled from rest by the end of the 1st, 2nd, 3rd, 4th, and 5th seconds.

Draw a graph plotting distance against time, taking 2 cm to represent both 1 m and 1 s. Draw a tangent to your graph at the point where $t = 3.5$ and measure its slope; deduce the velocity of the moving point when $t = 3.5$.

2 A point moves along a straight line so that, at the end of t s, its distance from a fixed point on the line is $t^3 - 2t^2 + t$ m. Find the velocity and acceleration at the end of 3 s.

3 A particle moves in a straight line and its distance (s m) from the point at which it is situated at zero time is given in terms of the time (t s) by the formula $s = 45t + 11t^2 - t^3$. Find the velocity and acceleration after 3 s, and prove that the particle will come to rest after 9 s. (C)

4 A particle moves along the x-axis in such a way that its distance x cm from the origin after t s is given by the formula $x = 27t - 2t^2$. What are its velocity and acceleration after 6.75 s? How long does it take for the velocity to be reduced from 15 cm/s to 9 cm/s, and how far does the particle travel meanwhile?

5 A point moves along a straight line OX so that its distance x cm from the point O at time t s is given by the formula $x = t^3 - 6t^2 + 9t$. Find
(a) at what times and in what positions the point will have zero velocity,

(b) its acceleration at those instants,
(c) its velocity when its acceleration is zero.

6 A particle moves in a straight line so that its distance x cm from a fixed point O on the line is given by $x = 9t^2 - 2t^3$ where t is the time in seconds measured from O. Find the speed of the particle when $t = 3$. Also find the distance from O of the particle when $t = 4$, and show that it is then moving towards O.

7 A particle moves along the x-axis in such a way that its distance x cm from the origin after t s is given by the formula $x = 7t + 12t^2$. What distance does it travel in the nth second? What are its velocity and acceleration at the end of the nth second?

Chapter 5

Maxima and minima

The symbols δx, δy and $\frac{dy}{dx}$

5.1 In Chapter 4 we met the symbols δs and δt, and to extend the scope of differentiation it is convenient to denote small increases in x and y as δx and δy in the same way. If P is the point (x, y) on a curve, and Q is another point, and if the increase in x in moving from P to Q is δx, then the corresponding increase in y is δy; thus Q is the point $(x + \delta x, y + \delta y)$ (Fig. 5.1).

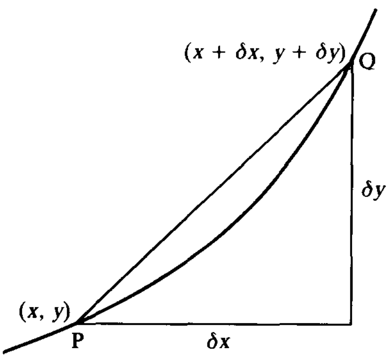


Figure 5.1

The gradient of the chord PQ is $\frac{\delta y}{\delta x}$, and the gradient of the curve at P is the limit of $\frac{\delta y}{\delta x}$, as $\delta x \to 0$. Up to now we have denoted this limit as 'grad y' to keep in mind the fundamental idea of gradient in relation to differentiation. We will in future adopt the usual practice of writing this limit as $\frac{dy}{dx}$, the symbol $\frac{d}{dx}$

being an instruction to differentiate.* Thus, if $y = x^2$, $\frac{dy}{dx} = 2x$; or we may write $\frac{d}{dx}(x^2) = 2x$. The gradient function will also be referred to in future as the *derived function*, or *derivative* (see §3.8).

Qu. 1 Find $\frac{dy}{dx}$ when

(a) $y = x^2 - 4x$, (b) $y = 3x^2 - 3$, (c) $y = 2x^3 - 5x^2 + 1$,
(d) $y = x(x - 2)$, (e) $y = x(x + 1)(x - 3)$.

The notation $\frac{dy}{dx}$ is often called 'Leibnitz notation' after Gottfried Leibnitz (1646–1716), who invented it.

Greatest and least values

5.2 Fig. 5.2 represents the path of a stone thrown from O, reaching its greatest height AB, and striking the ground at C. Between O and A, when the stone is climbing, the gradient is positive but steadily decreases to zero at A. Past A the stone is descending, and the path has a negative gradient.

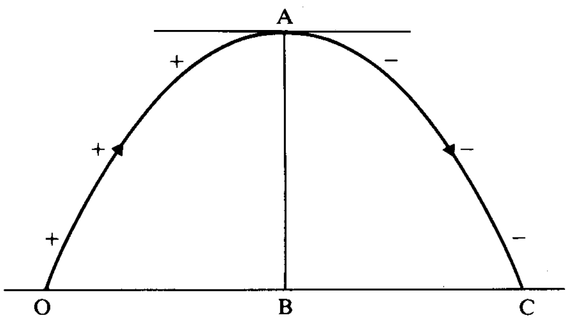


Figure 5.2

The curve $y = x^2$ of which we made much use earlier on, is called a parabola. A more general equation of this type of curve is of the form $y = ax^2 + bx + c$. When a is positive, we get a curve like a valley, such as DEF in Fig. 5.3, on which y has a least value (GE); when a is negative, we get a curve like a mountain top, such as OAC in Fig. 5.3, on which y has a greatest value (BA).

If we allow our eye to travel along each curve in Fig. 5.3 from left to right (the

**Note.* This notation $\frac{d}{dx}$ serves to indicate that we are differentiating with respect to x. Thus $\frac{d}{dy}(y^3) = 3y^2$, and $\frac{d}{dt}(2t^2) = 4t$.

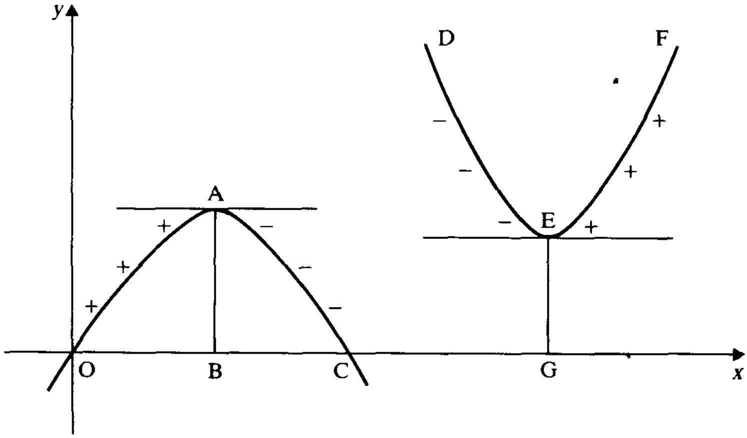


Figure 5.3

direction in which x increases), we notice that in passing through A, where y has a greatest value, the gradient is zero and is changing sign *from positive to negative*; on the other hand in passing through E, where y has a least value, the gradient is zero and is changing sign *from negative to positive*. This distinction enables us to investigate the highest or lowest point on a parabola without going to the length of plotting the curve in detail.

Example 1 *Find the greatest or least value of y on the curve $y = 4x - x^2$. Sketch the curve.*

$$y = 4x - x^2$$

$$\frac{dy}{dx} = 4 - 2x$$

$$= 2(2 - x)$$

The gradient is zero when

$$2(2 - x) = 0$$

$$x = 2$$

and $$y = 4 \times 2 - 2^2 = 4$$

We must now investigate the sign of the gradient on either side of the point (2, 4) to discover whether it is a highest (Fig. 5.4) or lowest (Fig. 5.5) point on the curve. We look back to the gradient in the form $2(2 - x)$.

Figure 5.4

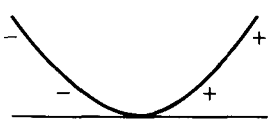


Figure 5.5

Just to the left of (2, 4), x is just less than 2, and $\frac{dy}{dx}$ is positive.

Just to the right of (2, 4), x is just greater than 2, and $\frac{dy}{dx}$ is negative.

Thus Fig. 5.4 gives the shape of the curve at (2, 4), and the greatest value of y is $+4$.

To make a rough sketch of the curve, we find where it cuts the axes.

$$y = 4x - x^2$$

When $x = 0$,

$$y = 0$$

$\therefore$ the curve passes through (0, 0).

When $y = 0$,

$$4x - x^2 = 0$$
$$x(4 - x) = 0$$
$$x = 0 \text{ or } 4$$

$\therefore$ the curve passes through (0, 0) and (4, 0).

From this information we can make the sketch (Fig. 5.6).

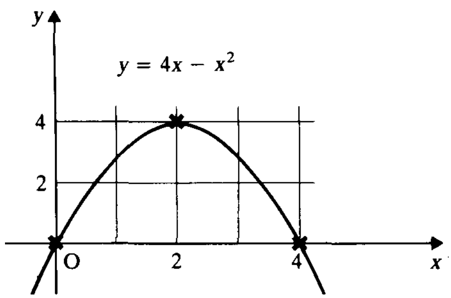


Figure 5.6

Qu. 2 Find the coordinates of the points on the following curves where the gradient is zero:
(a) $y = 4x - 2x^2$, (b) $y = 3x^2 + 2x - 5$, (c) $y = 4x^2 - 6x + 2$.

At this stage the reader must be clear about the meaning of 'greater than' and 'less than' in respect of negative numbers. For example, -3.1 is *less* than -3, and -2.9 is *greater* than -3.

In Qu. 3 and Example 2, we use the notation $f'(x)$ for the derived function; it is a useful alternative to the $\frac{dy}{dx}$ notation and the reader should be prepared to use it.

Qu. 3 Find the values of x for which the following derived functions are zero, and determine whether the corresponding graphs have a highest or a lowest point for these values of x:
(a) $f'(x) = 5 - 3x$, (b) $f'(x) = 6x - 7$,
(c) $f'(x) = 2x + 3$, (d) $f'(x) = -4 - 5x$.

The investigation of the sign of the gradient may be conveniently laid out in the way shown in the following example.

Example 2 *Find the greatest or least value of the function* $f(x) = x^2 + 4x + 3$ *and the value of* x *for which it occurs.*

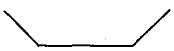
$$f(x) = x^2 + 4x + 3$$
$$f'(x) = 2x + 4$$
$$= 2(x + 2)$$

The gradient is zero when $f'(x) = 0$, i.e. when $x = -2$ and

$$f(-2) = (-2)^2 + 4(-2) + 3 = -1$$

Value of x	L	-2	R
Sign of $f'(x)$	$-$	0	$+$

[L for 'left', R for 'right']



When $x = -2$, $x^2 + 4x + 3$ has the least value -1.

This method can be used to solve some practical problems, as in the following example.

Example 3 1000 m *of fencing is to be used to make a rectangular enclosure. Find the greatest possible area, and the corresponding dimensions.*

If the length is x m, the width will be $(500 - x)$ m, and the area, A m^2, is given by

$$A = x(500 - x)$$
or $$A = 500x - x^2$$

[This problem could now be solved by drawing accurately the graph of area plotted against length (Fig. 5.7), and reading off the greatest area (NM) and the corresponding length (ON). In practice it is, of course, much quicker to continue, along the lines of Example 2, by finding the greatest value of $500x - x^2$, without plotting a graph.]

$$\frac{dA}{dx} = 500 - 2x$$

$$= 2(250 - x)$$

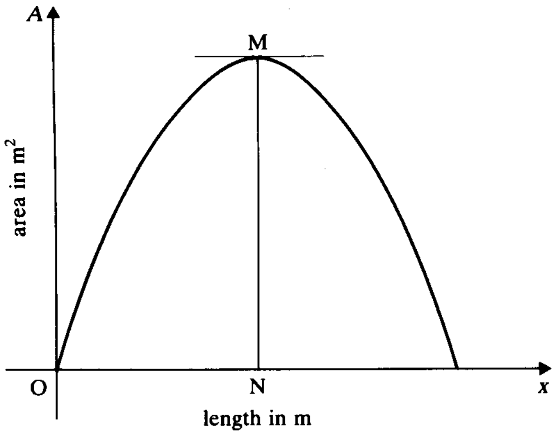


Figure 5.7

which is zero when

$$x = 250$$

and $A = 250(500 - 250) = 62\,500$

Value of x	L	250	R
Sign of $\frac{dA}{dx}$	+	0	−

The greatest area is 62 500 m^2, when the length is 250 m, and the width is 250 m.

Exercise 5a

1 Find $\frac{dy}{dx}$ when

(a) $y = 3x^2 - 2x + 5$, (b) $y = 5x^2 + 4x - 6$, (c) $y = 2x(1 - x)$,
(d) $y = (x + 1)(3x - 2)$, (e) $y = 3(2x - 1)(4x + 3)$.

2 Find the coordinates of the points on the following curves where the gradient is zero:
(a) $y = x^2 + 5x - 2$, (b) $y = 5 + 9x - 7x^2$,
(c) $y = x(3x - 2)$, (d) $y = (2 + x)(3 - 4x)$.

3 Find the values of x for which the following derived functions are zero, and determine whether the corresponding graphs have a highest or a lowest point for these values of x:
(a) $f'(x) = 2x - 5$, (b) $f'(x) = \frac{1}{2}x + 3$,
(c) $f'(x) = \frac{1}{3} - \frac{1}{4}x$, (d) $f'(x) = -5 - \frac{1}{5}x$.

4 Find the greatest or least values of the following functions:
(a) $x^2 - x - 2$, (b) $x(4 - x)$,
(c) $15 + 2x - x^2$, (d) $(2x + 3)(x - 2)$.

5 Sketch the graphs of the functions in No. 4.

6 A ball is thrown vertically upwards from ground level and its height after t s is $(15.4t - 4.9t^2)$ m. Find the greatest height it reaches, and the time it takes to get there.

7 A farmer has 100 m of metal railing with which to form two adjacent sides of a rectangular enclosure, the other two sides being two existing walls of the yard, meeting at right angles. What dimensions will give him the maximum possible area?

8 A stone is thrown into a mud bank and penetrates $(1200t - 36\,000t^2)$ cm in t s after impact. Calculate the maximum depth of penetration.

9 A rectangular sheep pen is to be made out of 1000 m of fencing, using an existing straight hedge for one of the sides. Find the maximum area possible, and the dimensions necessary to achieve this.

10 An aeroplane flying level at 250 m above the ground suddenly swoops down to drop supplies, and then regains its former altitude. It is h m above the ground t s after beginning its dive, where $h = 8t^2 - 80t + 250$. Find its least altitude during this operation, and the interval of time during which it was losing height.

11 Fig. 5.8 represents the end view of the outer cover of a match box, AB and EF being gummed together, and assumed to be the same length. If the total length of edge (ABCDEF) is 12 cm, calculate the lengths of AB and BC which will give the maximum possible cross-section area.

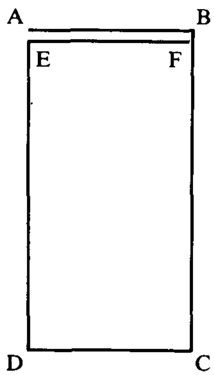


Figure 5.8

To differentiate the function $f(x) = x^{-1}$

5.3 In §3.6 we reached the conclusion that if $f(x) = x^n$, where $n \in \mathbb{Z}^+$, then $f'(x) = nx^{n-1}$, although we only *proved* that this was so for $n = 1, 2, 3$ and 4. In this section we shall prove that it is also true when $n = -1$, that is, we shall prove that if $f(x) = 1/x = x^{-1}$, then $f'(x) = -x^{-2} = 1/x^2$. We start by quoting the expression for $f'(x)$ in §3.8,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Now, in this case,

$$\begin{aligned} f(x+h) - f(x) &= \frac{1}{x+h} - \frac{1}{x} \\ &= \frac{x-(x+h)}{(x+h)x} \\ &= \frac{-h}{x(x+h)} \end{aligned}$$

Hence

$$\frac{f(x+h) - f(x)}{h} = \frac{-1}{x(x+h)}$$

and thus

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{-1}{x(x+h)} \\ &= -\frac{1}{x^2} \\ &= -x^{-2} \end{aligned}$$

We have proved that if $f(x) = x^{-1}$, then $f'(x) = -x^{-2}$ and this verifies that the general result, namely that if $f(x) = x^n$, then $f'(x) = nx^{n-1}$, is true when $n = -1$. We shall now assume that it is true for $n \in \mathbb{Z}$, that is, when n is a positive or negative integer, or zero.*

Qu. 4 Write down the derivative of

(a) x^{-4}, (b) $\frac{3}{x^2}$, (c) $\frac{2}{x^3}$, (d) $\frac{1}{2x^3}$, (e) $\frac{1}{x^m}$,

(f) $2x^2 - 3x + 4 + \frac{5}{x}$, (g) $\frac{x^3 + 3x - 4}{x^2}$.

Maxima and minima

5.4 In §5.2 we were dealing with a type of curve whose gradient was zero only at one point. With a more complicated curve (Fig. 5.9) the gradient may be zero

**Note.* $n = 0$ is a special case. The rule suggests that the gradient of $y = x^0$ is zero. Now $x^0 = 1$, (see §9.4) so the graph of $y = x^0$ is a straight line parallel to the x-axis, i.e. its gradient is zero. Consequently the result predicted by the rule is correct.

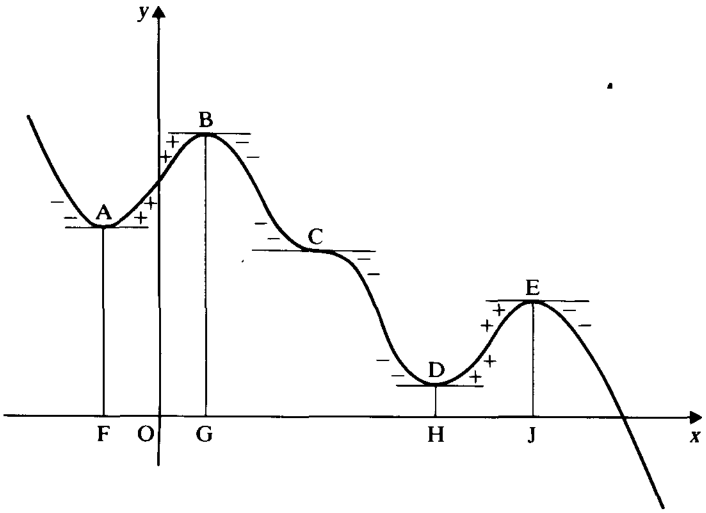


Figure 5.9

at a number of points, and the possible shapes fall into three categories. In this case, moving along the curve from left to right, that is with x increasing,

(a) at A and D, the gradient is changing from negative to positive, and these are called **minimum points**; FA and HD are **minimum values** of y (or **minima**),
(b) at B and E, the gradient is changing from positive to negative, and these are called **maximum points**; GB and JE are **maximum values** of y (or **maxima**).

The reader will note that the words maximum and minimum are used in the sense of greatest and least only in the immediate vicinity of the point; this local meaning is brought out clearly in this curve, since a maximum value, JE, is in fact less than a minimum value, FA, and for this reason the expressions *local* maximum and *local* minimum are often used.

(c) At C the gradient is zero, but is *not* changing sign; this is a **point of inflexion**, which may be likened to the point on an S-bend at which a road stops turning left and begins to turn right, or vice versa. The gradient of a curve at a point of inflexion need not be zero (the reader should be able to spot four more in Fig. 5.9); however at this stage we are concerned only with searching for maxima and minima, and we need to bear in mind points of inflexion only as a third possibility at points where the gradient is zero.

At any point where the gradient of a curve is zero, y is said to have a **stationary value**. Any maximum or minimum point is called a **turning point**, and y is said to have a **turning value** there.

Qu. 5 Copy Figs. 5.10–5.12, and on each draw the tangents at all points where the gradient is zero, and mark in the sign of the gradient for each segment of the curve. State whether the points marked are maxima, minima, or points of inflexion.

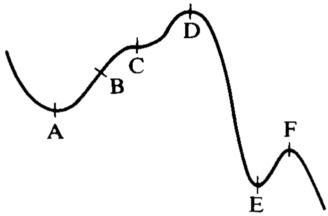


Figure 5.10

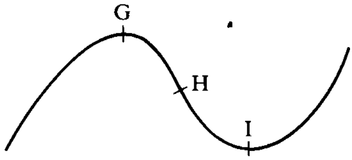


Figure 5.11

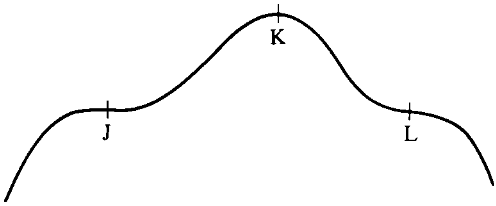


Figure 5.12

Consider the functions $f(x) = x^3$ and $g(x) = x^4$; sketches of their graphs are shown in Fig. 5.13.

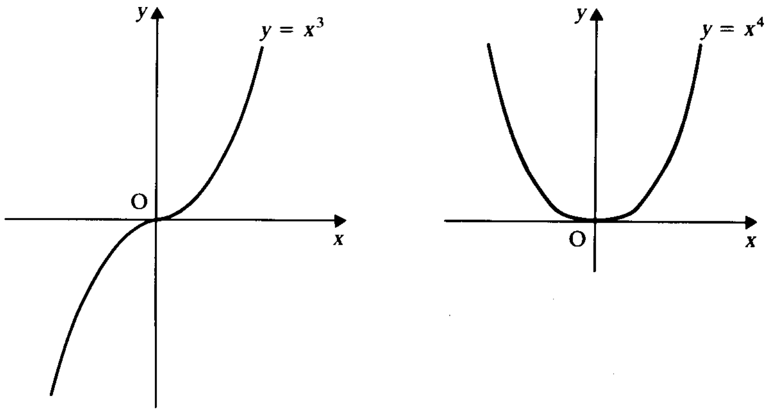


Figure 5.13

The derived functions are $f'(x) = 3x^2$ and $g'(x) = 4x^3$ and, in both cases, the derivative is zero when $x = 0$; this is confirmed by the graphs which both have zero gradient at the origin. Notice, however, that $f'(x) = 3x^2$ is never negative, which is in accordance with the observation that the graph of $y = x^3$ (see Fig. 5.13) always slopes upwards to the right, and has a point of inflexion at (0, 0). On the other hand, $g'(x)$ is negative for $x < 0$ and positive for $x > 0$. This also is in accordance with the graph of $y = x^4$ (see Fig. 5.13) which slopes downwards on the left and upwards on the right, and has a local minimum at (0, 0).

Example 4 *Investigate the stationary values of the function* $x^4 - 4x^3$.*

Let $y = x^4 - 4x^3$

$$\frac{dy}{dx} = 4x^3 - 12x^2$$

$$= 4x^2(x-3)$$

which is zero when $x = 0$ or $+3$.

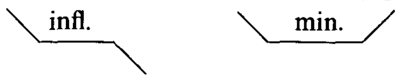
When $x = 0$, $y = 0$, and when $x = 3$, $y = -27$. Thus the stationary values of the function occur at $(0, 0)$ and at $(+3, -27)$.

[We now find the shape of the curve at these points by investigating the sign of the gradient just to the left and just to the right of each. Looking back to the factorised form of $\frac{dy}{dx}$, we see that $4x^2$ is positive for all values of x other than zero, so we are concerned with the sign of the factor $x - 3$ only.

When x is just less than 0, $x - 3$ is negative,
and when x is just greater than 0, $x - 3$ is negative.
When x is just less than $+3$, $x - 3$ is negative,
and when x is just greater than $+3$, $x - 3$ is positive.

These signs are entered in the table.]

Value of x	L	0	R	L	$+3$	R
Sign of $\frac{dy}{dx}$	$-$	0	$-$	$-$	0	$+$



The stationary values of $x^4 - 4x^3$ are 0 and -27; $(0, 0)$ is a point of inflexion; $(3, -27)$ is a minimum point.

The following example further illustrates the advisability of arranging the gradient function in a convenient factorised form, and brings out an important point in the investigation of the sign of the gradient for negative values of x.

Example 5 *Find the turning values of* y *on the graph* $y = f(x)$, *where*

$$f(x) = 5 + 24x - 9x^2 - 2x^3$$

and distinguish between them.

**Note.* The wording of this example illustrates that questions will often not specify the symbol for the dependent variable. The solution to such a question should normally start with a phrase like 'Let $y = x^4 - 4x^3$', as in this example, or, alternatively, 'Let $f(x) = \dots$'.

$$f(x) = 5 + 24x - 9x^2 - 2x^3$$
$$f'(x) = 24 - 18x - 6x^2 = -6(x^2 + 3x - 4)$$
$$= -6(x+4)(x-1)$$

which is zero when $x = -4$ or 1.

When $x = -4$,

$$y = 5 + 24 \times (-4) - 9 \times (-4)^2 - 2 \times (-4)^3 = -107$$

and when $x = 1$,

$$y = 5 + 24 - 9 - 2 = 18$$

Thus the stationary values of y occur at $(-4, -107)$ and $(1, 18)$.

[In completing the gradient table we must remember the negative factor -6, and find the sign of each factor $(x+4)$ and $(x-1)$; we shall then see if there are one, two or three negative factors, and so determine the sign of $f'(x)$.

Let us pay particular attention to the point $(-4, -107)$, and the sign of the factor $(x+4)$. To the *left*, when x is just *less* than -4 (e.g. -4.1), $(x+4)$ is negative, $(x-1)$ is also negative, thus $f'(x)$ has three negative factors and is negative. To the *right*, when x is just greater than -4 (e.g. -3.9), $(x+4)$ is now positive, $(x-1)$ is still negative, thus $f'(x)$ has two negative factors, and is positive.]

Value of x	L	-4	R	L	1	R
Sign of $f'(x)$	$-$	0	$+$	$+$	0	$-$
		min.			max.	

The turning values of y are -107 and 18; -107 is a minimum value; 18 is a maximum value.

Exercise 5b

1 Write down the values of x for which the following derived functions are zero, and prepare in each case a gradient table as in the foregoing examples, showing whether the corresponding points on the graphs are maxima, minima or points of inflexion:

(a) $f'(x) = 3x^2$, (b) $f'(x) = -4x^3$,
(c) $f'(x) = (x-2)(x-3)$, (d) $f'(x) = (x+3)(x-5)$,
(e) $f'(x) = (x+1)(x+6)$, (f) $f'(x) = -(x-1)(x-3)$,
(g) $f'(x) = -x^2 + x + 12$, (h) $f'(x) = -x^2 - 5x + 6$,
(i) $f'(x) = 15 - 2x - x^2$, (j) $f'(x) = 5x^4 - 27x^2$,
(k) $f'(x) = 1 - 4/x^2$.

2 Find any maximum or minimum values of the following functions:

(a) $f(x) = 4x - 3x^3$, (b) $f(x) = 2x^3 - 3x^2 - 12x - 7$,
(c) $f(x) = x^2(x-4)$, (d) $f(x) = x + 1/x$,
(e) $f(x) = x(2x-3)(x-4)$.

3 Find the turning points on the following curves, and state whether y has a maximum or minimum value at each:
(a) $y = x(x^2 - 12)$, (b) $y = x^3 - 5x^2 + 3x + 2$,
(c) $y = x^2(3 - x)$, (d) $y = 4x^2 + 1/x$,
(e) $y = x(x - 8)(x - 15)$.

4 Investigate the stationary values of y on the following curves:
(a) $y = x^4$, (b) $y = 3 - x^3$,
(c) $y = x^3(2 - x)$, (d) $y = 3x^4 + 16x^3 + 24x^2 + 3$.

5 Fig. 5.14 represents a rectangular sheet of metal 8 cm by 5 cm. Equal squares of side x cm are removed from each corner, and the edges are then turned up to make an open box of volume V cm^3. Show that $V = 40x - 26x^2 + 4x^3$. Hence find the maximum possible volume, and the corresponding value of x.

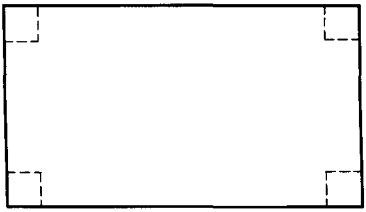


Figure 5.14

6 Repeat No. 5 when the dimensions of the sheet of metal are 8 cm by 3 cm, showing that in this case $V = 24x - 22x^2 + 4x^3$.

7 The size of a parcel despatched through the post used to be limited by the fact that the sum of its length and girth (perimeter of cross-section) must not exceed 6 feet. What was the volume of the largest parcel of square cross-section which was acceptable for posting? (Let the cross-section be a square of side x feet.)

8 Repeat No. 7 for a parcel of circular cross-section, leaving π in your answer.

9 A chemical factory wishes to make a cylindrical container, of thin metal, to hold 10 cm^3, using the least possible area of metal. If the outside surface is S cm^2, and the radius is r cm, show that $S = 2\pi r^2 + 20/r$ and hence find the required radius and height for the container. (Leave π in your answer.)

10 Repeat No. 9 showing that whatever may be the given volume, the area of metal will always be least when the height is twice the radius.

11 64 cm^3 of butter is to be made into a slab of square cross-section. Calculate the required length if the total surface area is to be as small as possible.

12 An open cardboard box with a square base is required to hold 108 cm^3. What should be the dimensions if the area of cardboard used is as small as possible?

Curve sketching

5.5 We have seen in §5.4 how maxima and minima problems may be solved without direct use of the relevant graph. Frequently however the determination of maximum and minimum points is a valuable aid in sketching a curve. (See §2.12 for a note on the difference between *sketching* and *plotting* a curve.)

Example 6 *Sketch the curve* $y = 4x^3 - 3x^4$.

(a) To find where the curve meets the x-axis, put $y = 0$, then

$$4x^3 - 3x^4 = 0$$
$$\therefore x^3(4 - 3x) = 0$$

Therefore the curve meets the x-axis at the points $(0, 0)$ and $(\frac{4}{3}, 0)$.

(b) To find where the curve meets the y-axis, put $x = 0$. The curve meets the y-axis at the origin.

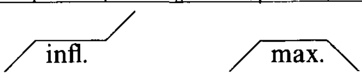
(c) To find stationary points:

$$y = 4x^3 - 3x^4$$
$$\therefore \frac{dy}{dx} = 12x^2 - 12x^3$$
$$= 12x^2(1 - x)$$

which is zero when $x = 0$ or 1.

Therefore (0, 0) and (1, 1) are stationary points.

Value of x	L	0	R	L	1	R
Sign of $\frac{dy}{dx}$	+	0	+	+	0	−



Hence (0, 0) is a point of inflexion and (1, 1) is a maximum.

These results may now be used to sketch the curve, as in Fig. 5.15.

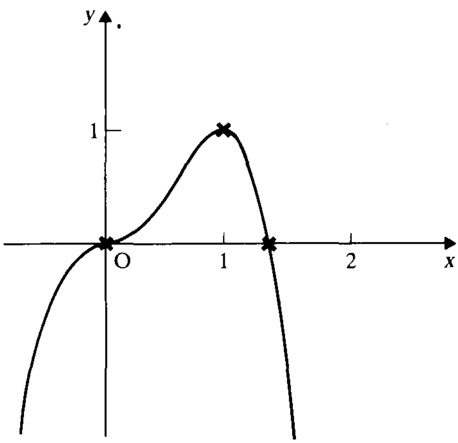


Figure 5.15

Exercise 5c

Find where the following curves meet the axes. Find, also, the coordinates of their stationary points and use these results to sketch the curves.

1 $y = 3x^2 - x^3$. **2** $y = x^3 - 6x^2$. **3** $y = x^3 - 2x^2 + x$.
4 $y = (x+1)^2(2-x)$. **5** $y = x^2(x-2)^2$. **6** $y = x^4 - 8x^3$.
7 $y = x^4 - 10x^2 + 9$. **8** $y = x^4 + 32x$. **9** $y = 4x^5 - 5x^4$.
10 $y = 3x^5 - 5x^3$. **11** $y = 2x^5 + 5x^2$.

Another useful approach to curve sketching is shown in the next example.

Example 7 *Sketch the curve* $y = (x+1)(x-1)(2-x)$.

(a) To find where the curve meets the x-axis, put $y = 0$, then

$$(x+1)(x-1)(2-x) = 0$$

Therefore the curve meets the x-axis at $(-1, 0)$, $(1, 0)$, $(2, 0)$.

(b) To find where the curve meets the y-axis, put $x = 0$. Thus the curve meets the y-axis at $(0, -2)$.

(c) To examine the behaviour of the curve 'at infinity', expand the R.H.S. of the equation:

$$y = (x^2 - 1)(2 - x) = -x^3 + 2x^2 + x - 2$$

Now, if x is large, the sign of y will be determined by the term of highest degree, $-x^3$. (If $x = 100$, say, $y = -1\,000\,000 + 20\,000 + 100 - 2$; or if $x = -100$, $y = 1\,000\,000 + 20\,000 - 100 - 2$. In either case the term in x^3 predominates.)

If x is large and positive, y is large and negative, and if x is large and negative, y is large and positive. Thus the behaviour of the curve as $x \to +\infty$ and $x \to -\infty$ is illustrated by Fig. 5.16.

The curve is then sketched, as in Fig. 5.17.

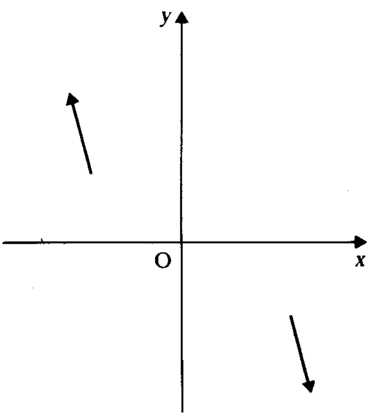


Figure 5.16

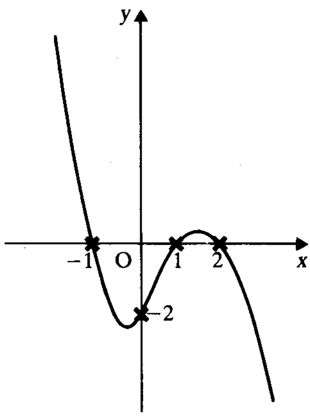


Figure 5.17

Distance, velocity and acceleration graphs

5.6 Useful physical interpretations of the graphical ideas discussed in §5.4 are obtained from the space-time, velocity-time, and acceleration-time graphs for the motion of a particle, if we plot one above the other as in the following example.

Example 8 *O is a point on a straight line. A particle moves along the line so that it is* s m *from* O, *t* s *after a certain instant, where* $s = t(t-2)^2$. *Describe the motion before and after* $t = 0$.

The *space-time graph* has the equation $s = t(t-2)^2$. By the methods of §5.5 we may determine that the graph has a max. point $(\frac{2}{3}, \frac{32}{27})$, a min. point (2, 0), and passes through (0, 0). We thus arrive at the upper sketch in Fig. 5.18.

The equation may be written $s = t^3 - 4t^2 + 4t$.

$$\therefore \frac{ds}{dt} = 3t^2 - 8t + 4 = (3t-2)(t-2)$$

Hence the *velocity-time graph* has the equation $v = (3t-2)(t-2)$. This graph has a min. point $(1\frac{1}{3}, -1\frac{1}{3})$, and passes through $(\frac{2}{3}, 0)$, (2, 0), and (0, 4); it is the middle sketch in Fig. 5.18.

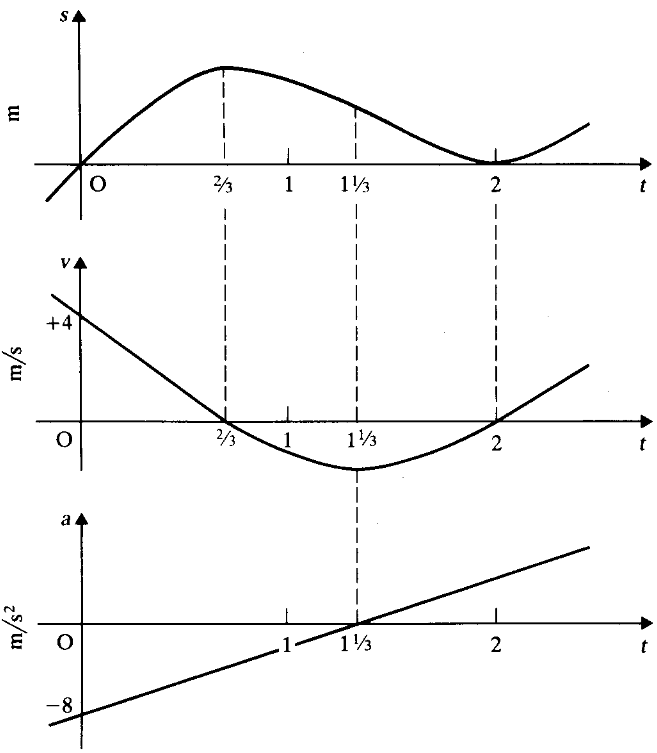


Figure 5.18

Differentiating once again, $\frac{dv}{dt} = 6t - 8$, and so the *acceleration-time graph* has the equation $a = 6t - 8$, and is the bottom sketch in Fig. 5.18.

Notice that the local max. and min. values of s occur when $v\left(\text{i.e. } \frac{ds}{dt}\right)$ is zero, and that the local min. value of v occurs when $a\left(\text{i.e. } \frac{dv}{dt}\right)$ is zero.

It is easy to visualise the motion of the particle as being along the Os axis of the space-time graph, its distance from O at any instant being given by the height of the graph for the corresponding value of t. Before $t = 0$, the particle is approaching O from the negative side; at $t = 0$, it is passing through O with velocity 4 m/s, and acceleration -8 m/s^2, hence its speed is decreasing. It comes momentarily to rest $\frac{32}{27}$ m from O (on the positive side) when $t = \frac{2}{3}$; it returns to O, where it is momentarily at rest when $t = 2$, and thereafter it moves away from O in the positive direction.

Some further points regarding the sign and direction of the velocity and acceleration deserve emphasis. Consider the three graphs between $t = 0$ and $t = 1\frac{1}{3}$; throughout this interval the acceleration is negative, and the velocity decreases from $+4$ m/s to $-1\frac{1}{3}$ m/s. The effect of the negative acceleration is to *decrease* the speed when the velocity is positive ($t = 0$ to $t = \frac{2}{3}$), and to *increase* the speed when the velocity is negative ($t = \frac{2}{3}$ to $t = 1\frac{1}{3}$). The reader should note the distinction between the *speed* and the *velocity*, the speed being the numerical value of the velocity, irrespective of direction.

Qu. 6 In Example 8, give the signs of the velocity, and acceleration, and state if the speed is increasing or decreasing, when (a) $t = 1\frac{1}{2}$, (b) $t = 3$, (c) $t = 1\frac{1}{3}$.

Exercise 5d

1 Make a rough sketch of each of the following curves by finding the points of intersection with the axes, and by investigating the behaviour of y as $x \to +\infty$ and as $x \to -\infty$. (Do *not* find maximum and minimum points).

(a) $y = (x + 2)(x - 3)$, (b) $y = (5 + x)(1 - x)$,
(c) $y = x(x + 1)(x + 2)$, (d) $y = (2 + x)(1 + x)(3 - x)$,
(e) $y = (x - 1)(x - 3)^2$, (f) $y = (x + 4)^2(x - 3)$,
(g) $y = -x(x - 7)^2$, (h) $y = x^2(5 - x)$,
(i) $y = (x - 2)^3$, (j) $y = (x - 3)^4$,
(k) $y = -x(x - 4)^3$.

2 A particle moves along a straight line OB so that t s after passing O it is s m from O, where $s = t(2t - 3)(t - 4)$. Deduce expressions for the velocity and acceleration in terms of t, and sketch the space-, velocity-, and acceleration-time graphs as in Fig. 5.18. Briefly describe the motion, and when $t = 2$ find
(a) where the particle is,
(b) if it is going towards or away from B,

(c) its speed,
(d) if its speed is increasing or decreasing,
(e) the rate of change of the speed.

3 Answer the questions in No. 2 for the instant when $t = 1$.

4 With the data of No. 2, when is the particle moving at its greatest speed away from B, and where is it then?

5 A particle is moving along a straight line OA in such a way that t s after passing through O for the first time it is s m from O where

$$s = -t(t-8)(t-15)$$

A is taken to be on the positive side of O. Deduce expressions for the velocity and acceleration in terms of t, and sketch the three graphs as in Fig. 5.18. Briefly describe the motion.
(a) Describe in detail the motion and position of the particle when $t = 10$.
(b) When is it moving towards A?
(c) When is it travelling at its greatest speed towards A?

6 A car in a traffic jam starts from rest with constant acceleration 2 m/s^2, and when its velocity reaches 6 m/s it remains constant at that figure for 4 s, and it is then reduced to zero in 6 s at a constant retardation. Sketch the space-, velocity-, and acceleration-time graphs for this motion.

Exercise 5e (Miscellaneous)

1 Find the coordinates of the points on the following curves at which y is a local maximum or a local minimum:
(a) $y = x^3 - 6x^2 + 9x + 2$, (b) $y = 2x^3 - 3x^2 - 12x + 8$,
(c) $y = x^3 - 3x$, (d) $y = 4x^3 - 3x^2 - 6x + 4$,
(e) $y = x^2(x^2 - 8)$, (f) $y = 2(x+1)(x-1)^2 + 1$.

2 Find the turning points of the graph $y = 2x^3 + 3x^2 - 12x + 7$, distinguishing between maximum and minimum values. Show that the graph passes through (1, 0) and one other point on the x-axis. Draw a rough sketch of the curve.

3 If $y = x^4 - 2x^2 + 1$, find the values of x for which y is a minimum and draw a rough sketch of the curve.

4 The equation of a curve is $y = x^3 - x^4 - 1$. Has y a maximum or a minimum value (a) when $x = \frac{3}{4}$, (b) when $x = 0$?

5 Prove that there are two points on the curve $y = 2x^2 - x^4$ at which y has a maximum value, and one point at which y has a minimum value. Give the equations of the tangents to the curve at these three points.

6 A point P whose x-coordinate is a is taken on the line $y = 3x - 7$. If Q is the point (4, 1) show that $PQ^2 = 10a^2 - 56a + 80$. Find the value of a which will make this expression a minimum. Hence show that the coordinates of N, the foot of the perpendicular from Q to the line, are $(2\frac{4}{5}, 1\frac{2}{5})$. Find the equation of QN.

7 The tangent to the curve of $y = ax^2 + bx + c$ at the point where $x = 2$ is parallel to the line $y = 4x$. Given that y has a minimum value of -3 where $x = 1$ find the values of a, b and c.

8 Find the equation of the tangent to the curve $xy = 4$ at the point P whose coordinates are $(2t, 2/t)$. If O is the origin and the tangent at P meets the x-axis at A and the y-axis at B, prove
(a) that P is the mid-point of AB,
(b) that the area of the triangle OAB is the same for all positions of P.

9 Find the equations of the normals to the curve $xy = 4$ which are parallel to the line $4x - y - 2 = 0$.

10 A solid rectangular block has a square base. Find its maximum volume if the sum of the height and any one side of the base is 12 cm.

11 A man wishes to fence in a rectangular enclosure of area 128 m^2. One side of the enclosure is formed by part of a brick wall already in position. What is the least possible length of fencing required for the other three sides?

12 The angle C of triangle ABC is always a right angle.
(a) If the sum of CA and CB is 6 cm, find the maximum area of the triangle.
(b) If, on the other hand, the hypotenuse AB is kept equal to 4 cm, and the sides CA, CB allowed to vary, find the maximum area of the triangle.

13 A piece of wire of length l is cut into two parts of lengths x and $l - x$. The former is bent into the shape of a square, and the latter into a rectangle of which the base is double the height. Find an expression for the sum of the areas of these two figures. Prove that the only value of x for which this sum is a maximum or a minimum is $x = 8l/17$, and find which it is.

14 A farmer has a certain length of fencing and uses it all to fence in two square sheep-folds. Prove that the sum of the areas of the two folds is least when their sides are equal.

15 Prove that, if the sum of the radii of two circles remains constant, the sum of the areas of the circles is least when the circles are equal.

16 An open tank is to be constructed with a horizontal square base and four vertical rectangular sides. It is to have a capacity of 32 m^3. Find the least area of sheet metal of which it can be made.

17 A sealed cylindrical jam tin is of height h cm and radius r cm. The area of its total outer surface is A cm^2 and its volume is V cm^3. Find an expression for A in terms of r and h. Taking $A = 24\pi$, find
(a) an expression for h in terms of r, and hence an expression for V in terms of r;
(b) the value of r which will make V a maximum.

18 (a) A variable rectangle has a constant perimeter of 20 cm. Find the lengths of the sides when the area is a maximum.
(b) A variable rectangle has a constant area 36 cm^2. Find the lengths of the sides when the perimeter is a minimum.

19 A cylinder is such that the sum of its height and the circumference of its base is 5 m. Express the volume (V m^3) in terms of the radius of the base (r m). What is the greatest volume of the cylinder?

20 An open tank is to be constructed with a square base and vertical sides so as to contain 500 m^3 of water. What must be the dimensions if the area of sheet metal used in its construction is to be a minimum?

21 The length of a rectangular block is twice the width, and the total surface area is 108 cm^2. Show that, if the width of the block is x cm, the volume is $\frac{4}{3}x(27-x^2)$ cm^3. Find the dimensions of the block when its volume is a maximum.

22 A circular cylinder open at the top is to be made so as to have a volume of 1 m^3. If r m is the radius of the base, prove that the total outside surface area is $(\pi r^2 + 2/r)$ m^2. Hence prove that this surface area is a minimum when the height equals the radius of the base.

23 A match box consists of an outer cover, open at both ends, into which slides a rectangular box without a top. The length of the box is one and a half times its breadth, the thickness of the material is negligible, and the volume of the box is 25 cm^3. If the breadth of the box is x cm, find, in terms of x, the area of material used. Hence show that, if the least area of material is to be used to make the box, the length should be 3.7 cm approximately.

24 Two opposite ends of a closed rectangular tank are squares of side x m and the total area of sheet metal forming the tank is S m^2. Show that the volume of the tank is $\frac{1}{4}x(S-2x^2)$ m^3. If the value of S is 2400, find the value of x for which the volume is a maximum.

25 The point P(x, y) lies on the curve $y = x^2$; the point A has coordinates (0, 1). Express AP^2 in terms of x. Hence find the positions of P for which AP^2 is least, and verify that for each of these positions the line AP is perpendicular to the tangent to the curve at P.

Chapter 6

Integration

The reverse of differentiation — geometrical interpretation

6.1 Suppose that instead of an equation of a curve, we take as our starting point a gradient function. For example, what is represented geometrically by the equation $\frac{dy}{dx} = \frac{1}{3}$?

The constant gradient $\frac{1}{3}$ indicates a straight line; $y = \frac{1}{3}x$ is the equation of the straight line of this gradient through the origin, and, on differentiation, it leads to $\frac{dy}{dx} = \frac{1}{3}$. But $y = \frac{1}{3}x$ is not the only possibility; any straight line of gradient $\frac{1}{3}$ may be written as $y = \frac{1}{3}x + c$, where c is a constant, and this is the most general equation which gives $\frac{dy}{dx} = \frac{1}{3}$.

Thus the equation $\frac{dy}{dx} = \frac{1}{3}$ represents the same as the equation $y = \frac{1}{3}x + c$, namely *all straight lines of gradient* $\frac{1}{3}$ (Fig. 6.1).

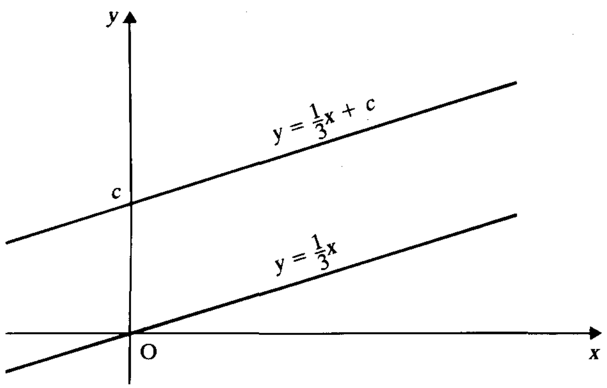


Figure 6.1

Let us take another example, $\frac{dy}{dx} = 2x$. We know that $y = x^2$ is a curve with this gradient function; but the most general equation leading to $\frac{dy}{dx} = 2x$ on differentiation is $y = x^2 + c$, where c is a constant.

Thus the equation $\frac{dy}{dx} = 2x$ represents the same as the equation $y = x^2 + c$, namely the family of curves 'parallel' to $y = x^2$ (see Fig. 3.8).

We have found that

$$\text{if } \frac{dy}{dx} = \tfrac{1}{3}, \quad \text{then} \quad y = \tfrac{1}{3}x + c$$

Also

$$\text{if } \frac{dy}{dx} = 2x, \quad \text{then} \quad y = x^2 + c$$

This process of finding the expression for y in terms of x when given the gradient function — in other words, the reverse of differentiation — is called **integration**.

$x^2 + c$ is called the **integral** of $2x$ with respect to x.

The constant c, which, unless further data is given, cannot be determined, is called the **arbitrary constant** of integration.

We know that when we differentiate a power of x, the index is reduced by 1, since $\frac{d}{dx}(x^n) = nx^{n-1}$. In this reverse process of integration we must therefore increase the index by 1, thus

$$\text{if } \frac{dy}{dx} = x, \qquad y = \frac{x^2}{2} + c$$

and

$$\text{if } \frac{dy}{dx} = 5x^2, \quad y = 5 \times \frac{x^3}{3} + c$$

The reader should check these by differentiating, and it will then be clear why the denominators 2 and 3 arise. The rule for integrating a power of x is seen to be 'increase the index by 1, and divide by the new index'.

Qu. 1 Integrate with respect to x:

(a) 2, (b) m, (c) $3x^2$, (d) $3x$,
(e) $3x^4$, (f) $3 + 2x$, (g) $x - x^2$, (h) $ax + b$.

Just as we have assumed that the rule for differentiating x^n is valid for $n \in \mathbb{Z}$, i.e. when n is any integer, positive or negative, so we shall make a similar assumption about the rule for integrating x^n, with the notable exception of x^{-1}.

In other words, for all positive and negative integral values of n, other than -1,

$$\text{if } \frac{dy}{dx} = x^n, \quad \text{then} \quad y = \frac{x^{n+1}}{n+1} + c$$

Thus if $\frac{dy}{dx} = 1/x^2 = x^{-2}$, then

$$y = \frac{x^{-2+1}}{-2+1} + c = \frac{x^{-1}}{-1} + c = -\frac{1}{x} + c$$

The reader should check this last result by differentiating, and in fact should make a habit of doing this always. It is important to remember that the arbitrary constant is an essential part of each integral.

Qu. 2 Integrate with respect to x:

(a) $\frac{1}{x^3}$, (b) x^{-4}, (c) $\frac{2}{x^2}$, (d) $\frac{1}{x^n}$.

Qu. 3 Why is the rule for integrating not valid when $n = -1$?

Reverting to our earlier examples, $\frac{dy}{dx} = \frac{1}{3}$ and $\frac{dy}{dx} = 2x$ are called **differential equations**, and $y = \frac{1}{3}x + c$ and $y = x^2 + c$ respectively are the **general solutions.** We saw that the differential equation $\frac{dy}{dx} = \frac{1}{3}$ represents all straight lines of gradient $\frac{1}{3}$; to be able to find the equation of a particular straight line of gradient $\frac{1}{3}$, we must find the appropriate value of c in the general solution $y = \frac{1}{3}x + c$, and to do this we need to know one point through which the line passes. The reader should now read again the alternative solution of Example 9 in §1.9; it will be seen that the process of finding the equation of a straight line of given gradient passing through a given point may be thought of as finding a particular solution of a differential equation.

Qu. 4 $\frac{dy}{dx} = 4$. Find y in terms of x, given that $y = 10$ when $x = -2$. What does the solution represent graphically?

Exercise 6a

1 Integrate:

(a) with respect to x: $\frac{1}{2}$, $\frac{1}{2}x^2$, $x^2 + 3x$, $(2x+3)^2$, x^{-5}, $\frac{-2}{x^4}$;

(b) with respect to t: at, $\frac{1}{3}t^3$, $(t+1)(t-2)$, $\frac{1}{t^{n+1}}$, $\frac{1}{t^2} + 3 + 2t$;

(c) with respect to y: $-ay^{-2}$, $\frac{k}{y^2}$, $\frac{(y^2+2)(y^2-3)}{y^2}$.

2 Solve the following differential equations:

(a) $\dfrac{dy}{dx} = 3ax^2$, (b) $\dfrac{ds}{dt} = 3t^3$,

(c) $\dfrac{ds}{dt} = u + at$, (d) $\dfrac{dx}{dt} = \left(1 + \dfrac{1}{t}\right)\left(1 - \dfrac{1}{t}\right)$,

(e) $\dfrac{dy}{dt} = \dfrac{t^3 - 3t + 4}{t^3}$, (f) $\dfrac{dA}{dx} = \dfrac{(1 + x^2)(1 - 2x^2)}{x^2}$.

3 What is the gradient function of a straight line passing through $(-4, 5)$ and $(2, 6)$? Find its equation.

4 A curve passes through the point $(3, -1)$ and its gradient function is $2x + 5$. Find its equation.

5 A curve passes through the point $(2, 0)$ and its gradient function is $3x^2 - 1/x^2$. Find its equation.

6 The gradient of a curve at the point (x, y) is $3x^2 - 8x + 3$. If it passes through the origin, find the other points of intersection with the x-axis.

7 The gradient of a curve at the point (x, y) is $8x - 3x^2$, and it passes through the origin. Find where it cuts the x-axis, and find the equation of the tangent parallel to the x-axis.

8 Find s in terms of t if $\dfrac{ds}{dt} = 3t - 8/t^2$, given that $s = 1\frac{1}{2}$ when $t = 1$.

9 Find A in terms of x if $\dfrac{dA}{dx} = (3x + 1)(x^2 - 1)/x^5$. What is the value of A when $x = 2$, if $A = 0$ when $x = 1$?

Velocity and acceleration

6.2 In Chapter 4 we used the formula $s = 4.9t^2$ for a stone falling from rest, and it was explained that this is based on the assumption that the acceleration of the stone is 9.8 metres per second per second, or 9.8 m/s^2. We are now in a position to see how the formula is deduced from this assumption by the process of integration.

If the acceleration is given by

$$\frac{dv}{dt} = 9.8$$

then

$$v = 9.8t + c$$

Now if the stone falls from rest at the instant from which we measure the time, $v = 0$ when $t = 0$, and substituting these values in the last equation we get $c = 0$.

$$\therefore v = 9.8t$$

This may be written

$$\frac{ds}{dt} = 9.8t$$

from which

$$s = 4.9t^2 + k$$

If s measures the distance below the initial position of the stone, $s = 0$ when $t = 0$, and substituting these values in the last equation, we get $k = 0$.

$$\therefore s = 4.9t^2$$

Qu. 5 A stone is thrown vertically downwards from the top of a cliff at 15 m/s. Assuming that its acceleration due to gravity is 9.81 m/s^2, find expressions for its velocity and position t s later, by solving the differential equation $\frac{dv}{dt} = 9.81$.

It again needs emphasising that displacement (s), velocity and acceleration in a straight line are positive in one direction, negative in the other, and it is important to decide at the outset which is to be taken as the positive direction. The reader should take upwards as positive in Qu. 6.

Qu. 6 A stone is thrown vertically upwards from the edge of a cliff at 19.6 m/s. Assuming that gravity produces a downwards acceleration of 9.8 m/s^2, deduce the velocity and position of the stone after 1, 3 and 5 s. Explain the sign of each answer, taking upwards as positive.

Example 1 *Fig.* 6.2 *represents part of a conveyor belt, the dots being small articles on it at* 1 m *spacing. Initially the belt is at rest with the article* R 7 m *short of* O, *a fixed mark on a wall. The belt is accelerated from rest so that its velocity is* $0.1t$ m/s, t s *after starting. Find* (a) *the position of* R *when* $t = 10$, *and* (b) *the distance moved by* R *between* $t = 3$ *and* $t = 5$.

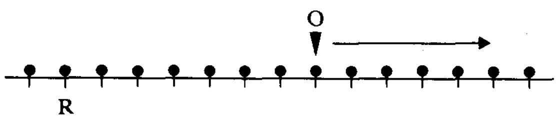


Figure 6.2

(a) If the distance from O at time t s is s m (positive to the right of O, negative to the left), then it is true of each article that its velocity, $\frac{ds}{dt} = 0.1t$, and also, by integration, that

$$s = 0.05t^2 + c$$

However, this last equation does not give us the distance of any particular article from O, until we have discovered the appropriate value of c. Since when $t = 0$, $s = c$, the arbitrary constant of integration in this case represents the initial position of an article.

In the case of R, when $t=0$,

$$s=-7$$

Substituting in the last equation, $-7=0+c$,

$$\therefore c=-7$$

Therefore the distance of R from O at time t s is s m where

$$s=0.05t^2-7$$

When $t=10$,

$$s=0.05\times 100-7=-2$$

$\therefore$ R is 2 m short of O at this instant.

(b) The distance moved by each article in any given interval is the same, therefore we are not concerned with any particular numerical value for the constant of integration, and we shall leave c in our working.

As before, since $\dfrac{ds}{dt}=0.1t$,

$$s=0.05t^2+c$$

When $t=3$,

$$s=0.05\times 3^2+c$$

When $t=5$,

$$s=0.05\times 5^2+c$$

The distance moved between $t=3$ and $t=5$ is

$$\begin{aligned}&(0.05\times 5^2+c)-(0.05\times 3^2+c)\text{ m}\\=&\,0.05\times 25+c-0.05\times 9-c\text{ m}\\=&\,0.8\text{ m}\end{aligned}$$

(b) (*Alternative layout*) The following square bracket notation is an instruction to substitute and subtract, and shortens the working.

$$\frac{ds}{dt}=0.1t$$

$$\therefore s=0.05t^2+c$$

The distance moved between $t=3$ and $t=5$ is

$$\begin{aligned}\Big[0.05t^2+c\Big]_3^5\text{ m}&=(0.05\times 25+c)-(0.05\times 9+c)\text{ m}\\&=1.25+c-0.45-c\text{ m}\\&=0.8\text{ m}\end{aligned}$$

Qu. 7 Evaluate:

(a) $\left[3t+8\right]_2^5$, (b) $\left[3t^2-t+k\right]_1^4$,

(c) $\left[t^2-t\right]_{-2}^{+1}$, (d) $\left[t^3-3t^2+t\right]_{-3}^{-2}$.

Qu. 8 A particle moves in a straight line with velocity $2t^2$ m/s, t s after the start. Find the distance moved in the 3rd second.

Qu. 9 With the data of Example 1, answer the following questions.
(a) Find the position of R when $t = 20$.
(b) Find the position when $t = 10$ of the article initially at O.
(c) An article N is 2.2 m past O when $t = 2$; find its position when $t = 10$.
(d) An article T is 99.95 m short of O when $t = 1$; find its initial position.

Exercise 6b

1 A stone is thrown vertically downwards at 20 m/s from the top of a cliff. Assuming that gravity produces on it an acceleration of 9.81 m/s^2, deduce, from the differential equation $\dfrac{dv}{dt} = 9.81$, expressions for its velocity and position t s later.

2 A stone is thrown vertically upwards from ground level at 12 m/s, at a point immediately above a well. Taking the downwards direction as positive, deduce, from the differential equation $\dfrac{dv}{dt} = 9.8$, expressions for the stone's velocity and position t s later. Find the velocity and position after 1, 2, 3 s, explaining the sign of each answer.

3 Find the displacement (s) in terms of time (t) from the following data:

(a) $\dfrac{ds}{dt} = 3$, $s = 3$ when $t = 0$,

(b) $v = 4t - 1$, $s = 0$ when $t = 2$,

(c) $v = (3t-1)(t+2)$, $s = 1$ when $t = 2$,

(d) $v = t^2 + 5 - \dfrac{2}{t^2}$, $s = \frac{1}{3}$ when $t = 1$.

4 Evaluate:

(a) $\left[8t+c\right]_1^5$, (b) $\left[3t^2+5t\right]_2^{10}$,

(c) $\left[t^2-4t\right]_{-3}^{0}$, (d) $\left[2t^3-t^2-t\right]_{-2}^{-1}$.

5 Find s in terms of t, and the distance moved in the stated interval (the units being metres and seconds), given that

(a) $\frac{ds}{dt} = 4t + 3$, $t = 0$ to $t = 2$,

(b) $v = t^2 - 3$, $t = 2$ to $t = 3$,

(c) $v = (t-1)(t-2)$, $t = -1$ to $t = 0$,

(d) $v = t + 3 - \frac{1}{t^2}$, $t = 10$ to $t = 20$.

6 If a particle moves in a straight line so that its acceleration in terms of the time is At (A being a constant), deduce expressions for the velocity and displacement at time t.

7 Deduce expressions for v and s from the following data, determining the constants of integration whenever possible:
(a) $a = 3t$, $s = 0$ and $v = 3$ when $t = 0$,
(b) $a = 2 + t$, $s = -3$ and $v = 0$ when $t = 0$,
(c) $a = 10 - t$, $v = 2$ when $t = 1$, $s = 0$ when $t = 0$,
(d) $a = \frac{1}{2}t$, $v = 5$ when $t = 0$,
(e) $a = t^2$, $s = 10$ when $t = 1$.

8 A system of particles moves along a straight line OA so that t s after a certain instant their velocity is v m/s where $v = 3t$.
(a) One of the particles is at O when $t = 0$. Find its position when $t = 3$.
(b) A second particle is 4 m past O when $t = 1$. Find its position when $t = 0$.
(c) A third particle is 10 m short of O when $t = 2$. Find its position when $t = 4$.
(d) Find the distance moved by the particles during the 3rd second.

9 A particle moves along a straight line OA with velocity $(6 - 2t)$ m/s. When $t = 1$ the particle is at O.
(a) Find an expression for its distance from O in terms of t, and deduce the net change in position which takes place between $t = 0$ and $t = 5$.
(b) By finding the time at which it is momentarily at rest, calculate the actual distance through which it moves during the same interval.
(c) Sketch the space-time and velocity-time graphs from $t = 0$ to $t = 6$.

10 A stone is thrown vertically upwards from ground level with a velocity of 12.6 m/s. If the acceleration due to gravity is 9.8 m/s^2, deduce, from the differential equation $\frac{dv}{dt} = -9.8$, expressions for its velocity and its height t s later. Find
(a) the time to the highest point,
(b) the greatest height reached,
(c) the distance moved through by the stone during each of the first two seconds of motion.

11 A train runs non-stop between two stations P and Q, and its velocity t hours after leaving P is $60t - 30t^2$ km/h. Find
(a) the distance between P and Q,

(b) the average velocity for the journey,
(c) the maximum velocity attained.

12 A stopping train travels between two adjacent stations so that its velocity is v km/min, t min after leaving the first, where $v = \frac{4}{3}t(1-t)$. Find
(a) the average velocity for the journey in km/h,
(b) the maximum velocity in km/h.

13 The formula connecting the velocity and time for the motion of a particle is $v = 1 + 4t + 6t^2$. Find the average velocity and the average acceleration for the interval $t = 1$ to $t = 3$, the units being metres and seconds.

14 A racing car starts from rest and its acceleration after t s is $(k - \frac{1}{6}t)$ m/s^2 until it reaches a velocity of 60 m/s at the end of 1 min. Find the value of k, and the distance travelled in this first minute.

15 A particle starting from rest at O moves along a straight line OA so that its acceleration after t s is $(24t - 12t^2)$ m/s^2.
(a) Find when it again returns to O and its velocity then.
(b) Find its maximum displacement from O during this interval.
(c) What is its maximum velocity and its greatest speed during this interval?

16 P and R are two adjacent railway stations, and Q is a signal box on the line between them. A train which stops at P and R has a velocity of $(\frac{3}{8} + \frac{1}{2}t - \frac{1}{2}t^2)$ km/min at t min past noon, and it passes Q at noon. Find
(a) the times of departure from P and arrival at R,
(b) an expression for the distance of the train from P in terms of t,
(c) the average velocity between P and R, in km/h,
(d) the maximum velocity attained, in km/h.

The area under a curve

6.3 Another important aspect of integration is that it enables us to calculate exactly the areas enclosed by curves.

Let us consider the area enclosed by the axes, the line $x = 3$, and part of the curve $y = 3x^2 + 2$. This is the area TUVO in Fig. 6.3.

P is the point (x, y) on the curve, PM is its y-coordinate, and the area TPMO we shall call A. Now if we move P along the curve, A increases or decreases as x increases or decreases; clearly the size of A depends upon the value of x, i.e. A is a function of x, and our present aim is to find an expression for A in terms of x.

With the usual notation Q is the point $(x + \delta x, y + \delta y)$ adjacent to P, and QN is its y-coordinate. If we move the right-hand boundary of A from PM to QN, we increase x by δx, and the resulting increase in A, the shaded area PQNM, we call δA. In other words δA is the increment in A corresponding to the increment δx in x. It can be seen from Fig. 6.3 that δA lies between the areas of the two rectangles PRNM, $y\,\delta x$, and SQNM, $(y + \delta y)\delta x$. This may be written*

$$y\,\delta x < \delta A < (y + \delta y)\delta x$$

*This statement is called an inequality. $<$ means 'is less than'; $>$ means 'is greater than'. The reader should note in passing that an inequality is reversed by changing the sign of each term. Thus $1 < 2 < 3$, but $-1 > -2 > -3$; this explains the reference to δx being positive.

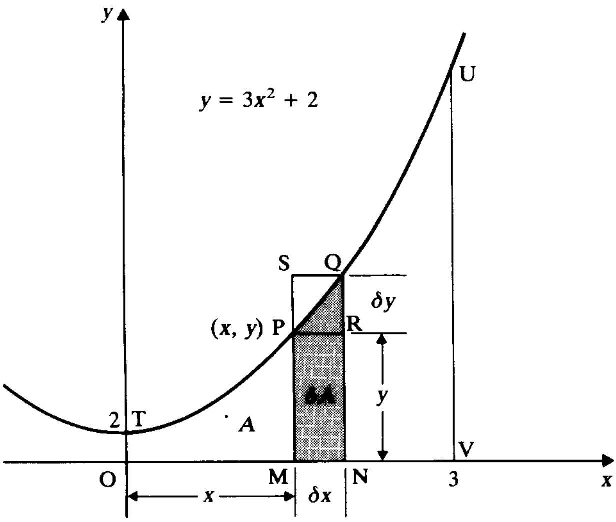


Figure 6.3

and dividing by δx, which is positive,

$$y < \frac{\delta A}{\delta x} < (y + \delta y)$$

Now as $\delta x \to 0$, $\delta y \to 0$, and so $(y + \delta y) \to y$. Thus we find that $\frac{\delta A}{\delta x}$ lies between y and something which we can make as near to y as we please, by making δx sufficiently small. Therefore the limit of $\frac{\delta A}{\delta x}$ is y, and writing the limit of $\frac{\delta A}{\delta x}$ as $\frac{dA}{dx}$, we get

$$\mathbf{\frac{dA}{dx} = y}$$

$$\therefore \frac{dA}{dx} = 3x^2 + 2$$

and by integration,

$$A = x^3 + 2x + c$$

If we were to bring in the right-hand boundary of the area A from PM to TO, we should reduce A to zero; that is to say, when $x = 0$, $A = 0$. Substituting these values in the last equation we find that $c = 0$.

$$\therefore A = x^3 + 2x$$

and we have achieved our immediate aim of expressing A in terms of x; now to find the area TUVO. In this case, the right-hand boundary of A has been pushed out from PM to UV, and x is increased to 3.

When $x = 3$,

$$A = 3^3 + 2 \times 3 = 33$$

$\therefore$ the area TUVO = 33.

Example 2 *Find the area enclosed by the x-axis, the curve $y = 3x^2 + 2$ and the straight lines $x = 3$ and $x = 5$.*

The required area is UWZV in Fig. 6.4, and it may be found as the difference between the areas TWZO and TUVO. Using A as above,

$$\frac{dA}{dx} = y = 3x^2 + 2$$

$$\therefore A = x^3 + 2x$$

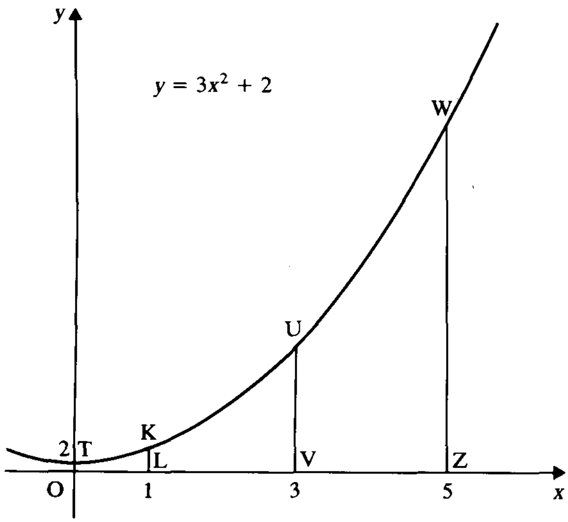


Figure 6.4

(We have shown above that the constant of integration is zero.)
When $x = 5$,

$$A = 5^3 + 2 \times 5 = 135 \quad \text{(Area TWZO)}$$

and when $x = 3$,

$$A = 3^3 + 2 \times 3 = 33 \quad \text{(Area TUVO)}$$

$\therefore$ the area UWZV = 135 − 33 = 102.

Qu. 10 Find the area enclosed by the x-axis, the curve $y = 3x^2 + 2$, and the following straight lines:
(a) the y-axis and $x = 4$, (b) $x = 1$ and $x = 2$,
(c) $x = -1$ and $x = 3$, (d) $x = -3$ and $x = -2$.

In all the working so far in this chapter we have used the symbol A to denote an area having the y-axis as its left-hand boundary. Suppose that instead we had, in Fig. 6.3, defined a similar area A' having the line $x = 1$ as its left-hand boundary. By the same process of reasoning we should arrive at the result

$$\frac{dA'}{dx} = y = 3x^2 + 2$$

$$\therefore A' = x^3 + 2x + k$$

But $A' = 0$ when $x = 1$, and substituting these values we get $k = -3$.

$$\therefore A' = x^3 + 2x - 3$$

Now A' is measured to the right from the line KL $(x = 1)$ in Fig. 6.4, and Example 2 might just as well be done using A' instead of A, finding the area UWZV as the difference between the areas KWZL and KUVL. Thus, when $x = 5$,

$$A' = 5^3 + 2 \times 5 - 3 = 135 - 3$$

and when $x = 3$,

$$A' = 3^3 + 2 \times 3 - 3 = 33 - 3$$

$\therefore$ the area UWZV $= (135 - 3) - (33 - 3) = 102$.

In each solution we have determined the constant of integration; using A, it is zero, and using A', it is -3. But as is clear from the second solution, the constant drops out on subtraction. We could in fact have measured A from any convenient left-hand boundary, and found the area UWZV by subtraction, without evaluating the constant of integration.

We shall from now onwards assume the relationship $\frac{dA}{dx} = y$ to calculate areas of this nature, and the square bracket notation introduced in §6.2 may now be put to further use, as is illustrated in the next example.

Example 3 *Find the area enclosed by the x-axis, $x = 1$, $x = 3$ and the graph $y = x^3$.* (Fig. 6.5).

$$\frac{dA}{dx} = y = x^3$$

$$\therefore A = \tfrac{1}{4}x^4 + c$$

The required area $= \left[\tfrac{1}{4}x^4 + c\right]_1^3$,

$$= (\tfrac{81}{4} + c) - (\tfrac{1}{4} + c)$$
$$= \tfrac{81}{4} + c - \tfrac{1}{4} - c$$
$$= 20$$

The area evaluated in Example 3 is called *the area under the curve $y = x^3$ from $x = 1$ to $x = 3$.*

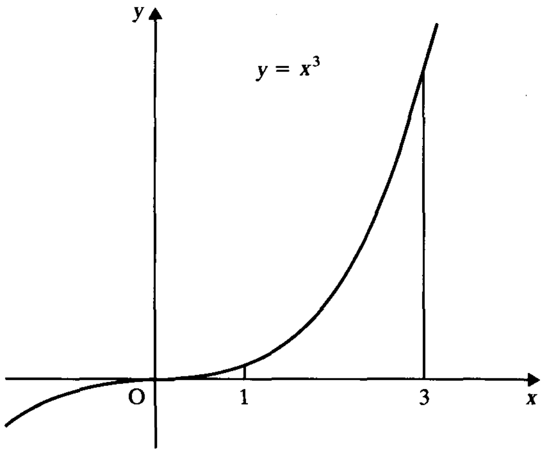


Figure 6.5

1 and 3 are called, respectively, the *lower* and *upper limits of integration.*

The integral $\frac{1}{4}x^4 + c$, involving the arbitrary constant of integration, is called an **indefinite integral.**

When however limits are given, and the integral may be evaluated, e.g. $\left[\frac{1}{4}x^4 + c\right]_1^3$, it is called a **definite integral.** Since the constant of integration drops out in a definite integral, it is not necessary to write it in the bracket.

Qu. 11 Evaluate the following definite integrals:

(a) $\left[3x^2 + 2x\right]_{1/2}^{1}$, (b) $\left[x^4 - 2x^2\right]_{-1}^{2}$,

(c) $\left[x^3 - 3x\right]_{-2}^{0}$, (d) $\left[2x^2 + 4x\right]_{-3}^{-1}$,

(e) $\left[x^4 - 2x^3 + x^2 - x\right]_{-2}^{0}$, (f) $\left[x^2 + 3x - \frac{1}{x^3}\right]_{+1/2}^{+1}$.

Qu. 12 Find the area under $y = \frac{1}{2}x$ from $x = 0$ to $x = 10$ by integration. Check by another method.

Qu. 13 Find the area under

(a) $y = x^2$ from $x = 0$ to $x = 3$, (b) $y = 2x^2 + 1$ from $x = 2$ to $x = 5$.

Two further examples will illustrate the advisability of making a rough sketch in this work if the reader is in doubt as to the shape and position of any curve; they also bring out two important points.

Example 4 *Find the area under the curve* $y = x^2(x - 2)$ (a) *from* $x = 0$ *to* $x = 2$, *and* (b) *from* $x = 2$ *to* $x = \frac{8}{3}$.

Consideration of the sign of the highest degree term, and the points of intersection with the x-axis, enables an adequate sketch to be made (Fig. 6.6).

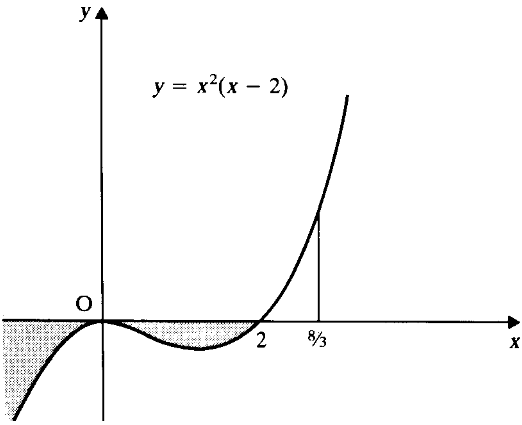


Figure 6.6

$$\frac{dA}{dx} = y = x^3 - 2x^2$$

$$\therefore A = \tfrac{1}{4}x^4 - \tfrac{2}{3}x^3 + c$$

(a) The required area

$$= \left[\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3\right]_0^2$$

$$= (\tfrac{1}{4} \times 2^4 - \tfrac{2}{3} \times 2^3) - (0)$$

$$= -1\tfrac{1}{3}$$

(b) The required area

$$= \left[\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3\right]_2^{8/3}$$

$$= \left(\frac{1}{4} \times \frac{8^4}{3^4} - \frac{2}{3} \times \frac{8^3}{3^3}\right) - \left(\frac{1}{4} \times 2^4 - \frac{2}{3} \times 2^3\right)$$

$$= (0) - (-1\tfrac{1}{3})$$

$$= +1\tfrac{1}{3}$$

Part (a) of this example illustrates that *the area under a curve is negative below the x-axis*. The reader should verify that $\left[\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3\right]_0^{8/3}$ is zero, and now that we have the convention about the sign of an area, we see that this is because it represents the sum of the two areas we have evaluated, numerically equal but of opposite sign.

The reader should now appreciate that a sketch of the relevant curve may help to avoid misleading results arising from perfectly correct calculation.

Qu. 14 Confirm that the total area enclosed by $y = x^2(x-2)$, the x-axis, $x = 1$ and $x = 3$ is $4\frac{1}{2}$.

What is the value of $\left[\frac{1}{4}x^4 - \frac{2}{3}x^3\right]_1^3$?

Qu. 15 Sketch the curve $y = x(x-1)(x-2)$. Find the total area enclosed between this curve and the x-axis.

Example 5 (a) *Find the area under $y = 1/x^2$ from $x = 1$ to $x = 2$.* (b) *Can any meaning be attached to the phrase 'the area under $y = 1/x^2$ from $x = -1$ to $x = +2$'?*

(a) $\dfrac{dA}{dx} = y = \dfrac{1}{x^2} = x^{-2}$

$\therefore A = -x^{-1} + c$

The required area

$= \left[-\dfrac{1}{x}\right]_1^2$

$= (-\frac{1}{2}) - (-1)$

$= -\frac{1}{2} + 1$

$= \frac{1}{2}$

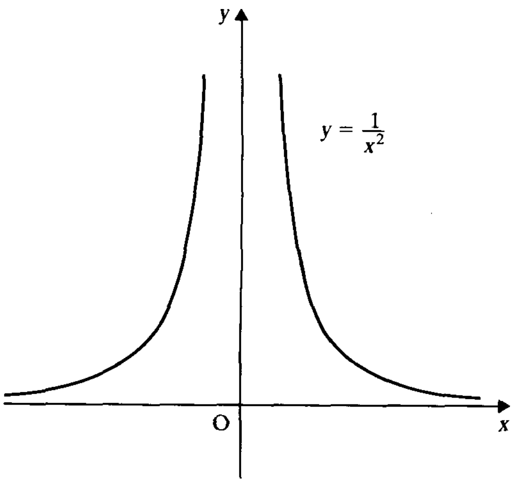


Figure 6.7

(b) Fig. 6.7 is a sketch of $y = \dfrac{1}{x^2}$, and we see that if we try to find the area under the graph from $x = -1$ to $x = 2$, between these limits is the value $x = 0$ for which y has no value, and the curve consists of two separate branches. It is possible to go through the motions of evaluating $\left[-\frac{1}{x}\right]_{-1}^{+2}$ but the result, $-1\frac{1}{2}$, is meaningless. If we break up the area into two parts and integrate from -1 to 0 and from 0 to 2, in each case we get the meaningless term $\frac{1}{0}$. (See §2.5.)

The second part of Example 5 illustrates that in order that we may calculate the area under a curve, the curve must have no breaks between the limits of x involved, i.e. the function must be continuous (see §2.19) for all values of x between these limits.

Exercise 6c

1 Evaluate:

(a) $\left[\frac{x^4}{4}\right]_{1/2}^{2}$, (b) $\left[3x^3 - 4x\right]_{-1}^{+1}$,

(c) $\left[\frac{1}{6}x^3 - 3x^2 + \frac{1}{2}x\right]_{-2}^{-1}$, (d) $\left[x^3 - \frac{1}{x^2}\right]_{-4}^{-3}$.

2 Find the area enclosed by $x + 4y - 20 = 0$ and the axes, by integration. Check by another method.

3 Find the areas enclosed by the x-axis, and the following curves and straight lines:
(a) $y = 3x^2$, $x = 1$, $x = 3$,
(b) $y = x^2 + 2$, $x = -2$, $x = 5$,
(c) $y = x^2(x-1)(x-2)$, $x = -2$, $x = -1$,
(d) $y = 3/x^2$, $x = 1$, $x = 6$.

4 Find the area under $y = 4x^3 + 8x^2$ from $x = -2$ to $x = 0$.

5 Sketch the curve $y = x^2 - 5x + 6$ and find the area cut off below the x-axis.

6 Sketch the curve $y = x(x+1)(2-x)$, and find the area of each of the two segments cut off by the x-axis.

7 Sketch the following curves and find the areas enclosed by them, and by the x-axis, and the given straight lines:
(a) $y = x(4-x)$, $x = 5$, (b) $y = -x^3$, $x = -2$,
(c) $y = x^3(x-1)$, $x = 2$, (d) $y = 1/x^2 - 1$, $x = 2$.

8 Find the area of the segment cut off from $y = x^2 - 4x + 6$ by the line $y = 3$.

9 Repeat No. 8 for the curve $y = 7 - x - x^2$, and $y = 5$.

10 Find the points of intersection of the following curves and straight lines, and find the area of the segment cut off from each curve by the corresponding straight line:
(a) $y = \frac{1}{2}x^2$, $y = 2x$,
(b) $y = 3x^2$, $3x + y - 6 = 0$,
(c) $y = (x+1)(x-2)$, $x - y + 1 = 0$.

11 Find the areas enclosed by the following curves and straight lines:
(a) $y = \frac{1}{2}x^3$, the y-axis, and $y = 32$,

(b) $y = x^3 - 1$, the axes and $y = 26$,
(c) $y = 1/x^2 - 1$, $y = -1$, $x = \frac{1}{2}$ and $x = 2$.

12 Find the area enclosed by the curves $y = 2x^2$ and $y = 12x^2 - x^3$.

Exercise 6d (Miscellaneous)

1 If $\dfrac{dy}{dx} = (3x - 2)/x^3$ find y in terms of x, if $y = 1$ when $x = 1$.

2 If $f'(x) = 2x - 1/x^2$ and if $f(1) = 1$, find $f(x)$.

3 The curve $y = 6 - x - x^2$ cuts the x-axis in two points A and B. By integration find the area enclosed by the x-axis and that portion of the curve which lies between A and B.

4 Sketch the curve $y = x^2 - x - 2$ from $x = -2$ to $x = 3$. Find the area bounded by the curve and the x-axis.

5 Sketch roughly the curve $y = x^2(3 - x)$ between $x = -1$ and $x = 4$. Calculate the area bounded by the curve and the x-axis.

6 For the curve $y = 12x - x^3$, find the area bounded by the curve and the positive x-axis.

7 The velocity v of a point moving along a straight line is given in terms of the time t by the formula $v = 2t^2 - 9t + 10$, the point being at the origin when $t = 0$. Find expressions in terms of t for the distance from the origin, and the acceleration. Show that the point is at rest twice, and find its distances from the origin at those instants.

8 The velocity v of a point moving along a straight line is connected with the time t by the formula $v = t^2 - 3t + 2$, the units being metres and seconds. If the distance of the point from the origin is 5 m when $t = 1$, find its position and acceleration when $t = 2$.

9 A particle moves in a straight line with a velocity of v m/s after t s, where $v = 3t^2 + 2t$. Find the acceleration at the end of 2 s, and the distance it travels in the 4th second.

10 Find the equation of the curve which passes through the point $(-1, 0)$ and whose gradient at any point (x, y) is $3x^2 - 6x + 4$. Find the area enclosed by the curve, the axis of x and the ordinates $x = 1$ and $x = 2$.

11 Draw in the same figure, for values of x from 0 to 6, a sketch of the curve $y = 6x - x^2$ and the line $y = 2x$. Calculate the area enclosed by them.

12 The parabola $y = 6x - x^2$ meets the x-axis at O and A. The tangents at O and A meet at T. Show that the curve divides the area of the triangle OAT into two parts in the ratio 2:1.

13 The curve $y = x(x - 1)^2$ touches the x-axis at the point A. B is the point (2, 2) on the curve and N is the foot of the perpendicular from B to the x-axis. Prove that the tangent at B divides the area between the arc AB, BN, and AN in the ratio 11:24.

14 The point P moves in a straight line with an acceleration of $(2t - 4)$ m/s^2 after t s. When $t = 0$, P is at O and its velocity is 3 m/s. Find
(a) the velocity of P after t s,
(b) the value of t when P starts to return to O,

(c) the distance of P from O at this moment.

15 A train starts from rest and its acceleration t s after the start is $0.1(20-t)$ m/s^2. What is its speed after 20 s? Acceleration ceases at this instant and the train proceeds at this uniform speed. What is the total distance covered 30 s after the start from rest, to the nearest metre?

16 A particle moves in a straight line with velocity $(7t-t^2-6)$ m/s at the end of t s. What is its acceleration when $t=2$ and when $t=4$? When $t=3$ the particle is at A; when $t=5$ the particle is at B. Find the length of AB. For what values of t is the particle momentarily at rest?

17 A particle, starting from rest, moves along a straight line with a velocity of $(8t-t^2)$ m/s at the end of t s. Find its velocity when its acceleration vanishes and the distance travelled up to that time. What distance will have been travelled when the velocity vanishes instantaneously?

18 The velocity of a train starting from rest is proportional to t^2, where t is the time which has elapsed since it started. If the distance it has covered at the end of 6 s is 18 m, find the velocity and the acceleration at that instant.

19 A car starts from rest with an acceleration proportional to the time. It travels 9 m in the first 3 s. Calculate its velocity and acceleration at the end of this time. Also find the distance travelled up to the instant when the velocity and acceleration are numerically equal.

20 A particle starts from rest and moves in a straight line. Its speed for the first 3 s is proportional to $(6t-t^2)$, where t is the time in seconds from the commencement of motion, and thereafter it travels with uniform speed at the rate it had acquired at the end of the 3rd second. Prove that the distance travelled in the first 3 s is two-thirds of the distance travelled in the next 3 s.

Chapter 7

Further differentiation

To differentiate the function $f(x) = x^n$ ($n \in \mathbb{Q}$)

7.1 In this chapter we shall use fractional and negative indices, and any reader not prepared for this should first read §9.2–§9.4. We are already familiar with the rule that the derivative of x^n is nx^{n-1}, but so far we have used it only when n has been a positive or negative integer or zero, i.e. for $n \in \mathbb{Z}$. We now need to extend this rule. First we shall prove its validity for the special case $n = \frac{1}{2}$.

Fig. 7.1 shows the graphs of the function $f(x) = x^{1/2} = \sqrt{x}$ and its inverse function $f^{-1}(x) = x^2$, $x \geqslant 0$. We saw in Chapter 2 that the graph of the inverse function is the reflection of the graph of $y = f(x)$ in the line $y = x$. The point $Q(b, a)$ on the graph $y = x^2$ is the reflection of the point $P(a, b)$ on the graph $y = x^{1/2}$. Notice, in particular, that the tangent at P to $y = x^{1/2}$ is inclined at an angle α to the x-axis, whereas the tangent at Q to $y = x^2$ is inclined at an angle α to the y-axis. Thus, in Fig. 7.1 (ii), α is equal to $(90° - \beta)$. Also notice that, since $P(a, b)$ is on $y = \sqrt{x}$, and $Q(b, a)$ is on $y = x^2$, $a = b^2$, or $\sqrt{a} = b$.

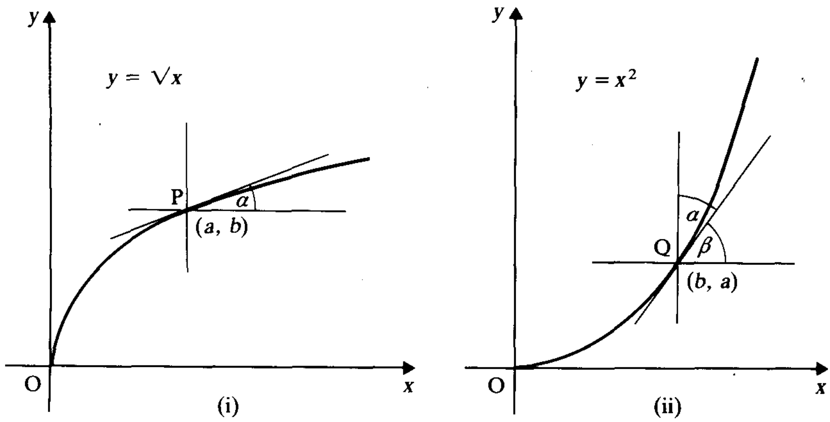


Figure 7.1

The gradient of $y = \sqrt{x}$ at P, $f'(a)$, is equal to $\tan \alpha$, but at the moment we do not know how to find $f'(x)$. However if we consider the graph of $y = x^2$, we know that $\tan \beta$ is given by the derivative of $y = x^2$ when $x = b$, and this we can find. In fact the derivative is $2x$ and hence

$$\tan \beta = 2b$$

But $\tan \alpha = \tan (90° - \beta) = 1/(2b)$, (see Fig. 7.2), therefore

$$f'(a) = \frac{1}{2b} = \frac{1}{2\sqrt{a}} = \tfrac{1}{2}a^{-1/2}$$

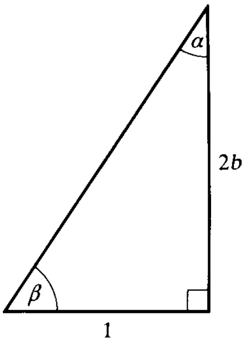


Figure 7.2

So we have proved that, for the function $f(x) = x^{1/2}$, the derivative

$$f'(x) = \tfrac{1}{2}x^{-1/2}$$

and this is in accordance with the general rule we have previously been using for differentiating x^n. From now on we shall assume that

if $f(x) = x^n$, $f'(x) = nx^{n-1}$ when $n \in \mathbb{Q}$,

i.e. when n is any rational number. It is important that the reader should bear in mind that, although this assumption is indeed valid, we have on each occasion so far justified the use of a general rule for differentiation simply by demonstrating its truth for particular values of n. At a more advanced level of study a proof can be provided.

Example 1 *Differentiate* (a) $\dfrac{2}{x^3}$, (b) $\dfrac{1}{\sqrt{x}}$.

(a) Let $\quad y = \dfrac{2}{x^3} = 2x^{-3}$

$$\therefore \frac{dy}{dx} = 2(-3)x^{-4}$$

$$\therefore \frac{d}{dx}\left(\frac{2}{x^3}\right) = \frac{-6}{x^4}$$

(b) Let $\quad y = \dfrac{1}{\sqrt{x}} = x^{-1/2}$

$$\therefore \frac{dy}{dx} = -\frac{1}{2}x^{-3/2}$$

$$\therefore \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = -\frac{1}{2\sqrt{x^3}}$$

Example 2 *Integrate* $\dfrac{3}{\sqrt[3]{x}}$.

$$\text{If } \frac{dy}{dx} = \frac{3}{\sqrt[3]{x}} = 3x^{-1/3}$$

$$y = 3\frac{x^{-1/3+1}}{-1/3+1} + c$$

$$= \tfrac{9}{2}\sqrt[3]{x^2} + c$$

Qu. 1 Differentiate: (a) x^{-4}, (b) $2x^{-3}$, (c) $\dfrac{1}{x^2}$, (d) $\dfrac{4}{x}$, (e) $-\dfrac{2}{x^2}$, (f) $\dfrac{1}{3x^3}$, (g) $-\dfrac{1}{x^4}$, (h) $\dfrac{3}{5x^5}$.

Integrate: (i) x^{-3}, (j) $2x^{-2}$, (k) $\dfrac{1}{x^2}$, (l) $\dfrac{2}{x^3}$, (m) $\dfrac{1}{3x^3}$, (n) $\dfrac{2}{5x^4}$.

Qu. 2 Differentiate: (a) $x^{1/2}$, (b) $2x^{-1/3}$, (c) $\sqrt{x}$, (d) $\sqrt[3]{x}$, (e) $\dfrac{1}{\sqrt[3]{x}}$, (f) $\dfrac{-2}{\sqrt[3]{x}}$, (g) $2\sqrt{x^3}$, (h) $\dfrac{2}{3\sqrt{x}}$.

Integrate: (i) $x^{-1/4}$, (j) $2x^{3/2}$, (k) $\sqrt{x}$, (l) $\sqrt[3]{x}$, (m) $\dfrac{1}{\sqrt{x}}$, (n) $\dfrac{1}{\sqrt{x^3}}$.

The chain rule

7.2 The process of differentiating a function has already been dealt with in this book and the reader faced with a simple expression will differentiate it term by term after expansion and know he is quite in order. If

$$y = (x+3)^2 = x^2 + 6x + 9$$

then

$$\frac{dy}{dx} = 2x + 6 = 2(x+3)$$

Quite obviously this expansion process leads to laborious multiplication when something like $(x+3)^7$ is met. The more venturesome reader might hazard a guess that $7(x+3)^6$ would be its derivative — and he would be right merely because $x+3$ has the same derivative as x. Guessing is rather apt to grow indiscriminate, however, and is entirely untrustworthy!

The derivative of $(3x+2)^4$ is *not* $4(3x+2)^3$.
The derivative of $(x^2+3x)^7$ is *not* $7(x^2+3x)^6$.

Qu. 3 In each part of this question, find $\frac{dy}{dx}$ by removing the brackets and then differentiating. Factorise each answer and try to guess its relationship to the original expression.

(a) $y=(x+4)^2$, (b) $y=(x+2)^3$, (c) $y=(3x+1)^2$,
(d) $y=(5-2x)^2$, (e) $y=(x+4)^3$, (f) $y=(x^3+1)^2$,
(g) $y=(5+x^2)^3$, (h) $y=(2+1/x)^2$, (i) $y=(1-x^3)^2$,
(j) $y=(\frac{1}{2}x-7)^3$.

Suppose y is a function of t, and t is itself a function of x. If δy, δt, and δx are corresponding small increments in the variables y, t, and x, then

$$\frac{\delta y}{\delta x}=\frac{\delta y}{\delta t}\times\frac{\delta t}{\delta x} \qquad (1)$$

When δy, δt, and δx tend to zero,

$$\frac{\delta y}{\delta x}\to\frac{dy}{dx}, \qquad \frac{\delta y}{\delta t}\to\frac{dy}{dt}, \qquad \frac{\delta t}{\delta x}\to\frac{dt}{dx}$$

and equation (1) becomes

$$\mathbf{\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}}$$

This important result is known as the **chain rule**. It will affect almost every exercise in differentiation which the reader will meet from here onwards, so it is most important to master it. The following examples are intended to give the reader some practice in its use.

Example 3 *Differentiate* $(3x+2)^4$.

Let $y=(3x+2)^4$ and $t=3x+2$, then $y=t^4$.

$$\frac{dt}{dx}=3, \qquad \frac{dy}{dt}=4t^3$$

But, by the chain rule,

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$$

$$\therefore \frac{dy}{dx}=4t^3\times 3$$

$$\therefore \frac{d}{dx}\{(3x+2)^4\}=12(3x+2)^3$$

Example 4 *Differentiate* $(x^2+3x)^7$.

Let $y=(x^2+3x)^7$ and $t=x^2+3x$, then $y=t^7$.

$$\frac{dt}{dx} = 2x + 3 \qquad \frac{dy}{dt} = 7t^6$$

$$\therefore \frac{dy}{dx} = 7t^6(2x+3)$$

$$\therefore \frac{d}{dx}\{(x^2+3x)^7\} = 7(2x+3)(x^2+3x)^6$$

In the very simple instance of Example 3 a similar method will apply for integration, i.e. $\int (3x+2)^4\,dx$ does equal $\frac{1}{5}(3x+2)^5 \times \frac{1}{3}$, but this is a special case. A corresponding division rule in integration does *not* apply. The integration of these awkward composite functions is dealt with in Book 2.

It is not necessary to show the actual substitution, as has been done in the examples above, but it is advisable, until practice has made perfect this art of substitution. The bracket is really treated as a single term — the t of our formula — and then the reader must remember to 'multiply by the derivative of the bracket'. Differentiation of reciprocals and roots of functions is pure chain rule technique.

In the function notation, the chain rule becomes $(fg)'(x) = f'(g(x)) \times g'(x)$, but this lacks the elegant simplicity of the statement

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

(This may be remembered as 'differentiate y with respect to t and then multiply by $\frac{dt}{dx}$'.)

Example 5 *Differentiate* $\frac{1}{1+\sqrt{x}}$.

Let $y = (1+\sqrt{x})^{-1}$

$$\therefore \frac{dy}{dx} = -1 \times (1+\sqrt{x})^{-2} \times \left\{\frac{d}{dx}(1+\sqrt{x})\right\}$$

$$= -1 \times (1+\sqrt{x})^{-2} \times (\tfrac{1}{2}x^{-1/2})$$

$$\therefore \frac{d}{dx}\left(\frac{1}{1+\sqrt{x}}\right) = \frac{-1}{2\sqrt{x}(1+\sqrt{x})^2}$$

Example 6 *Differentiate* $\sqrt{(1+x^2)}$.

Let $y = (1+x^2)^{1/2}$

$$\therefore \frac{dy}{dx} = \tfrac{1}{2}(1+x^2)^{-1/2} \times 2x$$

$$\therefore \frac{d}{dx}\{\sqrt{(1+x^2)}\} = \frac{x}{\sqrt{(1+x^2)}}$$

Exercise 7a

1 Differentiate:

(a) $(2x+3)^2$, (b) $2(3x+4)^4$, (c) $(2x+5)^{-1}$,
(d) $(3x-1)^{2/3}$, (e) $(3-2x)^{-1/2}$, (f) $(3-4x)^{-3}$.

2 Integrate:

(a) $(3x+2)^3$, (b) $(2x+3)^2$, (c) $(3x-4)^{-2}$, (d) $(2x+3)^{1/2}$.

3 Differentiate:

(a) $\dfrac{1}{(3x+2)}$, (b) $\dfrac{1}{(2x+3)^2}$, (c) $\dfrac{1}{\sqrt{(3x+1)}}$, (d) $\dfrac{1}{(2x-1)^{2/3}}$.

4 Integrate:

(a) $\dfrac{1}{(2x-3)^2}$, (b) $\dfrac{1}{\sqrt{(3x+2)}}$, (c) $\dfrac{1}{(2x-1)^{3/4}}$.

5 Differentiate:

(a) $(3x^2+5)^3$, (b) $(3x^3+5x)^2$, (c) $(7x^2-4)^{1/3}$,
(d) $(6x^3-4x)^{-2}$, (e) $(3x^2-5x)^{-2/3}$.

6 Differentiate:

(a) $\dfrac{1}{(3x^2+2)}$, (b) $\dfrac{3}{\sqrt{(2+x^2)}}$, (c) $\dfrac{-1}{(1+\sqrt{x})^2}$,

(d) $\left(1-\dfrac{1}{x}\right)^3$, (e) $\dfrac{1}{(x^2-1)^{1/3}}$.

7 Differentiate:

(a) $(3\sqrt{x}-2x)^3$, (b) $\left(\dfrac{2}{\sqrt{x}}-1\right)^{-1}$,

(c) $\left(2x^2-\dfrac{3}{x^2}\right)^{1/3}$, (d) $\left(x-\dfrac{1}{x}\right)^{1/2}$.

8 Differentiate:

(a) $\dfrac{1}{x^{3/2}-1}$, (b) $\sqrt{\left(1-\dfrac{1}{x}\right)}$,

(c) $\sqrt[3]{(1-\sqrt{x})}$, (d) $\left(\sqrt{x}-\dfrac{1}{\sqrt{x}}\right)^2$.

9 Differentiate:

(a) $\dfrac{1}{(x^2-7x)^3}$, (b) $\dfrac{1}{(x^2-\sqrt{x})^2}$,

(c) $\sqrt{\left(\dfrac{1}{1-x^2}\right)}$, (d) $\left(\dfrac{1}{1-\sqrt{x}}\right)^2$.

10 Differentiate:

(a) $\sqrt{\left(x^2 - \frac{1}{x^2}\right)}$, (b) $\frac{2}{x + 2\sqrt{x}}$,

(c) $\left(1 - \frac{2}{\sqrt{x}}\right)^{1/3}$, (d) $\sqrt{\left(1 - \frac{1}{\sqrt{x}}\right)}$.

Rates of change

7.3 The chain rule can be used to investigate related rates of change. Suppose a spherical balloon is inflated at the rate of 2 cm^3 every second. What is the rate of growth of the radius?

The solution of this type of problem has obvious calculus possibilities because $\frac{dy}{dx}$ is the rate of change of y with respect to x, and with the formula of the preceding section we have a ready means of connecting rates of change of dependent variables.

If the radius of the balloon is r, then the volume, $V = \frac{4}{3}\pi r^3$.

The fact we are given is that $\frac{dV}{dt}$, the rate of change of the volume with respect to time, t, is 2 cm^3/s, but, by the chain rule,

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} \quad \text{and} \quad \frac{dV}{dr} = 4\pi r^2$$

which leads to

$$\frac{dr}{dt} = \frac{2}{4\pi r^2}$$

i.e. the rate of change of the radius is $1/(2\pi r^2)$ cm/s. Any reader will surely at some time have blown up a balloon and noticed that the radius grows much more quickly at the beginning than near the end — sudden though the latter may sometimes be! The rate of change of the radius at any particular time could be calculated when the value of r is known. In the problem chosen, the radius after t s could be calculated from $\frac{4}{3}\pi r^3 = 2t$. The arithmetic is harder than the calculus.

Example 7 *A container in the shape of a right circular cone of height* 10 cm *and base radius* 1 cm *is catching the drips from a tap leaking at the rate of* 0.1 cm^3/s. *Find the rate at which the surface area of water is increasing when the water is half-way up the cone.*

Suppose the height of the water at any time is h cm, and that the radius of the surface of water at that time is r cm (Fig. 7.3).

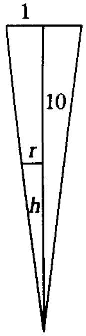


Figure 7.3

By similar triangles,

$$\frac{r}{1} = \frac{h}{10}$$

$$\therefore r = \tfrac{1}{10}h$$

The surface area of water, $A = \pi r^2 = \pi h^2/100$ and we wish to find $\frac{dA}{dt}$ when $h = 5$. By the chain rule,

$$\frac{dA}{dt} = \frac{dA}{dh} \times \frac{dh}{dt} = \frac{2\pi h}{100} \times \frac{dh}{dt} \tag{1}$$

The volume of water, $V = \frac{1}{3}\pi r^2 h = \pi h^3/300$, and using the chain rule again,

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt} = \frac{3\pi h^2}{300} \times \frac{dh}{dt}$$

But we are given that $\frac{dV}{dt} = 0.1$,

$$\therefore \frac{dh}{dt} = \frac{dV}{dt} \times \frac{300}{3\pi h^2} = 0.1 \times \frac{100}{\pi h^2} = \frac{10}{\pi h^2} \tag{2}$$

From (1) and (2)

$$\frac{dA}{dt} = \frac{2\pi h}{100} \times \frac{10}{\pi h^2} = \frac{1}{5h}$$

and, when $h = 5$,

$$\frac{dA}{dt} = \frac{1}{25} = 0.04$$

$\therefore$ when the water is half-way up, the rate of change of the surface area is equal to 0.04 cm^2/s.

Exercise 7b

1 The side of a cube is increasing at the rate of 6 cm/s. Find the rate of increase of the volume when the length of a side is 9 cm.

2 The area of surface of a sphere is $4\pi r^2$, r being the radius. Find the rate of change of the area in square cm per second when $r = 2$ cm, given that the radius increases at the rate of 1 cm/s.

3 The volume of a cube is increasing at the rate of 2 cm^3/s. Find the rate of change of the side of the base when its length is 3 cm.

4 The area of a circle is increasing at the rate of 3 cm^2/s. Find the rate of change of the circumference when the radius is 2 cm.

5 At a given instant the radii of two concentric circles are 8 cm and 12 cm. The radius of the outer circle increases at the rate of 1 cm/s and that of the inner at 2 cm/s. Find the rate of change of the area enclosed between the two circles.

6 If $y = (x^2 - 3x)^3$, find $\dfrac{dy}{dt}$ when $x = 2$, given $\dfrac{dx}{dt} = 2$.

7 A hollow right circular cone is held vertex downwards beneath a tap leaking at the rate of 2 cm^3/s. Find the rate of rise of water level when the depth is 6 cm given that the height of the cone is 18 cm and its radius 12 cm.

8 An ink blot on a piece of paper spreads at the rate of $\frac{1}{2}$ cm^2/s. Find the rate of increase of the radius of the circular blot when the radius is $\frac{1}{2}$ cm.

9 A hemispherical bowl is being filled with water at a uniform rate. When the height of the water is h cm the volume is $\pi(rh^2 - \frac{1}{3}h^3)$ cm^3, r cm being the radius of the hemisphere. Find the rate at which the water level is rising when it is half way to the top, given that $r = 6$ and that the bowl fills in 1 min.

10 An inverted right circular cone of vertical angle 120° is collecting water from a tap at a steady rate of 18π cm^3/min. Find
(a) the depth of the water after 12 min,
(b) the rate of increase of the depth at this instant.

11 From the formula $v = \sqrt{(60s + 25)}$ the velocity, v, of a body can be calculated when its distance, s, from the origin is known. Find the acceleration when $v = 10$.

12 If $y = (x - 1/x)^2$, find $\dfrac{dx}{dt}$ when $x = 2$, given $\dfrac{dy}{dt} = 1$.

13 A rectangle is twice as long as it is broad. Find the rate of change of the perimeter when the breadth of the rectangle is 1 m and its area is changing at the rate of 18 cm^2/s, assuming the expansion uniform.

14 A horse-trough has triangular cross-section of height 25 cm and base 30 cm, and is 2 m long. A horse is drinking steadily, and when the water level is 5 cm below the top it is being lowered at the rate of 1 cm/min. Find the rate of consumption in litres per minute.

Products and quotients

7.4 The reader is now able to differentiate quite elaborate functions, but no method has been suggested for a product such as $f(x) = (x + 1)^7(x - 3)^4$. We

Example 8 *Differentiate the expression $y=(x^2-3)(x+1)^2$ and simplify the result.*

Let $u=(x^2-3)$ and let $v=(x+1)^2$, then

$$\frac{du}{dx}=2x \qquad \text{and} \qquad \frac{dv}{dx}=2(x+1)$$

$$\therefore \frac{dy}{dx}=(x+1)^2\times 2x+(x^2-3)\times 2(x+1)$$

$$=2(x+1)\{x(x+1)+(x^2-3)\}$$

$$=2(x+1)\{2x^2+x-3\}$$

$$=2(x+1)(2x+3)(x-1)$$

Example 9 *Differentiate $(x^2+1)^3(x^3+1)^2$.*

If $u=(x^2+1)^3$ and $v=(x^3+1)^2$, then let $y=uv$.

$$\frac{du}{dx}=3(x^2+1)^2\times 2x \qquad \text{and} \qquad \frac{dv}{dx}=2(x^3+1)\times 3x^2$$

$$\therefore \frac{dy}{dx}=(x^3+1)^2\times 6x(x^2+1)^2+(x^2+1)^3\times 6x^2(x^3+1)$$

$$=6x(x^3+1)(x^2+1)^2\{(x^3+1)+x(x^2+1)\}$$

$$\therefore \frac{d}{dx}\{(x^2+1)^3(x^3+1)^2\}=6x(x^3+1)(x^2+1)^2(2x^3+x+1)$$

Example 10 *Find the x-coordinates of the stationary points of the curve $y=(x^2-1)\sqrt{(1+x)}$.*

$$y=(x^2-1)(x+1)^{1/2}$$

$$\therefore \frac{dy}{dx}=(x+1)^{1/2}\times 2x+(x^2-1)\times \tfrac{1}{2}(x+1)^{-1/2}$$

$$=\frac{2(x+1)\times 2x+(x^2-1)}{2(x+1)^{1/2}}$$

$$=\frac{(x+1)(4x+x-1)}{2(x+1)^{1/2}}$$

$$=\frac{(5x-1)(x+1)}{2(x+1)^{1/2}}$$

$$=\tfrac{1}{2}(5x-1)(x+1)^{1/2}$$

$\therefore$ for stationary points $x=\frac{1}{5}$ or -1.

could multiply out the brackets and differentiate each term separately, but this would be extremely laborious. Although it is possible to differentiate each of the factors, we have, as yet, no method for tackling the product as it stands. (We must not simply write down the product of the two derivatives. A reader tempted to do so should consider the product $f(x) = x^3 \times x^4$, which is equal to x^7 and hence its derivative is $f'(x) = 7x^6$; but this is plainly not the same as the product of the two derivatives $3x^2$ and $4x^3$.)

A further brief return to fundamental ideas will produce a formula to help us with functions of this kind.

Let y be the product of two functions u and v of a variable x. Then $y = uv$ and

$$y + \delta y = (u + \delta u)(v + \delta v)$$

where a small increment δx in x produces increments δu in u, δv in v and δy in y.

$$y + \delta y = uv + v\delta u + u\delta v + \delta u\delta v$$

and since $y = uv$,

$$\delta y = v\delta u + u\delta v + \delta u\delta v$$

Dividing by δx,

$$\frac{\delta y}{\delta x} = v\frac{\delta u}{\delta x} + u\frac{\delta v}{\delta x} + \frac{\delta u}{\delta x} \times \delta v$$

As $\delta x \to 0$, δu, δv and δy also approach 0,

$$\frac{\delta y}{\delta x} \to \frac{dy}{dx} \qquad \frac{\delta u}{\delta x} \to \frac{du}{dx} \qquad \frac{\delta v}{\delta x} \to \frac{dv}{dx}$$

$$\therefore \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} + \frac{du}{dx} \times 0$$

$$\therefore \mathbf{\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}}$$

This formula must be remembered, and this is perhaps most easily done in words,

> 'To differentiate the product of two factors, differentiate the first factor, leaving the second one alone and then differentiate the second, leaving the first one alone,'

and it is necessary to remember also that, should one of the factors in the product be a composite function, its derivative must be found as carefully as those in §7.2 before insertion in this product formula.

Qu. 4 Use this method to differentiate the following functions:
(a) $(x + 1)(x + 2)$, (b) $(x^2 + 1)x^2$,
(c) $(x - 2)^2(x^2 - 2)$, (d) $(x + 1)^2(x + 2)^2$.
Check your results by multiplying out and then differentiating.

The most common mistakes made in this type of question are due to careless algebra and so particular attention should be paid to details of simplification.

There is a formula for quotients corresponding to that for products and it is proved in a similar way.

If $y = u/v$ then

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

If the reader wishes to ignore this formula and to deal with the quotient u/v as the product uv^{-1} he is at liberty to do so — it is merely a matter of preference.

Example 11 *Differentiate* $\dfrac{(x-3)^2}{(x+2)^2}$.

Let $y = (x-3)^2/(x+2)^2$ and let $u = (x-3)^2$ and $v = (x+2)^2$, then $y = u/v$.

$$\frac{du}{dx} = 2(x-3) \qquad \text{and} \qquad \frac{dv}{dx} = 2(x+2)$$

$$\therefore \frac{dy}{dx} = \frac{(x+2)^2 \times 2(x-3) - (x-3)^2 \times 2(x+2)}{(x+2)^4}$$

$$= \frac{2(x+2)(x-3)\{(x+2)-(x-3)\}}{(x+2)^4}$$

$$= \frac{2(x-3) \times 5}{(x+2)^3}$$

$$\therefore \frac{d}{dx}\left\{\frac{(x-3)^2}{(x+2)^2}\right\} = \frac{10(x-3)}{(x+2)^3}$$

Example 12 *Differentiate* $\dfrac{x}{\sqrt{(1+x^2)}}$.

Let $y = x/\sqrt{(1+x^2)}$ and let $u = x$ and $v = \sqrt{(1+x^2)}$, then $y = u/v$.

$$\frac{du}{dx} = 1 \qquad \text{and} \qquad \frac{dv}{dx} = \frac{2x}{2\sqrt{(1+x^2)}}$$

$$\therefore \frac{dy}{dx} = \frac{\sqrt{(1+x^2)} \times 1 - x \times \dfrac{x}{\sqrt{(1+x^2)}}}{(1+x^2)}$$

$$= \frac{1 + x^2 - x^2}{(1+x^2)^{3/2}}$$

$$\therefore \frac{d}{dx}\left\{\frac{x}{\sqrt{(1+x^2)}}\right\} = \frac{1}{(1+x^2)^{3/2}}$$

Qu. 5 Prove the formula for quotients by the δu, δv method.

Qu. 6 Differentiate:

(a) $(x^2+1)(x+3)^{-2}$ as a product, (b) $\dfrac{x^2+1}{(x+3)^2}$ as a quotient.

Simplify the results and compare them.

*Simplification was an essential part of answering the question in Example 11 and, since the gradient of a function is often needed for a specific purpose, the reader should get into the habit of factorising and simplifying as far as possible. It will be necessary, in any case, in order to check the answers with those at the back of the book!

Exercise 7c

Differentiate with respect to x the following functions:

1 $x^2(x+1)^3$. **2** $x(x^2+1)^4$. **3** $(x+1)^2(x^2-1)$.

4 $\dfrac{x}{x+1}$. **5** $\dfrac{1-x^2}{1+x^2}$. **6** $\dfrac{x^2+1}{(x+1)^2}$.

7 $(1+x^2)^2(1-x^2)$. **8** $x^2\left(1-\dfrac{1}{\sqrt{x}}\right)$. **9** $(1-x^2)^2(1-x^3)$.

10 $(x-1)\sqrt{(x^2+1)}$. **11** $x^2\sqrt{(1+x^2)}$. **12** $\dfrac{x^2}{\sqrt{(1+x^2)}}$.

13 $\dfrac{(x-1)^2}{\sqrt{x}}$. **14** $\dfrac{2x^2-x^3}{\sqrt{(x^2-1)}}$. **15** $\sqrt{(x+2)}\sqrt{(x+3)}$.

16 $\dfrac{\sqrt{x}}{\sqrt{(x+1)}}$. **17** $\dfrac{1-\sqrt{x}}{1+\sqrt{x}}$. **18** $\sqrt{\left(\dfrac{1+x}{2+x}\right)}$.

19 $\sqrt{(x+1)}\sqrt{(x+2)^3}$. **20** $\sqrt{\left\{\dfrac{(x+1)^3}{x+2}\right\}}$.

Implicit functions

7.5 Up to the present we have dealt only with *explicit* functions of x, e.g. $y=x^2-5x+4/x$. Here y is given as an expression in x. If, however, y is given *implicitly* by an equation such as $x=y^4-y-1$, we cannot express y in terms of x.

Consider an easier case. If $x=y^2$, $y=x^{1/2}$.

$$\therefore \frac{dy}{dx}=\tfrac{1}{2}x^{-1/2}=\frac{1}{2x^{1/2}}=\frac{1}{2y}$$

*Practice in the algebra involved in differentiating a quotient is given in Exercise 9d, No. 5, and in the Appendix.

But $\frac{dx}{dy} = 2y$, so *in this case,*

$$\frac{dy}{dx} = 1 \bigg/ \frac{dx}{dy}$$

(Strictly speaking, the equation $x = y^2$, does not define y as a *function* of x, since, for each positive value of x, there are *two* values of y, namely the positive and negative square roots of x.)

Now consider the general case.

$$\frac{\delta y}{\delta x} = 1 \bigg/ \frac{\delta x}{\delta y}$$

where δx and δy are the increments in x and y respectively.

Now as $\delta x, \delta y \to 0$, $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$, and $\frac{\delta x}{\delta y} \to \frac{dx}{dy}$

$$\therefore \mathbf{\frac{dy}{dx} = 1 \bigg/ \frac{dx}{dy}}.$$

When it is impracticable to express either variable explicitly in terms of the other, we can still differentiate both sides with respect to x, as in Example 13 below. A term like y^n can be differentiated by first differentiating with respect to y then, as the chain rule demands, multiplying by $\frac{dy}{dx}$. Thus

$$\frac{d}{dx}(y^n) = \frac{d}{dy}(y^n)\frac{dy}{dx} = ny^{(n-1)}\frac{dy}{dx}$$

Similarly, if we have a term of the form $x^m y^n$, then we use the product rule and obtain

$$\begin{aligned}\frac{d}{dx}(x^m y^n) &= x^m \frac{d}{dx}(y^n) + y^n \frac{d}{dx}(x^m)\\ &= nx^m y^{(n-1)}\frac{dy}{dx} + mx^{(m-1)}y^n\end{aligned}$$

Example 13 *Find the gradient of the curve*

$$x^2 + 2xy - 2y^2 + x = 2$$

at the point $(-4, 1)$.

To find the gradient, differentiate with respect to x.

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(2xy) - \frac{d}{dx}(2y^2) + \frac{d}{dx}(x) = \frac{d}{dx}(2)$$

$$\therefore 2x + \left(2y + 2x\frac{dy}{dx}\right) - 4y\frac{dy}{dx} + 1 = 0$$

$$\therefore \frac{dy}{dx}(2x - 4y) = -1 - 2x - 2y$$

When $x = -4$, $y = 1$,

$$\frac{dy}{dx}(-8 - 4) = -1 + 8 - 2$$

$$\therefore \frac{dy}{dx} = \frac{+5}{-12}$$

$\therefore$ the gradient at $(-4, 1)$ is $-\frac{5}{12}$.

Qu. 7 Differentiate with respect to x:
(a) x, (b) y, (c) x^2, (d) y^2, (e) xy, (f) x^2y, (g) xy^2.

Qu. 8 Find $\frac{dy}{dx}$ if $x^2 + y^2 - 6xy + 3x - 2y + 5 = 0$.

Parameters

7.6 Sometimes both x and y are given as functions of another variable, a **parameter**. In such cases the gradient is given in terms of the variable parameter.

Example 14 *If $x = t^3 + t^2$, $y = t^2 + t$ find $\frac{dy}{dx}$ in terms of t.*

$$\frac{dx}{dt} = 3t^2 + 2t \qquad \frac{dy}{dt} = 2t + 1$$

Now $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$, but $\frac{dt}{dx} = 1 \Big/ \frac{dx}{dt}$.

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$\therefore \frac{dy}{dx} = \frac{2t + 1}{t(3t + 2)}$$

Qu. 9 Show that the above parametric representation is of the curve $y^3 = x^2 + xy$. Find $\frac{dy}{dx}$ for this curve and show that it agrees with the above result.

Example 15 *Find the gradient of the curve $x = \frac{t}{1+t}$, $y = \frac{t^3}{1+t}$ at the point $(\frac{1}{2}, \frac{1}{2})$.*

$$\frac{dx}{dt} = \frac{(1 + t) \times 1 - t \times 1}{(1 + t)^2} = \frac{1}{(1 + t)^2}$$

$$\frac{dy}{dt} = \frac{(1+t)\times 3t^2 - t^3 \times 1}{(1+t)^2} = \frac{3t^2+2t^3}{(1+t)^2}$$

$$\therefore \frac{dy}{dx} = 3t^2 + 2t^3$$

At $(\frac{1}{2}, \frac{1}{2})$, $t = 1$,

$$\therefore \frac{dy}{dx} = 3 + 2$$

$\therefore$ the gradient at $(\frac{1}{2}, \frac{1}{2})$ is 5.

Exercise 7d

1 Find the gradient of the ellipse $2x^2 + 3y^2 = 14$ at the points where $x = 1$.

2 Find the x-coordinates of the stationary points of the curve represented by the equation $x^3 - y^3 - 4x^2 + 3y = 11x + 4$.

3 Find the gradient of the ellipse $x^2 - 3yx + 2y^2 - 2x = 4$ at the point $(1, -1)$.

4 Find the gradient of the tangent at the point (2, 3) to the hyperbola $xy = 6$.

5 (a) If $x = t^2$, $y = t^3$ find $\frac{dy}{dx}$ in terms of t. (b) If $y = x^{3/2}$ find $\frac{dy}{dx}$.

Is there any connection between these two results?

6 At what points are the tangents to the circle $x^2 + y^2 - 6y - 8x = 0$ parallel to the y-axis?

7 Find $\frac{dy}{dx}$ when (a) $x^2y^3 = 8$, (b) $xy(x - y) = 4$.

8 Find $\frac{dy}{dx}$, in terms of t, when

(a) $x = at^2$, $y = 2at$; (b) $x = (t+1)^2$, $y = (t^2 - 1)$.

9 If $x = t/(1-t)$ and $y = t^2/(1-t)$ find $\frac{dy}{dx}$ in terms of t.

10 Find $\frac{dy}{dx}$ in terms of x, y when $x^2 + y^2 - 2xy + 3y - 2x = 7$.

11 If $x = 2t/(t+2)$, $y = 3t/(t+3)$, find $\frac{dy}{dx}$ in terms of t.

12 Find $\frac{dy}{dx}$ in terms of x, y when $3(x - y)^2 = 2xy + 1$.

Small changes

7.7 We have seen that, as $\delta x \to 0$, $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$. Therefore, if δx is small,

$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$$

$$\therefore \boldsymbol{\delta y \approx \frac{dy}{dx}\delta x}$$

Three applications of this formula follow in Examples 16–18.

Example 16 *The side of a square is* 5 cm. *Find the increase in the area of the square when the side expands* 0.01 cm.

Let the area of the square be A cm^2 when the side is x cm. Then $A = x^2$. Now

$$\delta A \approx \frac{dA}{dx}\delta x \quad \text{and} \quad \frac{dA}{dx} = 2x$$

$$\therefore \delta A \approx 2x\delta x$$

When $x = 5$ and $\delta x = 0.01$,

$$\delta A \approx 2 \times 5 \times 0.01 = 0.1$$

$\therefore$ the increase in area ≈ 0.1 cm^2.

In this case the increase in area can be found accurately very easily:

$$\delta A = 5.01^2 - 5^2 = 0.1001$$

The reader is strongly advised to use the calculus method, for the moment, even if he sees a quicker way, since it is an important introduction to certain topics which he may meet later.

Note that the error by the calculus method is, *in this case*, $(0.01)^2 = (\delta x)^2$.

Example 17 *A 2% error is made in measuring the radius of a sphere. Find the percentage error in surface area.*

Let the surface area be S and the radius be r, then

$$S = 4\pi r^2 \qquad \therefore \frac{dS}{dr} = 8\pi r$$

$$\therefore \delta S \approx 8\pi r \delta r$$

But the error in r is 2%, therefore $\delta r = \frac{2}{100} \times r$.

$$\therefore \delta S \approx 8\pi r \times \frac{2r}{100} = \frac{16\pi r^2}{100}$$

$$\therefore \frac{\delta S}{S} \approx \frac{16\pi r^2}{100} \div 4\pi r^2 = \frac{4}{100}$$

$\therefore$ the error in the surface area $\approx 4\%$.

Example 18 *Find an approximation for* $\sqrt{9.01}$.

Let $y = \sqrt{x}$, so $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$.

$$\therefore \delta y \approx \frac{1}{2\sqrt{x}} \times \delta x$$

When $x = 9$, and $\delta x = 0.01$,

$$\delta y \approx \tfrac{1}{6} \times 0.01 \approx 0.001\,67$$

$\therefore \sqrt{9.01} \approx 3.001\,67.$

Exercise 7e

1 The surface area of a sphere is $4\pi r^2$. If the radius of the sphere is increased from 10 cm to 10.1 cm, what is the approximate increase in surface area?
2 An error of 3% is made in measuring the radius of the sphere. Find the percentage error in volume.
3 Find (a) $\sqrt[3]{8.01}$, (b) $\sqrt{25.1}$ by the method of Example 18.
4 If l cm is the length of a pendulum and t s the time of one complete swing, it is known that $l = kt^2$. If the length of the pendulum is increased by x%, x being small, find the corresponding percentage increase in time of swing.
5 If the pressure and volume of a gas are p and v then Boyle's law states $pv =$ constant (k). If δp and δv denote corresponding small changes in p and v express $\dfrac{\delta p}{p}$ in terms of $\dfrac{\delta v}{v}$.
6 An error of $2\frac{1}{2}$% is made in the measurement of the area of a circle. What percentage error results in (a) the radius, (b) the circumference?
7 The height of a cylinder is 10 cm and its radius is 4 cm. Find the approximate increase in volume when the radius increases to 4.02 cm.
8 One side of a rectangle is three times the other. If the perimeter increases by 2% what is the percentage increase in area?
9 The radius of a closed cylinder is equal to its height. Find the percentage increase in total surface area corresponding to unit percentage increase in height.
10 Find (a) $\sqrt{627}$, (b) $\sqrt[3]{1005}$, by the method of Example 18.
11 The volume of a sphere increases by 2%. Find the corresponding percentage increase in surface area.
12 As x increases, prove that the area of a circle of radius x and the area of a square of side x increase by the same percentage, provided that the increase in x is small.

Second derivative

7.8 We know that velocity, v, is the rate of change of displacement, s, with respect to time, t, and may be denoted by $\dfrac{\mathrm{d}s}{\mathrm{d}t}$. Acceleration is the rate of change of velocity with respect to time, and we have up to now denoted this by $\dfrac{\mathrm{d}v}{\mathrm{d}t}$; but $\dfrac{\mathrm{d}}{\mathrm{d}t}(v)$ may also be written as $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}s}{\mathrm{d}t}\right)$, and thus acceleration is seen to be the second derivative of s with respect to t.

The second derivative arises in a wide variety of contexts as well as in kinematics, of course, and we need a less cumbersome notation.

$$\frac{\mathbf{d}}{\mathbf{d}x}\left(\frac{\mathbf{d}y}{\mathbf{d}x}\right) \textbf{ is written as } \frac{\mathbf{d}^2y}{\mathbf{d}x^2}$$

which is spoken 'd two y by d x squared'.

Acceleration, $\frac{d^2s}{dt^2}$, may be written in yet another way by using the fact that

$$\frac{dv}{dt} = \frac{dv}{ds} \times \frac{ds}{dt} = \frac{dv}{ds} \times v$$

thus we have arrived at the following alternative notations,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = v\frac{dv}{ds}$$

the last form, $v\frac{dv}{ds}$, being applicable when velocity or acceleration is a function of s rather than of t.

Remember that if $y = f(x)$, $\frac{dy}{dx}$ is written as $\mathbf{f'(x)}$, and $\frac{d^2y}{dx^2}$ as $\mathbf{f''(x)}$.

Qu. 10 (a) If $y = x^2 - 1/x^2$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.
(b) Given that $v = 3(4 - s^2)^{1/2}$, show that $a = -9s$.
(c) If $f(x) = x/(x - 1)$, find $f'(x)$ and $f''(x)$.

If $\frac{dy}{dx}$ is found in terms of a parameter t, $\frac{d^2y}{dx^2}$ requires differentiation with respect to x, so

$$\frac{\mathbf{d}^2y}{\mathbf{d}x^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} = \frac{\mathbf{d}}{\mathbf{d}t}\left(\frac{\mathbf{d}y}{\mathbf{d}x}\right) \div \frac{\mathbf{d}x}{\mathbf{d}t}$$

Qu. 11 If $x = a(t^2 - 1)$, $y = 2a(t + 1)$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t.

Exercise 7f (Miscellaneous)

These questions are in two stages. The first 25 are set in the order of the sections in this chapter in order to give the reader further practice in the fundamentals. The later questions are 'mixed' and also include ones involving a knowledge of other sections of this book.

1 Differentiate:

(a) $\frac{1}{x^{2n}}$, (b) x^{n+1}, (c) $\frac{1}{2}x^{2a-1}$, (d) $(x^2)^m$, (e) $\sqrt{x^n}$.

2 Integrate:
(a) $(x^2)^{k-1}$, (b) $(x^n)^{-2}$, (c) $nx^{-1/n-1}$, (d) x^{-1+k}.

3 Differentiate:

(a) $\sqrt[n]{x}$, (b) $(\sqrt{x})^n$, (c) $\dfrac{2x}{x^n}$, (d) $\dfrac{1}{\sqrt{x^n}}$, (e) $\dfrac{1}{\sqrt[3]{x^{2n}}}$.

4 Differentiate:

(a) $\dfrac{2}{\sqrt[n]{x^3}}$, (b) $\sqrt[3]{\dfrac{1}{x}}$, (c) $\left(\dfrac{1}{\sqrt{x^n}}\right)^3$, (d) $\sqrt{(2x^3)}$, (e) $x^{n/(n+1)}$.

In Nos. 5–12 differentiate with respect to x and simplify.

5 (a) $(x^2+3)^4$, (b) $\sqrt{(2x^3-3)}$, (c) $(\sqrt{x}+1)^3$, (d) $\left(\dfrac{x}{2}-\dfrac{2}{x}\right)^n$.

6 (a) $\dfrac{1}{x+\sqrt{x}}$, (b) $\dfrac{1}{x^2-1}$, (c) $\dfrac{1}{(\sqrt{x}-1)^2}$, (d) $\dfrac{1}{\sqrt[3]{(2x-x^2)}}$.

7 (a) $x^2(x-1)^3$, (b) $(x+1)^{3/2}(x-1)^{5/2}$, (c) $(x-2)^{1/2}(x^2+3)$.

8 (a) $\sqrt{\{(x+1)(x-2)^3\}}$, (b) $(1-x^2)\sqrt[3]{(1-2x)}$, (c) $x\sqrt{(x^2-1)}$.

9 (a) $\dfrac{x}{x^2-1}$, (b) $\dfrac{x}{\sqrt{(x-1)}}$, (c) $\dfrac{\sqrt{x}}{x-1}$, (d) $\dfrac{\sqrt{(x-1)}}{\sqrt{x}-1}$.

10 (a) $\dfrac{x^2+2}{(x+2)^2}$, (b) $\dfrac{(x-1)^3}{(x^3-1)}$.

11 (a) $\sqrt{\left(\dfrac{x+1}{x+2}\right)}$, (b) $\sqrt{\left\{\dfrac{(x+2)^3}{x-1}\right\}}$.

12 (a) $\sqrt{\dfrac{x^2+1}{x^2-1}}$, (b) $\dfrac{(1-\sqrt{x})^2}{\sqrt{(x^2-1)}}$.

13 Find $\dfrac{dy}{dx}$ when $x^2+2xy+y^2=3$.

14 Find $\dfrac{dy}{dx}$ when $x^2-3xy+y^2-2y+4x=0$.

15 Find $\dfrac{dy}{dx}$ when $3x^2-4xy=7$.

16 If $x=2t/(1+t^2)$, $y=(1-t^2)/(1+t^2)$ find $\dfrac{dy}{dx}$ in terms of t.

17 If $x=1/\sqrt{(1+t^2)}$, $y=t/\sqrt{(1+t^2)}$ find $\dfrac{dy}{dx}$ in terms of t.

18 If $x=t/(1-t)$, $y=(1-2t)/(1-t)$ find $\dfrac{dy}{dx}$.

19 When measuring the area of a circle, 2% error is made. Find the percentage error in the radius.

20 When measuring the dimensions of cubical box 1% error was made — all measurements being too large. Find the percentage error in volume.

21 The circumference of a circle is measured with a piece of string which stretches 1%. What is the percentage error in the area of the circle?

22 Calculate (a) $\sqrt[3]{65}$, (b) $\sqrt{37}$, without using tables or a calculator.

23 If $y = 4x^3 - 6x^2 - 9x + 1$, find $\frac{dy}{dx}$ and hence find the values of $\frac{d^2y}{dx^2}$ when the gradient is zero.

24 If $x = at^2$, $y = 2at$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t.

25 If $f(x) = 8x^3 - 11x^2 - 30x + 9$ for what values of x is $f'(x) = 0$?

---

26 Find the equation of the tangent to the curve $x^2 - y^2 = 9$ at the point (5, 4).

27 Prove from first principles that the derivative of $1/x^2$ is $-2/x^3$.

28 If $y = x^2/\sqrt{(x+1)}$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

29 Find what values of x give stationary values of the function $(2x-3)^2(x-2)^3$.

30 Differentiate with respect to x (a) $\sec 2x$, (b) $\sin^2 x$, (c) $x \cos x$, (d) $\tan^3 x$, (e) $\sin \sqrt{x}$.

31 A curve is represented parametrically by $x = (t^2 - 1)^2$, $y = t^3$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t.

32 The volume of a sphere is increased by 3%. Find the percentage increase in the radius.

33 A curve called the Witch of Agnesi has for its equation $y = \sqrt{(3/x - 1)}$. Find its gradient when $x = \frac{1}{2}$.

34 Show that of all rectangles with given perimeter the square has maximum area.

35 Find the equation of the tangent to the parabola $y^2 = 4x$ which is parallel to the line $y = 3x - 4$. What are the coordinates of the point of contact?

Find also the equation of the normal at this point.

36 The distance, s, of a particle from a point after time t is given by the formula $s^2 = a + bt^2$.

Find the velocity and acceleration in terms of s, t, a, b.

37 Find the equation of the tangent and normal to the curve $x = a \cos^3 t$, $y = a \sin^3 t$, at the point whose parameter is t.

38 If $R = ar^n$ and an error of x% is made in measuring r, prove that an error of nx% will result in R.

39 Find the maximum and minimum values of $x^2\sqrt{(2-x)}$.

40 Find the equation of the tangent and normal to the cycloid

$$x = a(2\theta + \sin 2\theta), \qquad y = a(1 - \cos 2\theta)$$

at the point whose parameter is θ.

41 If the radius of a spherical soap bubble increases from 1 cm to 1.02 cm, find the approximate increase in volume.

42 If the velocity v is given by the formula $v = u/(1 + ks)$ where u is the initial velocity, s is the distance and k is a constant prove that the acceleration varies as v^3.

43 Differentiate:

(a) $\sin^2(3x^2 + 4)$, (b) $\tan\sqrt{(x + 1)}$,

(c) $\dfrac{\sin x}{1 + \cos x}$, (d) $\sqrt{\dfrac{\tan x}{1 - \tan^2 x}}$.

44 If $x^2 + 3xy - y^2 = 3$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the point (1, 1).

45 Differentiate $\sqrt{(x^2 + 1)}$ with respect to x.

Why does the substitution $x = \tan\theta$ in this function and its derivative not give you the derivative of $\sec\theta$?

46 Draw a sketch of the graph of $y = x^{1/3}$ to illustrate that this function is odd and continuous. Draw also a sketch of the graph of its gradient function showing that this is even and discontinuous.

47 Given that $f(x) = x - 1 + 1/(x + 1)$, x real, $x \neq -1$, find the values of x for which $f'(x) = 0$.

Sketch the graph of f, showing the coordinates of the turning points and indicating clearly the form of the graph when $|x|$ becomes large. (JMB)

48 (a) Show that, when k is constant, the curve

$$y = 3x^4 - 8x^3 - 6x^2 + 24x + k$$

has a stationary point when $x = 1$ and find the values of x at the other two stationary points on the curve. Find the values of k for which the curve touches the x-axis.

(b) A spherical balloon is being inflated at a constant rate. Show that the rate of increase of the surface area is inversely proportional to the radius. (L)

49 The functions f and g with domains $\{x: x \text{ real}, x \neq 0\}$ are defined as follows:

$$f: x \mapsto 1 + x - \frac{6}{x}, \qquad g: x \mapsto \frac{1}{x}$$

Find a and b so that the composite function $h = gf$ is defined on the set

$$\{x: x \text{ real}, x \neq 0, x \neq a, x \neq b\}$$

Verify that

$$h(x) = \frac{1}{5}\left(\frac{3}{x + 3} + \frac{2}{x - 2}\right)$$

and show that if $h'(x)$ is the derived function of h, then $h'(x) < 0$ at all points in the domain of h.

Sketch the graph of h, marking all the asymptotes and showing how the graph approaches the asymptotes. (L)

50 Prove that the curve with the equation

$$y = \frac{(x+1)^2}{(x-1)(x+2)}$$

has two points at which $\frac{dy}{dx} = 0$. Find the coordinates of these points and determine the nature of each point.

Sketch the curve. (C)

Chapter 8

Further integration

Some standard curves

8.1 In §5.5 we dealt with some simple aids to curve sketching. By this stage, the reader should be thoroughly familiar with some standard curves which will be frequently occurring in the work which follows.

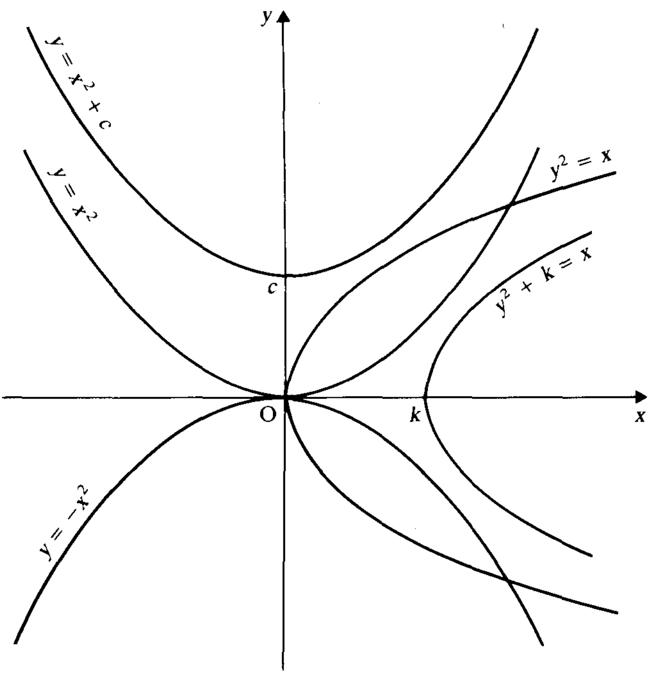


Figure 8.1

Fig. 8.1 shows some variations on the curve $y = x^2$, which is a *parabola*. The line about which the curve is symmetrical is called the *axis*, and it cuts the curve at the *vertex*. Thus for the curve $y = x^2 + c$, the axis is the y-axis, and the vertex is $(0, c)$. Any equation of the form $y = ax^2 + bx + c$, where a, b, and c are constants (a not being zero), represents a parabola with the axis parallel to the y-axis (see Chapter 10).

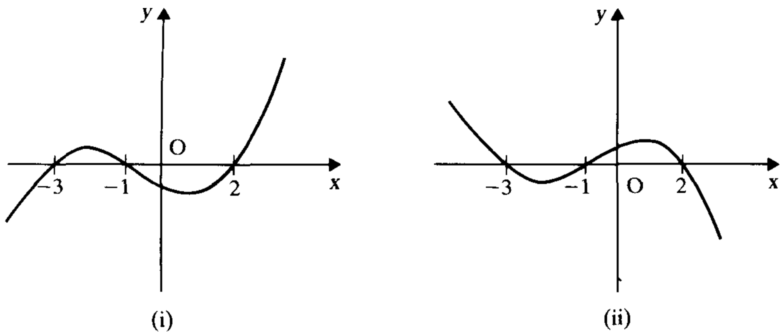


Figure 8.2

Typical shapes of curves for which y is given as a cubic function of x are shown in Fig. 8.2. (i) represents $y = (x + 3)(x + 1)(x - 2)$, the x^3 term in the expansion being positive; (ii) represents $y = (3 + x)(1 + x)(2 - x)$, the x^3 term in the expansion being negative (see §5.5, Example 7).

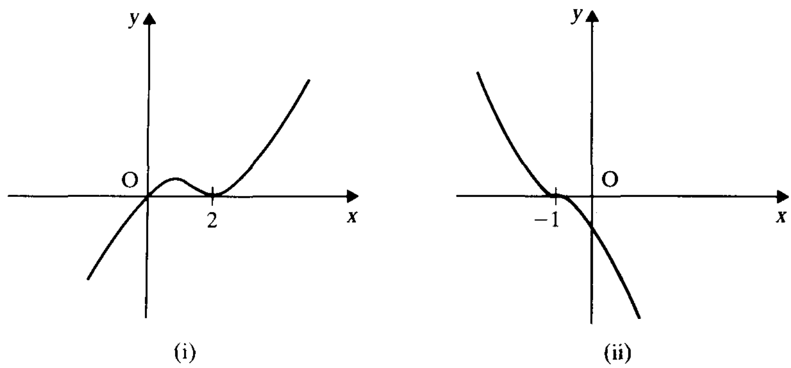


Figure 8.3

Fig. 8.3 shows (i) $y = x(x - 2)^2$, and (ii) $y = -(x + 1)^3$, illustrating that when the function of x has a squared factor, the curve touches the x-axis; and with a cubed factor, the curve touches *and crosses* the x-axis.

Fig. 8.4 illustrates how a sketch of the curve $y = x^2 + 1/x$ may be built up by adding the y-coordinates of the two known curves $y = x^2$ and $y = 1/x$.

The integration of x^n ($n \in \mathbb{Q}$)

8.2 In Chapter 7 the differentiation of x^n was assumed to include cases where n is a fraction, and so we can now integrate powers of x with fractional indices.

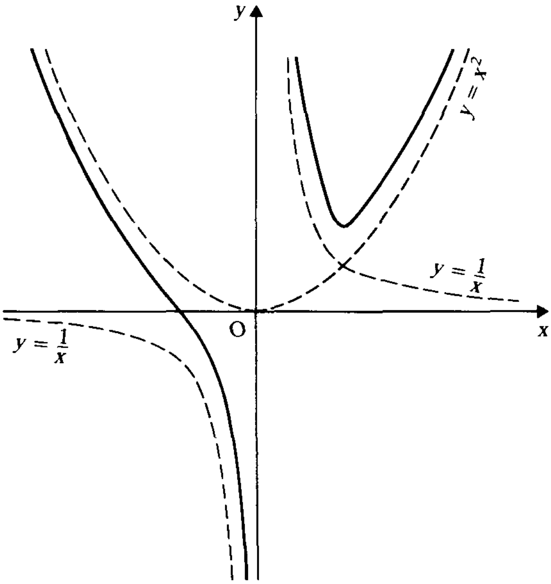


Figure 8.4

Thus, if

$$\frac{dy}{dx} = \sqrt{x} = x^{1/2}$$

then

$$y = \frac{x^{3/2}}{3/2} + c = \tfrac{2}{3}x^{3/2} + c$$

Exercise 8a

1 Sketch the following curves:
(a) $y = 4x^2$, (b) $y = -x^2 + 9$, (c) $y - 1 = x^2$,
(d) $x = -y^2$, (e) $x - y^2 + 4 = 0$, (f) $2x + y^2 + 16 = 0$.

2 Sketch the following curves showing where each meets the x-axis:
(a) $y = (x-1)(x-2)(x-3)$, (b) $y = (1-x)(x-2)(x-3)$,
(c) $y = (x+1)(x-2)^2$, (d) $y = x^2(3-x)$,
(e) $y = (x+2)(1-x)^2$, (f) $y^2 = x^6$,
(g) $x = y^3$, (h) $x + y^3 = 0$,
(i) $x = y(y-3)^2$.

3 Sketch the following curves:

(a) $y = -x^4$, (b) $y = \dfrac{1}{x^2}$, (c) $y = x^2 + \dfrac{1}{x^2}$, (d) $y = x^3 + \dfrac{1}{x}$,

(e) $y = x^3 + \dfrac{1}{x^2}$, (f) $y = x^2 - \dfrac{1}{x}$, (g) $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$.

4 Integrate with respect to x:

(a) $x^{1/3}$, (b) $\sqrt[4]{x}$, (c) $2x^{1/5}$,

(d) $k\sqrt[3]{x}$, (e) $x^{-1/2}$, (f) $\dfrac{1}{\sqrt[3]{x}}$,

(g) $x^{-1/6}$, (h) $\dfrac{2}{\sqrt[5]{x}}$, (i) $\sqrt[3]{x^2}$,

(j) $x^{7/3}$, (k) $(\sqrt{x})^3$, (l) $x^{-4/3}$,

(m) $x^{1/a}$, (n) $\dfrac{1}{\sqrt[n]{x}}$, (o) $\dfrac{x^3+2x^2-3x}{\sqrt{x}}$,

(p) $\sqrt{x}+\dfrac{2}{\sqrt{x}}$, (q) $(\sqrt{x}+2)(\sqrt{x}-3)$, (r) $\sqrt{(x+2)}$,

(s) $x\sqrt{(x^2-3)}$.

5 Evaluate the following:

(a) $\left[x^{-1/2}\right]_1^4$, (b) $\left[x^{3/2}+2x^{1/2}\right]_4^9$, (c) $\left[\tfrac{2}{3}(x+4)^{3/2}\right]_0^5$.

Area as the limit of a sum

8.3 We have already discussed the use of integration in finding the area under a curve (§6.3). The word *integration* implies the putting together of parts to make up a whole, and this fundamental aspect of the process is brought out in the following alternative approach to the area under a curve.

Suppose that we wish to find the area under the curve in Fig. 8.5 from $x = 0$ to $x = 3$. We divide this area into three equal strips by the lines $x = 1$ and $x = 2$.

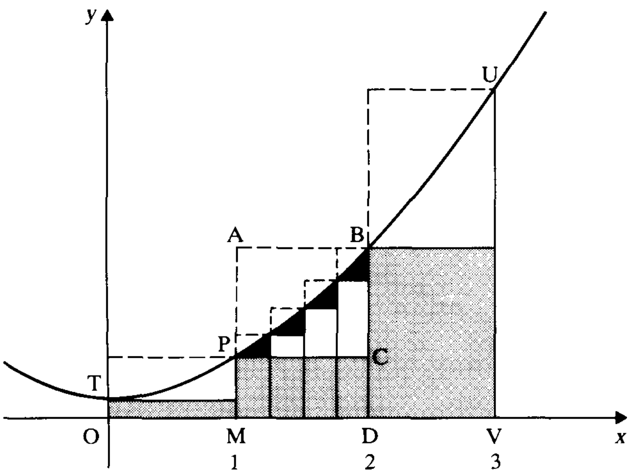


Figure 8.5

The required area TUVO lies between the sum of the areas of the three shaded 'inside' rectangles, and the sum of the three 'outside' rectangles bounded at the top by the broken lines; for example, the middle strip PBDM lies between the areas PCDM and ABDM.

We shall for the time being confine our attention to the 'inside' rectangles; the sum of these falls short of the required area by the sum of PBC and the two corresponding areas. We now divide TUVO into 12 strips (for clarity only 4 of these are shown in Fig. 8.5). The sum of the 12 'inside' rectangles is clearly a better approximation to the area under the curve, since an error such as PBC has been reduced to a much smaller error represented by the 4 black roughly triangular areas. Thus by taking a sufficient number of strips (in other words, by making the width of each strip sufficiently small) we can make the sum of the areas of the 'inside' rectangles as near as we please to the area under the curve.

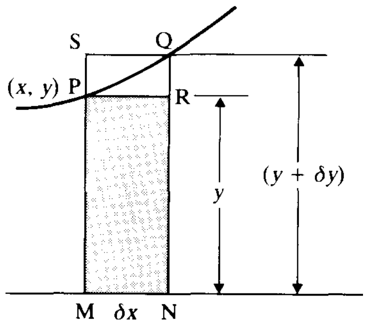


Figure 8.6

If we were to divide the area TUVO into a very large number of strips, then a typical one would be PQNM (Fig. 8.6), where $P(x, y)$ and $Q(x+\delta x, y+\delta y)$ are two points on the curve. A typical 'inside' rectangle is PRNM, of area $y\delta x$, and the process of increasing the number of strips is the same as letting $\delta x \to 0$. The required area TUVO is found by adding all the 'inside' rectangular areas $y\delta x$ between $x=0$ and $x=3$, and then finding the *limit* of this sum as $\delta x \to 0$. Using the symbol $\sum$ to denote 'the sum of' (see §13.8),

$$\text{as } \delta x \to 0, \qquad \sum_{x=0}^{x=3} y\delta x \to \text{the area TUVO}$$

*Hence area TUVO = the *limit*, as $\delta x \to 0$, of $\sum_{x=0}^{x=3} y\delta x$.

Example 1 *Calculate the area under $y = x + 1$ from $x = 0$ to $x = 10$.*

Divide the area into n strips of equal width parallel to Oy (Fig. 8.7); the width of each strip will be $10/n$. To find the sum of the areas of the inner shaded rectangles we must first calculate their heights.

*For simplicity we have confined our attention to the 'inside' rectangles. Fig. 8.6 also shows a typical 'outside' rectangle SQNM of area $(y+\delta y)\delta x$; as $\delta x \to 0$, $\sum_{x=0}^{x=3} (y+\delta y)\delta x$ tends to the same limit.

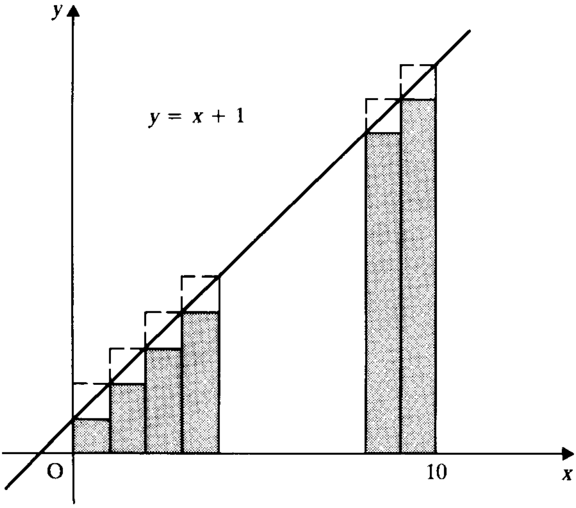


Figure 8.7

For the three smallest,

when $x = 0$, $\quad y = x + 1 = 1$

when $x = \dfrac{10}{n}$, $\quad y = \dfrac{10}{n} + 1$

when $x = 2 \times \dfrac{10}{n}$, $\quad y = \dfrac{20}{n} + 1$

and for the largest,

when $x = 10 - \dfrac{10}{n}$, $y = 11 - \dfrac{10}{n}$

The sum of the areas of the inner rectangles is

$$\left\{\frac{10}{n}(1) + \frac{10}{n}\left(\frac{10}{n} + 1\right) + \frac{10}{n}\left(\frac{20}{n} + 1\right) + \dots + \frac{10}{n}\left(11 - \frac{10}{n}\right)\right\}$$

$$= \frac{10}{n}\left\{1 + \left(\frac{10}{n} + 1\right) + \left(\frac{20}{n} + 1\right) + \dots + \left(11 - \frac{10}{n}\right)\right\}$$

The dots have been used to signify the terms corresponding to all the intermediate rectangles; we know that there are as many terms in the curly brackets as there are strips, namely n, and they form an arithmetic progression (see §13.2) with common difference $10/n$. We can now sum the terms in the brackets using the formula

$$S_n = \frac{n}{2}(a + l) \qquad \text{(See §13.4)}$$

$$= \frac{n}{2}\left(1 + 11 - \frac{10}{n}\right)$$

$$= \frac{n}{2}\left(12 - \frac{10}{n}\right)$$

$\therefore$ the sum of the 'inside' rectangles

$$= \frac{10}{n} \times \frac{n}{2}\left(12 - \frac{10}{n}\right)$$

$$= 60 - \frac{50}{n}$$

As $n \to \infty$, the limit of the sum is 60,
$\therefore$ the area under $y = x + 1$ from $x = 0$ to $x = 10$ is 60.

Qu. 1 Calculate the sum of the areas of the n 'outside' rectangles in Example 1, and find the limit of this sum as $n \to \infty$.

Example 2 *Calculate the area under the curve $y = x^2$, from $x = 0$ to $x = a$.*

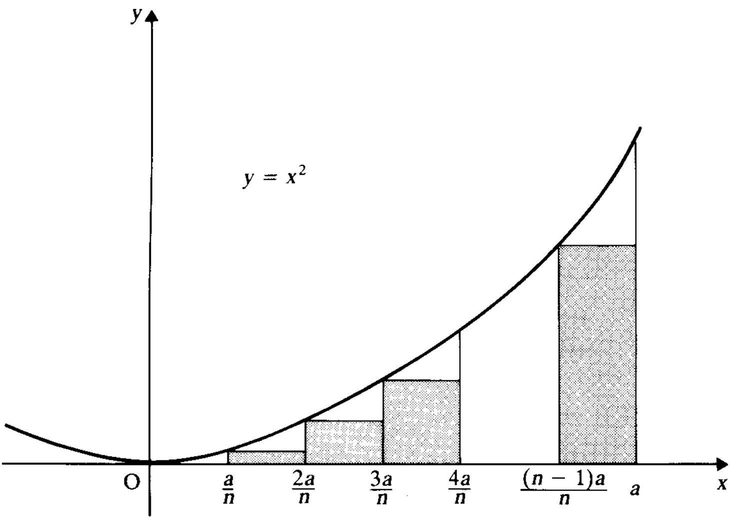


Figure 8.8

Here again we divide the interval $0 \leqslant x \leqslant a$, into n equal sub-intervals, each of length a/n (Fig. 8.8). To find the area inside the shaded region, we must first calculate the heights of the $(n - 1)$ rectangles. Since the equation of the curve is $y = x^2$, these heights are

$$\left(\frac{a}{n}\right)^2, \quad \left(\frac{2a}{n}\right)^2, \quad \left(\frac{3a}{n}\right)^2, \quad \ldots \quad \left(\frac{(n-1)a}{n}\right)^2$$

and, since the width of each rectangle is a/n, the sum of the areas of the rectangles is

$$\frac{a}{n} \times \frac{a^2}{n^2} + \frac{a}{n} \times \frac{4a^2}{n^2} + \frac{a}{n} \times \frac{9a^2}{n^2} + \ldots + \frac{a}{n} \times \frac{(n-1)^2a^2}{n^2}$$

$$= \frac{a^3}{n^3} + \frac{4a^3}{n^3} + \frac{9a^3}{n^3} + \ldots + \frac{(n-1)^2a^3}{n^3}$$

$$= \frac{a^3}{n^3}(1 + 4 + 9 + \ldots (n-1)^2)$$

Now, it can be shown (see §13.7) that

$$1 + 4 + 9 + \ldots (n-1)^2 = \frac{1}{6}(n-1) \times n \times (2n-1)$$

$$= \frac{1}{6}(2n^3 - 3n^2 + n)$$

Hence the sum of the areas of these rectangles is

$$S = \frac{a^3}{n^3} \times \frac{1}{6}(2n^3 - 3n^2 + n)$$

$$= \frac{a^3}{6}\left(2 - \frac{3}{n} + \frac{1}{n^2}\right)$$

and hence, when $n \to \infty$,

$$S \to \frac{a^3}{3}$$

Hence the area under the curve $y = x^2$, from $x = 0$ to $x = a$, is $a^3/3$.

It is interesting to note that this result, proved by a similar method, was known to the ancient Greeks, long before the invention of calculus.

The integral notation

8.4 Example 1 could be done by integration. Before doing this, we introduce the symbol $\int (\ldots)\,dx$ to denote integration with respect to x. The symbol $\int$, which is an elongated S, for 'sum', is a reminder that integration is essentially summation.

The area under $y = x + 1$ from $x = 0$ to $x = 10$ is

$$\int_0^{10} y\,dx = \int_0^{10} (x+1)\,dx$$

$$= \left[\tfrac{1}{2}x^2 + x\right]_0^{10}$$

$$= (\tfrac{1}{2} \times 10^2 + 10) - (0)$$
$$= 60$$

Similarly the result of Example 2 can be obtained by integration, as follows:

$$\int_0^a x^2\,dx = \Big[\tfrac{1}{3}x^3\Big]_0^a$$
$$= \tfrac{1}{3}a^3$$

For indefinite integrals, where there are no limits, a similar notation is used. Thus

$$\int (3x^2 + 4)\,dx = x^3 + 4x + c$$

Qu. 2 Find the following indefinite integrals:

(a) $\int (3x - 4)\,dx$, (b) $\displaystyle\int \frac{8x^5 - 3x}{x^3}\,dx$,

(c) $\int \sqrt[7]{x}\,dx$, (d) $\int (2\sqrt{t} - 3)(1 - \sqrt{t})\,dt$.

Qu. 3 Evaluate the following definite integrals:

(a) $\displaystyle\int_{1/2}^{1} (60t - 16t^2)\,dt$, (b) $\displaystyle\int_1^2 \frac{1}{2x^4}\,dx$, (c) $\displaystyle\int_1^4 \frac{(y+3)(y-3)}{\sqrt{y}}\,dy$.

We have shown above that when $y = x + 1$, the limit of $\displaystyle\sum_{x=0}^{x=10} y\delta x$, as $\delta x \to 0$, is identical with, and is more readily evaluated as $\displaystyle\int_0^{10} y\,dx$.

We shall now assume that for any curve which is continuous between $x = a$ and $x = b$, the area under the curve from $x = a$ to $x = b$ is

the limit, as $\delta x \to 0$, of $\displaystyle\sum_{x=a}^{x=b} y\delta x = \int_a^b y\,dx$*

Notice that, in general, if $f(x)$ is a continuous function and $F(x)$ is the function whose derivative is $f(x)$, i.e. $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b$$
$$= F(b) - F(a)$$

If, in addition, $f(x) \geqslant 0$, when $a \leqslant x \leqslant b$, then this integral gives the area under the curve $y = f(x)$, from $x = a$ to $x = b$. If, however, $f(x)$ is not always positive in

*The reader may be interested to note the parallel between this statement and that concerning gradient, namely the limit, as $\delta x \to 0$, of $\dfrac{\delta y}{\delta x} = \dfrac{dy}{dx}$.

this interval, then the graph of $y = f(x)$ must be consulted, in order to distinguish between the positive and negative areas.

The reader should in future think of every area bounded by a curve as a summation, first writing down the area of one of the typical strips, or *elements of area*, into which it is most conveniently divided, and then evaluating the limit of the sum of those strips by integration. A convenient way of laying out the working is shown in the following examples; these extend the work of Chapter 6 in the following ways:

(a) by using elements of area parallel to the x-axis, we may integrate with respect to y;

(b) by finding the element of area cut off between two curves we may evaluate in only one step the area enclosed between them.

Example 3 *Find the area enclosed by $y = 4x - x^2$, $x = 1$, $x = 2$ and the x-axis* (Fig. 8.9).

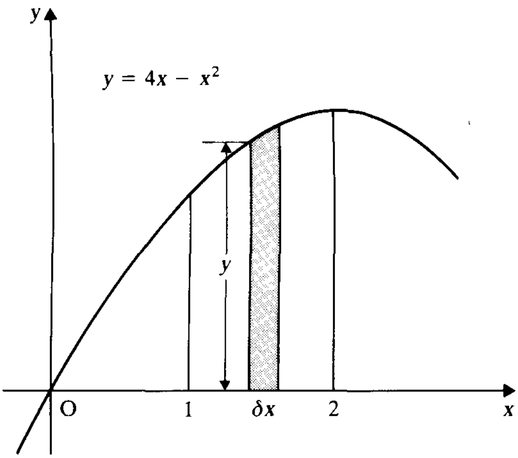


Figure 8.9

The element of area is $y\delta x = (4x - x^2)\delta x$

$$\therefore \text{ the required area} = \int_1^2 (4x - x^2)\,dx$$

$$= \left[2x^2 - \tfrac{1}{3}x^3\right]_1^2$$

$$= (8 - \tfrac{8}{3}) - (2 - \tfrac{1}{3})$$

$$= 3\tfrac{2}{3}$$

Example 4 *Find the area enclosed by that part of $y = x^2$ for which x is positive, the y-axis, and the lines $y = 1$ and $y = 4$.*

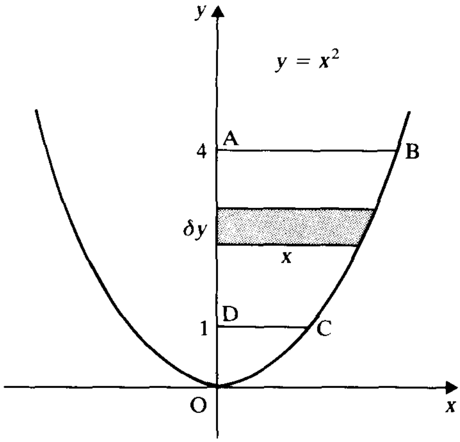


Figure 8.10

The required area is ABCD in Fig. 8.10. The equation may be written $x = \pm\sqrt{y}$, and for the part of the curve with which we are concerned $x = +\sqrt{y} = +y^{1/2}$.

The element of area is $x\delta y$.

$$\therefore \text{ the required area} = \int_1^4 x\,dy$$

$$= \int_1^4 y^{1/2}\,dy$$

$$= \left[\tfrac{2}{3}y^{3/2}\right]_1^4$$

$$= (\tfrac{2}{3} \times 8) - (\tfrac{2}{3})$$

$$= 4\tfrac{2}{3}$$

Example 5 *Find the area enclosed between the two curves $y = 4 - x^2$ and $y = x^2 - 2x$.*

We must first sketch the curves, and to find the limits of integration we must find the x-coordinates of the points of intersection.

When $x^2 - 2x = 4 - x^2$,

$$2x^2 - 2x - 4 = 0$$
$$x^2 - x - 2 = 0$$
$$(x - 2)(x + 1) = 0$$
$$x = -1 \quad \text{or} \quad +2$$

The element of area is shown shaded in Fig. 8.11.

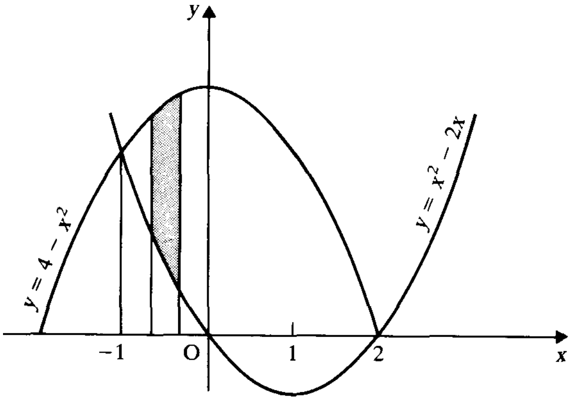


Figure 8.11

If we write $Y = 4 - x^2$ and $y = x^2 - 2x$, the element of area is $(Y - y)\delta x$.

$$\therefore \text{ the required area} = \int_{-1}^{+2} (Y - y)\,dx$$

$$= \int_{-1}^{+2} \{(4 - x^2) - (x^2 - 2x)\}\,dx$$

$$= \int_{-1}^{+2} (4 + 2x - 2x^2)\,dx$$

$$= \left[4x + x^2 - \tfrac{2}{3}x^3\right]_{-1}^{+2}$$

$$= (4 \times 2 + 2^2 - \tfrac{2}{3} \times 2^3) - (-4 + 1 + \tfrac{2}{3})$$

$$= 8 + 4 - 5\tfrac{1}{3} + 4 - 1\tfrac{2}{3}$$

$$= 9$$

Exercise 8b

1 Find the following integrals:

(a) $\int x(x - 3)\,dx$, (b) $\int \frac{2(x-1)}{x^3}\,dx$,

(c) $\int \left(at^2 + b + \frac{c}{t^2}\right)dt$, (d) $\int \left(x^4 - \sqrt[3]{x^2} + 2 - \frac{1}{x^2}\right)dx$,

(e) $\int \left(y + \frac{1}{\sqrt{y}}\right)\left(y + \frac{1}{y}\right)dy$, (f) $\int \frac{(s+1)^2}{\sqrt[3]{s}}\,ds$.

2 Evaluate:

(a) $\int_{-2}^{+3} (v^2 + 3)\,dv$, (b) $\int_1^4 (y^2 + \sqrt{y})\,dy$,

(c) $\int_0^1 \sqrt{x}(x + 2)\,dx$, (d) $\int_1^2 \left(3 + \frac{1}{t^2} + \frac{1}{t^4}\right) dt$,

(e) $\int_1^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$, (f) $\int_4^{11} \sqrt{(x + 5)}\,dx$.

3 Find the area under each of the following curves between the given limits:
(a) $y = x^2 + 3$, $x = -1$ to $x = 2$;
(b) $y = x^2(3 - x)$, $x = 4$ to $x = 5$;
(c) $y = x^2 + 1/x^2$, $x = \frac{1}{2}$ to $x = 1$.

4 Find the area enclosed by the y-axis and the following curves and straight lines:
(a) $x = y^2$, $y = 3$; (b) $y = x^3$, $y = 1$, $y = 8$;
(c) $x - y^2 - 3 = 0$, $y = -1$, $y = 2$; (d) $x = 1/\sqrt{y}$, $y = 2$, $y = 3$.

5 Find the area enclosed by each of the following curves and the y-axis:
(a) $x = (y - 1)(y - 4)$ (Why is this negative?),
(b) $x = 3y - y^2$,
(c) $x = y(y - 2)^2$.

6 Find the area enclosed by $y^2 = 4x$ and the straight lines $x = 1$ and $x = 4$.

7 Find the area enclosed by $y^2 = x + 9$ and the y-axis, by taking an element of area (a) parallel to the y-axis, and (b) parallel to the x-axis.

8 Find the area enclosed by $9x^2 + y - 16 = 0$ and the x-axis, by integrating (a) with respect to x, and (b) with respect to y.

9 Calculate the areas enclosed by
(a) $y = 1/x^2$, $y = 1$ and $y = 4$; (b) $x = 1/y^2$, $y = 1$, $y = 4$, and $x = 0$.

10 Find the area of the segment cut off from each of the following curves by the given straight line:
(a) $y = x^2 - 2x + 2$, $y = 5$; (b) $y = x^2 - 6x + 9$, $y = 1$;
(c) $y = -x^2 + 3x - 4$, $y = -4$; (d) $y = x(x - 2)$, $y = x$;
(e) $y = 4 - 3x - x^2$, $2x + y + 2 = 0$; (f) $y = x^2 - 6x + 2$, $x + y - 2 = 0$.

11 Find the area enclosed by each of the following pairs of curves:
(a) $y = x(x - 1)$ and $y = x(2 - x)$,
(b) $y = x(x + 3)$ and $y = x(5 - x)$,
(c) $y = x^2 - 5x$ and $y = 3x^2 - 6x$,
(d) $y^2 = 4x$ and $x^2 = 4y$,
(e) $y = x^2 - 3x - 7$ and $y = 5 - x - x^2$,
(f) $y = 2x^2 + 7x + 3$ and $y = 9 + 4x - x^2$.

12 Find the area of the segment cut off from $y = 1/x^2$ by $10x + 4y - 21 = 0$, given that one of the points of intersection of the straight line and the curve is $(-\frac{2}{5}, \frac{25}{4})$.

13 By reference to a clear diagram, show that if f(x) is an odd function, then

$$\int_{-a}^{+a} f(x)\,dx = 0$$

Show also that if g(x) is an even function, then

$$\int_{-a}^{+a} g(x)\,dx = 2\int_{0}^{+a} g(x)\,dx$$

14 Prove, using the method of Example 2 in the text, that

$$\int_{0}^{a} x^3\,dx = \frac{a^4}{4}$$

[You will need to quote that $1 + 8 + 27 + \ldots + (N-1)^3 = N^2(N-1)^2/4$.]

Solids of revolution

8.5 If we take a triangular piece of cardboard ABC with a right angle at B, and rotate it through 360 degrees about AB, we sweep out the volume of a right circular cone (Fig. 8.12). The cone can thus be thought of as the **solid of revolution** generated by rotating the area ABC about the line AB.

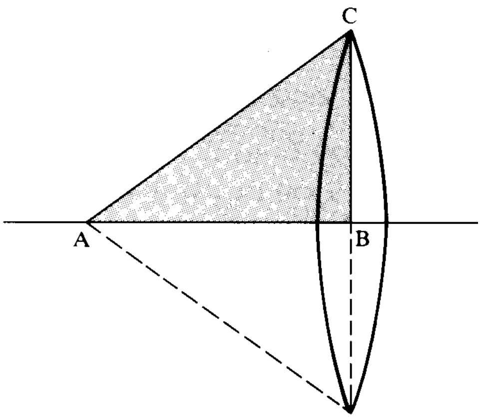


Figure 8.12

Qu. 4 State the solid generated by rotating through 360 degrees:
(a) the above triangle ABC (i) about BC, (ii) about AC,
(b) the area of a semi-circle about the bounding diameter,
(c) a quadrant of a circle about a boundary radius,
(d) the area of a circle centre (3, 3) radius 1, about the y-axis,
(e) a rectangle about one of its sides.

The method of calculating the volume of a solid of revolution is best illustrated by discussing an example; the ideas involved are the same as those of §8.3.

Example 6 *Find the volume of the solid generated by rotating about the x-axis the area under $y=\frac{3}{4}x$ from $x=0$ to $x=4$.*

A typical element of area under $y=\frac{3}{4}x$ is $y\delta x$, shown shaded in Fig. 8.13; rotating this area about the x-axis we generate the typical *element of volume*, a cylinder of volume $\pi y^2\delta x$. The corresponding 'slice' of the solid (Fig. 8.14) has one circular face of radius y, and the other of radius $y+\delta y$, and its volume lies between that of an 'inside' cylinder $\pi y^2\delta x$, and an 'outside' cylinder $\pi(y+\delta y)^2\delta x$.

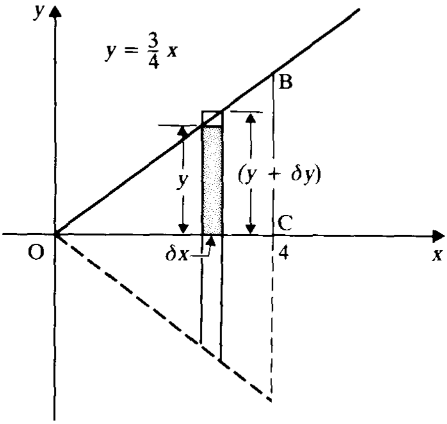


Figure 8.13

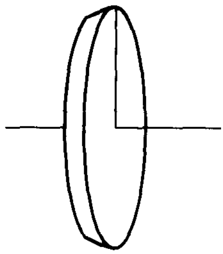


Figure 8.14

The sum of the volumes of all the 'inside' (or 'outside') cylinders is an approximation to the volume required, and, by making δx sufficiently small, we can make this sum approach as close as we please to the volume of the solid of revolution, which may therefore be written as

the limit, as $\delta x\to 0$, of $\sum_{x=0}^{x=4}\pi y^2\delta x$

This may be evaluated as $\int_0^4 \pi y^2\,dx$; thus the solution of this example may be presented as follows.

The element of volume $=\pi y^2\delta x=\pi\dfrac{9x^2}{16}\delta x$

$$\therefore \text{ the required volume} = \int_0^4 \pi\frac{9x^2}{16}\,dx$$

$$=\left[\pi\frac{3x^3}{16}\right]_0^4$$

$$=\pi\frac{3\times 4^3}{16}$$

$$=12\pi$$

Qu. 5 Find the volume of the solid generated by rotating about the x-axis
(a) the area under $y = x^2$ from $x = 1$ to $x = 2$,
(b) the area under $y = x^2 + 1$ from $x = -1$ to $x = +1$.

The volumes of solids generated by rotating areas about the y-axis may be evaluated by integration with respect to y. This, and other aspects of this work, are illustrated by the following examples.

Example 7 *Find the volume of the solid generated by rotating about the y-axis the area in the first quadrant enclosed by* $y = x^2$, $y = 1$, $y = 4$ *and the y-axis* (Fig. 8.15).

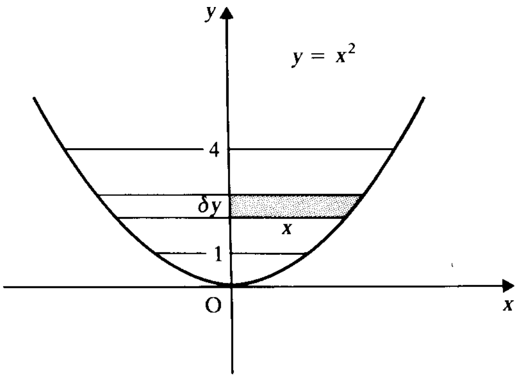


Figure 8.15

The element of volume $= \pi x^2 \delta y = \pi y \delta y$

$$\therefore \text{ the required volume} = \int_1^4 \pi y \, dy$$

$$= \left[\tfrac{1}{2}\pi y^2\right]_1^4$$

$$= \tfrac{1}{2}\pi \times 16 - \tfrac{1}{2}\pi$$

$$= \frac{15\pi}{2}$$

Example 8 *The area of the segment cut off by* $y = 5$ *from the curve* $y = x^2 + 1$ *is rotated about* $y = 5$; *find the volume generated* (Fig. 8.16).

The points of intersection occur when

$$x^2 + 1 = 5$$
$$x^2 = 4$$
$$x = -2 \quad \text{or} \quad +2$$

The element of volume $= \pi(5 - y)^2 \delta x$

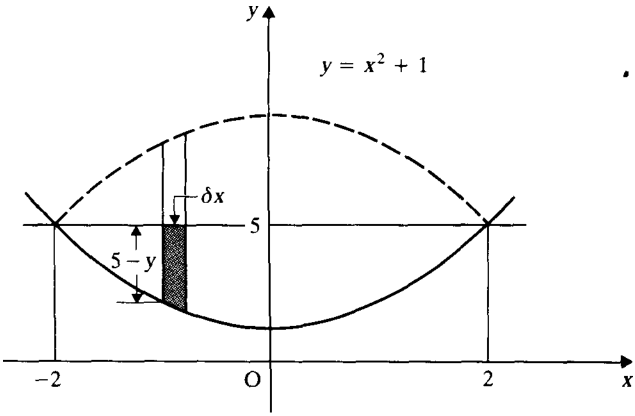


Figure 8.16

$$= \pi(5 - x^2 - 1)^2 \delta x$$
$$= \pi(16 - 8x^2 + x^4)\delta x$$

$$\therefore \text{ the required volume} = \int_{-2}^{+2} \pi(16 - 8x^2 + x^4)\,dx$$

$$= \left[\pi(16x - \tfrac{8}{3}x^3 + \tfrac{1}{5}x^5)\right]_{-2}^{+2}$$

$$= \pi(32 - 21\tfrac{1}{3} + 6\tfrac{2}{5}) - \pi(-32 + 21\tfrac{1}{3} - 6\tfrac{2}{5})$$

$$= 34\tfrac{2}{15}\pi$$

Example 9 *The area of the segment cut off by $y = 5$ from the curve $y = x^2 + 1$ is rotated about the x-axis; find the volume generated* (Fig. 8.17).

The solid generated is a cylinder fully open at each end, but with the internal diameter decreasing towards the middle; its volume is found by subtracting the volume of the cavity from the volume of the solid cylinder of the same external dimensions.

The required volume = the volume generated by rotation, about the x-axis, of the rectangle ABDE (1)
minus the volume generated by rotation, about the x-axis, of the area under $y = x^2 + 1$ from $x = -2$ to $x = +2$, i.e. ABCDE (2)

Volume (1) $= \pi r^2 h = \pi \times 5^2 \times 4 = 100\pi$

Element of Volume (2) $= \pi y^2 \delta x$
$= \pi(x^4 + 2x^2 + 1)\delta x$

$$\therefore \text{ Volume (2)} = \int_{-2}^{+2} \pi(x^4 + 2x^2 + 1)\,dx$$

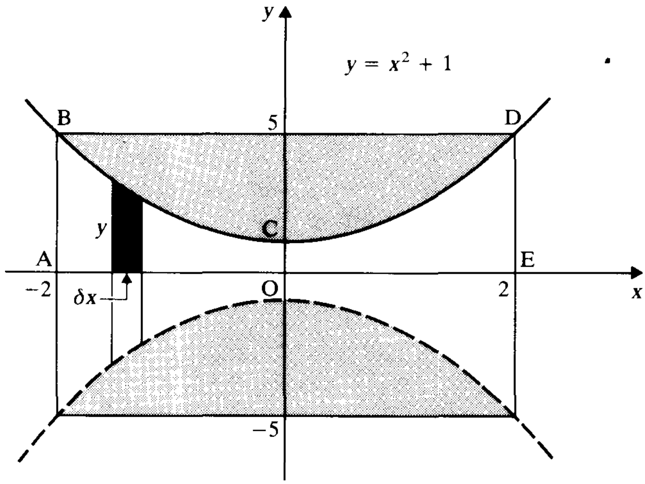


Figure 8.17

$$\therefore \text{ Volume (2)} = \left[\pi\left(\frac{x^5}{5} + \frac{2}{3}x^3 + x\right)\right]_{-2}^{+2}$$

$$= \pi(6\tfrac{2}{5} + 5\tfrac{1}{3} + 2) - \pi(-6\tfrac{2}{5} - 5\tfrac{1}{3} - 2)$$

$$= 27\tfrac{7}{15}\pi$$

$\therefore$ the required volume $= 100\pi - 27\frac{7}{15}\pi$

$= 72\frac{8}{15}\pi$

Exercise 8c

Leave π in the answers.

1 Find the volumes of the solids generated by rotating about the x-axis each of the areas bounded by the following curves and lines:
(a) $x + 2y - 12 = 0$, $x = 0$, $y = 0$; (b) $y = x^2 + 1$, $y = 0$, $x = 0$, $x = 1$;
(c) $y = \sqrt{x}$, $y = 0$, $x = 2$; (d) $y = x(x - 2)$, $y = 0$;
(e) $y = x^2(1 - x)$, $y = 0$; (f) $y = 1/x$, $y = 0$, $x = 1$, $x = 4$.

2 Find the volumes of the solids generated by rotating about the y-axis each of the areas bounded by the following curves and lines:
(a) $y = 2x - 4$, $y = 2$, $x = 0$; (b) $x = \sqrt{(y - 1)}$, $x = 0$, $y = 4$;
(c) $x - y^2 - 2 = 0$, $x = 0$, $y = 0$, $y = 3$; (d) $y^2 = x + 4$, $x = 0$;
(e) $y = 1 - x^3$, $x = 0$, $y = 0$; (f) $xy = 1$, $x = 0$, $y = 2$, $y = 5$.

3 Find the volumes of the solids generated when each of the areas enclosed by the following curves and lines is rotated about the given line:
(a) $y = x$, $x = 0$, $y = 2$, about $y = 2$;
(b) $y = \sqrt{x}$, $y = 0$, $x = 4$, about $x = 4$;
(c) $y^2 = x$, $x = 0$, $y = 2$, about $y = 2$;

(d) $y = 2 - x^2$, $y = 1$, about $y = 1$;
(e) $y = x^3 - 2x^2 + 3$, $y = 3$, about $y = 3$;
(f) $y = 1/x^2$, $y = 4$, $x = 1$, about $y = 4$.

4 Repeat No. 3 for the following areas:
(a) $x - 3y + 3 = 0$, $x = 0$, $y = 2$, about the x-axis;
(b) $x - y^2 - 1 = 0$, $x = 2$, about the y-axis;
(c) $y^2 = 4x$, $y = x$, about $y = 0$;
(d) $y = 1/x$, $y = 1$, $x = 2$, about $y = 0$.

5 Obtain, by integration, the formula for the volume of a right circular cone of base radius r, height h. (Consider the area enclosed by $y = (h/r)x$, $x = 0$ and $y = h$.)

6 The equation of a circle centre the origin and radius r is $x^2 + y^2 = r^2$. By considering the area of this circle cut off in the first quadrant being rotated about either the x- or y-axis, deduce the formula for the volume of a sphere radius r.

7 A hemispherical bowl of internal radius 13 cm contains water to a maximum depth of 8 cm. Find the volume of the water.

8 A goldfish bowl is a glass sphere of inside diameter 20 cm. Calculate the volume of water it contains when the maximum depth is 18 cm.

9 A wall vase has one plane face, and its volume is equivalent to that generated when the area enclosed by $x = \frac{1}{64}y^3 + 1$, the y-axis and $y = 8$ is rotated through 2 right angles about the y-axis, the units being cm. Calculate its volume.

10 The area under $y = \frac{1}{9}x^2 + 1$ from $x = 0$ to $x = 3$, and the area enclosed by $y = 0$, $y = 2$, $x = 3$, and $x = 4$, are rotated about the y-axis, and the solid generated represents a metal ash tray, the units being cm. Calculate the volume of metal.

11 The area enclosed by $y = x^2 - 6x + 18$ and $y = 10$ is rotated about $y = 10$. Find the volume generated.

12 The area enclosed by $y = x^2 + 1/x$, the x-axis and $x = -2$, is rotated about the x-axis; find the volume generated.

13 The area enclosed by $y = 4/x^2$, $y = 1$ and $y = 4$ is rotated about the x-axis; find the volume generated.

14 The area enclosed by $y = x^2 - 6x + 18$ and $y = 10$ is rotated about the y-axis; find the volume generated. [Take an element of area parallel to the x-axis of length $(x_2 - x_1)$; express the typical element of volume in terms of y by using the fact that x_1 and x_2 are the roots of $x^2 - 6x + (18 - y) = 0$; see §9.7.]

15 Repeat No. 14 for the area enclosed by $4y = 4x^2 - 20x + 25$ and $4y = 9$.

Centre of gravity

8.6 The reader who has dealt with this topic in mechanics will be familiar with the fact that, for a system of bodies whose centres of gravity lie in a plane, taking moments about any line in the plane,

the moment of their total weight acting at the centre of gravity of the system = the sum of the moments of the weight of each body

If n bodies of weight $w_1, w_2, w_3, \dots w_n$ have their centres of gravity at (x_1, y_1), (x_2, y_2), $(x_3, y_3) \dots (x_n, y_n)$ respectively, writing the coordinates of the centre of gravity of the system as $(\bar{x}, \bar{y})$, and taking moments about the y-axis,

$$\bar{x}(w_1 + w_2 + w_3 + \dots + w_n) = x_1w_1 + x_2w_2 + x_3w_3 + \dots + x_nw_n$$

Using the $\sum$ notation,

$$\bar{x}\sum w = \sum xw$$

Similarly, taking moments about the x-axis,

$$\bar{y}\sum w = \sum yw$$

If, instead of separate bodies, we consider the elements of area of a uniform lamina, then $\sum xw$ and $\sum yw$ become the sums of the moments of the weights of the elements about the axes, and these can be evaluated by integration.

Example 10 *Find the centre of gravity of a uniform lamina whose shape is the area bounded by* $y^2 = 4x$ *and* $x = 9$.

By symmetry the centre of gravity lies on the x-axis, hence $\bar{y} = 0$.

Consider the lamina as made up of strips parallel to the y-axis, then if the weight per unit area is ρ, a typical element (Fig. 8.18) at a distance x from the y-axis has weight $\rho \times 2y \times \delta x$ and its moment about the y-axis is $x \times 2\rho y\delta x$.

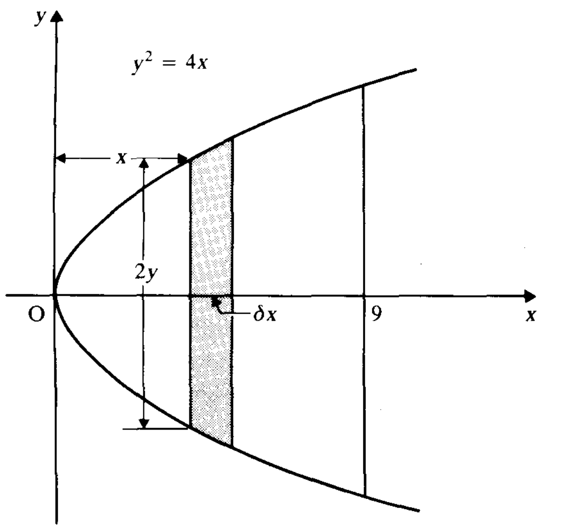


Figure 8.18

The sum of the moments of the weights of the elements

$$= \sum_{x=0}^{x=9} 2\rho xy\delta x$$

and the limit of this, as $\delta x \to 0$, is evaluated as

$$2\rho \int_0^9 xy\,dx$$

The weight of the whole lamina

$= \rho \times$ twice the area under $y = 2x^{1/2}$ from $x = 0$ to $x = 9$

$$= 2\rho \int_0^9 y\,dx$$

Since $\quad \bar{x}\sum w = \sum xw$

$$\bar{x} \times 2\rho \int_0^9 y\,dx = 2\rho \int_0^9 xy\,dx$$

$$\therefore \bar{x} \int_0^9 y\,dx = \int_0^9 xy\,dx$$

But $y = 2x^{1/2}$,

$$\therefore \bar{x} \int_0^9 x^{1/2}\,dx = \int_0^9 x^{3/2}\,dx$$

$$\therefore \bar{x}\left[\tfrac{2}{3}x^{3/2}\right]_0^9 = \left[\tfrac{2}{5}x^{5/2}\right]_0^9$$

$$\therefore \bar{x} \times \tfrac{2}{3} \times 3^3 = \tfrac{2}{5} \times 3^5$$

$$\therefore \bar{x} = \tfrac{27}{5}$$

$\therefore$ the centre of gravity of the lamina is at $(\frac{27}{5}, 0)$.

Example 11 *Find the centre of gravity of a uniform lamina whose shape is the area bounded by $y = x^2$, the x-axis and $x = 4$* (Fig. 8.19).

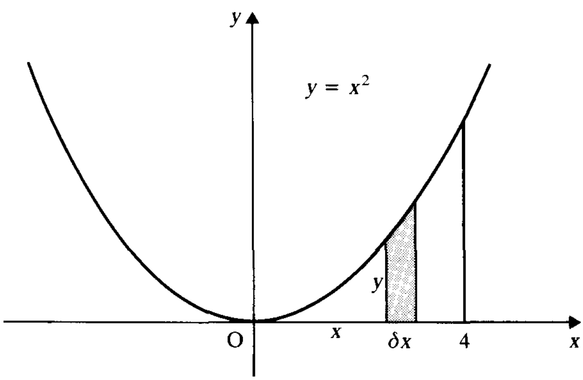


Figure 8.19

Let the weight per unit area be ρ.
Taking moments about the y-axis,

$$\bar{x} \times \rho \int_0^4 y \, dx = \rho \int_0^4 xy \, dx$$

$$\therefore \bar{x} \int_0^4 y \, dx = \int_0^4 xy \, dx$$

But $y = x^2$,

$$\therefore \bar{x} \int_0^4 x^2 \, dx = \int_0^4 x^3 \, dx$$

$$\therefore \bar{x} \left[\tfrac{1}{3}x^3 \right]_0^4 = \left[\tfrac{1}{4}x^4 \right]_0^4$$

$$\therefore \bar{x} \times \tfrac{1}{3} \times 4^3 = \tfrac{1}{4} \times 4^4$$

$$\therefore \bar{x} = 3$$

The centre of gravity of the element is at its mid-point, thus the moment of its weight about the x-axis is $\frac{1}{2}y \times \rho y \delta x$.
Taking moments about the x-axis,

$$\bar{y} \times \rho \int_0^4 y \, dx = \rho \int_0^4 \tfrac{1}{2}y^2 \, dx$$

$$\therefore \bar{y} \int_0^4 y \, dx = \int_0^4 \tfrac{1}{2}y^2 \, dx$$

$$\therefore \bar{y} \int_0^4 x^2 \, dx = \int_0^4 \tfrac{1}{2}x^4 \, dx$$

$$\therefore \bar{y} \left[\tfrac{1}{3}x^3 \right]_0^4 = \left[\tfrac{1}{10}x^5 \right]_0^4$$

$$\therefore \bar{y} \times \tfrac{1}{3} \times 4^3 = \tfrac{1}{10} \times 4^5$$

$$\therefore \bar{y} = \tfrac{24}{5}$$

$\therefore$ the centre of gravity of the lamina is at $(3, \frac{24}{5})$.

Qu. 6 Find the centre of gravity of the lamina whose area is bounded by
(a) $y^2 = x$ and $x = 2$, (b) $y = \sqrt{x}$, $y = 0$ and $x = 2$.

The centre of gravity of a solid of revolution may be found in the same way, since the centre of gravity of each element of volume lies in the plane of the axes.

Example 12 *Find the centre of gravity of the solid generated by rotating about the x-axis the area under $y = x$ from $x = 0$ to $x = 3$* (Fig. 8.20).

The solid is a cone, vertex O, and axis Ox. By symmetry, the centre of gravity lies on the x-axis.

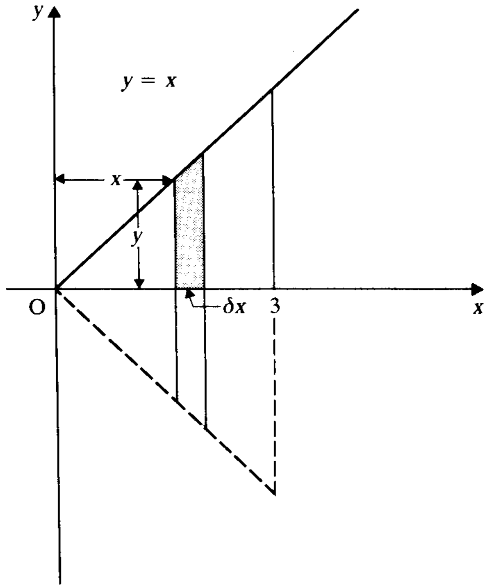


Figure 8.20

Let the weight per unit volume be ρ.

The centre of gravity of the element of volume is on the x-axis, thus the moment of its weight about the y-axis is $x \times \rho\pi y^2 \delta x$.

Taking moments about the y-axis,

$$\bar{x} \times \rho \int_0^3 \pi y^2 \, dx = \int_0^3 x\rho\pi y^2 \, dx$$

$$\therefore \bar{x} \int_0^3 y^2 \, dx = \int_0^3 xy^2 \, dx$$

But $y = x$,

$$\therefore \bar{x} \int_0^3 x^2 \, dx = \int_0^3 x^3 \, dx$$

$$\therefore \bar{x}\left[\tfrac{1}{3}x^3\right]_0^3 = \left[\tfrac{1}{4}x^4\right]_0^3$$

$$\therefore \bar{x} \times \tfrac{1}{3} \times 3^3 = \tfrac{1}{4} \times 3^4$$

$$\therefore \bar{x} = \tfrac{9}{4}$$

$\therefore$ the centre of gravity of the cone is at $(\frac{9}{4}, 0)$.

Exercise 8d

In Nos. 1 to 3, find the coordinates of the centre of gravity of the uniform lamina whose area is bounded by the given straight lines and curves.

1 (a) $y^2 = 9x$, $x = 4$; (b) $y = \frac{1}{4}x^2$, $y = 1$;
(c) $y^2 = 4 - x$, the y-axis; (d) $y = 1/x^2$, $y = 1$, $y = 4$.

2 (a) $x + y^2 - 1 = 0$, the y-axis;
(b) $y = x^2 + 2$, $x = -1$, $x = +1$ and $y = 0$.

3 (a) $y = \frac{1}{3}x$, $y = 0$, $x = 12$; (b) $x = 2\sqrt{y}$, $y = 1$, $x = 0$;
(c) $y = x^2$, $y = 0$, $x = 3$; (d) $y = x^3$, $y = 0$, $x = 2$.

4 Find the centres of gravity of the solids of revolution generated when the areas bounded by the following straight lines and curves are rotated about the given axes:
(a) $x + 3y - 6 = 0$, $x = 0$, $y = 0$, about the x-axis;
(b) $y = 2\sqrt{x}$, $y = 0$, $x = 4$, about the x-axis;
(c) $y^2 = 4x$, $y = 4$, $x = 0$, about the y-axis;
(d) $y = x^2(2 - x)$, $y = 0$, about the x-axis;
(e) $y = 1/x^2$, $y = 0$, $x = 1$, $x = 2$, about the x-axis;
(f) $y = x^3$, $y = 1$, $y = 8$, $x = 0$, about the y-axis.

5 By considering the solid generated by rotating, about the x-axis, the area enclosed by $y = (r/h)x$, the x-axis and $x = h$, deduce the position of the centre of gravity of a right circular cone.

6 The equation of the circle centre the origin, radius r, is $x^2 + y^2 = r^2$. By considering the solid generated by rotating about either axis the area of one quadrant, deduce the distance of the centre of gravity of a solid hemisphere from its plane surface.

7 A goldfish bowl consists of a sphere of inside radius 10 cm. If it contains water to a maximum depth of 16 cm, find the height of the centre of gravity of the water above the lowest point.

8 A uniform lamina is of the shape of the quadrant of the circle $x^2 + y^2 = r^2$ cut off by the positive axes. Find the coordinates of its centre of gravity.

Exercise 8e (Miscellaneous)

1 Calculate $\displaystyle\int_{-1}^{1} x(x^2 - 1)\,dx$.

Find the area bounded by the curve $y = x(x^2 - 1)$ and the x-axis (a) between $x = -1$ and $x = 0$, and (b) between $x = 0$ and $x = 1$.

2 Find the area between the curve $y = x(x - 1)^2(2 - x)$ and the portion of the x-axis between $x = 1$ and $x = 2$.

3 The line $y = \frac{1}{2}x + 1$ meets the curve $y = \frac{1}{4}(7x - x^2)$ at the points A and B. Calculate the coordinates of A and B and the length of the line AB. Prove that the segment of the curve cut off by the line has an area $1\frac{1}{8}$.

4 The area enclosed between the line $x = 1$, the x-axis, the line $x = 3$ and the line $3x - y + 2 = 0$, is rotated through four right angles about the x-axis.

Find the volume generated.

5 Solids of revolution are generated by rotating

(a) about the x-axis the area bounded by the arc of the curve $y = 2x^2$ between $(0, 0)$ and $(2, 8)$, the line $x = 2$ and the x-axis;

(b) about the y-axis the area bounded by the same arc, the line $y = 8$ and the y-axis.

Calculate the volumes of the two solids so formed.

6 The corners of a trapezium are at the points $(0, 2)$, $(2, 2)$, $(0, 4)$, $(3, 4)$. Find the volume of the solid formed by revolving the area about the y-axis.

7 Sketch the curve $y = x^2(1 - x)$. The area between the curve and the part of the x-axis from $x = 0$ to $x = 1$ is rotated about the x-axis. Find the volume swept out.

8 The portion of the parabola $y = \frac{2}{3}\sqrt{x}$ between $x = \frac{1}{4}$ and $x = 2$ is revolved about the x-axis so as to obtain a parabolic cup with a circular base and top. Show that the volume of the cup is approximately 2.75.

9 Find the equation of the tangent to the curve $y = x - 1/x$ at the point $(1, 0)$.

The area between the curve, the x-axis and the ordinate $x = 2$ is rotated about the x-axis. Prove that the volume thus obtained is $\frac{5}{6}\pi$.

10 The curve $y = x^2 + 4$ meets the axis of y at the point A, and B is the point on the curve where $x = 2$. Find the area between the arc AB, the axes, and the line $x = 2$.

If this area is revolved about the x-axis, prove that the volume swept out is approximately 188.

11 The area bounded by the x-axis, the line $x = 1$, the line $x = 4$, and the curve $y^2 = 4x^3$ is rotated about the x-axis. Find the volume of the resulting solid. (C)

12 A cylindrical hole of radius 4 cm is cut from a sphere of radius 5 cm, the axis of the cylinder coinciding with a diameter of the sphere. Prove that the volume of the remaining portion of the sphere is 36π cm^3.

13 Find the area bounded by the curve $y = 3x^2 - x^3$ and the x-axis. Find the x-coordinate of the centre of gravity of this area.

14 Find the area bounded by the x-axis and the arc of the curve

$$y = x^2(x - 1)(3 - x)$$

from $x = 1$ to $x = 3$. Find also the x-coordinate of the centre of gravity of this area.

15 Find the area and the x-coordinate of the centre of gravity of the lamina whose edges are formed by the lines $x = 0$, $y = 0$, and the part of the curve $y = (1 - x)(5 + 4x + x^2)$ which is cut off by these lines in the first quadrant.

16 (a) Find the area bounded by the curve $y = x^2$, the x-axis and the ordinates $x = 1$ and $x = 2$.

(b) Find the x- and y-coordinates of the centre of gravity of this area.

17 Find the coordinates of the centre of gravity of the area enclosed by the x-axis and the curve $y = x^2(3 - x)$.

18 Find the area bounded by the curve $y = (x + 1)(x - 2)^2$ and the x-axis from $x = -1$ to $x = 2$. Also find the x-coordinate of the centre of

Chapter 9

Some useful topics in algebra

Surds

9.1 It is not immediately obvious that

$$\frac{3\sqrt{5}}{2\sqrt{7}},\quad \frac{\sqrt{45}}{\sqrt{28}},\quad \frac{3}{14}\sqrt{35},\quad \frac{15}{2\sqrt{35}},\quad \frac{3}{2}\sqrt{\frac{5}{7}},\quad \frac{\sqrt{45}}{2\sqrt{7}},\quad \frac{3\sqrt{5}}{\sqrt{28}},\quad \sqrt{\frac{45}{28}}$$

all represent the same number. Again, it may not be clear on first sight that $1/(\sqrt{2}-1)$ and $\sqrt{2}+1$ are equal.

Since square roots frequently occur in trigonometry and coordinate geometry, it is useful to be able to recognise a number when it is written in different ways, and the purpose of this section is to give the reader practice in this.

The reader may have found an approximate value of $\sqrt{2} \approx 1.414\,213\,562$ on a calculator and may know that this decimal does not terminate or recur. The ancient Greeks did not use decimals, but they discovered that $\sqrt{2}$ could not be expressed as a fraction of two integers (see §2.4). Such a root ($\sqrt{3}$, $\sqrt{5}$, $\sqrt[3]{6}$ are other examples) is called a **surd**. In general, a number which cannot be expressed as a fraction of two integers is called an **irrational** number.

Qu. 1 Square: (a) $\sqrt{2}$, (b) $\sqrt{6}$, (c) $\sqrt{a}$, (d) $\sqrt{(ab)}$, (e) $3\sqrt{2}$, (f) $4\sqrt{5}$, (g) $2\sqrt{a}$, (h) $\sqrt{2}\times\sqrt{3}$, (i) $\sqrt{5}\times\sqrt{7}$, (j) $\sqrt{2}\times\sqrt{8}$, (k) $\sqrt{12}\times\sqrt{3}$, (l) $\sqrt{a}\times\sqrt{b}$.

Note that the answers to parts (d) and (l) are the same, i.e.

$$\sqrt{(ab)} = \sqrt{a}\times\sqrt{b}$$

This result will be used in the next example.

Example 1 *Write $\sqrt{63}$ as the simplest possible surd.*

The factors of 63 are $3^2 \times 7$.

$$\therefore\ \sqrt{63} = \sqrt{(3^2\times 7)} = \sqrt{3^2}\times\sqrt{7} = 3\sqrt{7}$$

Example 2 *Express $6\sqrt{5}$ as a simple square root.*

$$6\sqrt{5} = \sqrt{36}\times\sqrt{5} = \sqrt{(36\times 5)} = \sqrt{180}$$

Example 3 *Simplify* $\sqrt{50}+\sqrt{2}-2\sqrt{18}+\sqrt{8}$.

$$\begin{aligned}\sqrt{50}+\sqrt{2}-2\sqrt{18}+\sqrt{8} &= 5\sqrt{2}+\sqrt{2}-2\times 3\sqrt{2}+2\sqrt{2}\\ &= 8\sqrt{2}-6\sqrt{2}\\ &= 2\sqrt{2}\end{aligned}$$

It is usual not to write surds in the denominator of a fraction when this can be avoided. The process of clearing *irrational* numbers is called **rationalisation.**

Example 4 *Rationalise the denominators of* (a) $\dfrac{1}{\sqrt{2}}$, (b) $\dfrac{1}{3-\sqrt{2}}$.

(a) Multiply numerator and denominator by $\sqrt{2}$. Thus

$$\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$

(b) Multiply numerator and denominator by the denominator with the sign of $\sqrt{2}$ changed:

$$\begin{aligned}\frac{1}{3-\sqrt{2}} &= \frac{1}{3-\sqrt{2}}\times\frac{3+\sqrt{2}}{3+\sqrt{2}}\\ &= \frac{3+\sqrt{2}}{9-2}\\ &= \frac{1}{7}(3+\sqrt{2})\end{aligned}$$

Exercise 9a (Oral)

1 Square:

(a) $\sqrt{5}$, (b) $\sqrt{\frac{1}{2}}$, (c) $4\sqrt{3}$, (d) $\frac{1}{2}\sqrt{2}$, (e) $\sqrt{\dfrac{a}{b}}$,

(f) $\sqrt{3}\times\sqrt{5}$, (g) $\sqrt{3}\times\sqrt{7}$, (h) $\dfrac{\sqrt{p}}{\sqrt{q}}$, (i) $\dfrac{1}{2\sqrt{p}}$, (j) $\dfrac{3\sqrt{a}}{\sqrt{(2b)}}$.

2 Express in terms of the simplest possible surds:
(a) $\sqrt{8}$, (b) $\sqrt{12}$, (c) $\sqrt{27}$, (d) $\sqrt{50}$,
(e) $\sqrt{45}$, (f) $\sqrt{1210}$, (g) $\sqrt{75}$, (h) $\sqrt{32}$,
(i) $\sqrt{72}$, (j) $\sqrt{98}$, (k) $\sqrt{60}$, (l) $\sqrt{512}$.

3 Express as square roots:
(a) $3\sqrt{2}$, (b) $2\sqrt{3}$, (c) $4\sqrt{5}$, (d) $2\sqrt{6}$,
(e) $3\sqrt{8}$, (f) $6\sqrt{6}$, (g) $8\sqrt{2}$, (h) $10\sqrt{10}$,

(i) $\dfrac{\sqrt{2}}{2}$, (j) $\dfrac{\sqrt{3}}{3}$, (k) $\dfrac{\sqrt{2}}{2\sqrt{3}}$, (l) $\dfrac{2}{\sqrt{6}}$.

4 Rationalise the denominators of the following fractions:

(a) $\frac{1}{\sqrt{5}}$, (b) $\frac{1}{\sqrt{7}}$, (c) $-\frac{1}{\sqrt{2}}$, (d) $\frac{2}{\sqrt{3}}$,

(e) $\frac{3}{\sqrt{6}}$, (f) $\frac{1}{2\sqrt{2}}$, (g) $-\frac{3}{2\sqrt{3}}$, (h) $\frac{9}{4\sqrt{6}}$,

(i) $\frac{1}{\sqrt{2}+1}$, (j) $\frac{1}{2-\sqrt{3}}$, (k) $\frac{1}{4-\sqrt{10}}$, (l) $\frac{2}{\sqrt{6}+2}$,

(m) $\frac{1}{\sqrt{5}-\sqrt{3}}$, (n) $\frac{3}{\sqrt{6}-\sqrt{5}}$, (o) $\frac{1}{3-2\sqrt{2}}$, (p) $\frac{1}{3\sqrt{2}-2\sqrt{3}}$.

Exercise 9b

Calculators should not be used in this exercise.

1 Simplify:
(a) $\sqrt{8}+\sqrt{18}-2\sqrt{2}$, (b) $\sqrt{75}+2\sqrt{12}-\sqrt{27}$,
(c) $\sqrt{28}+\sqrt{175}-\sqrt{63}$, (d) $\sqrt{1000}-\sqrt{40}-\sqrt{90}$,
(e) $\sqrt{512}+\sqrt{128}+\sqrt{32}$, (f) $\sqrt{24}-3\sqrt{6}-\sqrt{216}+\sqrt{294}$.

2 Given that $\sqrt{2}=1.414\ldots$ and $\sqrt{3}=1.732\ldots$, evaluate correct to 3 significant figures:

(a) $\sqrt{648}$, (b) $\sqrt{5.12}$, (c) $\frac{1}{\sqrt{3}-\sqrt{2}}$,

(d) $(3+\sqrt{2})^2$, (e) $\sqrt{\tfrac{1}{2}}-\sqrt{\tfrac{1}{8}}$, (f) $\sqrt{0.0675}$.

3 Express in the form $A+B\sqrt{C}$:

(a) $\frac{2}{3-\sqrt{2}}$, (b) $(\sqrt{5}+2)^2$, (c) $(1+\sqrt{2})(3-2\sqrt{2})$,

(d) $(\sqrt{3}-1)^2$, (e) $(1-\sqrt{2})(3+2\sqrt{2})$, (f) $\sqrt{\tfrac{1}{2}}+\sqrt{\tfrac{1}{4}}+\sqrt{\tfrac{1}{8}}$,

(g) $\sqrt{\tfrac{1}{3}}-\sqrt{\tfrac{1}{27}}$, (h) $\frac{1}{\sqrt{5}}+\sqrt{\frac{1}{125}}$, (i) $\frac{\sqrt{3}+2}{2\sqrt{3}-1}$,

(j) $\frac{\sqrt{5}+1}{\sqrt{5}-1}$, (k) $\frac{\sqrt{8}+3}{\sqrt{18}+2}$, (l) $\sqrt{3}+2+\frac{1}{\sqrt{3}-2}$.

4 Rationalise the denominators of

(a) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$, (b) $\frac{\sqrt{5}+1}{\sqrt{5}-\sqrt{3}}$, (c) $\frac{2\sqrt{2}-\sqrt{3}}{\sqrt{2}+\sqrt{3}}$,

(d) $\frac{\sqrt{2}+2\sqrt{5}}{\sqrt{5}-\sqrt{2}}$, (e) $\frac{\sqrt{6}+\sqrt{3}}{\sqrt{6}-\sqrt{3}}$, (f) $\frac{\sqrt{10}+2\sqrt{5}}{\sqrt{10}+\sqrt{5}}$.

5 Express in surd form and rationalise the denominators:

(a) $\frac{1}{1+\cos 45^\circ}$, (b) $\frac{2}{1-\cos 30^\circ}$, (c) $\frac{1+\tan 60^\circ}{1-\tan 60^\circ}$,

(d) $\dfrac{1+\tan 30^\circ}{1-\tan 30^\circ}$, (e) $\dfrac{1+\sin 45^\circ}{1-\sin 45^\circ}$, (f) $\dfrac{1}{(1-\sin 45^\circ)^2}$.

Laws of indices

9.2 It is assumed that the reader knows the three laws of indices for positive integers:

(1) $a^m \times a^n = a^{m+n}$
(2) $a^m \div a^n = a^{m-n}$, $(m > n)$
(3) $(a^m)^n = a^{mn}$

We shall now assume that these laws hold for *any* indices, and see what meaning must be assigned to fractional and negative indices as a result of this assumption.

Rational indices

9.3 We know that $4^3 = 4 \times 4 \times 4$, but so far $4^{1/2}$ has not been given any meaning. If rational indices are to be used, clearly it is an advantage if they are governed by the laws of indices. This being so, what meaning should be given to $4^{1/2}$? By the first law of indices,

$$4^{1/2} \times 4^{1/2} = 4^1 = 4$$

Therefore $4^{1/2}$ is defined as the square root of 4 (to avoid ambiguity we take it to be the positive square root) and so $4^{1/2} = 2$. Similarly, $a^{1/2} = \sqrt{a}$.

To see what value should be given to $8^{1/3}$, consider

$$8^{1/3} \times 8^{1/3} \times 8^{1/3} = 8^1 = 8$$

Therefore $8^{1/3}$ is defined as $\sqrt[3]{8}$, which is 2. Similarly, $a^{1/3} = \sqrt[3]{a}$.

In general, taking n factors of $a^{1/n}$,

$$a^{1/n} \times a^{1/n} \times \ldots \times a^{1/n} = a$$

so that

$$\mathbf{a^{1/n} = \sqrt[n]{a}}$$

Next consider $8^{2/3}$. We know that $8^{1/3} = 2$, so

$$8^{2/3} = 8^{1/3} \times 8^{1/3} = 2 \times 2 = 4$$

Therefore we must take $8^{2/3}$ to be the square of the cube root of 8, and in general $a^{m/n}$ must be taken to be the mth power of $\sqrt[n]{a}$ (or the nth root of a^m), and we may write

$$\boldsymbol{a^{m/n} = \sqrt[n]{a^m}}$$

Qu. 2 Find the values of
(a) $9^{1/2}$, (b) $27^{1/3}$, (c) $27^{2/3}$, (d) $4^{1/2}$,
(e) $4^{3/2}$, (f) $9^{5/2}$, (g) $8^{4/3}$, (h) $16^{3/4}$.

(a) $\frac{1}{\sqrt{5}}$, (b) $\frac{1}{\sqrt{7}}$, (c) $-\frac{1}{\sqrt{2}}$, (d) $\frac{2}{\sqrt{3}}$,

(e) $\frac{3}{\sqrt{6}}$, (f) $\frac{1}{2\sqrt{2}}$, (g) $-\frac{3}{2\sqrt{3}}$, (h) $\frac{9}{4\sqrt{6}}$,

(i) $\frac{1}{\sqrt{2}+1}$, (j) $\frac{1}{2-\sqrt{3}}$, (k) $\frac{1}{4-\sqrt{10}}$, (l) $\frac{2}{\sqrt{6}+2}$,

(m) $\frac{1}{\sqrt{5}-\sqrt{3}}$, (n) $\frac{3}{\sqrt{6}-\sqrt{5}}$, (o) $\frac{1}{3-2\sqrt{2}}$, (p) $\frac{1}{3\sqrt{2}-2\sqrt{3}}$.

Exercise 9b

Calculators should not be used in this exercise.

1 Simplify:
(a) $\sqrt{8}+\sqrt{18}-2\sqrt{2}$, (b) $\sqrt{75}+2\sqrt{12}-\sqrt{27}$,
(c) $\sqrt{28}+\sqrt{175}-\sqrt{63}$, (d) $\sqrt{1000}-\sqrt{40}-\sqrt{90}$,
(e) $\sqrt{512}+\sqrt{128}+\sqrt{32}$, (f) $\sqrt{24}-3\sqrt{6}-\sqrt{216}+\sqrt{294}$.

2 Given that $\sqrt{2}=1.414\ldots$ and $\sqrt{3}=1.732\ldots$, evaluate correct to 3 significant figures:

(a) $\sqrt{648}$, (b) $\sqrt{5.12}$, (c) $\frac{1}{\sqrt{3}-\sqrt{2}}$,

(d) $(3+\sqrt{2})^2$, (e) $\sqrt{\tfrac{1}{2}}-\sqrt{\tfrac{1}{8}}$, (f) $\sqrt{0.0675}$.

3 Express in the form $A+B\sqrt{C}$:

(a) $\frac{2}{3-\sqrt{2}}$, (b) $(\sqrt{5}+2)^2$, (c) $(1+\sqrt{2})(3-2\sqrt{2})$,

(d) $(\sqrt{3}-1)^2$, (e) $(1-\sqrt{2})(3+2\sqrt{2})$, (f) $\sqrt{\tfrac{1}{2}}+\sqrt{\tfrac{1}{4}}+\sqrt{\tfrac{1}{8}}$,

(g) $\sqrt{\tfrac{1}{3}}-\sqrt{\tfrac{1}{27}}$, (h) $\frac{1}{\sqrt{5}}+\sqrt{\frac{1}{125}}$, (i) $\frac{\sqrt{3}+2}{2\sqrt{3}-1}$,

(j) $\frac{\sqrt{5}+1}{\sqrt{5}-1}$, (k) $\frac{\sqrt{8}+3}{\sqrt{18}+2}$, (l) $\sqrt{3}+2+\frac{1}{\sqrt{3}-2}$.

4 Rationalise the denominators of

(a) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$, (b) $\frac{\sqrt{5}+1}{\sqrt{5}-\sqrt{3}}$, (c) $\frac{2\sqrt{2}-\sqrt{3}}{\sqrt{2}+\sqrt{3}}$,

(d) $\frac{\sqrt{2}+2\sqrt{5}}{\sqrt{5}-\sqrt{2}}$, (e) $\frac{\sqrt{6}+\sqrt{3}}{\sqrt{6}-\sqrt{3}}$, (f) $\frac{\sqrt{10}+2\sqrt{5}}{\sqrt{10}+\sqrt{5}}$.

5 Express in surd form and rationalise the denominators:

(a) $\frac{1}{1+\cos 45^\circ}$, (b) $\frac{2}{1-\cos 30^\circ}$, (c) $\frac{1+\tan 60^\circ}{1-\tan 60^\circ}$,

Zero and negative indices

9.4 So far 2^0 has been given no meaning. Again it is desirable for it to be given a meaning consistent with the laws of indices, so we divide 2^1 by 2^1 using the second law:

$$2^1 \div 2^1 = 2^0$$

But $2^1 \div 2^1 = 1$, so 2^0 must be taken to be 1. In the same way, $a^n \div a^n = a^0$, so

$$\boldsymbol{a^0 = 1} \quad (a \neq 0)$$

Qu. 3 Why does the above not hold for $a = 0$?

To find what meaning must be given to 2^{-1}, divide 2^0 by 2^1, using the second law of indices:

$$2^0 \div 2^1 = 2^{-1}$$

But $2^0 \div 2^1 = 1 \div 2 = \frac{1}{2}$, therefore we must take 2^{-1} to be $\frac{1}{2}$.

Similarly,

$$2^{-3} = 2^0 \div 2^3 = \frac{1}{2^3}$$

Thus 2^{-3} is the reciprocal of 2^3.

In the same way

$$\boldsymbol{a^{-n} = \frac{1}{a^n}}$$

that is, a^{-n} is the reciprocal of a^n.

Example 5 *Find the value of* $(27/8)^{-2/3}$.

Using the last result, $(27/8)^{-2/3} = (8/27)^{2/3}$.

Taking the cube root,

$$\left(\frac{8}{27}\right)^{2/3} = \left(\frac{2}{3}\right)^2$$

$$\therefore \left(\frac{27}{8}\right)^{-2/3} = \frac{4}{9}$$

Example 6 *Simplify* $\dfrac{(1+x)^{1/2} - \frac{1}{2}x(1+x)^{-1/2}}{1+x}$.

Multiply numerator and denominator by $2(1+x)^{1/2}$.

$$\frac{(1+x)^{1/2} - \frac{1}{2}x(1+x)^{-1/2}}{1+x} = \frac{2(1+x) - x}{2(1+x)^{3/2}}$$

$$= \frac{2+x}{2(1+x)^{3/2}}$$

Exercise 9c (Oral)

1 Find the values of

(a) $25^{1/2}$, (b) $27^{1/3}$, (c) $64^{1/6}$, (d) $49^{1/2}$,

(e) $\left(\frac{1}{4}\right)^{1/2}$, (f) $1^{1/4}$, (g) $(-8)^{1/3}$, (h) $(-1)^{1/5}$,

(i) $8^{4/3}$, (j) $27^{2/3}$, (k) $25^{3/2}$, (l) $49^{3/2}$,

(m) $\left(\frac{1}{4}\right)^{3/2}$, (n) $\left(\frac{4}{9}\right)^{1/2}$, (o) $\left(\frac{27}{8}\right)^{1/3}$, (p) $\left(\frac{16}{81}\right)^{1/4}$.

2 Find the values of

(a) 7^0, (b) 3^{-1}, (c) 5^0, (d) 4^{-1},

(e) 2^{-3}, (f) $\left(\frac{1}{2}\right)^{-1}$, (g) $\left(\frac{1}{3}\right)^{-2}$, (h) $\left(\frac{4}{9}\right)^{0}$,

(i) 3^{-3}, (j) $(-6)^{-1}$, (k) $\left(-\frac{1}{6}\right)^{0}$, (l) $\left(\frac{2}{3}\right)^{-2}$,

(m) $\left(-\frac{1}{2}\right)^{-2}$, (n) $\frac{1}{3^{-1}}$, (o) $\frac{2^{-1}}{3^{-2}}$, (p) $\frac{2^0 \times 3^{-2}}{5^{-1}}$.

3 Find the values of

(a) $8^{-1/3}$, (b) $8^{-2/3}$, (c) $4^{-1/2}$, (d) $4^{-3/2}$,

(e) $27^{-2/3}$, (f) $\left(\frac{1}{4}\right)^{-1/2}$, (g) $\left(\frac{1}{8}\right)^{-1/3}$, (h) $\left(\frac{1}{27}\right)^{-2/3}$,

(i) $\left(\frac{4}{9}\right)^{-1/2}$, (j) $\left(\frac{8}{27}\right)^{-1/3}$, (k) $\left(\frac{16}{81}\right)^{-1/4}$, (l) $\left(\frac{27}{8}\right)^{-4/3}$.

Exercise 9d

1 Find the values of

(a) $256^{1/2}$, (b) $1296^{1/2}$, (c) $64^{1/3}$, (d) $216^{1/3}$,

(e) $(2\frac{1}{4})^{1/2}$, (f) $(1\frac{7}{9})^{1/2}$, (g) $8^{-1/3}$, (h) $4^{-3/2}$,

(i) $64^{-2/3}$, (j) $81^{-3/4}$, (k) $\left(\frac{121}{16}\right)^{1/2}$, (l) $\left(\frac{1}{16}\right)^{-3/2}$,

(m) $\left(\frac{8}{27}\right)^{2/3}$, (n) $1.331^{1/3}$, (o) $0.04^{-3/2}$, (p) $\frac{4^{-3/2}}{8^{-2/3}}$.

2 Find the values of

(a) $\frac{16^{1/3} \times 4^{1/3}}{8}$, (b) $\frac{27^{1/2} \times 243^{1/2}}{243^{4/5}}$, (c) $\frac{32^{3/4} \times 16^0 \times 8^{5/4}}{128^{3/2}}$,

(d) $\dfrac{6^{1/2} \times 96^{1/4}}{216^{1/4}}$, (e) $\dfrac{12^{1/3} \times 6^{1/3}}{81^{1/6}}$, (f) $\dfrac{8^{1/6} \times 4^{1/3}}{32^{1/6} \times 16^{1/12}}$.

3 Simplify:
(a) $8^n \times 2^{2n} \div 4^{3n}$,
(b) $3^{n+1} \times 9^n \div 27^{(2/3)n}$,
(c) $16^{(3/4)n} \div 8^{(5/3)n} \times 4^{n+1}$,
(d) $9^{-(1/2)n} \times 3^{n+2} \times 81^{-1/4}$,
(e) $6^{(1/2)n} \times 12^{n+1} \times 27^{-(1/2)n} \div 32^{(1/2)n}$,
(f) $10^{(1/3)n} \times 15^{(1/2)n} \times 6^{(1/6)n} \div 45^{(1/3)n}$.

4 Simplify:

(a) $\dfrac{x^{-2/3} \times x^{1/4}}{x^{1/6}}$, (b) $\dfrac{\sqrt{(xy)} \times x^{1/3} \times 2y^{1/4}}{(x^{10}y^9)^{1/12}}$,

(c) $\dfrac{x^{2n+1} \times x^{1/2}}{\sqrt{x^{3n}}}$, (d) $\dfrac{x^{3n+1}}{x^{2n+2\frac{1}{2}} \times \sqrt{x^{2n-3}}}$,

(e) $\dfrac{x^{p+(1/2)q} \times y^{2p-q}}{(xy^2)^p \times \sqrt{x^q}}$, (f) $\dfrac{x^{-2/3} \times y^{-1/3}}{(x^4y^2)^{-1/6}}$.

5 Simplify:

(a) $\dfrac{x^2(x^2+1)^{-1/2} - (x^2+1)^{1/2}}{x^2}$,

(b) $-\dfrac{\frac{1}{2}x(1-x)^{-1/2} + (1-x)^{1/2}}{x^2}$,

(c) $\dfrac{\frac{1}{2}x^{1/2}(1+x)^{-1/2} - \frac{1}{2}x^{-1/2}(1+x)^{1/2}}{x}$,

(d) $\dfrac{(1+x)^{1/3} - \frac{1}{3}x(1+x)^{-2/3}}{(1+x)^{2/3}}$,

(e) $\dfrac{\sqrt{(1-x)}\frac{1}{2}(1+x)^{-1/2} + \frac{1}{2}(1-x)^{-1/2}\sqrt{(1+x)}}{1-x}$.

Logarithms

9.5 Readers will probably be familiar with the use of logarithms for multiplication and division, but there are certain properties of logarithms that are useful in more advanced work. Having just considered indices, this is the appropriate place for logarithms because *a logarithm is an index*.

From a calculator we can see that the logarithm of 3, to base 10, is 0.47712 (correct to five decimal places). This means that $10^{0.47712} = 3$, working to five significant figures. The statement 'the logarithm of 3, to the base 10, is 0.47712' is abbreviated to $\log_{10} 3 = 0.47712$.

Similarly, $10^{0.90309} \approx 8$, which may be expressed as $\log_{10} 8 \approx 0.90309$.

Now $2^3 = 8$, and this statement may also be written in logarithmic notation. Here the base is 2 and the index (i.e. logarithm) is 3, thus $\log_2 8 = 3$.

Qu. 4 What are the bases and logarithms in the following statements?
(a) $10^2 = 100$, (b) $10^{1.6021} \approx 40$, (c) $9 = 3^2$,
(d) $4^3 = 64$, (e) $1 = 2^0$, (f) $8 = (1/2)^{-3}$,
(g) $a^b = c$.

Exercise 9e (Oral)

1 Express the following statements in logarithmic notation:
(a) $2^4 = 16$, (b) $27 = 3^3$, (c) $125 = 5^3$,
(d) $10^6 = 1\,000\,000$, (e) $1728 = 12^3$, (f) $64 = 16^{3/2}$,
(g) $10^4 = 10\,000$, (h) $4^0 = 1$, (i) $0.01 = 10^{-2}$,
(j) $\frac{1}{2} = 2^{-1}$, (k) $9^{3/2} = 27$, (l) $8^{-2/3} = \frac{1}{4}$,
(m) $81 = (1/3)^{-4}$, (n) $e^0 = 1$, (o) $16^{-1/4} = \frac{1}{2}$,
(p) $(1/8)^0 = 1$, (q) $27 = 81^{3/4}$, (r) $4 = (1/16)^{-1/2}$,
(s) $(-2/3)^2 = 4/9$, (t) $(-3)^{-1} = -\frac{1}{3}$, (u) $c = a^5$,
(v) $a^3 = b$, (w) $p^q = r$, (x) $a = b^c$.

2 Express in index notation:
(a) $\log_2 32 = 5$, (b) $\log_3 9 = 2$, (c) $2 = \log_5 25$,
(d) $\log_{10} 100\,000 = 5$, (e) $7 = \log_2 128$, (f) $\log_9 1 = 0$,
(g) $-2 = \log_3 \frac{1}{9}$, (h) $\log_4 2 = \frac{1}{2}$, (i) $\log_e 1 = 0$,
(j) $\log_{27} 3 = \frac{1}{3}$, (k) $2 = \log_a x$, (l) $\log_3 a = b$,
(m) $\log_a 8 = c$, (n) $y = \log_x z$, (o) $p = \log_q r$.

3. Evaluate:
(a) $\log_2 64$, (b) $\log_{10} 100$, (c) $\log_{10} 10^7$, (d) $\log_a a^2$,
(e) $\log_8 2$, (f) $\log_4 1$, (g) $\log_{27} 3$, (h) $\log_{2/3} \frac{4}{9}$,
(i) $\log_5 125$, (j) $\log_{0.1} 10$, (k) $\log_e e^3$, (l) $\log_e \frac{1}{e}$.

Two numbers can be multiplied by adding their logarithms and divided by subtracting them. The rules are familiar, but it is worth while proving them as an example of logarithmic notation.

Qu. 5 Write in logarithmic notation: $a = c^x$, $b = c^y$, $ab = c^{x+y}$, $a/b = c^{x-y}$.

Deduce that

$\log_c a + \log_c b = \log_c ab$, and that

$\log_c a - \log_c b = \log_c (a/b)$

The logarithm of the nth power of a number is obtained by multiplying its logarithm by n. A method of proving this rule is suggested in the next question.

Qu. 6 Write in logarithmic notation: $a = c^x$, $a^n = c^{nx}$.
Deduce that $\log_c a^n = n \log_c a$.

In Qu. 5 and Qu. 6, the suffix c has been used to denote the base of the logarithms. However, when the same base is used throughout a piece of work (for example the answer to a single question or exercise) the suffix may be omitted. Using this convention, the results we have found above can be summarised as follows:

$$\begin{aligned} \log a + \log b &= \log (a \times b) \\ \log a - \log b &= \log (a/b) \\ n \times \log a &= \log (a^n) \end{aligned}$$

These three results are used in the next example.

Example 7 *Express* $\log_{10} \dfrac{a^2b^3}{100\sqrt{c}}$ *in terms of* $\log_{10} a$, $\log_{10} b$, $\log_{10} c$.

First note that $\sqrt{c} = c^{1/2}$.
Using the two rules of Qu. 5,

$$\log_{10} \frac{a^2b^3}{100c^{1/2}} = \log_{10} a^2 + \log_{10} b^3 - \log_{10} 100 - \log_{10} c^{1/2}$$

Then by the rule of Qu. 6, and writing $\log_{10} 100 = 2$,

$$\log_{10} \frac{a^2b^3}{100c^{1/2}} = 2 \log_{10} a + 3 \log_{10} b - 2 - \frac{1}{2} \log_{10} c$$

The logarithm, to base *ten*, of x is frequently written $\lg x$. This abbreviation is used in the next example and in the exercise which follows.

Example 8 *Simplify* $\dfrac{\lg 125}{\lg 25}$.

[Note that 125 and 25 are both powers of 5, so their logarithms can be expressed in terms of lg 5.]

$$\frac{\lg 125}{\lg 25} = \frac{\lg 5^3}{\lg 5^2} = \frac{3 \lg 5}{2 \lg 5} = \frac{3}{2}$$

Example 9 *Use tables or a calculator to find an approximate value of* $\log_2 7$.

Write $x = \log_2 7$, then $2^x = 7$. Since $2^x = 7$, their logarithms to the base of ten are equal, therefore

$$\begin{aligned} \lg 2^x &= \lg 7 \\ \therefore x \lg 2 &= \lg 7 \\ \therefore x &= \frac{\lg 7}{\lg 2} \\ &= 2.8074 \quad \text{(correct to five significant figures)} \end{aligned}$$

Therefore $\log_2 7 \approx 2.8074$.

Exercise 9f

Note $\lg x = \log_{10} x$.

1 Express in terms of $\log a$, $\log b$, $\log c$:

(a) $\log ab$, (b) $\log \frac{a}{c}$, (c) $\log \frac{1}{b}$,

(d) $\log a^2 b^{3/2}$, (e) $\log \frac{1}{b^4}$, (f) $\log \frac{a^{1/3} b^4}{c^3}$,

(g) $\log \sqrt{a}$, (h) $\log \sqrt[3]{b}$, (i) $\log \sqrt{(ab)}$,

(j) $\lg (10a)$, (k) $\lg \frac{1}{100b^2}$, (l) $\log \sqrt{\left(\frac{a}{b}\right)}$,

(m) $\log \sqrt{\left(\frac{ab^3}{c}\right)}$, (n) $\log \frac{b\sqrt{a}}{\sqrt[3]{c}}$, (o) $\lg \sqrt{\left(\frac{10a}{b^5 c}\right)}$.

2 Express as single logarithms:

(a) $\log 2 + \log 3$, (b) $\log 18 - \log 9$,
(c) $\log 4 + 2 \log 3 - \log 6$, (d) $3 \log 2 + 2 \log 3 - 2 \log 6$,
(e) $\log c + \log a$, (f) $\log x + \log y - \log z$,
(g) $2 \log a - \log b$, (h) $2 \log a + 3 \log b - \log c$,
(i) $\frac{1}{2} \log x - \frac{1}{2} \log y$, (j) $\log p - \frac{1}{3} \log q$,
(k) $2 + 3 \lg a$, (l) $1 + \lg a - \frac{1}{2} \lg b$,
(m) $2 \lg a - 3 - \lg 2c$, (n) $3 \lg x - \frac{1}{2} \lg y + 1$.

3 Simplify:

(a) $\lg 1000$, (b) $\frac{1}{2} \log_3 81$, (c) $\frac{1}{3} \log_2 64$,

(d) $-\log_2 \frac{1}{2}$, (e) $\frac{1}{3} \log 8$, (f) $\frac{1}{2} \log 49$,

(g) $-\frac{1}{2} \log 4$, (h) $3 \log 3 - \log 27$, (i) $5 \log 2 - \log 32$,

(j) $\frac{\log 8}{\log 2}$, (k) $\frac{\log 81}{\log 9}$, (l) $\frac{\log 49}{\log 343}$.

4 Solve the equations:

(a) $2^x = 5$, (b) $3^x = 2$, (c) $3^{4x} = 4$,
(d) $2^x \times 2^{x+1} = 10$, (e) $(1/2)^x = 6$, (f) $(2/3)^x = 1/16$.

5 Evaluate, taking $\log \pi = 0.4971$ and $e = 2.718$:

(a) $\log_2 9$, (b) $\log_{12} 6$, (c) $\log_3 \pi$,
(d) $\log_e 10$, (e) $\log_e \pi$, (f) $\log_3 \frac{1}{2}$.

6 Show that $\log_a b = 1/\log_b a$,

(a) using the result $\log_a b \times \log_b c = \log_a c$,
(b) from first principles.

7 Evaluate:

(a) $2.56^{1.21}$, (b) $1.57^{0.576}$, (c) $2.718^{3.142}$,
(d) $0.561^{2/5}$, (e) $0.513^{3/2}$, (f) $0.0057^{1.39}$.

The functions $x \mapsto a^x$ and $x \mapsto \log_a x$

9.6 We can legitimately use the word function to describe $x \mapsto 10^x$ and $x \mapsto \log_{10} x$, because, in each case, for a given value of x, the rule will produce a unique result. In the case of $x \mapsto 10^x$, the domain is $\mathbb{R}$ and the range is $\mathbb{R}^+$, and for $x \mapsto \log_{10} x$, the domain is $\mathbb{R}^+$ and the range is $\mathbb{R}$. In most instances, the actual calculation of 10^x or $\log_{10} x$ will be very complicated, but this does not matter; a calculator can be used where it is appropriate (the same remarks apply to the function $x \mapsto \sqrt{x}$). More generally, if a is a fixed, positive, real number, $x \mapsto a^x$ and $x \mapsto \log_a x$, are perfectly satisfactory functions. (Note that the domains are $\mathbb{R}$ and $\mathbb{R}^+$, respectively.)

Qu. 7 If $f(x) = 10^x$ and $g(x) = \log_{10} x$, find the values of
(a) $f(1)$, (b) $f(2)$, (c) $f(-1)$, (d) $g(10)$, (e) $g(1)$, (f) $g(\sqrt{10})$.
Qu. 8 If $F(x) = a^x$ and $G(x) = \log_a x$, find
(a) $F(1)$, (b) $F(2)$, (c) $F(-1)$, (d) $G(a)$, (e) $G(1)$, (f) $G(\sqrt{a})$.

The following special cases are very common and the reader is advised to commit them to memory:

$$\log_a 1 = 0$$
$$\log_a a = 1$$
$$\log_a (1/a) = -1$$

Remember that a logarithm is an index; the logarithm of q to base a is the power to which a must be raised to equal q, e.g. $\log_{10} 1000 = 3$, and $\log_2 (1/8) = -3$. Thus if $a^p = q$, then $\log_a q = p$, and these are equivalent statements, being simply alternative ways of stating the relationship between a, p and q. We can combine these statements in two ways:

$$\log_a (a^p) = \log_a (q) = p$$

and

$$a^{\log_a q} = a^p = q$$

So, if $f(x) = \log_a x$ and $g(x) = a^x$, then the composite functions fg and gf are given by

$$fg(x) = f(a^x) = \log_a (a^x) = x$$

and

$$gf(x) = g(\log_a x) = a^{\log_a x} = x$$

In other words, the composite function merely gives the original value of x; the function f 'undoes' the effect of function g, and function g 'undoes' the effect of function f; that is the functions f and g are inverses of one another.

This effect can easily be observed on a pocket calculator. Enter any positive number, say 5, press the 'log' function key (the display should show 0.69897), and then press the '10^x' function key. The display should return to the value originally entered, i.e. 5. Repeat this with other numbers; try it also with the functions in the reverse order. If your calculator is equipped with function keys for e^x and $\log_e x$ (these appear as exp and ln on some calculators) try the same routine with this pair of inverse functions.

Sketches of the graphs of $y = a^x$ and $\log_a x$ are shown in Fig. 9.1. As with all inverse functions, the graphs are reflections of one another in the line $y = x$.

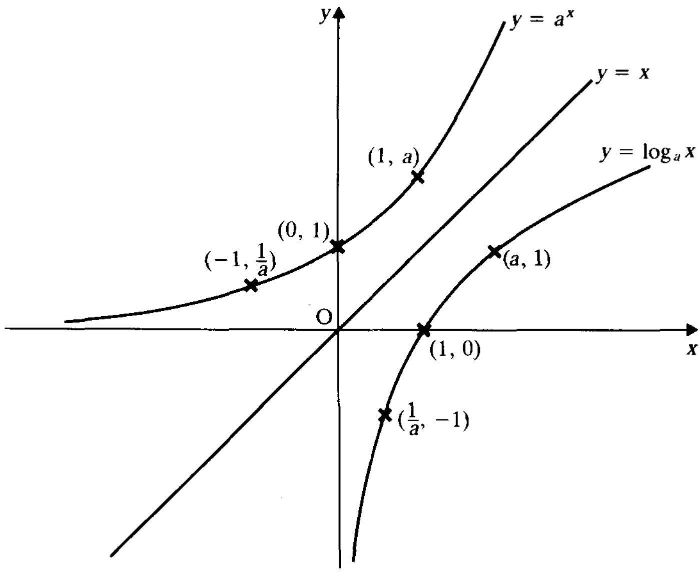


Figure 9.1

Roots of quadratic equations

9.7 If an algebraic equation, in which the unknown quantity is x, is satisfied by putting $x = c$, we say that c is a **root** of the equation. For example $x^2 - 5x + 6 = 0$ is satisfied by putting $x = 2$, so one root of this equation is 2 (the other is 3).

It is often useful to be able to obtain information about the roots of an equation without actually solving it. For instance, if α and β are the roots of the equation $3x^2 + x - 1 = 0$, the value of $\alpha^2 + \beta^2$ can be found without first finding the values of α and β. This is done by finding the values of $\alpha + \beta$ and $\alpha\beta$, and expressing $\alpha^2 + \beta^2$ in terms of $\alpha + \beta$ and $\alpha\beta$.

The equation whose roots are α and β may be written

$$(x - \alpha)(x - \beta) = 0$$
$$\therefore x^2 - \alpha x - \beta x + \alpha\beta = 0$$
$$\therefore x^2 - (\alpha + \beta)x + \alpha\beta = 0 \qquad (1)$$

But suppose that α and β are also the roots of the equation

$$\boldsymbol{ax^2 + bx + c = 0}$$

which may be written

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad (2)$$

Now equations (1) and (2), having the same roots, must be precisely the same equation, written in two different ways, since the coefficients of x^2 are both 1. Therefore

(a) the coefficients of x must be equal,

$$\therefore \boldsymbol{\alpha + \beta = -\frac{b}{a}}$$

(b) the constant terms must be equal,

$$\therefore \boldsymbol{\alpha\beta = \frac{c}{a}}$$

Note. If it is required to write down an equation whose roots are known, equation (1) gives it in a convenient form. It may be written:

$$x^2 - (\text{sum of the roots})x + (\text{product of the roots}) = 0$$

Qu. 9 Write down the sums and products of the roots of the following equations:
(a) $3x^2 - 2x - 7 = 0$, (b) $5x^2 + 11x + 3 = 0$,
(c) $2x^2 + 5x = 1$, (d) $2x(x + 1) = x + 7$.

Qu. 10 Write down equations, the sums and products of whose roots are respectively:
(a) 7, 12; (b) 3, -2; (c) $-\frac{1}{2}, -\frac{3}{8}$; (d) $\frac{2}{3}$, 0.

Qu. 11 Write down the sum and product of the roots of the equation

$$3x^2 + 9x + 7 = 0.$$

Example 10 *The roots of the equation* $3x^2 + 4x - 5 = 0$ *are* α, β. *Find the values of* (a) $1/\alpha + 1/\beta$, (b) $\alpha^2 + \beta^2$.

Both $1/\alpha + 1/\beta$ and $\alpha^2 + \beta^2$ can be expressed in terms of $\alpha + \beta$ and $\alpha\beta$.

$$\alpha + \beta = -\tfrac{4}{3}, \qquad \alpha\beta = -\tfrac{5}{3}$$

(a) $$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{-\frac{4}{3}}{-\frac{5}{3}} = \frac{4}{5}.$$

(b) $$\alpha^2 + \beta^2 = \alpha^2 + 2\alpha\beta + \beta^2 - 2\alpha\beta$$
$$= (\alpha + \beta)^2 - 2\alpha\beta = (-\tfrac{4}{3})^2 - 2(-\tfrac{5}{3})$$

$$\therefore \alpha^2 + \beta^2 = \frac{16}{9} + \frac{10}{3} = \frac{46}{9}$$

Alternatively, since α and β are roots of the equation $3x^2 + 4x - 5 = 0$,

$$3\alpha^2 + 4\alpha - 5 = 0$$
$$3\beta^2 + 4\beta - 5 = 0$$

Adding,

$$3(\alpha^2+\beta^2)+4(\alpha+\beta)-10=0$$
$$\therefore\ 3(\alpha^2+\beta^2)-\tfrac{16}{3}-10=0$$
$$\therefore\ \alpha^2+\beta^2=\tfrac{16}{9}+\tfrac{10}{3}=\tfrac{46}{9}.$$

Example 11 *The roots of the equation* $2x^2-7x+4=0$ *are* α, β. *Find an equation with integral coefficients whose roots are* α/β, β/α.

Since α, β are the roots of the equation $2x^2-7x+4=0$, we have

$$\alpha+\beta=\tfrac{7}{2}, \qquad \alpha\beta=2$$

Then the required equation may be formed from equation (3) above, if the sum and product of α/β, β/α are expressed in terms of $\alpha+\beta$ and $\alpha\beta$.

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$$
$$=\frac{\frac{49}{4}-4}{2}=\frac{33}{8}$$

Therefore the sum of the roots is $\frac{33}{8}$.

$$\frac{\alpha}{\beta}\times\frac{\beta}{\alpha}=1$$

Therefore the product of the roots is 1.

Hence the equation with roots α/β, β/α is

$$x^2-\tfrac{33}{8}x+1=0$$

Multiplying through by 8, in order to obtain integral coefficients, the required equation is

$$8x^2-33x+8=0$$

Symmetrical functions

9.8 The functions of α and β that have been used in this chapter all show a certain symmetry. Consider, for example,

$$\alpha+\beta,\quad \alpha\beta,\quad \frac{1}{\alpha}+\frac{1}{\beta},\quad \alpha^2+\beta^2,\quad \frac{\alpha}{\beta}+\frac{\beta}{\alpha}$$

Notice that if α and β are interchanged:

$$\beta+\alpha,\quad \beta\alpha,\quad \frac{1}{\beta}+\frac{1}{\alpha},\quad \beta^2+\alpha^2,\quad \frac{\beta}{\alpha}+\frac{\alpha}{\beta}$$

the resulting functions are the same. When a function of α and β is unchanged when α and β are interchanged, it is called a **symmetrical** function of α and β. Such functions occurring in this chapter may be expressed in terms of $\alpha+\beta$ and $\alpha\beta$, as in the next example.

Example 12 *Express in terms of $\alpha+\beta$ and $\alpha\beta$:* (a) $\alpha^3+\beta^3$, (b) $(\alpha-\beta)^2$.

(a) α^3 and β^3 occur in the expansion of $(\alpha+\beta)^3$.

$$(\alpha+\beta)^3 = \alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3$$
$$\therefore \alpha^3+\beta^3 = (\alpha+\beta)^3-3\alpha^2\beta-3\alpha\beta^2$$
$$\therefore \alpha^3+\beta^3 = (\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$$

(b) $(\alpha-\beta)^2 = \alpha^2-2\alpha\beta+\beta^2$.
α^2 and β^2 occur in the expansion of $(\alpha+\beta)^2$.

$$(\alpha+\beta)^2 = \alpha^2+2\alpha\beta+\beta^2$$
$$\therefore (\alpha-\beta)^2 = (\alpha+\beta)^2-4\alpha\beta$$

Exercise 9g

1 Find the sums and products of the roots of the following equations:

(a) $2x^2-11x+3=0$, (b) $2x^2+x-1=0$, (c) $3x^2=7x+6$,

(d) $x^2+x=1$, (e) $t(t-1)=3$, (f) $y(y+1)=2y+5$,

(g) $x+\dfrac{1}{x}=4$, (h) $\dfrac{1}{t}+\dfrac{1}{t+1}=\dfrac{1}{2}$.

2 Find equations, with integral coefficients, the sums and products of whose roots are respectively:
(a) 3, 4; (b) -5, 6; (c) $\frac{3}{2}, -\frac{5}{2}$; (d) $-\frac{7}{3}$, 0;
(e) 0, -7; (f) 1.2, 0.8; (g) $-\frac{1}{3}, \frac{1}{36}$; (h) -2.5, -1.6.

3 The roots of the equation $2x^2+3x-4=0$ are α, β. Find the values of
(a) $\alpha^2+\beta^2$, (b) $1/\alpha+1/\beta$, (c) $(\alpha+1)(\beta+1)$, (d) $\beta/\alpha+\alpha/\beta$.

4 If the roots of the equation $3x^2-5x+1=0$ are α, β, find the values of
(a) $\alpha\beta^2+\alpha^2\beta$, (b) $\alpha^2-\alpha\beta+\beta^2$, (c) $\alpha^3+\beta^3$, (d) $\alpha^2/\beta+\beta^2/\alpha$.

5 The equation $4x^2+8x-1=0$ has roots α, β. Find the values of

(a) $1/\alpha^2+1/\beta^2$, (b) $(\alpha-\beta)^2$, (c) $\alpha^3\beta+\alpha\beta^3$, (d) $\dfrac{1}{\alpha^2\beta}+\dfrac{1}{\alpha\beta^2}$.

6 If the roots of the equation $x^2-5x-7=0$ are α, β, find equations whose roots are
(a) α^2, β^2; (b) $\alpha+1$, $\beta+1$; (c) $\alpha^2\beta$, $\alpha\beta^2$.

7 The roots of the equation $2x^2-4x+1=0$ are α, β. Find equations with integral coefficients whose roots are
(a) $\alpha-2$, $\beta-2$; (b) $1/\alpha$, $1/\beta$; (c) α/β, β/α.

8 Find an equation, with integral coefficients, whose roots are the squares of the roots of the equation $2x^2+5x-6=0$.

9 The roots of the equation $x^2+6x+q=0$ are α and $\alpha-1$. Find the value of q.

10 The roots of the equation $x^2-px+8=0$ are α and $\alpha+2$. Find two possible values of p.

11 The roots of the equation $x^2 + 2px + q = 0$ differ by 2. Show that $p^2 = 1 + q$.

12 If the roots of the equation $ax^2 + bx + c = 0$ are α, β, find expressions in terms of a, b, c for
(a) $\alpha^2\beta + \alpha\beta^2$, (b) $\alpha^2 + \beta^2$, (c) $\alpha^3 + \beta^3$,
(d) $1/\alpha + 1/\beta$, (e) $\alpha/\beta + \beta/\alpha$, (f) $\alpha^4 + \beta^4$.

13 The equation $ax^2 + bx + c = 0$ has roots α, β. Find equations whose roots are
(a) $-\alpha, -\beta$; (b) $\alpha + 1, \beta + 1$; (c) α^2, β^2;
(d) $-1/\alpha, -1/\beta$; (e) $\alpha - \beta, \beta - \alpha$; (f) $2\alpha + \beta, \alpha + 2\beta$.

14 Prove that, if the difference between the roots of the equation

$$ax^2 + bx + c = 0$$

is 1, then $a^2 = b^2 - 4ac$.

15 Prove that, if one root of the equation $ax^2 + bx + c = 0$ is twice the other, then $2b^2 = 9ac$.

16 Prove that, if the sum of the squares of the roots of the equation

$$ax^2 + bx + c = 0$$

is 1, then $b^2 = 2ac + a^2$.

17 Prove that, if the sum of the reciprocals of the roots of the equation

$$ax^2 + bx + c = 0$$

is 1, then $b + c = 0$. If, in addition, one root of the equation is twice the other, use the result of No. 15 to find one set of values of a, b, c. Solve the equation.

18 In the equation $ax^2 + bx + c = 0$, make the substitutions
(a) $x = y - 1$, (b) $x = y^2$, (c) $x = \sqrt{y}$,
and simplify the equations.

If the roots of the equation $ax^2 + bx + c = 0$ are α, β, what are the roots of the three equations in y? [Express y in terms of x, and give your answers in terms of α, β.]

19 If the roots of the equation $ax^2 + bx + c = 0$ are α, β, make substitutions, as in No. 18, to find equations whose roots are
(a) $\alpha + 2, \beta + 2$; (b) $1/\alpha, 1/\beta$; (c) $1 \pm \sqrt{\alpha}, 1 \pm \sqrt{\beta}$.

The remainder theorem

9.9 An expression of the form

$$ax^n + bx^{n-1} + \ldots + k$$

where $a, b, \ldots k$ are real numbers and n is a positive integer is called a **polynomial** of degree n. (The expression $5x^7 - 3x^2 + 1.5x - 0.3$, for example, is a polynomial of degree 7.)

If we divide the polynomial $x^3 - 3x^2 + 6x + 5$ by $x - 2$:

$$
\begin{array}{r}
x^2 - x + 4 \\
x-2 \overline{)\, x^3 - 3x^2 + 6x + 5} \\
\underline{x^3 - 2x^2 } \\
-x^2 + 6x \\
\underline{-x^2 + 2x } \\
4x + 5 \\
\underline{4x - 8} \\
13
\end{array}
$$

the result may be expressed in the identity

$$x^3 - 3x^2 + 6x + 5 = (x-2)(x^2 - x + 4) + 13$$

Here $x^2 - x + 4$ is called the **quotient** and 13 the **remainder**.

The remainder theorem gives a method of finding the remainder without going through the process of division.

Suppose it is required to find the remainder when $x^4 - 5x + 6$ is divided by $x - 2$. If the division were performed, we could write

$$x^4 - 5x + 6 = (x-2) \times \text{quotient} + \text{remainder}$$

Now if we put $x = 2$ in this identity we obtain

$$16 - 10 + 6 = 0 \times \text{quotient} + \text{remainder}$$

$\therefore$ the remainder $= 12$

Applying this process to any such expression divided by $x - a$, we may write

$$\text{expression} = (x-a) \times \text{quotient} + \text{remainder}$$

Putting $x = a$ in this identity, it follows that

the remainder = the value of the expression when $x = a$.

The function notation may be used to state the remainder theorem. **If a polynomial f(x) is divided by $x - a$, the remainder is f(a).**

Qu. 12 For what type of expression is the above method valid?

Example 13 *Find the remainder when*

$$x^5 - 4x^3 + 2x + 3$$

is divided by (a) $x - 1$, (b) $x + 2$.

Let $f(x) = x^5 - 4x^3 + 2x + 3$, then
(a) the remainder when f(x) is divided by $x - 1$ is

$$f(1) = 1 - 4 + 2 + 3 = 2$$

(b) the remainder when f(x) is divided by $x + 2$ is

$$f(-2) = -32 + 32 - 4 + 3 = -1$$

Example 14 *Find the remainder when* $4x^3 - 6x + 5$ *is divided by* $2x - 1$.

As $2x - 1$ is not in the form $x - a$, imagine the division to have been performed, then

$$4x^3 - 6x + 5 = (2x - 1) \times \text{quotient} + \text{remainder}$$

Putting $x = \frac{1}{2}$ in this identity,

$$\tfrac{1}{2} - 3 + 5 = 0 \times \text{quotient} + \text{remainder}$$

Therefore the remainder is $2\frac{1}{2}$.

Example 15 *Factorise the expression* $2x^3 + 3x^2 - 32x + 15$.

Let $f(x) = 2x^3 + 3x^2 - 32x + 15$.

[$x - a$ will be a factor of f(x) only if there is no remainder on division, i.e. if $f(a) = 0$.]

$f(1) = 2 + 3 - 32 + 15 \neq 0$ $\therefore$ $x - 1$ is not a factor.
$f(-1) = -2 + 3 + 32 + 15 \neq 0$ $\therefore$ $x + 1$ is not a factor.

$x - 2$ and $x + 2$ cannot be factors, as 2 is not a factor of the constant term 15.

$f(3) = 54 + 27 - 96 + 15 = 0$ $\therefore$ $x - 3$ is a factor.

On division (or by inspection),

$$2x^3 + 3x^2 - 32x + 15 = (x - 3)(2x^2 + 9x - 5)$$

Therefore the factors of $2x^3 + 3x^2 - 32x + 15$ are $(x - 3)(x + 5)(2x - 1)$.

Example 16 *When the expression* $x^5 + 4x^2 + ax + b$ *is divided by* $x^2 - 1$, *the remainder is* $2x + 3$. *Find the values of a and b.*

Suppose the division to have been performed, then

$$x^5 + 4x^2 + ax + b = (x^2 - 1) \times \text{quotient} + 2x + 3$$

Putting $x = 1$, $1 + 4 + a + b = 2 + 3$.
Putting $x = -1$, $-1 + 4 - a + b = -2 + 3$.

These equations may be rewritten $a + b = 0$ and $-a + b = -2$.

Adding,

$$2b = -2$$
$$\therefore b = -1 \quad \text{and} \quad a = 1$$

Exercise 9h

1 Find the values of f(0), f(1), f(−1), f(2), f(−2) when
(a) $f(x) = x^3 + 3x^2 - 4x - 12$, (b) $f(x) = 3x^3 - 2x - 1$,
(c) $f(x) = x^5 + 2x^4 + 3x^3$, (d) $f(x) = x^4 - 4x^2 + 3$.
State one factor of each expression.

2 Find the remainders when
(a) $x^3 + 3x^2 - 4x + 2$ is divided by $x - 1$,
(b) $x^3 - 2x^2 + 5x + 8$ is divided by $x - 2$,
(c) $x^5 + x - 9$ is divided by $x + 1$,
(d) $x^3 + 3x^2 + 3x + 1$ is divided by $x + 2$,
(e) $4x^3 - 5x + 4$ is divided by $2x - 1$,
(f) $4x^3 + 6x^2 + 3x + 2$ is divided by $2x + 3$.

3 Find the values of a in the expressions below when the following conditions are satisfied:
(a) $x^3 + ax^2 + 3x - 5$ has remainder -3 when divided by $x - 2$,
(b) $x^3 + x^2 + ax + 8$ is divisible by $x - 1$,
(c) $x^3 + x^2 - 2ax + a^2$ has remainder 8 when divided by $x - 2$,
(d) $x^4 - 3x^2 + 2x + a$ is divisible by $x + 1$,
(e) $x^3 - 3x^2 + ax + 5$ has remainder 17 when divided by $x - 3$,
(f) $x^5 + 4x^4 - 6x^2 + ax + 2$ has remainder 6 when divided by $x + 2$.

4 Show that $2x^3 + x^2 - 13x + 6$ is divisible by $x - 2$, and hence find the other factors of the expression.

5 Show that $12x^3 + 16x^2 - 5x - 3$ is divisible by $2x - 1$ and find the factors of the expression.

6 Factorise:
(a) $x^3 - 2x^2 - 5x + 6$, (b) $x^3 - 4x^2 + x + 6$,
(c) $2x^3 + x^2 - 8x - 4$, (d) $2x^3 + 5x^2 + x - 2$,
(e) $2x^3 + 11x^2 + 17x + 6$, (f) $2x^3 - x^2 + 2x - 1$.

7 Find the values of a and b if $ax^4 + bx^3 - 8x^2 + 6$ has remainder $2x + 1$ when divided by $x^2 - 1$.

8 The expression $px^4 + qx^3 + 3x^2 - 2x + 3$ has remainder $x + 1$ when divided by $x^2 - 3x + 2$. Find the values of p and q.

9 The expression $ax^2 + bx + c$ is divisible by $x - 1$, has remainder 2 when divided by $x + 1$, and has remainder 8 when divided by $x - 2$. Find the values of a, b, c.

10 $x - 1$ and $x + 1$ are factors of the expression $x^3 + ax^2 + bx + c$, and it leaves a remainder of 12 when divided by $x - 2$. Find the values of a, b, c.

Exercise 9i (Miscellaneous)

Calculators should not be used in this exercise.

1 Write in terms of the simplest possible surds:

(a) $\sqrt{180} + \sqrt{1125} - \sqrt{1280}$, (b) $\dfrac{3\sqrt{2} - 4}{3 - 2\sqrt{2}}$, (c) $(\sqrt{3} + \sqrt{2})^3 + (\sqrt{3} - \sqrt{2})^3$.

2 Given that $\sqrt{2} \approx 1.414$ and $\sqrt{3} \approx 1.732$, evaluate correct to three significant figures:

(a) $\sqrt{48} + \sqrt{72} + \sqrt{12.5}$, (b) $\dfrac{2}{\sqrt{3} - \sqrt{2}}$, (c) $\sqrt{\tfrac{1}{8}} + \sqrt{\tfrac{1}{12}}$.

3 Express in the form $a+b\sqrt{2}$:

(a) $\dfrac{3+\sqrt{2}}{3-\sqrt{2}}$, (b) $(3+\sqrt{2})(5-\sqrt{2})$.

4 (a) Find the values of $8^{-4/3}$, $(4/9)^{3/2}$, $512^{-2/9}$.
(b) Solve the equation $x^{2/3}-5x^{1/3}+6=0$.

5 Multiply $x^{2/3}+2x^{1/3}+1$ by $x^{1/3}-2$. Check your answer by substituting $x=8$.

6 Without using tables, find the values of

(a) $\dfrac{12^{3/2}\times 16^{1/8}}{27^{1/6}\times 18^{1/2}}$, (b) $\lg 75+2\lg 2-\lg 3$.

7 Given that $\lg 2=0.301\ 030$ and $\lg 3=0.477\ 121$, find, the values correct to five places of decimals of:
(a) $\lg 12$, (b) $2\lg 21-\lg 98$, (c) $\lg\sqrt[3]{60}$.

8 (a) Express $\lg\dfrac{100a^2}{b^3\sqrt{c}}$ in terms of $\lg a$, $\lg b$, $\lg c$.
(b) Given that $\lg 5\approx 0.698\ 970\ 0$, find correct to six decimal places the value of $\lg 40$.

9 Taking $\lg 2\approx 0.301\ 030\ 0$ and $\lg 3\approx 0.477\ 121\ 3$, find the values of (a) $\lg 5$, (b) $\lg 18$, (c) $\lg 1.5$, correct to six decimal places.

10 Solve the equations:
(a) $9^x=27^{3/4}$, (b) $2^x=9$.

11 On a slide rule the distance from mark '1' to mark 'n' is proportional to $\lg n$. If the distance from mark '1' to mark '10' is 25 cm, calculate the distances
(a) from the mark '1' to the mark '2',
(b) from the mark '2' to the mark '3'.

12 Find the sum and product of the roots of the equation $3x^2+5x-1=0$. Also find the equation whose roots are the squares of the roots of this equation.

13 Find the values of m for which the equation $x^2+(m+3)x+4m=0$ has equal roots. For what value of m is the sum of the roots zero?

14 If α and β are the roots of the equation $2x^2-5x-1=0$, find
(a) the value of $\alpha^2+\beta^2$,
(b) an equation with integral coefficients whose roots are $1/\alpha$ and $1/\beta$.

15 What are the values of a and b if $x-3$ and $x+7$ are factors of the quadratic $ax^2+12x+b$?

16 Show that $3x^3+x^2-8x+4$ is zero when $x=\frac{2}{3}$, and hence factorise the expression.

17 What is the value of a if $2x^2-x-6$, $3x^2-8x+4$ and $ax^3-10x-4$ have a common factor?

18 Factorise the expression $3x^3-11x^2-19x-5$.

19 If the expression $ax^4+bx^3-x^2+2x+3$ has remainder $3x+5$ when it is divided by x^2-x-2, find the values of a and b.

20 Find the values of p and q which make $x^4+6x^3+13x^2+px+q$ a perfect square.

Chapter 10

Quadratic equations and complex numbers

The quadratic equation $ax^2 + bx + c = 0$

10.1 It is assumed that readers are familiar with solving quadratic equations by factorisation, as in Example 1 below, and that some will be familiar with 'completing the square', which is illustrated in Example 2.

Example 1 *Solve* $2x^2 + 7x - 15 = 0$.

$$2x^2 + 7x - 15 = 0$$
$$(2x - 3)(x + 5) = 0$$

hence,

either $2x - 3 = 0, \quad x = 1\tfrac{1}{2}$
or $\quad x + 5 = 0, \quad x = -5$

When it is difficult to factorise, the technique of completing the square can be used. This method depends on the identity

$$(x + k)^2 = x^2 + 2kx + k^2$$

Example 2 *Solve* $5x^2 - 6x - 2 = 0$.

$$5x^2 - 6x - 2 = 0$$

Add 2 to both sides,

$$5x^2 - 6x \quad = 2$$

divide through by 5,

$$x^2 - \frac{6}{5}x \quad = \frac{2}{5}$$

complete the square, by adding $(\frac{3}{5})^2$ to both sides,

$$x^2 - \left(\frac{6}{5}\right)x + \left(\frac{3}{5}\right)^2 = \left(\frac{3}{5}\right)^2 + \frac{2}{5} = \frac{9 + 10}{25} = \frac{19}{25}$$

factorise the left-hand side,

$$\left(x - \frac{3}{5}\right)^2 = \frac{19}{25}$$

take the square root of both sides,

$$\left(x - \frac{3}{5}\right) = \pm\sqrt{\frac{19}{25}} = \pm\frac{\sqrt{19}}{5}$$

and finally, add $\frac{3}{5}$ to both sides,

$$x = \frac{3}{5} \pm \frac{\sqrt{19}}{5} = \frac{3 \pm \sqrt{19}}{5}$$

[Answers to such questions should be left with surds in them, unless a specified degree of accuracy is demanded by the question.]

Notice that the roots can be used to find the factors of the original expression. Thus in Example 2,

$$\begin{aligned} 5x^2 - 6x - 2 &= 5(x^2 - \tfrac{6}{5}x - 2) \\ &= 5\left(x - \frac{3 + \sqrt{19}}{5}\right)\left(x - \frac{3 - \sqrt{19}}{5}\right) \end{aligned}$$

The quadratic formula

10.2 The procedure illustrated in Example 2, above, can be generalised, as follows. To solve

$$ax^2 + bx + c = 0$$

subtract c from both sides,

$$ax^2 + bx = -c$$

divide through by a,

$$x^2 + \left(\frac{b}{a}\right)x = -\left(\frac{c}{a}\right)$$

complete the square (by adding $b^2/(4a^2)$ to both sides),

$$x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

and factorise the left-hand side, which gives

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Take the square root (but note this is only possible if the right-hand side is non-negative i.e. if $b^2 - 4ac \geqslant 0$),

$$\left(x+\frac{b}{2a}\right)=\pm\sqrt{\left(\frac{b^2-4ac}{4a^2}\right)}=\frac{\pm\sqrt{(b^2-4ac)}}{2a} \qquad (1)$$

Now subtract $b/(2a)$ from both sides

$$x=-\frac{b}{2a}\pm\frac{\sqrt{(b^2-4ac)}}{2a}$$

So, if $\boldsymbol{ax^2+bx+c=0}$ then $\boldsymbol{x=\dfrac{-b\pm\sqrt{(b^2-4ac)}}{2a}}$

This formula is usually the most convenient way of solving quadratic equations which cannot readily be solved by factorisation.

Example 3 *Solve* $2x^2-6x-3=0$.

In this example $a=2$, $b=-6$ and $c=-3$, hence $b^2-4ac=36-4\times2\times(-3)$, that is, $b^2-4ac=60$. Substituting these figures into the formula

$$x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}$$

gives

$$x=\frac{+6\pm\sqrt{60}}{4}$$

$$=\frac{6\pm2\sqrt{15}}{4}$$

$$\therefore x=\frac{3\pm\sqrt{15}}{2}$$

Notice the importance of the step marked (1) in the proof of the quadratic formula. Three possibilities can arise:

(i) $b^2-4ac>0$; a *real* value of $\sqrt{(b^2-4ac)}$ can be found and so the equation has two real distinct roots,

(ii) $b^2-4ac=0$; the solution is $x=-b/(2a)$,

(iii) $b^2-4ac<0$; there is no real value of $\sqrt{(b^2-4ac)}$ and so there are no real roots.

In case (ii), the expression $x^2+(b/a)x+(c/a)$ is the square of $x+b/(2a)$ and it is convenient to say that the quadratic equation has 'two identical roots'. We shall return to case (iii) in §10.6.

Because of its important role in determining the nature of the roots, the term (b^2-4ac) is called the **discriminant** of the equation.

Qu. 1 Calculate the discriminant of each of the quadratics below and state whether the equation has (i) two distinct real roots, (ii) two identical roots, or (iii) no real roots.

(a) $3x^2+5x-1=0$, (b) $49x^2+42x+9=0$,
(c) $2x^2+8x+9=0$, (d) $2x^2+7x+4=0$.

The quadratic function $f(x) = ax^2 + bx + c$

10.3 Using the method of completing the square, the form $ax^2 + bx + c$ can always be reduced to the form $a(x - p)^2 + q$. This is illustrated in Example 4 below.

Example 4 *Express the function* $f(x) = 2x^2 - 12x + 23$ *in the form* $a(x - p)^2 + q$.

$$\begin{aligned} 2x^2 - 12x + 23 &= 2(x^2 - 6x + 11.5) \\ &= 2[(x - 3)^2 - 9 + 11.5] \\ &= 2[(x - 3)^2 + 2.5] \\ &= 2(x - 3)^2 + 5 \end{aligned}$$

In this example, $a = 2, p = 3$ and $q = 5$. One advantage of this form is that, since $(x - 3)^2 \geqslant 0$, we can read off that $f(x) \geqslant 5$ and that the least value of the function occurs when $x = 3$.

Example 5 *Find, by completing the square, the greatest value of the function* $f(x) = 1 - 6x - x^2$.

$$\begin{aligned} f(x) &= 1 - 6x - x^2 \\ &= 10 - (9 + 6x + x^2) \\ &= 10 - (3 + x)^2 \end{aligned}$$

Since $(3 + x)^2$ is the square of a real number it cannot be negative; it is zero when $x = -3$, otherwise it is positive. Consequently $10 - (3 + x)^2$ is always less than or equal to 10.

$\therefore$ the greatest value of the function is 10 and this occurs when $x = -3$.

In general

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right] \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] \\ &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a} \end{aligned}$$

and thus $f(x) = ax^2 + bx + c$ may be written $a(x - p)^2 + q$, where

$$p = -\frac{b}{2a} \quad \text{and} \quad q = -\frac{b^2 - 4ac}{4a}$$

The least (or greatest) value of f(x) is $f(p) = q$. If $a > 0$, f(p) is the least value; if $a < 0$, it is the greatest value.

Qu. 2 Find, by completing the square, the range of the function

$$f(t) = 10 + 20t - 5t^2$$

The graph of $y = ax^2 + bx + c$

10.4 We have seen in §10.3 that this equation can be expressed in the form

$$y = a(x - p)^2 + q$$

Now, we know that the graph of $y = x^2$ is a parabola and that the graph of $y = (x - p)^2$ is the same shape, but it is displaced p units to the right (Fig. 10.1).

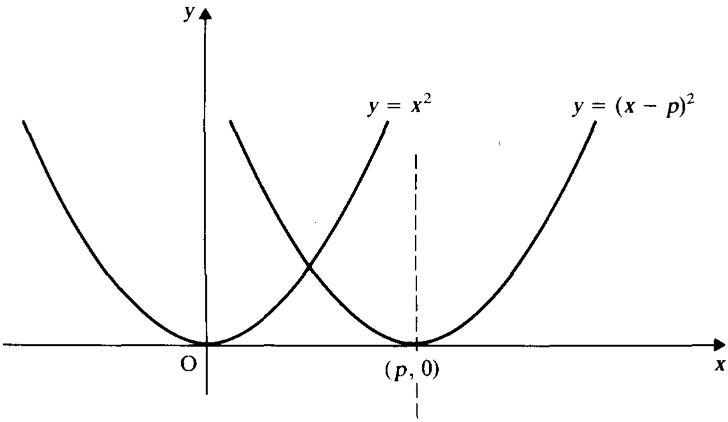


Figure 10.1

Multiplying $(x - p)^2$ by a merely 'stretches' the graph parallel to the y-axis, although if a is negative it will also turn it upside down. Adding q to $a(x - p)^2$ translates the graph q units vertically upwards. Thus the graph of

$$y = a(x - p)^2 + q$$

looks like Fig. 10.2. In this diagram $a > 0$, $p > 0$ and $q > 0$ (i.e. $b^2 - 4ac < 0$).

Notice that if $q < 0$ (i.e. $b^2 - 4ac > 0$) but a and p are positive, then the graph would look like Fig. 10.3.

In this case, M is the point $(-b/(2a), 0)$ and H and K are the points where $ax^2 + bx + c = 0$ and, as we have seen in §10.2, at these points

$$x = -\frac{b}{2a} \pm \frac{\sqrt{(b^2 - 4ac)}}{2a}$$

Notice that these values of x can only be real if $b^2 - 4ac \geqslant 0$.

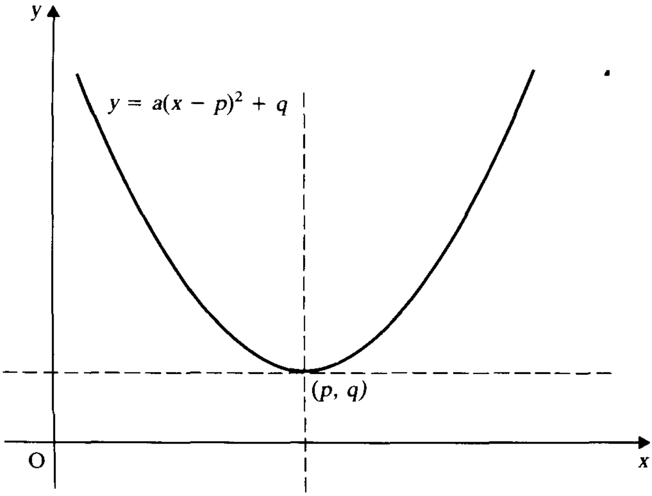


Figure 10.2

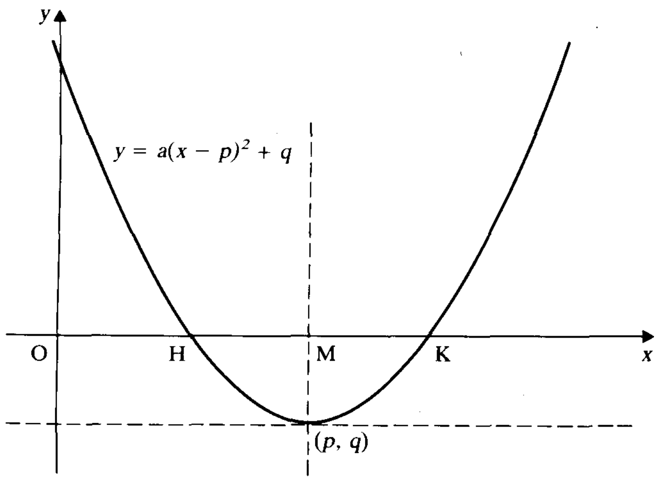


Figure 10.3

Qu. 3 Sketch the graph of $y = ax^2 + bx + c$, when
(a) $b^2 > 4ac$ and $a < 0$, (b) $b^2 = 4ac$ and $a > 0$.
In each diagram mark clearly the coordinates of the vertex.

Exercise 10a

Leave surds in the answers.

1 Solve, by factorisation:
(a) $2x^2 - 5x + 3 = 0$, (b) $x^2 + 4x - 21 = 0$,

(c) $4x^2 - 25 = 0$, (d) $7x^2 + 5x = 0$.

2 Solve, by completing the square:
(a) $2x^2 - 6x - 1 = 0$, (b) $5x^2 + 12x + 6 = 0$,
(c) $x^2 + 7x - 3 = 0$, (d) $10 + 3x - 2x^2 = 0$.

3 Solve, by using the formula:
(a) $3t^2 - 7t - 1 = 0$, (b) $5z^2 + 3z - 7 = 0$,
(c) $4 + 13y + y^2 = 0$, (d) $3p^2 = 7p + 2$.

4 Solve, where possible, by any suitable method:
(a) $15 - 30x + 4x^2 = 0$, (b) $11x^2 = 48x$,
(c) $9x^2 = 8x - 2$, (d) $7x^2 - 38x + 15 = 0$.

5 Using the results of No. 2, factorise:
(a) $2x^2 - 6x - 1$, (b) $5x^2 + 12x + 6$,
(c) $x^2 + 7x - 3$, (d) $10 + 3x - 2x^2$.

6 Sketch the graphs of
(a) $y = 2x^2 - 5x + 3$, (b) $y = 2x^2 - 6x - 1$,
(c) $y = 3x^2 - 7x - 1$, (d) $y = 3x^2 - 7x + 5$.
[Hint: use the answers to 1(a), 2(a) and 3(a).]

7 Sketch the graphs of
(a) $y = 9x^2 - 30x + 25$, (b) $y = x^2 - 6x + 13$,
(c) $y = 5 - x^2$, (d) $y = 36 + 48x - 9x^2$.

8 Given that $3x^2 - kx + 12$ is positive for all values of x, find the range of possible values for k.

9 Given that α and β are the roots of the quadratic equation, $x^2 - 7x + 3 = 0$, find α and β from the formula, and verify that $\alpha + \beta = 7$ and $\alpha\beta = 3$.

10 By completing the square, find the greatest values of
(a) $2 - 2x - x^2$, (b) $-7 + 12x - 3x^2$,
and the least values of
(c) $13 + 6x + 3x^2$, (d) $15 + 8x + \frac{1}{2}x^2$.

Imaginary numbers

10.5 We have seen that the equation $x^2 + 1 = 0$, or $x^2 = -1$, has no real roots. For the moment, let us not worry too much about this; instead, we will write i for $\sqrt{(-1)}$. We could then say that $x^2 + 1 = 0$ has two roots, namely $x = \pm i$. Historically, this *is* how the subject developed. The sixteenth century mathematicians Cardano and Bombelli started to use symbols for square roots of negative numbers even though they knew they were not *real* numbers. Later Descartes started to call these numbers '*imaginary numbers*'. Then, in the eighteenth century, the Swiss mathematician Euler introduced the symbol i for $\sqrt{(-1)}$.

Having introduced i there is no need to invent further symbols for the square roots of other negative numbers. Consider, for example, $\sqrt{(-25)}$.

$$\begin{aligned}\sqrt{(-25)} &= \sqrt{(25 \times -1)} \\ &= \sqrt{25} \times \sqrt{(-1)} \\ &= 5i\end{aligned}$$

So an equation in the form

$$x^2 + n^2 = 0 \quad \text{or} \quad x^2 = -n^2, \quad \text{where } n \in \mathbb{R},$$

has two roots, $x = \pm n\mathrm{i}$.

[In some contexts, especially electricity where i is used to represent the current in an electrical circuit, the symbol j is used instead of i.]

Qu. 4 Solve the equations:
(a) $x^2 + 64 = 0$, (b) $x^2 + 7 = 0$, (c) $4x^2 + 9 = 0$, (d) $(x + 3)^2 = -25$.

Complex numbers

10.6 We can now return to the problem of solving

$$ax^2 + bx + c = 0 \text{ when } b^2 < 4ac$$

Previously (§10.2) we decided that no real roots exist in this case.

First, we consider a particular example; we shall try to solve

$$x^2 - 4x + 5 = 0$$

Completing the square gives

$$\begin{aligned} x^2 - 4x &= -5 \\ (x-2)^2 - 4 &= -5 \\ (x-2)^2 &= -1 \end{aligned}$$

Previously, at this stage we were unable to proceed further because we could not find the square root of -1. Now, we can use our imaginary numbers. Hence

$$\begin{aligned} (x-2) &= \pm \mathrm{i} \\ \therefore\ x &= 2 \pm \mathrm{i} \end{aligned}$$

Qu. 5 Solve $x^2 - 6x + 34 = 0$.

The general solution of the equation $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

When $b^2 < 4ac$ this can be written

$$x = \frac{-b \pm \sqrt{\{-1(4ac - b^2)\}}}{2a}$$

$$= \frac{-b}{2a} \pm \frac{\sqrt{(4ac - b^2)}}{2a}\mathrm{i}$$

Notice that both $-b/(2a)$ and $\sqrt{(4ac - b^2)}/(2a)$ are real numbers.

Numbers of the form $p + \mathrm{i}q$, where p and q are real numbers, are called **complex numbers**. The standard symbol for the set of complex numbers is $\mathbb{C}$.

Example 6 *Solve* $x^2 - 6x + 13 = 0$, *where* $x \in \mathbb{C}$.

$$x^2 - 6x + 13 = 0$$

Using the formula

$$x = \frac{+6 \pm \sqrt{(36 - 4 \times 1 \times 13)}}{2}$$

$$= \frac{6 \pm \sqrt{(-16)}}{2}$$

$$= \frac{6 \pm 4\mathrm{i}}{2}$$

$$\therefore x = 3 \pm 2\mathrm{i}.$$

In the complex number $p + \mathrm{i}q$, the number p is called the **real part** of the complex number and q is called its **imaginary part**. (Thus the real part of $5 + 4\mathrm{i}$ is 5, and the imaginary part is 4.) It is frequently convenient to have a single letter to represent a complex number, and the normal choice for this is z, although w is also sometimes used. The real part of a complex number z can then be abbreviated to $\mathrm{Re}(z)$ and the imaginary part is written $\mathrm{Im}(z)$. Thus if $z = 2 + 7\mathrm{i}$, then $\mathrm{Re}(z) = 2$ and $\mathrm{Im}(z) = 7$, or again, if $w = 4 - 3\mathrm{i}$, then we can write $\mathrm{Re}(w) = 4$ and $\mathrm{Im}(w) = -3$.

It is important to notice that two complex numbers are equal if, and only if, their real parts are equal and their imaginary parts are equal, for if

$$a + \mathrm{i}b = c + \mathrm{i}d$$

then

$$a - c = \mathrm{i}(d - b)$$

and, squaring both sides,

$$(a - c)^2 = -(d - b)^2$$

Now, since a, b, c and d are real numbers, $(a - c)^2$ and $(d - b)^2$ are either positive or they are zero. It is impossible for them to be positive, because we would then have a positive number on the left-hand side and a negative number on the right. Therefore $(a - c)^2$ and $(d - b)^2$ are both zero, i.e.

$$a = c \quad \text{and} \quad b = d$$

[The reader may feel that this is a rather trivial point, but, as we will see later, this is a very valuable feature of complex numbers. It may seem less trivial if it is compared with a similar situation in rational numbers. Here it *is* possible to have $a/b = c/d$, even though $a \neq c$ and $b \neq d$, for example, $2/3 = 10/15$.]

Since it was necessary to introduce complex numbers in order to include the roots of all quadratic equations, it might be thought that yet further types of number might be necessary in order to find the roots of equations of higher degree. However this is not so; it can be proved that a polynomial equation of degree n has exactly n roots (possibly repeated) in $\mathbb{C}$, but the proof is beyond the scope of this book.

Qu. 6 Solve the following equations with the quadratic formula or by completing the square:
(a) $z^2 - 4z + 13 = 0$, (b) $9z^2 + 25 = 0$,
(c) $2z^2 = 2z - 13$ (d) $34z^2 - 6z + 1 = 0$.

The algebra of complex numbers

10.7 In the course of learning elementary arithmetic, one has to learn how to add, subtract, multiply, and divide fractions: we are now faced with the problem of manipulating complex numbers. The operations addition, subtraction, multiplication, and division which we have used so far are concerned with real numbers, hence it is necessary to define what we mean by these operations with regard to complex numbers. It is easiest for us to define these operations by saying that we shall use the usual laws of algebra together with the relation $i^2 = -1$. Thus

$$\begin{aligned}(a + ib) + (c + id) &= (a + c) + i(b + d)\\ (a + ib) - (c + id) &= (a - c) + i(b - d)\\ (a + ib) \times (c + id) &= ac + aid + ibc + i^2bd\\ &= (ac - bd) + i(ad + bc)\end{aligned}$$

At this stage it is worth comparing the corresponding operations with real numbers in the form $a + \sqrt{2}b$ (a, b rational):

$$\begin{aligned}(a + b\sqrt{2}) + (c + d\sqrt{2}) &= (a + c) + \sqrt{2}(b + d)\\ (a + b\sqrt{2}) - (c + d\sqrt{2}) &= (a - c) + \sqrt{2}(b - d)\\ (a + b\sqrt{2}) \times (c + d\sqrt{2}) &= ac + ad\sqrt{2} + bc\sqrt{2} + 2bd\\ &= (ac + 2bd) + \sqrt{2}(ad + bc)\end{aligned}$$

This helps us to find a way of expressing $(a + ib)/(c + id)$ in the form $p + iq$. The reader will probably recall the corresponding process with $(a + b\sqrt{2})/(c + d\sqrt{2})$. The method is to multiply numerator and denominator in such a way that the new denominator involves a difference of two squares:

$$\begin{aligned}\frac{a + b\sqrt{2}}{c + d\sqrt{2}} \times \frac{c - d\sqrt{2}}{c - d\sqrt{2}} &= \frac{(ac - 2bd) + \sqrt{2}(bc - ad)}{c^2 - 2d^2}\\ &= \frac{ac - 2bd}{c^2 - 2d^2} + \sqrt{2}\frac{bc - ad}{c^2 - 2d^2}\end{aligned}$$

Similarly, the expression $(a + ib)/(c + id)$ may be expressed in the form $p + iq$ by multiplying numerator and denominator by $c - id$ because

$$(c + id) \times (c - id) = c^2 - i^2d^2 = c^2 + d^2$$

In other words,

$$\frac{a+\mathrm{i}b}{c+\mathrm{i}d}=\frac{a+\mathrm{i}b}{c+\mathrm{i}d}\times\frac{c-\mathrm{i}d}{c-\mathrm{i}d}$$

$$=\frac{(ac+bd)+\mathrm{i}(bc-ad)}{c^2+d^2}$$

Definition

Two complex numbers in the form $x+\mathrm{i}y$, $x-\mathrm{i}y$ *are called* **conjugate** ***complex numbers.***

The symbol z^* is used to represent the complex conjugate of z, so if $z=x+\mathrm{i}y$, then we write

$$z^*=x-\mathrm{i}y$$

Qu. 7 Express $(2+3\mathrm{i})/(1+\mathrm{i})$ in the form $p+\mathrm{i}q$ $(p, q\in\mathbb{R})$. [Multiply numerator and denominator by $1-\mathrm{i}$.]

Do not attempt to memorise expressions for the sum, difference, product, and quotient of two complex numbers: simply use the usual laws of algebra, together with the relation $\mathrm{i}^2=-1$.

Exercise 10b

Simplify:

1 (a) i^3, (b) i^4, (c) i^5, (d) i^6, (e) $\frac{1}{\mathrm{i}^2}$, (f) $\frac{1}{\mathrm{i}}$, (g) $\frac{1}{\mathrm{i}^3}$.

2 (a) $(3+\mathrm{i})+(1+2\mathrm{i})$, (b) $(5-3\mathrm{i})+(4+3\mathrm{i})$,
(c) $(2-3\mathrm{i})-(1+2\mathrm{i})$, (d) $(1+\mathrm{i})-(1-\mathrm{i})$.

3 (a) $(2+3\mathrm{i})(4+5\mathrm{i})$, (b) $(2-\mathrm{i})(3+2\mathrm{i})$,
(c) $(1+\mathrm{i})(1-\mathrm{i})$, (d) $(3+4\mathrm{i})(3-4\mathrm{i})$,
(e) $(u+\mathrm{i}v)(u-\mathrm{i}v)$, (f) $(x+2\mathrm{i}y)(2x+\mathrm{i}y)$,
(g) $\mathrm{i}(2p+3\mathrm{i}q)$, (h) $(p+2\mathrm{i}q)(p-2\mathrm{i}q)$.

4 Express with real denominators:

(a) $\frac{1-\mathrm{i}}{1+\mathrm{i}}$, (b) $\frac{1}{2-3\mathrm{i}}$, (c) $\frac{3\mathrm{i}-2}{1+2\mathrm{i}}$, (d) $\frac{5+4\mathrm{i}}{5-4\mathrm{i}}$,

(e) $\frac{1}{x+\mathrm{i}y}$, (f) $\frac{1}{x-\mathrm{i}y}$, (g) $\frac{1}{2+3\mathrm{i}}+\frac{1}{2-3\mathrm{i}}$.

Simplify the expressions in Nos. 5 and 6:

5 (a) $(2+3\mathrm{i})^2$, (b) $(4-5\mathrm{i})^2$, (c) $(x+\mathrm{i}y)^2$.
6 (a) $(1+\mathrm{i})^3$, (b) $(1-\mathrm{i})^3$, (c) $1/(1+\mathrm{i})^3$.

7 Solve the quadratic equations:
(a) $z^2 - 4z + 29 = 0$, (b) $4z^2 + 7 = 0$,
(c) $2z^2 + 3z + 5 = 0$, (d) $4z^2 + 4z + 5 = 0$.

8 If α and β are the roots of $z^2 - 10z + 29 = 0$, find α and β by using the formula. Verify that $\alpha + \beta = 10$ and $\alpha\beta = 29$.

9 If α and β are the roots of $az^2 + bz + c = 0$, find, by using the formula, expressions for α and β, in terms of a, b and c. Verify that $\alpha + \beta = -b/a$ and that $\alpha\beta = c/a$.

10 Solve the cubic equation $2z^3 + 3z^2 + 8z - 5 = 0$.

Complex numbers as ordered pairs

10.8 To see how a satisfactory definition of complex numbers can be given, consider the problem of defining rational numbers in terms of the integers.
Note. (i) A rational number is formed from a *pair* of integers, e.g. 2/3, 7/5, 4/1 (the last of which is commonly abbreviated to 4). (See §2.3.)

(ii) The position of the integers is important because in general $\frac{a}{b} \neq \frac{b}{a}$.

We therefore say that a rational number is an *ordered pair* of integers — but this, by itself, is not enough. To complete the definition, we must say how numbers of this type are to be added, subtracted, multiplied and divided.

We know that for rational numbers

$$\text{`}\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}\text{'}$$

but this is by no means the only possible way of defining addition of the ordered pairs a/b, c/d. For instance, it would be much simpler to define addition by the rule

$$\text{`}\frac{a}{b} + \frac{c}{d} = \frac{a + c}{b + d}\text{'}$$

As to multiplication, with rational numbers,

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

but multiplication of the ordered pairs a/b, c/d might have been defined by the rule

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac - bd}{ad + bc}$$

We need not go through the process of defining subtraction and division: the point to note about defining the various operations on ordered pairs is that the properties of the numbers so defined will depend on the rules chosen.

Now consider complex numbers. We have seen that a complex number involves a *pair* of real numbers and that the *order* of the pair is important because in general $a + ib \neq b + ia$. We therefore define a complex number as an ordered pair of real numbers which we shall write as $[a, b]$. The fundamental operations of addition and multiplication are defined by the rules:

$$[a, b] + [c, d] = [a + c, b + d]$$
$$[a, b] \times [c, d] = [ac - bd, ad + bc]$$

Subtraction and division are defined in terms of addition and multiplication thus, for any type of number,

$p - q$ is the number x such that $q + x = p$ and
$p \div q$ is the number y such that $q \times y = p$

Now

$$[c, d] + [a - c, b - d] = [a, b]$$
$$\therefore [a, b] - [c, d] = [a - c, b - d].$$

Qu. 8 Use the definition of division above to show that

(a) for the rational numbers $\frac{a}{b}, \frac{c}{d}$,

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$$

(b) for the complex numbers $[a, b]$, $[c, d]$,

$$[a, b] \div [c, d] = \left(\frac{ac + bd}{c^2 + d^2}, \frac{bc - ad}{c^2 + d^2}\right)$$

Qu. 9 Note that to every real number a there corresponds a unique complex number $[a, 0]$. Find, from the definitions of the four operations on complex numbers
(a) $[a, 0] + [c, 0]$, (b) $[a, 0] \times [c, 0]$,
(c) $[a, 0] - [c, 0]$, (d) $[a, 0] \div [c, 0]$.
Compare these results with the corresponding operations on the real numbers a, c.

The next stage would be to show that these ordered pairs obey the laws of arithmetic. This would justify the use of the term *complex numbers*. However, we shall not pursue this argument.

The definition of a complex number as an ordered pair was first given by Hamilton in 1835.

The Argand diagram

10.9 The last section was written to show the reader that complex numbers can be put on a satisfactory logical basis. However, manipulation of complex numbers is most easily carried out as before: the ordered pair notation is simply a device for defining these numbers without reference to $\sqrt{(-1)}$. We could write $\sqrt{(-1)}$ as the ordered pair $[0, 1]$ but this would be rather clumsy and it is easier to write $\sqrt{(-1)} = \mathrm{i}$.

Qu. 10 Prove from the definition of multiplication of complex numbers that

$$[0, 1] \times [0, 1] = [-1, 0]$$

Although the idea of an ordered pair may appear to some readers to have been a digression, it leads us to the next step in our treatment of the subject. The Argand diagram is named after J. R. Argand, who published his work on the graphical representation of complex numbers in 1806.

Corresponding to every complex number $[x, y]$ or $x + iy$, there is a point (x, y) in the Cartesian plane; and corresponding to any point (x, y) in the plane, there is a complex number $x + iy$. (Here it is worth comparing the equivalent situation with real numbers. Corresponding to every real number x there is a point on the x-axis. What is less easy to prove is that corresponding to every point on the x-axis there is a real number). At first this correspondence between complex numbers and points on the plane may appear to be rather obvious and not very useful, but in fact it proves to be a considerable importance to the theory of complex numbers.

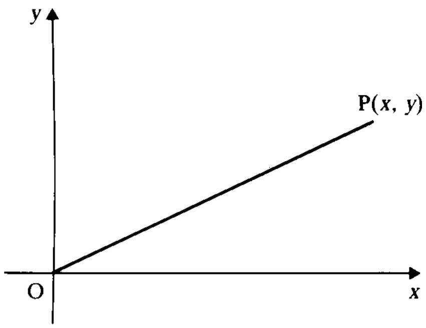


Figure 10.4

The value of this correspondence is increased by the fact that with every point P(x, y) in the plane there is associated a *radius vector* OP (see Fig. 10.4). This means that corresponding to every complex number $x + iy$ there is a radius vector OP where P is (x, y). Further, corresponding to every radius vector OP in the plane there is complex number $x + iy$.

Look at Fig. 10.5. The points A, B, A′, B′ are respectively $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$. Corresponding to

OA there is the complex number	$1 + 0i$	or	1
OB there is the complex number	$0 + 1i$	or	i
OA′ there is the complex number	$-1 + 0i$	or	-1
OB′ there is the complex number	$0 + (-1)i$	or	$-i$

Looking down the right-hand side of the last four lines, each number is equal to the previous one multiplied by i. Meanwhile, the corresponding radius vector has rotated in the positive (anti-clockwise) sense through one right angle. Would the same thing happen if any complex number were multiplied by i?

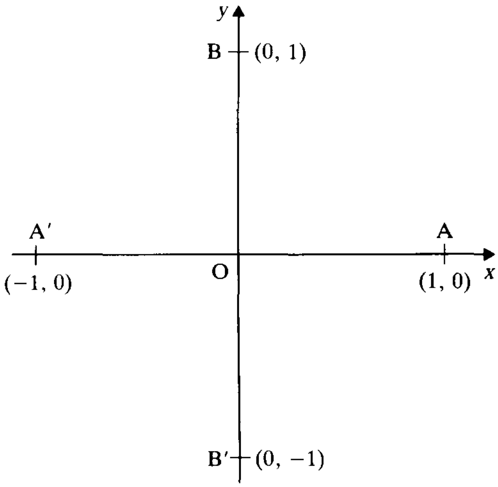


Figure 10.5

Qu. 11 Find the complex numbers obtained by multiplying $x + iy$ once, twice and three times by i. Does the corresponding radius vector rotate through one right angle each time?

Two quantities are required to specify a vector through the origin: magnitude and direction. The magnitude r of OP (Fig. 10.6) presents no difficulty

$$r = \sqrt{(x^2 + y^2)}$$

This quantity is called the **modulus** of the complex number $x + iy$. 'The modulus of $x + iy$' is abbreviated to $|x + iy|$ hence

$$|x + iy| = \sqrt{(x^2 + y^2)}$$

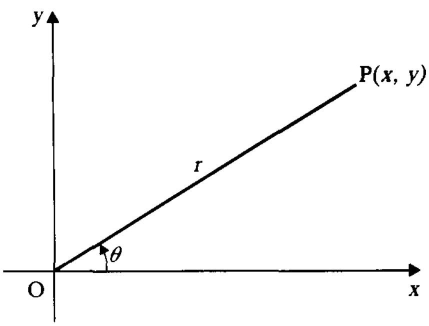


Figure 10.6

Qu. 12 Write down the moduli of
(a) $3 + 4i$, (b) $-i$, (c) $\cos\theta + i\sin\theta$,
(d) $\frac{1}{2} - \frac{1}{2}\sqrt{3}i$, (e) -3, (f) $1 + i$.

The direction specifying the radius vector OP is not quite so easy to deal with because there are infinitely many positive and negative angles which would do.

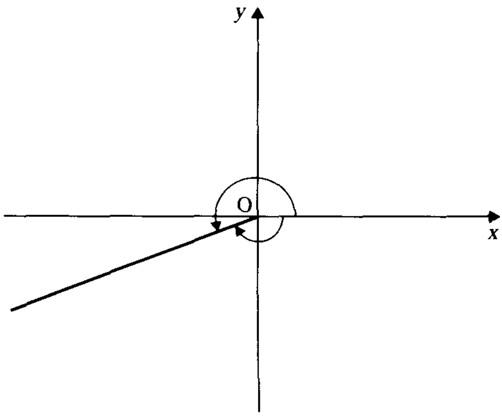


Figure 10.7

The problem of which angle to choose is well illustrated by a radius vector in the third quadrant (Fig. 10.7). It is simply a matter of convention whether we take the positive reflex angle or the negative obtuse angle. In fact the numerically smaller angle is used. The angle between the radius vector OP and the positive x-axis is called the **argument** of the complex number $x + iy$. This is abbreviated to $\arg(x + iy)$ and has, as we have said before, infinitely many values. The value uniquely specified by the above convention is called the **principal value** of the argument and is written $\arg(x + iy)$, so that

$$-180° < \arg(x + iy) \leqslant 180°$$

[In some textbooks, the argument is called the amplitude but this term is less acceptable because of possible confusion with the amplitude of a current, motion, or wave.]

Qu. 13 Find the principal values of the arguments of
(a) $\cos 45° + i \sin 45°$, (b) $+1$,
(c) $-i$, (d) $1 - i$,
(e) $\frac{1}{2} + \frac{1}{2}\sqrt{3}i$, (f) $\cos 120° + i \sin 120°$,
(g) $\cos 20° - i \sin 20°$, (h) $\sin 20° + i \cos 20°$.

A complex number can be completely specified by its modulus and argument, because, as we can see from Fig. 10.8, $x = r \cos \theta$ and $y = r \sin \theta$. Thus if $|z| = r$ and $\arg z = \theta$, then

$$\begin{aligned} z &= r \cos \theta + ir \sin \theta \\ &= r(\cos \theta + i \sin \theta) \end{aligned}$$

Notice, also, that if we are given a complex number $z = x + iy$, then its complex conjugate, $z^* = x - iy$. In other words z^* is the reflection of z in the real axis. Hence $|z^*| = |z|$ and $\arg(z^*) = -\arg z$. ($\bar{z}$ may also be used to denote the complex conjugate of z.)

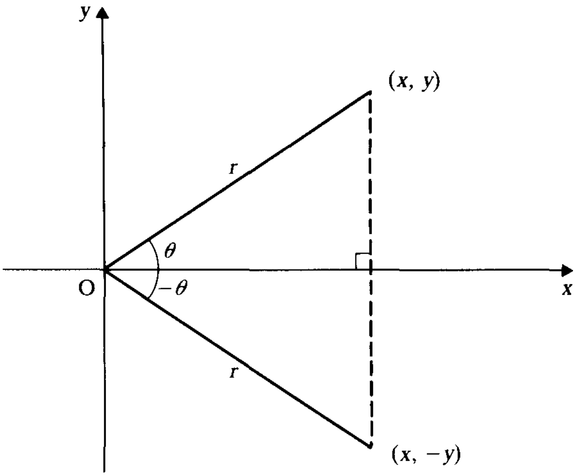


Figure 10.8

Example 7 *Given* $|z| = 10$ *and* $\arg z = 120°$, *write down* z.

$$z = 10(\cos 120° + \mathrm{i} \sin 120°)$$

$$= 10\left(-\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}\right)$$

$$= -5 + 5\sqrt{3}\,\mathrm{i}$$

Exercise 10c

1 Mark on the Argand diagram, the radius vectors corresponding to
(a) $1 + \mathrm{i}$, (b) $-3 + 2\mathrm{i}$,
(c) $-3 - 2\mathrm{i}$, (d) $3 - 4\mathrm{i}$,
(e) $-4 + 3\mathrm{i}$, (f) $\cos 60° + \mathrm{i} \sin 60°$,
(g) $\cos 120° + \sin 120°$, (h) $\cos 180° + \mathrm{i} \sin 180°$.
Write down the moduli of these complex numbers and give the principal values of their arguments.

2 Write down, in the form $x + \mathrm{i}y$, the complex numbers whose moduli are equal to one and whose arguments are
(a) 0°, (b) 90°, (c) 180°, (d) 270°, (e) 360°,
(f) 30°, (g) −30°, (h) 120°, (i) −120°, (j) 150°.

3 Given that $z = 3 + 4\mathrm{i}$ and $w = 12 + 5\mathrm{i}$, write down the moduli and arguments of
(a) z, (b) w, (c) $1/z$, (d) $1/w$, (e) zw,
(f) z^*, (g) w^*, (h) $(zw)^*$, (i) z^2, (j) w^2.

4 Simplify: $(1 + \mathrm{i})^2$, $(1 + \mathrm{i})^3$, $(1 + \mathrm{i})^4$.

Draw in the Argand diagram the radius vectors corresponding to $(1+i)$, $(1+i)^2$, $(1+i)^3$, $(1+i)^4$. Find the principal values of the arguments of these complex numbers.

5 Repeat No. 4
(a) for the complex number $\frac{1}{2}\sqrt{3}+\frac{1}{2}i$,
(b) for the complex number $\sqrt{3}+i$.

6 Given the complex number $z = a + ib$, where a and $b \in \mathbb{R}$, find z^2 and $1/z$ in terms of a and b. Verify that $|z^2| = |z|^2$ and $|1/z| = 1/|z|$.

7 Prove that if $|z| = r$, then $zz^* = r^2$.

8 Given that $z = a + ib$ and $w = c + id$, where a, b, c and $d \in \mathbb{R}$, find zw in terms of a, b, c, and d, and verify that $|zw| = |z| \times |w|$.

Exercise 10d (Miscellaneous)

1 Prove that $3x - 2$ is a factor of $3x^3 - 2x^2 + 3x - 2$.
Find the solution set of the equation $3x^3 - 2x^2 + 3x - 2 = 0$, when x belongs to the set of (a) integers, $\mathbb{Z}$; (b) rational numbers, $\mathbb{Q}$; (c) real numbers, $\mathbb{R}$; (d) complex numbers, $\mathbb{C}$.

2 Solve each of the equations

(i) $(x+4)(5x-7)=0$ (ii) $(x^2+4)(5x^2-7)=0$

when x belongs to the set of (a) integers, (b) rational numbers, (c) real numbers, (d) complex numbers.

3 Given that $z = 3 + i$ and $w = 1 + 3i$, express in the form $a + ib$, where $a, b \in \mathbb{R}$, the complex numbers (a) zw, (b) z/w, (c) $z^2 - w^2$ and find their moduli and arguments in degrees, correct to the nearest 1°.

4 (a) Express the following complex numbers in a form having a real denominator:

$$\frac{1}{3-2i}, \qquad \frac{1}{(1-i)^2}$$

(b) Find the modulus and principal argument of each of the complex numbers $z = 1 + 2i$ and $w = 2 - i$, and represent z and w clearly by points A and B in an Argand diagram. Find also the sum and product of z and w and mark the corresponding points C and D in your diagram. (C)

5 If the complex number $x + iy$ is denoted by z, then the complex conjugate number $x - iy$ is denoted by z^*.
(a) Express $|z^*|$ and $\arg(z^*)$ in terms of $|z|$ and $\arg(z)$.
(b) If a, b, and c are real numbers, prove that if $az^2 + bz + c = 0$ then

$$a(z^*)^2 + b(z^*) + c = 0$$

(c) If p and q are complex numbers and $q \neq 0$, prove that $\left(\frac{p}{q}\right)^* = \frac{p^*}{q^*}$. (C)

6 Find the values of a and b such that $(a+ib)^2 = i$. Hence or otherwise, solve the equation $z^2 + 2z + 1 - i = 0$, giving your answers in the form $p + iq$, where p and q are real numbers. (O)

7 (a) The equation $x^4 - 4x^3 + 3x^2 + 2x - 6 = 0$ has a root $1 - \mathrm{i}$. Find the other three roots.

(b) Given that 1, w_1, w_2 are the roots of the equation $z^3 = 1$ express w_1 and w_2 in the form $x + \mathrm{i}y$ and hence, or otherwise, show that

(i) $1 + w_1 + w_2 = 0$,

(ii) $1/w_1 = w_2$. (L)

8 (a) Given that the complex numbers w_1 and w_2 are the roots of the equation $z^2 - 5 - 12\mathrm{i} = 0$, express w_1 and w_2 in the form $a + ib$, where a and b are real.

(b) Indicate the point sets in an Argand diagram corresponding to the sets of complex numbers

$$A = \{z\colon |z| = 3,\ z \in \mathbb{C}\}$$
$$B = \{z\colon |z| = 2,\ z \in \mathbb{C}\}$$

Shade the region corresponding to values of z for which the inequalities

$$2 < |z| < 3 \quad \text{and} \quad 30° < \arg z < 60°$$

are simultaneously satisfied. (L)

9 If $z = \frac{1}{2}(1 + \mathrm{i})$, write down the modulus and argument for each of the numbers z, z^2, z^3, z^4. Hence, or otherwise, show in an Argand diagram, the point representing the number $1 + z + z^2 + z^3 + z^4$.

(O & C: SMP I, part of question)

10 If α and β are the roots of the quadratic equation

$$(1 + \mathrm{j})z^2 - 2\mathrm{j}z + (3 + \mathrm{j}) = 0$$

where $\mathrm{j} = \sqrt{(-1)}$, express each of $\alpha + \beta$ and $\alpha\beta$ in the form $a + \mathrm{j}b$, where a and b are real, and show, on an Argand diagram, the points representing the complex numbers $\alpha + \beta$ and $\alpha\beta$.

Find, in a form not involving α and β, the quadratic equation whose roots are $\alpha + 2\beta$ and $2\alpha + \beta$. (O & C: MEI)

Chapter 11

Matrices

Introduction

11.1 Readers who are already familiar with this topic are advised to check through §11.2 to §11.5 to ensure that their knowledge of the basic work is absolutely secure, while those to whom it is totally new may find it helpful to supplement the exercises in this book with further practice from a more elementary textbook.

A matrix is nothing more than a rectangular array of numbers. A matrix containing m rows and n columns is called an $\boldsymbol{m \times n}$ **matrix**. Matrices are often used to store information; the matrix **P**, below, records the sales of three books, labelled A, B and C, each of which is published in hardback and paperback form, on one particular day. (This is a 2×3 matrix.)

	A	B	C
Hardback	5	2	1
Paperback	10	7	4

This matrix tells us, for example, that, on the day in question, 7 copies of the paperback edition of book B were sold.

There are conventions in mathematics about the way matrices are written. Firstly, if the layout of the matrix, in a particular context, has been standardised, the labels of the rows and columns may be discarded. Secondly, the array of numbers should be enclosed in large round brackets (some writers use square brackets), and the letter used as the name of the matrix (**P** in the example above) should be printed in bold type (in manuscript it should be a capital letter with a wavy line underneath, i.e. $\utilde{P}$). So the matrix described in the preceding paragraph is written

$$\mathbf{P} = \begin{pmatrix} 5 & 2 & 1 \\ 10 & 7 & 4 \end{pmatrix}$$

The matrix **Q**, below, represents the sales of the same books on the following day:

$$\mathbf{Q}=\begin{pmatrix}3 & 0 & 1\\ 8 & 7 & 4\end{pmatrix}$$

On this day, for example, 3 copies of the hardback version of book *A* were sold.

One very common use for matrices in mathematics is to store the coordinates of points in coordinate geometry. In the example below, the first row of the 2×4 matrix **M** gives the x-coordinate and the second row gives the y-coordinate of four points, A, B, C and D, in order.

$$\mathbf{M}=\begin{pmatrix}0 & -3 & \frac{1}{2} & 5\\ 1 & 2 & -1 & 4\end{pmatrix}$$

This matrix tells us that A is the point (0, 1), B is $(-3, 2)$, C is $(\frac{1}{2}, -1)$ and D is (5, 4). Unlike the previous example, the entries in this matrix do not have to be whole numbers. In general, the elements in a matrix can be any real numbers (in more advanced work, even complex numbers may be used).

Matrix addition

11.2 In the last section, we used **P** and **Q** to represent the sales of books on two consecutive days. If the book shop owner wishes to know the number of books sold on the two days taken together, all he has to do is to add the corresponding elements, i.e. the numbers which appear in the corresponding positions in the two matrices. If he is good at arithmetic, he should obtain

$$\begin{pmatrix}8 & 2 & 2\\ 18 & 14 & 8\end{pmatrix}$$

It is natural to call the matrix obtained in this way the sum of **P** and **Q**, and so we write

$$\mathbf{P}+\mathbf{Q}=\begin{pmatrix}8 & 2 & 2\\ 18 & 14 & 8\end{pmatrix}$$

The difference of **P** and **Q** is obtained in a similar fashion:

$$\mathbf{P}-\mathbf{Q}=\begin{pmatrix}2 & 2 & 0\\ 2 & 0 & 0\end{pmatrix}$$

What meaning could the bookseller attach to this matrix?

In the preceding paragraphs we have described **P** and **Q** as '2×3 matrices' and **M** as 'a 2×4 matrix'. This was because **P** and **Q** each had two rows and three columns; **M**, on the other hand, had two rows and four columns. A matrix which has m rows and n columns is called an $m \times n$ matrix and we say that the **order** of the matrix is $m \times n$. It is only possible to add (or subtract) matrices which have the same order, i.e. they must each have the same number of rows and the same number of columns. If $m = n$, that is, the number of rows equals the number of columns, the matrix is called a **square** matrix.

Example 1 *Find* **A** + **B** *and* **A** − **B** *when*

(a) $\mathbf{A} = \begin{pmatrix} 3 & 5 & \frac{1}{2} & 4 \\ 4 & -1 & 2 & 0 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} -1 & 4 & 0 & 3 \\ 0 & 2 & 1 & 5 \end{pmatrix},$

(b) $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 3 & -6 \\ 5 & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 0 & -6 \\ 6 & 7 \\ 3 & 0 \end{pmatrix}.$

(a) $$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 3-1 & 5+4 & \frac{1}{2}+0 & 4+3 \\ 4+0 & -1+2 & 2+1 & 0+5 \end{pmatrix} = \begin{pmatrix} 2 & 9 & \frac{1}{2} & 7 \\ 4 & 1 & 3 & 5 \end{pmatrix}$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 3+1 & 5-4 & \frac{1}{2}-0 & 4-3 \\ 4-0 & -1-2 & 2-1 & 0-5 \end{pmatrix} = \begin{pmatrix} 4 & 1 & \frac{1}{2} & 1 \\ 4 & -3 & 1 & -5 \end{pmatrix}$$

(b) $$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2+0 & 0-6 \\ 3+6 & -6+7 \\ 5+3 & 1+0 \end{pmatrix} = \begin{pmatrix} 2 & -6 \\ 9 & 1 \\ 8 & 1 \end{pmatrix}$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 2-0 & 0+6 \\ 3-6 & -6-7 \\ 5-3 & 1-0 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ -3 & -13 \\ 2 & 1 \end{pmatrix}$$

A matrix in which every element is zero is called a **zero matrix**. When a zero matrix is added to another matrix with the same number of rows and columns, that matrix will be unchanged:

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

The zero matrix, then, has the property $\mathbf{A} + \mathbf{0} = \mathbf{A}$, which is very similar to the way the number zero behaves in ordinary algebra. (When you write $\mathbf{0}$ for the zero matrix, do not forget to put the wavy line under it to distinguish it from the number zero.)

Multiplication by a scalar

11.3 If **M** is the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ then, proceeding as in the last section,

$$\mathbf{M} + \mathbf{M} + \mathbf{M} + \mathbf{M} + \mathbf{M} = \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix}$$

In ordinary algebra we reduce $x + x + x + x + x$ to $5x$ and it is natural to do the same in matrix algebra, and so we write

$$5\mathbf{M} = \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix}$$

In general, to multiply a matrix **A** by a real number k (often called a **scalar** in this context), we multiply each number, or element, in the matrix **A** by k. Two examples are given below to illustrate this:

$$k\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} ka & kb & kc \\ kd & ke & kf \end{pmatrix}$$

and

$$x\begin{pmatrix} x+y & x-y \\ 2x & 3y \end{pmatrix} = \begin{pmatrix} x^2+xy & x^2-xy \\ 2x^2 & 3xy \end{pmatrix}$$

Qu. 1 Given that $\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, find $5\mathbf{A} + 4\mathbf{B}$.

Matrix multiplication

11.4 Returning to the illustration of the book sales in §11.2, suppose the matrix **S**, below, represents the total sales of the hardback books in one week,

$$\mathbf{S} = (20 \quad 25 \quad 10)$$

and, let us suppose the prices of the three books are £5, £6 and £7, respectively, then the total value of the books sold is

$$£(20 \times 5 + 25 \times 6 + 10 \times 7) = £320$$

Now, in any logical game, whether it is mathematics or chess or any similar intellectual pastime, it is necessary to define the basic rules of the game and adhere to them rigidly. (If we change the rule for moving a knight on a chessboard, we might have invented an interesting new game, but it is no longer chess!) In matrix algebra, the rule for multiplying matrices is very complicated and it requires care and patience to learn it and apply it accurately. In its simplest form, the rule for multiplying a single row by a single column, each containing the same number of elements can be expressed as follows:

$$(a \quad b \quad c \quad d)\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = (ap + bq + cr + ds)$$

If there are more than four elements, just continue to multiply each element of the row by the corresponding element in the column and add the product to the total. Notice that the result of the operation is a 1×1 matrix, that is, it is a single number (but it is still a matrix, so do not leave out the brackets).

The illustration of the book sales, above, can be expressed in matrix algebra as follows.

The *sales* are represented by the 1×3 matrix **S**, above, the *prices* are shown in a 3×1 column matrix **P**, where $\mathbf{P} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$ and the total value of the books sold is found by evaluating the matrix product **SP**.

$$\mathbf{SP} = (20 \quad 25 \quad 10)\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$$

$$= (100 + 150 + 70)$$

$$= (320)$$

Now suppose the sales of the same books in the following week are represented by the matrix **R**, where $\mathbf{R} = (30 \quad 15 \quad 5)$, then the value of the total sales in the second week is given by the matrix product **RP**.

$$\mathbf{RP} = (30 \quad 15 \quad 5)\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$$

$$= (150 + 90 + 35)$$

$$= (275)$$

We can combine these two sets of figures into a single matrix product, namely,

$$\begin{pmatrix} 20 & 25 & 10 \\ 30 & 15 & 5 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 320 \\ 275 \end{pmatrix}$$

When we read this, it must be clearly understood that the first row of the first matrix and the first row of the product represent the first week's figures and the second row in each case represents the second week's figures.

Let us now suppose that our bookseller discovered that the price list he had been using was out of date and the prices he should have been charging were £5.50, £6.50 and £7.50. He would, of course, want to know how much he should have got for his two weeks' sales. Proceeding as before, he would calculate the matrix product:

$$\begin{pmatrix} 20 & 25 & 10 \\ 30 & 15 & 5 \end{pmatrix} \begin{pmatrix} 5.50 \\ 6.50 \\ 7.50 \end{pmatrix} = \begin{pmatrix} 347.50 \\ 300.00 \end{pmatrix}$$

He could go a stage further and display both sets of figures side by side. Here, it must be understood, the second column of the price matrix corresponds to the second column of the product.

$$\begin{pmatrix} 20 & 25 & 10 \\ 30 & 15 & 5 \end{pmatrix} \begin{pmatrix} 5 & 5.50 \\ 6 & 6.50 \\ 7 & 7.50 \end{pmatrix} = \begin{pmatrix} 320 & 347.50 \\ 275 & 300.00 \end{pmatrix}$$

It is unlikely that there are many booksellers who bother to learn matrix algebra in order to do their accounts! Nevertheless this example will, it is hoped, serve to introduce the multiplication of matrices, which is absolutely fundamental in the study of matrix algebra. Matrix algebra was the brain-child of a Cambridge mathematician, Arthur Cayley (1821–1895). Cayley produced a paper on the subject in 1858; at the time he was working on the theory of transformations (see §11.6). The study of matrices has been one of the most significant factors in the development of mathematics in the twentieth century. Although it originated as a branch of pure mathematics it has turned out to be an extremely useful subject and today it is extensively used in applied mathematics and physics.

Let us now take another look at matrix multiplication. Here we multiply a 3×2 matrix $\mathbf{A}$ by a 2×1 matrix $\mathbf{B}$. (Notice that, for multiplication to be possible, it is essential that the number of *columns* in the *first* matrix should be the same as the number of *rows* in the *second* matrix.) Remember to work across each row and down each column.

If $\quad \mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \quad$ and $\quad \mathbf{B} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \times 7 & + & 2 \times 8 \\ 3 \times 7 & + & 4 \times 8 \\ 5 \times 7 & + & 6 \times 8 \end{pmatrix}$$

$$= \begin{pmatrix} 23 \\ 53 \\ 83 \end{pmatrix}$$

Now we examine the product of a 3×2 matrix **P** and a 2×2 matrix **Q**, bearing in mind that in picking out the pairs of corresponding elements for multiplying together, we work across each row of **P** and down each column of **Q**.

If $\mathbf{P}=\begin{pmatrix}1 & 2\\ 3 & 4\\ \mathbf{5} & \mathbf{6}\end{pmatrix}$ and $\mathbf{Q}=\begin{pmatrix}\mathbf{7} & 8\\ \mathbf{9} & 0\end{pmatrix}$

$$\mathbf{PQ}=\begin{pmatrix}1\times 7 + 2\times 9 & 1\times 8 + 2\times 0\\ 3\times 7 + 4\times 9 & 3\times 8 + 4\times 0\\ \mathbf{5\times 7} + \mathbf{6\times 9} & 5\times 8 + 6\times 0\end{pmatrix}$$

$$=\begin{pmatrix}25 & 8\\ 57 & 24\\ \mathbf{89} & 40\end{pmatrix}$$

It should be noted (a) that for each row of matrix **P** there is a row in the product **PQ**, and that for each column of matrix **Q** there is a column in the product **PQ**, and (b) that, for example, the element 89 in the third row and first column of **PQ** is the sum of the products of the corresponding elements of the third row of **P** and the first column of **Q**.

We can now set out the following general features of matrix multiplication:

(1) In any matrix product **CD**, if the first matrix **C** has m rows and n columns and the second matrix **D** has n rows and p columns, then the product **CD** has m rows and p columns.

(2) The element which lies in the ith row and jth column of **CD** is the sum of the products of the corresponding elements of the ith row of **C** and the jth column of **D**.

Example 2 *Find, where possible, the products* **PQ** *and* **MN**, *given that*

(a) $\mathbf{P}=\begin{pmatrix}2 & 3 & 4\\ 1 & 5 & 2\end{pmatrix}$, $\mathbf{Q}=\begin{pmatrix}1 & -1\\ 0 & 2\\ 1 & 3\end{pmatrix}$,

(b) $\mathbf{M}=\begin{pmatrix}2 & 1 & 0\\ 3 & 4 & 7\end{pmatrix}$, $\mathbf{N}=\begin{pmatrix}3 & 1\\ 4 & 5\end{pmatrix}$.

(a) $\mathbf{PQ}=\begin{pmatrix}2 & 3 & 4\\ 1 & 5 & 2\end{pmatrix}\begin{pmatrix}1 & -1\\ 0 & 2\\ 1 & 3\end{pmatrix}$

$$=\begin{pmatrix}2\times 1+3\times 0+4\times 1 & 2\times(-1)+3\times 2+4\times 3\\ 1\times 1+5\times 0+2\times 1 & 1\times(-1)+5\times 2+2\times 3\end{pmatrix}$$

$$=\begin{pmatrix}6 & 16\\ 3 & 15\end{pmatrix}$$

(b) It is impossible to form the product **MN**, because **M** has three columns, while **N** has only two rows.

Qu. 2 Find the following matrix products:

(a) $\begin{pmatrix} 3 & 1 \\ 2 & 0 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & 5 & 6 \\ 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$,

(c) $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 3 & 1 \\ 4 & 2 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$.

In the algebra of real numbers, the order of the terms in a product does not matter, for instance 3×5 and 5×3 both equal 15. We say that in the algebra of real numbers multiplication is *commutative*, that is, $ab = ba$ for any pair of real numbers. This is not the case in matrix algebra. For example, if $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$, then

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

but

$$\mathbf{BA} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 23 & 34 \\ 31 & 46 \end{pmatrix}$$

So in matrix algebra, the order of the matrices in a product *does* matter. We say that in matrix algebra, multiplication is *not commutative.*

Exercise 11a

1 Given that $\mathbf{A} = \begin{pmatrix} 3 & 1 & 2 \\ 5 & 1 & 7 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & -1 & 2 \\ 3 & 1 & 3 \end{pmatrix}$ evaluate:

(a) $3\mathbf{A}$, (b) $2\mathbf{B}$, (c) $3\mathbf{A} + 2\mathbf{B}$, (d) $3\mathbf{A} - 2\mathbf{B}$.

2 A newspaper agent records the number of papers sold on each day of one week, as follows:

	Mon	Tue	Wed	Thu	Fri	Sat
The Post	120	250	350	300	420	200
The News	120	300	420	200	300	500

Write this as a 2×6 matrix **S**.

The Post costs 12p and *The News* costs 15p. Write this information as a 1×2 row matrix **P**. It is only possible to form *one* of the products **PS** and **SP**. Evaluate the product which it is possible to form and explain the meaning of the first element in the product matrix.

3 Find, where possible, the following products. When it is not possible to form the product, state this clearly and give the reason for your conclusion.

(a) $(2 \quad 3 \quad 1)\begin{pmatrix}3 & 2\\4 & 7\\1 & 6\end{pmatrix}$, (b) $\begin{pmatrix}2 & 3\\1 & 4\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\end{pmatrix}$,

(c) $\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}(5 \quad 6)$, (d) $\begin{pmatrix}2 & 3\\4 & 5\end{pmatrix}\begin{pmatrix}7\\8\end{pmatrix}$.

4 Given that $\mathbf{A}=\begin{pmatrix}3 & -1\\4 & 5\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}\frac{1}{2} & 0\\1 & 2\end{pmatrix}$, find **AB** and **BA**. State the property of matrix multiplication which is illustrated by the answer.

5 Given that $\mathbf{I}=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ and $\mathbf{A}=\begin{pmatrix}a & b\\c & d\end{pmatrix}$, find **IA** and **AI**. In the algebra of real numbers there is a number which has a property which is very similar to the property shown by **I** in this question. State the number and describe this property.

6 Given that $\mathbf{A}=\begin{pmatrix}3 & 2\\5 & 4\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}4 & -2\\-5 & 3\end{pmatrix}$, find **AB** and **BA**.

7 Given that $\mathbf{A}=\begin{pmatrix}3 & 2\\5 & 4\end{pmatrix}$, $\mathbf{X}=\begin{pmatrix}x\\y\end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix}1\\-4\end{pmatrix}$, use the result of No. 6 to solve the matrix equation $\mathbf{AX}=\mathbf{C}$.
[Hint: multiply both sides of the equation by **B**.]

8 Repeat No. 6, given that $\mathbf{A}=\begin{pmatrix}5 & 2\\8 & 4\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}4 & -2\\-8 & 5\end{pmatrix}$.

9 Repeat No. 7, given that $\mathbf{A}=\begin{pmatrix}5 & 2\\8 & 4\end{pmatrix}$, $\mathbf{X}=\begin{pmatrix}x\\y\end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix}1\\-4\end{pmatrix}$.

10 Evaluate the matrix products:

(a) $\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\\1 & 0 & -1\end{pmatrix}\begin{pmatrix}2 & 1 & 0\\3 & -1 & 4\\1 & 0 & 7\end{pmatrix}$, (b) $\begin{pmatrix}2 & \frac{1}{2} & \frac{3}{4}\\1 & 0 & \frac{1}{4}\\3 & 1 & 0\end{pmatrix}\begin{pmatrix}8 & -4\\0 & 12\\4 & 0\end{pmatrix}$.

11 Matrices **M** and **N** are members of a set S which is defined as follows:

$$S=\left\{\begin{pmatrix}a & b\\-b & a\end{pmatrix}: a, b \in \mathbb{R}\right\}$$

Prove that the product **MN** is also a member of set S.

$\left[\text{Hint: let } \mathbf{M}=\begin{pmatrix}p & q\\-q & p\end{pmatrix} \text{ and } \mathbf{N}=\begin{pmatrix}r & s\\-s & r\end{pmatrix}.\right]$

12 Matrices **P** and **Q** are members of a set R which is defined as follows:

$$R=\left\{\begin{pmatrix}a & b\\c & d\end{pmatrix}: a,b,c,d \in \mathbb{R},\ ad-bc=1\right\}$$

Prove that the product **PQ** is also a member of set R.

13 S is the set of matrices of the form $\begin{pmatrix} a & -kb \\ b/k & a \end{pmatrix}$ where a and b can be any real numbers, but k is the *same* real number for *all* members of S. If **A** and **B** are two distinct members of set S, show that the product **AB** also belongs to set S.

14 Given that $\mathbf{P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and that $\mathbf{Q} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, evaluate the products **PQ** and **QP**. Comment on your answers.

Matrix algebra

11.5 The rules for adding and subtracting a pair of $m \times n$ matrices, which were introduced in §11.3, are very simple and unremarkable. The reader should have no difficulty convincing himself that, if **A** and **B** are a pair of such matrices,

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

so *matrix addition is commutative.* Also if **0** is the $m \times n$ zero matrix, then

$$\mathbf{A} + \mathbf{0} = \mathbf{A}$$

If **C** is another $m \times n$ matrix, then it follows from the associative property of real numbers under addition that

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$$

The technical term for this is that *matrix addition is associative.* (This terminology may be new to some readers. All it means is that the position of the brackets does not matter; and if this remark seems trivial, contrast it with $(24 \div 12) \div 2$ which does *not* equal $24 \div (12 \div 2)$. Division is *not* an associative operation in real numbers.)

Multiplication of matrices, which was introduced in §11.4, is a more complicated operation and, as a result, the rules of matrix multiplication are more interesting. We have already seen that it is possible to have a pair of matrices **A** and **B**, for which $\mathbf{AB} \neq \mathbf{BA}$, so *matrix multiplication is not commutative*.

We have also seen (Exercise 11a, No. 5) that if **A** is any 2×2 matrix and if $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$. This is very similar to the way the real number 1 behaves in ordinary algebra, that is, $1 \times x = x \times 1 = x$, where x is any real number. This matrix **I** is plainly a very special matrix and so it is given a special name; it is usually called the **unit matrix** (in recognition of its similarity to the number 1) or the **identity matrix**. More generally, if **A** is any $n \times n$ matrix, then the corresponding unit matrix is an $n \times n$ matrix, with 1's along the **leading diagonal** (the one that goes from the top left-hand corner to the bottom right-hand corner), and 0's elsewhere.

So the 3×3 unit matrix is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Qu. 3 If $\mathbf{A}=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$ and **I** is the 3×3 unit matrix, verify that $\mathbf{AI}=\mathbf{IA}=\mathbf{A}$.

In ordinary algebra, if we have a pair of numbers p and q such that $pq=1$ (for example $4\times\frac{1}{4}=1$) we say that q is the **inverse** of p, and conversely p is the inverse of q. (Similarly $\frac{1}{2}$ is the inverse of 2; 3/5 is the inverse of 5/3.) The same term is used in matrix algebra to describe a pair of matrices **A** and **B** such that $\mathbf{AB}=\mathbf{BA}=\mathbf{I}$. We say that **A** is the inverse of **B** and **B** is the inverse of **A**. For such a statement to be possible, both **A** and **B** must be square matrices which have the same number of rows and columns as each other. (If this is not obvious, write down a pair of matrices for which it is not true and try to evaluate both **AB** and **BA**.)

If we are given any square matrix **A**, the task of finding its inverse can be very difficult. In this section we shall tackle the simplest case, where **A** is a 2×2 matrix.

Suppose we are given a 2×2 matrix $\mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The problem is to find a 2×2 matrix **B**, such that $\mathbf{AB}=\mathbf{I}$. Let us write **B** as $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$. (In the work that follows, remember that a, b, c and d are known, but p, q, r and s are unknown; the task is to find p, q, r and s.)

$$\mathbf{AB}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}=\begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}$$

This product is to be equal to the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so we can write down four equations

$$ap+br=1 \qquad (1)$$
$$cp+dr=0 \qquad (2)$$
$$aq+bs=0 \qquad (3)$$
$$cq+ds=1 \qquad (4)$$

from which to find p, q, r and s.

Multiplying (1) by d and (2) by b, we have

$$adp+bdr=d$$
$$bcp+bdr=0$$

Subtracting,

$$(ad-bc)p=d$$

Provided $ad-bc$ is not zero we may divide by it, hence

$$p=\frac{d}{\Delta} \qquad (\text{where } \Delta=ad-bc)$$

Substituting this in equation (2) gives

$$\frac{cd}{\Delta} + dr = 0$$

$$\therefore r = -\frac{c}{\Delta}$$

The reader should now solve equations (3) and (4) to find q and s. The solutions are $q = -b/\Delta$ and $s = a/\Delta$.

Hence the inverse matrix **B** is given by

$$\mathbf{B} = \begin{pmatrix} d/\Delta & -b/\Delta \\ -c/\Delta & a/\Delta \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

This is the required inverse of the matrix **A** and the standard abbreviation of this is $\mathbf{A}^{-1}$. Consequently we write:

$$\text{if } \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ then } \mathbf{A}^{-1} = \frac{\mathbf{1}}{\boldsymbol{\Delta}}\begin{pmatrix} \boldsymbol{d} & \boldsymbol{-b} \\ \boldsymbol{-c} & \boldsymbol{a} \end{pmatrix}.$$

This is an important result and every effort should be made to memorise it.

The method for finding the inverse of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be summarised as follows:

> the elements on the leading diagonal, a and d, are interchanged, the elements on the other diagonal, b and c, have their signs changed, and the matrix is divided by $ad - bc$.

Qu. 4 Using the matrices **A** and $\mathbf{A}^{-1}$ above, verify that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

Notice that in finding the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the term $\Delta = ad - bc$ has a very important role to play. We shall be referring to this term quite frequently and so it is given a special name; it is called the **determinant** of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

It is convenient to reduce the phrase 'the determinant of matrix **M**' to det **M**, so if $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then we write $\det \mathbf{M} = ad - bc$.

Matrices for which $\Delta = 0$ are often called **singular** matrices. A singular matrix has no inverse because we cannot divide by zero.

Example 3 *Given that* $\mathbf{M} = \begin{pmatrix} 7 & 9 \\ 5 & 7 \end{pmatrix}$, *write the simultaneous equations*

$$7x + 9y = 3$$
$$5x + 7y = 1$$

in the form $\mathbf{MX} = \mathbf{C}$, *where* $\mathbf{X}$ *is the column matrix* $\begin{pmatrix} x \\ y \end{pmatrix}$ *and* $\mathbf{C}$ *is the column matrix* $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$. *Hence solve the equations.*

In matrix notation the equations can be expressed

$$\begin{pmatrix} 7 & 9 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

This is in the form

$$\mathbf{MX} = \mathbf{C}$$

as required. Multiply both sides of this matrix equation by $\mathbf{M}^{-1}$ and we have

$$\mathbf{M}^{-1}(\mathbf{MX}) = \mathbf{M}^{-1}\mathbf{C} \tag{1}$$

Now the left-hand side of this equation can be simplified, as follows:

$$\mathbf{M}^{-1}(\mathbf{MX}) = (\mathbf{M}^{-1}\mathbf{M})\mathbf{X}$$

using the associative property of matrix multiplication, and

$$(\mathbf{M}^{-1}\mathbf{M})\mathbf{X} = \mathbf{IX} = \mathbf{X}$$

using the properties of the inverse and identity matrices.

Equation (1) can now be reduced to

$$\begin{aligned} \mathbf{X} &= \mathbf{M}^{-1}\mathbf{C} \\ &= \frac{1}{4}\begin{pmatrix} 7 & -9 \\ -5 & 7 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 12 \\ -8 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -2 \end{pmatrix} \end{aligned}$$

Hence $x = 3$ and $y = -2$.

As a method for solving a pair of simultaneous equations, this is using a sledge-hammer to crack a nut. Nevertheless, it is a method which can be developed for tackling the more general problem of solving n simultaneous equations in n unknowns. It also gives an example of the way the basic properties of matrix algebra can be combined into a logical argument.

Exercise 11b (Oral)

Find the determinants of the following matrices:

1 (a) $\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$, (b) $\begin{pmatrix} 3 & 2 \\ 5 & 8 \end{pmatrix}$, (c) $\begin{pmatrix} 4 & -2 \\ 1 & 7 \end{pmatrix}$, (d) $\begin{pmatrix} 6 & \frac{1}{4} \\ 8 & \frac{1}{2} \end{pmatrix}$.

2 (a) $\begin{pmatrix} 3 & 12 \\ 2 & 8 \end{pmatrix}$, (b) $\begin{pmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{6} & \frac{1}{3} \end{pmatrix}$, (c) $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, (d) $\begin{pmatrix} a & b/k \\ ck & d \end{pmatrix}$, $k \neq 0$.

3 State which of these matrices are singular:

(a) $\begin{pmatrix} 3 & -2 \\ 9 & 6 \end{pmatrix}$, (b) $\begin{pmatrix} 3 & 2 \\ 9 & 6 \end{pmatrix}$, (c) $\begin{pmatrix} 34 & 119 \\ 26 & 91 \end{pmatrix}$, (d) $\begin{pmatrix} x & x^2 \\ x^2 & x^3 \end{pmatrix}$.

4 Find the values of x for which the following matrices have no inverse:

(a) $\begin{pmatrix} x & 7 \\ 8 & 2 \end{pmatrix}$, (b) $\begin{pmatrix} x & 8 \\ 2 & x \end{pmatrix}$, (c) $\begin{pmatrix} x-2 & 1 \\ 2 & x-3 \end{pmatrix}$, (d) $\begin{pmatrix} x & 2 \\ -2 & x \end{pmatrix}$.

5 State the inverse of each of these matrices (read each column in turn):

(a) $\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$, (c) $\begin{pmatrix} 6 & 11 \\ 2 & 7 \end{pmatrix}$, (d) $\begin{pmatrix} x & -1 \\ 1 & x \end{pmatrix}$.

Exercise 11c

1 Find, where possible, the inverses of the following matrices:

(a) $\begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}$, (b) $\begin{pmatrix} 8 & 2 \\ 11 & 3 \end{pmatrix}$, (c) $\begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 1 \end{pmatrix}$, (d) $\begin{pmatrix} 6 & 3 \\ 8 & 4 \end{pmatrix}$.

2 Find the inverses of the following matrices:

(a) $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$, (b) $\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$,

(c) $\begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}$, (d) $\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$.

3 Find the inverse of the matrix **M**, where $\mathbf{M} = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$ and hence solve the matrix equation $\mathbf{MX} = \mathbf{C}$, in which $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

4 Repeat No. 3 for $\mathbf{M} = \begin{pmatrix} 9 & 2 \\ 8 & 4 \end{pmatrix}$.

5 Write the simultaneous equations

$$7x + 9y = 1$$
$$10x + 13y = 2$$

in matrix form, and, using the method employed in Nos. 3 and 4, solve the equations.

6 Solve the matrix equation $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A} = \begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ to find the (unknown) matrix **X**.

7 Given that $\mathbf{P}=\begin{pmatrix}3 & 4\\ -4 & 3\end{pmatrix}$ and $\mathbf{A}=\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$, find the matrix $\mathbf{M}$, where $\mathbf{M}=\mathbf{P}^{-1}\mathbf{AP}$. Hence, or otherwise, find $\mathbf{M}^5$.

8 By writing $\mathbf{M}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and $\mathbf{N}=\begin{pmatrix}p & q\\ r & s\end{pmatrix}$, prove that, for any two 2×2 matrices $\mathbf{M}$ and $\mathbf{N}$, $\det \mathbf{MN}=\det \mathbf{M} \det \mathbf{N}$.

9 Verify that if $\mathbf{M}=\begin{pmatrix}-5 & 10 & 8\\ 4 & -7 & -6\\ -3 & 6 & 5\end{pmatrix}$ and $\mathbf{N}=\begin{pmatrix}-1 & 2 & 4\\ 2 & 1 & -2\\ -3 & 0 & 5\end{pmatrix}$, then $\mathbf{MN}=\mathbf{NM}=\mathbf{I}$, where $\mathbf{I}$ is the 3×3 unit matrix. Use this to solve the matrix equation

$$\begin{pmatrix}-5 & 10 & 8\\ 4 & -7 & -6\\ -3 & 6 & 5\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}-3\\ 3\\ 2\end{pmatrix}$$

10 Express the simultaneous equations

$$\begin{aligned}-x+2y+4z&=7\\ 2x+y-2z&=-2\\ -3x+5z&=7\end{aligned}$$

in the form of a matrix equation $\mathbf{NX}=\mathbf{C}$, where $\mathbf{N}$ is the 3×3 matrix in No. 9 and $\mathbf{X}$ and $\mathbf{C}$ are suitable column matrices. Hence, using the information from No. 9, solve these equations by the matrix method.

11 Given that $\mathbf{A}=\begin{pmatrix}2 & 1 & 0\\ 0 & 1 & 1\\ 0 & 4 & -3\end{pmatrix}$, verify that $\mathbf{A}^3=11\mathbf{A}-14\mathbf{I}$, where $\mathbf{I}$ is the 3×3 unit matrix. Hence find $\mathbf{A}^{-1}$.

12 Solve, by elimination, the simultaneous equations

$$\begin{aligned}2x+y&=a\\ y+z&=b\\ 4y-3z&=c\end{aligned}$$

in terms of a, b and c. Express the three simultaneous equations in the form $\mathbf{AX}=\mathbf{C}$, where $\mathbf{A}$ and $\mathbf{C}$ are suitably chosen matrices and $\mathbf{X}=\begin{pmatrix}x\\ y\\ z\end{pmatrix}$, and give your answer in the form $\mathbf{X}=\mathbf{BC}$. Hence write down the inverse of matrix $\mathbf{A}$.

Transformations and matrices

11.6 As mentioned earlier (§11.4), matrices were invented by Cayley in the course of his work on linear transformations. In this section we shall take a closer look at this topic. In two dimensions, a linear transformation is a transformation which moves any point P, with coordinates (x, y), to a new position P′, whose coordinates (x', y') are given by a pair of linear equations, that is equations of the form

$$x' = ax + by$$
$$y' = cx + dy$$

In matrix notation this can be written

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Example 4 *A transformation is defined by the matrix equation*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Draw a diagram showing the unit square OIRJ, *whose vertices are at* (0, 0), (1, 0), (1, 1) *and* (0, 1) *respectively, and its image* O′I′R′J′ *under the transformation. Describe in words the effect of the transformation on the unit square.*

$$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

It is worth noting that these four operations can be combined into a single one, in which the matrix $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ is applied to the 2×4 matrix $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, that is,

$$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & -2 & -2 \end{pmatrix}$$

From the diagram (Fig. 11.1) we can see that OIRJ has been enlarged by a scale-factor of 2 and it has been reflected in the x-axis.

If we are given a description, in words, of a certain transformation, it can be quite difficult to find the corresponding matrix, but in some simple cases the matrix can be found by considering the effect of the transformation on a triangle OPM, whose vertices are the points (0, 0), (x, y) and $(x, 0)$ respectively. It should be noted, at this stage, that the image of (0, 0) under this type of transformation is always (0, 0).

(a) Rotation, about O, through 90° anti-clockwise

From Fig. 11.2, we can see that the new y-coordinate is OM′ and that this is equal in length to OM (since OM′ is OM rotated through 90°) and OM is the

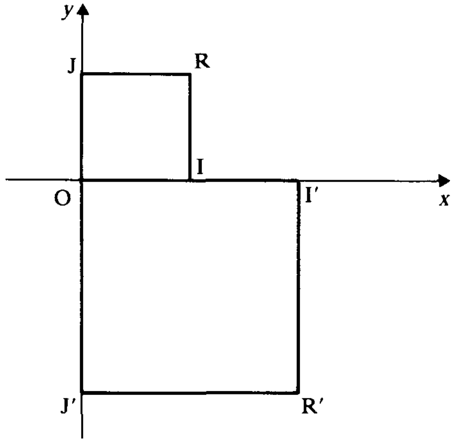


Figure 11.1

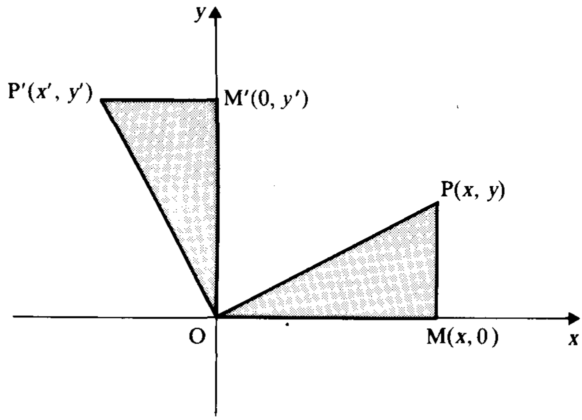


Figure 11.2

original x-coordinate, so $y' = x$. The new x-coordinate is equal to P′M′ in magnitude, but it is negative; however, P′M′ is equal in length to the original y-coordinate and so, $x' = -y$. Hence the new coordinates (x', y') are given by the pair of equations

$$\begin{aligned} x' &= \quad -y \\ y' &= x \end{aligned}$$

and these can be written in matrix form as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

In the next two cases the detailed explanation is omitted; the reader should make sure that he or she understands how the matrix equations are obtained from the diagram.

(b) Reflection in the *x*-axis (see Fig. 11.3)

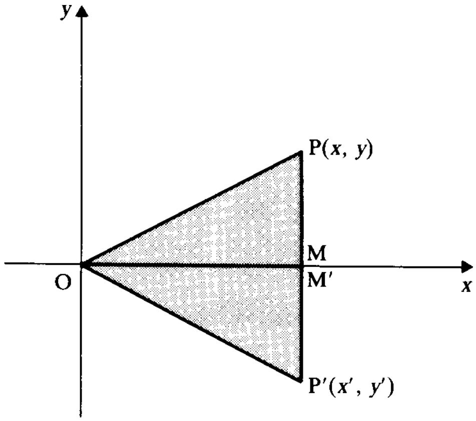


Figure 11.3

$$x' = x$$
$$y' = -y$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(c) Reflection in the line $y = x$ (see Fig. 11.4)

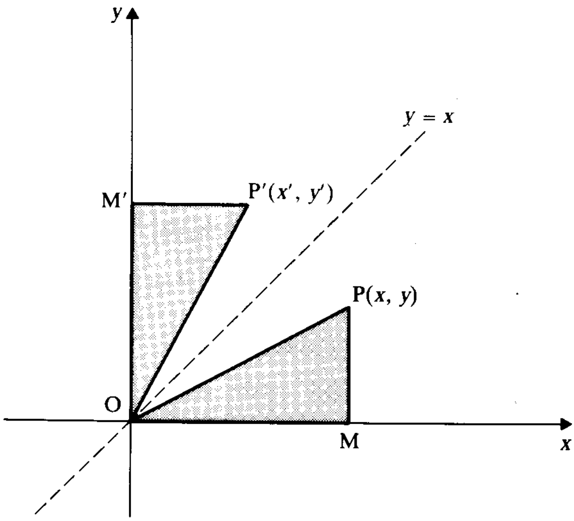


Figure 11.4

$$x' = y$$
$$y' = x$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Qu. 5 Find the matrices which correspond to the following transformations:
(a) a rotation about the origin, through 90°, clockwise,
(b) a reflection in the line $x + y = 0$,
(c) an enlargement by a factor of 5, with the origin as the centre of the enlargement.

General properties of linear transformations

11.7 In the last section we were able to look at some simple transformations and write down the corresponding matrices. Before we can tackle more complicated transformations, we must look more closely at the general properties of transformations which are defined by matrix equations of the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Where appropriate, the notation $(x, y) \mapsto (x', y')$ will be used to indicate that, under the transformation, the point (x, y) moves to the point (x', y'). It is the normal practice to say '(x', y') is the image of (x, y) under the transformation' and that '(x, y) is mapped onto the point (x', y')'.

The following four properties of such transformations are very important; the reader should make sure that they are understood before proceeding further.

(1) The image of (0, 0) is (0, 0)

We can see from the matrix product $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ that the image of (0, 0) is (0, 0), for all values of a, b, c and d. We say that the origin is *invariant* under any linear transformation; $(0, 0) \mapsto (0, 0)$.

(2) The images of (1, 0) and (0, 1) are (a, c) and (b, d) respectively

[Throughout this chapter the points (1, 0) and (0, 1) will be labelled I and J respectively; a similar convention is used in Chapter 15.]

As before we need only look at the matrix products

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

to see that $(1, 0) \mapsto (a, c)$ and $(0, 1) \mapsto (b, d)$.

This property is especially valuable because it means that, if we are given the description of a transformation, we only have to look at its effect on the unit square OIRJ, and in particular, the images of I and J, to find the values of a, b, c

and d. (At this stage the reader should look back at the transformations in §11.6 to confirm this.) Fig. 11.5 shows the unit square OIRJ and its image, for a general transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

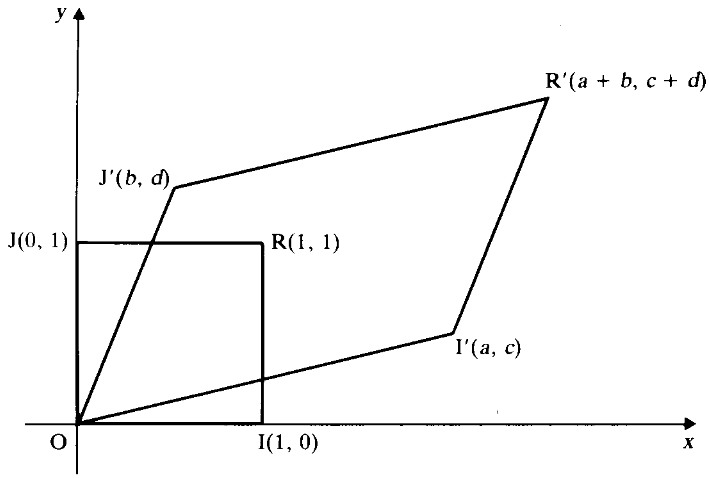


Figure 11.5

(3) The area of the parallelogram OI′R′J′ is ($ad - bc$)

This is left as an exercise for the reader. It can be proved fairly easily if the parallelogram is 'framed' in a rectangle which has O and R′ as a pair of diagonally opposite vertices. The region surrounding the parallelogram should then be dissected into suitable rectangles and right-angled triangles.

Notice that $(ad - bc)$ is Δ, the determinant of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Notice also that it is possible for $(ad - bc)$ to be negative. This will happen when the unit square is 'turned inside-out', as in a reflection.

(4) Any set of parallel lines is transformed into a set of lines which are also parallel to one another

Let the original set of lines have equations of the form $y = mx + k$, where m is constant, thereby ensuring that the lines in the original set all have the same gradient, i.e., they are parallel to one another. We shall show that these are transformed into a set of lines whose gradient does not depend upon the value of k, i.e. the gradient is the same for any line from the original set of lines.

The new coordinates (x', y') are given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Solving this equation, as in §11.5, Example 3, we obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

where $\Delta = ad - bc$. Hence,

$$x = \frac{dx' - by'}{\Delta} \qquad y = \frac{-cx' + ay'}{\Delta}$$

Now (x, y) is a point on the line $y = mx + k$, and consequently its coordinates satisfy this equation. Substituting for x and y we find

$$\frac{-cx' + ay'}{\Delta} = \frac{m(dx' - by')}{\Delta} + k$$

$$-cx' + ay' = mdx' - mby' + k\Delta$$

$$(a + bm)y' = (c + dm)x' + k\Delta$$

so the coordinates (x', y') of P′ satisfy the equation

$$(a + bm)y = (c + dm)x + k\Delta$$

This is the equation of a straight line and its gradient, $(c + dm)/(a + bm)$, does not depend on k. Consequently all members of the original set of lines are transformed into another set of lines, all of which have the same gradient as each other, namely $(c + dm)/(a + bm)$.

In Fig. 11.6 the first diagram shows the original plane with a set of equally spaced lines parallel to the x-axis and another set parallel to the y-axis. The second diagram shows these two sets of lines after the transformation. The unit square is labelled OI_1RJ_1 in the first diagram and its image $OI'_1R'J'_1$ appears in the second.

Notice that each little square in the original diagram has an area of one square unit and that each of these is transformed into a parallelogram whose area is $(ad - bc)$. Consequently any region in the original diagram will be transformed into a region whose area is $(ad - bc)$ times greater than the area of the original region.

Example 5 *A linear transformation is defined by*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Find the images of $(1, 0)$ *and* $(0, 1)$ *and find the factor by which areas are increased by the transformation. Find also the point whose image is* $(4, 6)$.

The image of $(1, 0)$ is given by the first column of the matrix. Hence $(1, 0) \mapsto (3, 5)$. The image of $(0, 1)$ is given by the second column. Hence $(0, 1) \mapsto (2, 4)$. The area is increased by a factor equal to the determinant, i.e. $(3 \times 4 - 2 \times 5) = 2$.

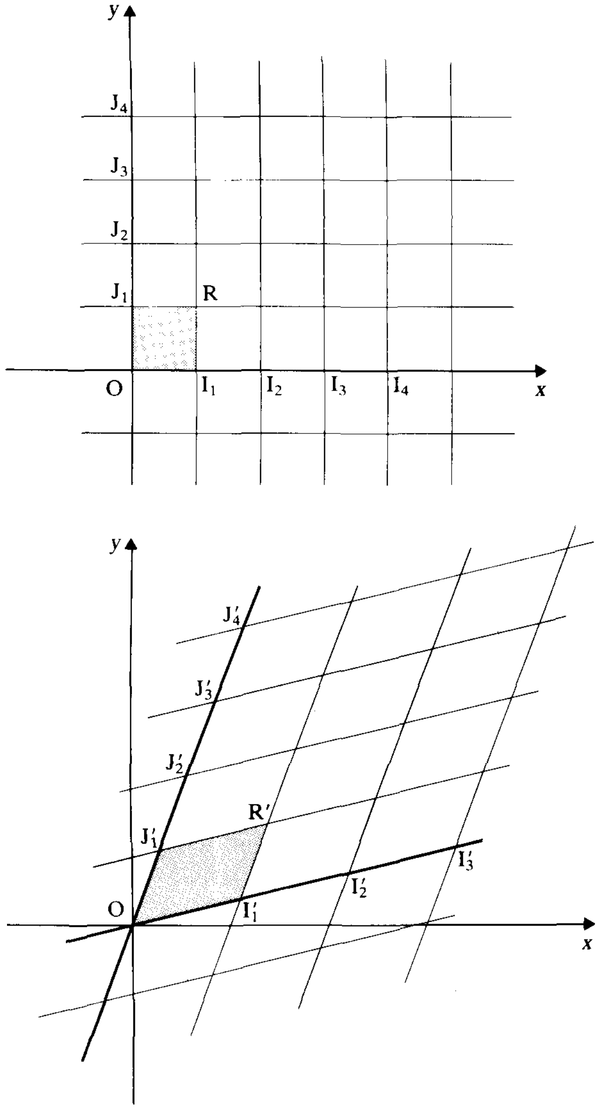


Figure 11.6

Let the point (4, 6) be the image of (x, y), then

$$\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

Multiplying both sides of this equation by the inverse matrix, we obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 6 \end{pmatrix} \qquad \text{where } \Delta = 12 - 10 = 2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Hence (4, 6) is the image of (2, −1).

Property (2), above, is especially useful because it enables us to write down, with very little working, the matrices which represent some common transformations, which we can add to our list (a), (b), (c) in §11.6.

(d) Rotation through an angle α about the origin

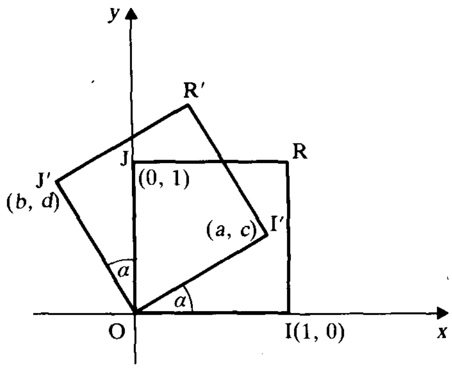


Figure 11.7

Since $OI' = 1$, we can see that $a = \cos\alpha$ and $c = \sin\alpha$. Also, since $OJ' = 1$, $b = -\sin\alpha$ and $d = \cos\alpha$. (See Fig. 11.7.) Hence the required matrix is

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

(e) Reflection in the line $y = mx$, where $m = \tan\alpha$

The required matrix is

$$\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$$

Proof of this is left to the reader; it is not difficult, provided a careful diagram is drawn.

(f) The transformation under which the unit square is mapped onto the parallelogram with vertices O, I′ (1, 0), R′ (3, 1) and J′ (2, 1)

(See Fig. 11.8; a transformation such as this is called a **shear**, parallel to the x-axis.)

Using the same method as before, the required matrix is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

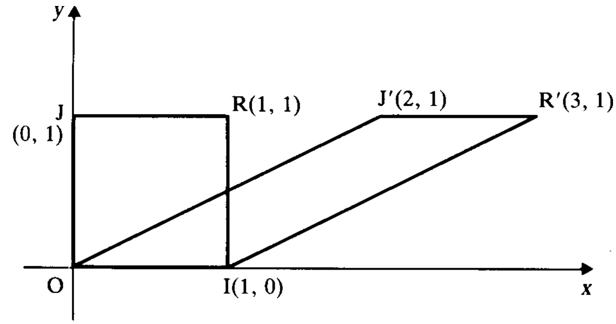


Figure 11.8

Qu. 6 Write down the matrix which represents the shear parallel to the y-axis, under which the unit square is mapped onto the parallelogram with vertices O, I′ (1, 5), R′ (1, 6) and J′ (0, 1).

It should be noticed that the same letter may be used to represent both the transformation and its corresponding matrix — indeed this causes less confusion than using two different letters. Thus we can say 'the transformation **E** is an enlargement with a scale factor k' and we can also say that the matrix representing this transformation is **E**, where $\mathbf{E} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$.

Composite transformations

11.8 Suppose that we have two transformations **P** and **Q**, which are given by the matrix equations

$$\mathbf{P}: \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \mathbf{Q}: \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and suppose that **P** is applied first, mapping (x, y) onto (x', y') and that **Q** is then applied, mapping (x', y') onto (x'', y'') i.e.

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

Then, substituting for $\begin{pmatrix} x' \\ y' \end{pmatrix}$ we obtain

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

So the matrix which represents the composite transformation 'do **P**, then do **Q**' is the matrix product

$$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

Notice that the matrix which represents the *first* transformation is the matrix on the *right* in this product. This composite matrix is always written **QP**. Remember that **P** is applied first and **Q** second. This may seem strange, but it is logical if we look at the way the matrix product, above, was formed. Notice also that it is the same convention as that used in forming composite functions (see §2.10).

Example 6 *Write down the matrices* **R** *and* **S**, *which represent a reflection in the line* $y = x$, *and a rotation through* 90°, *anti-clockwise about the origin, respectively. Find the matrix which represents the composite transformation* **SR** *and draw a diagram showing the unit square and its image under the transformation* **SR**. *Describe* **SR** *in words.*

$$\mathbf{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{(see §11.6)}$$

$$\mathbf{SR} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

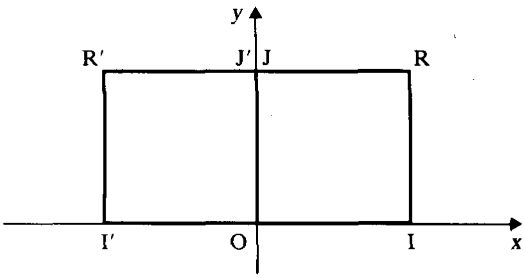


Figure 11.9

The transformation **SR** is a reflection in the line $x = 0$ (see Fig. 11.9).

Example 7 *Write down the matrices* **A** *and* **B** *which represent rotations about the origin, through angles* α *and* β, *respectively. Find the matrix which represents the transformation* **AB** *and describe this transformation in words. Write down another matrix which represents this transformation and hence find expressions, in terms of* $\sin\alpha$, $\cos\alpha$, $\sin\beta$ *and* $\cos\beta$, *for* $\sin(\alpha+\beta)$ *and* $\cos(\alpha+\beta)$.

$$\mathbf{A} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \qquad \text{(see §11.6).}$$

The composite transformation is given by the product

$$\mathbf{AB} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{pmatrix}$$

The composite transformation is a rotation through an angle β followed by a rotation through an angle α: this can be simplified by replacing it by a single rotation through an angle $(\alpha+\beta)$. (In this particular case, the order of the transformations is immaterial; in other words the transformations are commutative.) The single rotation through an angle $(\alpha+\beta)$ can be represented by the matrix

$$\begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$$

Comparing this with the matrix **AB**, above, we see that

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{aligned}$$

Example 8 *Write down the matrix* **R** *which represents a reflection in the line* $y = mx$, *where* $m = \tan\alpha$. *Prove that* $\mathbf{R}^2 = \mathbf{I}$, *and hence write down the inverse of the matrix* **R**. *Verify that this agrees with the result obtained by using the normal method for finding* $\mathbf{R}^{-1}$ *(see §11.5).*

$$\mathbf{R} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$$

$$\begin{aligned} \therefore\ \mathbf{R}^2 &= \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \\ &= \begin{pmatrix} \cos^2 2\alpha + \sin^2 2\alpha & \cos 2\alpha \sin 2\alpha - \sin 2\alpha \cos 2\alpha \\ \cos 2\alpha \sin 2\alpha - \sin 2\alpha \cos 2\alpha & \cos^2 2\alpha + \sin^2 2\alpha \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \mathbf{I} \end{aligned}$$

Since $\mathbf{R}^2 = \mathbf{I}$, the inverse of **R** is **R** itself, so

$$\mathbf{R}^{-1} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$$

(This is not very surprising because we have reflected an object in a given line, and then reflected it again in the same line; this would return the object to its original position. In other words $\mathbf{R}^2$ leaves the object unchanged. Any matrix **M** with the property $\mathbf{M}^{-1} = \mathbf{M}$ is called a **self-inverse matrix**.)

The determinant of **R** is given by

$$\begin{aligned} \det \mathbf{R} &= -\cos^2 2\alpha - \sin^2 2\alpha \\ &= -(\cos^2 2\alpha + \sin^2 2\alpha) \\ &= -1 \end{aligned}$$

Hence, applying the method in §11.5 for inverting a matrix, we obtain

$$\mathbf{R}^{-1} = \frac{1}{-1}\begin{pmatrix} -\cos 2\alpha & -\sin 2\alpha \\ -\sin 2\alpha & +\cos 2\alpha \end{pmatrix}$$

$$\mathbf{R}^{-1} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$$
$$= \mathbf{R}$$

Exercise 11d

1 Describe the transformations represented by

(a) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, (b) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, (c) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

(d) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, (e) $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$.

2 A certain transformation is represented by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Draw a diagram showing the unit square and its image under this transformation. The triangle whose vertices are A(3, 2), B(7, 2) and C(6, 5) is mapped onto A′B′C′, by this transformation. Find the coordinates of A′, B′ and C′. Find also the areas of the triangles ABC and A′B′C′.

3 Two matrices **P** and **Q** are given below:

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \qquad \mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Find the product **QPQ** and describe the transformation it represents.

4 A circle, centre O, radius a, is subject to a transformation whose matrix is $\begin{pmatrix} 1 & 0 \\ 0 & b/a \end{pmatrix}$. Draw a diagram showing the circle and its image and write down the area inside each of the curves.

5 Write down the matrices which represent

(a) an anti-clockwise rotation, about the origin, through an acute angle whose sine is 3/5,

(b) an enlargement by a factor of 5, followed by a reflection in the line $y = x$.

6 Describe the transformation which is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

where a and b are real numbers. State the condition required if this matrix represents a pure rotation.

7 By considering the effect on the unit square, describe the transformation which is represented by the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Hence, or otherwise, find λ

and m such that

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ m \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ m \end{pmatrix}$$

expressing m in the form $\tan \alpha$. Hence prove that $\tan 22\frac{1}{2}^\circ = \sqrt{2} - 1$.

8 Show that $\mathbf{A} = \begin{pmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{pmatrix}$ is 'self-inverse', that is, $\mathbf{A}^2 = \mathbf{I}$, the unit matrix. Hence describe the transformation which $\mathbf{A}$ represents.

9 Write down the matrix which represents a reflection in the line $y = (\tan \alpha)x$. Hence show that a reflection in a line which is inclined at an angle α to the x-axis, followed by a reflection in a line which is inclined at an angle β to the x-axis, is equivalent to a reflection. State the angle which the mirror line of this reflection makes with the x-axis.

[You will need the formulae

$$\sin (P - Q) = \sin P \cos Q - \cos P \sin Q$$
$$\cos (P - Q) = \cos P \cos Q + \sin P \sin Q$$

See Chapter 17.]

10 State the transformation which is represented by the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and find the matrix $\mathbf{A}^2$. Describe the transformation represented by $\mathbf{A}^2$ and hence write down expressions for $\cos 2\theta$ and $\sin 2\theta$, in terms of $\cos\theta$ and $\sin\theta$.

Exercise 11e (Miscellaneous)

1 Find, where possible, the following products:

(a) $\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 4 \\ -7 \end{pmatrix}$, (b) $(3 \quad 1 \quad 2)\begin{pmatrix} 5 & 2 \\ 1 & 1 \\ 3 & 4 \end{pmatrix}$,

(c) $\begin{pmatrix} 3 & 4 \\ 7 & 2 \end{pmatrix}(1 \quad 5)$, (d) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.

2 Find, where possible, the inverses of the following matrices:

(a) $\begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}$, (b) $\begin{pmatrix} 5 & 3 \\ 6 & 4 \end{pmatrix}$, (c) $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$, (d) $\begin{pmatrix} 39 & 91 \\ 51 & 119 \end{pmatrix}$.

3 Find, where possible, the inverses of the following matrices:

(a) $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, (b) $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, (c) $\begin{pmatrix} 0 & a \\ 1/a & 0 \end{pmatrix}$, (d) $\begin{pmatrix} b & b \\ b & b \end{pmatrix}$.

4 Solve the following matrix equations:

(a) $\begin{pmatrix} 7 & 6 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, (b) $\begin{pmatrix} 5 & 3 \\ 8 & 5 \end{pmatrix}\begin{pmatrix} u & x \\ v & y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$.

5 Find the equation of the line onto which the line $x + y = 0$ is mapped by the transformation $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 5 & 12 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

6 The transformation $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ maps the triangle A(3, 2), B(7, 2), C(3, 8) onto the triangle A′B′C′. Find the coordinates of A′, B′ and C′ and calculate the area of the triangle A′B′C′.

7 Under a certain transformation, the image of the point (x, y) is (X, Y), where $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. This transformation maps any point on the line $y = mx$ onto another point on the line $y = mx$. Find the (two) possible values of m.

8 Under a certain transformation, the images of the points (1, 0) and (0, 1) are (3, 5) and (5, 9) respectively. Find the image of the point (2, −5) under the same transformation. Find also the point whose image is (8, 6) under this transformation.

9 Given that **A** is the matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{pmatrix}$ and **B** is the matrix $\begin{pmatrix} 1 & -2 & -1 \\ -3 & 2 & 1 \\ 1 & 0 & -1 \end{pmatrix}$, find the product **AB**. Hence write down $\mathbf{A}^{-1}$, the inverse of **A**.

10 As a result of market research, it is known that a per cent of the population buys *Soft* shampoo and b per cent does not, and that if the product is advertised on television for a week, these percentages change from **C** to **AC**, where $\mathbf{C} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} \frac{3}{4} & \frac{4}{5} \\ \frac{1}{4} & \frac{1}{5} \end{pmatrix}$. However, if it is not advertised for a week, **C** changes to **BC**, where $\mathbf{B} = \begin{pmatrix} \frac{1}{2} & \frac{3}{10} \\ \frac{1}{2} & \frac{7}{10} \end{pmatrix}$.

At the start of week 1, $a = 20$ and $b = 80$. Find the values of a and b two weeks later, if *Soft* shampoo is advertised

(a) in both weeks,

(b) in week 1, but not in week 2,

(c) in week 2, but not in week 1.

11 Given that z is the complex number $x + iy$ and that the matrix $\mathbf{A}(z)$ is defined as $\mathbf{A}(z) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, prove that

$$\mathbf{A}(z_1 z_2) = \mathbf{A}(z_1)\mathbf{A}(z_2)$$

12 By considering the effect on the unit square, or otherwise, write down the matrices **M** and **R** which represent a reflection in the line $x = y$, and a rotation about the origin through an angle θ, respectively. Find the matrix $\mathbf{M}^{-1}\mathbf{RM}$ and describe it in words.

Find also the matrix product $\mathbf{R}^{-1}\mathbf{MR}$ and, by considering the effect of $\mathbf{R}^{-1}\mathbf{MR}$ on the unit square, show that

$$\mathbf{R}^{-1}\mathbf{MR} = \begin{pmatrix} \sin 2\theta & \cos 2\theta \\ \cos 2\theta & -\sin 2\theta \end{pmatrix}$$

Hence write down expressions, in terms of $\cos\theta$ and $\sin\theta$, for $\cos 2\theta$ and $\sin 2\theta$.

13 The matrix **A** is $\begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 3 & -2 \end{pmatrix}$. Show that **A** satisfies the matrix equation $\mathbf{A}^3 = 13\mathbf{A} - 12\mathbf{I}$. Assuming that $\mathbf{A}^{-1}$ exists, show that this equation can be written $\mathbf{A}^{-1} = \frac{1}{12}(13\mathbf{I} - \mathbf{A}^2)$, and hence find $\mathbf{A}^{-1}$.

14 The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a,b,c,d \in \mathbb{R}$. Find $\mathbf{M}^2$.

Given that $\mathbf{M}^2 = \mathbf{M}$ and that b and c are non-zero, prove that **M** is singular. Prove also that, in this case, the transformation **T**, defined by

$$\mathbf{T}: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \mathbf{M}\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$$

maps all points of the plane to points of the line $(1 - a)x = by$. (C)

15 Given the matrix $\mathbf{M} = \frac{1}{13}\begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix}$, evaluate $\mathbf{M}^2$ and the determinant of **M**.

Find a set of matrices $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ and also a set of matrices $\begin{pmatrix} u \\ v \end{pmatrix}$ such that $\mathbf{M}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -u \\ -v \end{pmatrix}$.

Describe, in geometrical terms, the transformation represented by the matrix **M**. (JMB)

16 The transpose of a matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the matrix $\mathbf{M}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, and **M** is said to be orthogonal when $\mathbf{M}^T\mathbf{M} = \mathbf{I}$, where **I** is the unit matrix. Given that the matrix $\mathbf{N} = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & k \end{pmatrix}$ is orthogonal, find the value of k. Describe geometrically the transformation of the x–y plane which is represented by **N**.

Under a transformation **S** of the real plane into itself, a point $\mathrm{P} = (x, y)$ is mapped onto the point $\mathrm{S(P)} = (ax + by, cx + dy)$. Show that, when **M** is orthogonal, the distance between any two points P and Q is the same as the distance between their images S(P) and S(Q). (L)

17 A transformation **T** is represented by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & 1/b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \text{ where } b \in \mathbb{R}^+.$$

(a) Draw a diagram showing the unit square and its image under **T**.
(b) Show that area is invariant under **T**.
(c) Show that **T** maps the curve $y = 1/x$ onto itself.
(d) Show that **T** maps the region bounded by the curve $y = 1/x$, the lines $x = 1$ and $x = a$, and the x-axis, onto the region bounded by the same curve, the lines $x = b$ and $x = ab$, and the x-axis.
(e) Hence show that $\displaystyle\int_1^a \frac{1}{x}\,dx = \int_b^{ab} \frac{1}{x}\,dx$.
(f) Given that $\displaystyle F(t) = \int_1^t \frac{1}{x}\,dx$, show that

$$F(ab) = F(a) + F(b)$$

(The reader should note several interesting and significant points about this question. The integral in (e) cannot be evaluated by methods which have been introduced so far, and the result of (f) looks very much like a standard property of logarithms. We shall return to these points in Book 2.)

Chapter 12

Permutations and combinations

Arrangements

12.1 This chapter aims at teaching a method of approach to certain problems involving arrangements and selections. In the course of the work, a notation is introduced, and a formula is obtained for use in the proof of the binomial theorem (Chapter 14).

Example 1 *From a pack of playing cards, the Ace, King, Queen, Jack, and Ten of Spades are taken. In how many ways can three of these five cards be placed in a row from left to right?*

The first card can be any one of the five, viz.:

A; K; Q; J; 10.

When the first card has been placed, there are four cards left to choose from, and so the possible ways of placing the first two cards are:

A K,	A Q,	A J,	A 10;
K A,	K Q,	K J,	K 10;
Q A,	Q K,	Q J,	Q 10;
J A,	J K,	J Q,	J 10;
10 A,	10 K,	10 Q,	10 J.

Thus, for *each* of the 5 ways of choosing the first, there are 4 ways in which the second card may be chosen; therefore there are 5×4 (i.e. 20) ways of choosing the first two cards.

Now for each of the 20 ways of placing the first two cards, there are 3 cards left to choose from (e.g. if the first two cards were A K, the third could be Q, J, or 10); therefore there are 20×3 ways of placing the third card.

Thus, three cards chosen from the Ace, King, Queen, Jack, and Ten of Spades may be placed in a row from left to right in 60 different ways.

Example 2 *Three schools have teams of six or more runners in a cross-country race. In how many ways can the first six places be taken by the three schools, if there are no dead heats?*

First it should be made clear that there is no question of the individuality of the runners, but only which school each of the first six runners belongs to.

The first place can be taken by any of the 3 schools.

When the first runner has come in, the second place can be taken by any of the 3 schools, so the first two places can be taken in 3×3, or 3^2, ways.

Similarly, the third place can be taken by any of the 3 schools, so the first three places can be taken in $3^2 \times 3$, or 3^3, ways.

Continuing the argument for the fourth, fifth and sixth places, it follows that the first six places may be taken in 3^6, or 729, ways by the three schools.

Example 3 *How many even numbers, greater than* 2000, *can be formed with the digits* 1, 2, 4, 8, *if each digit may be used only once in each number?*

If the number is greater than 2000, the first digit can be chosen in 3 ways (viz.: 2, 4, or 8).

Then, whichever has been chosen to be the first digit, there are 2 ways in which the last digit may be chosen, in order to make the number even. Thus there are 3×2 ways of choosing the first and last digits.

When the first and last digits have been chosen, there are 2 digits, either of which may be the second digit of the number. Thus there are $3 \times 2 \times 2$ ways of choosing the first, last, and second digit.

Now, when three digits have been chosen, there is only 1 left to fill the remaining place, and so there are $3 \times 2 \times 2 \times 1$, i.e. 12, even numbers greater than 2000 which may be formed from the digits 1, 2, 4, 8, without repetitions.

The following table is useful for showing the argument briefly:

Position of digit	First	Last	Second	Third
Number of possibilities	3	2	2	1

It is to be understood, in this and later tables, that the choice is made in the order of the first line.

Exercise 12a

1 Ten boys are running a race. In how many ways can the first three places be filled, if there are no dead heats?

2 In how many ways can four letters of the word BRIDGE be arranged in a row, if no letter is repeated?

3 Five letters from the word DRILLING are arranged in a row. Find the number of ways in which this can be done, when the first letter is I and the last is L,

(a) if no letter may be repeated,

(b) if each letter may occur as many times as it does in DRILLING.

4 A man, who works a five-day week, can travel to work on foot, by cycle or by bus. In how many ways can he arrange a week's travelling to work?

5 How many five-figure odd numbers can be made from the digits 1, 2, 3, 4, 5, if no digit is repeated?

6 A girl has two coats, four scarves and three pairs of gloves. How many different outfits, consisting of coat, scarf, and a pair of gloves, can she make out of these?

7 In a class of thirty pupils, one prize is awarded for English, another for French, and a third for mathematics. In how many ways can the recipients be chosen?

8 A man has five different flags. In how many ways can he fly them one above the other?

9 The computer department in a large company assigns a personal code number to each employee in the form of a three-digit number, using the digits 0 to 9 inclusive. Code numbers starting with 0 are reserved for members of the management. How many code numbers are available for non-management employees?

10 There are sixteen books on a shelf. In how many ways can these be arranged if twelve of them are volumes of a history, and must be kept together, in order?

11 A typist has six envelopes and six letters. In how many ways can one letter be placed in each envelope without getting every letter in the right envelope?

12 How many postal codes of the form AB1 2CD (i.e. two letters, followed by a single digit, a space, another digit and two more letters) can be formed from the symbols A, B, C, D, 1 and 2, if each symbol is used once only?

13 In how many ways can the letters of the word NOTATION be arranged?

14 How many odd numbers, greater than 500 000, can be made from the digits 2, 3, 4, 5, 6, 7, without repetitions?

15 Three letters from the word RELATION are arranged in a row. In how many ways can this be done? How many of these contain exactly one vowel?

16 Seven men and six women are to be seated in a row on a platform. In how many ways can they be arranged if no two men sit next to each other? In how many ways can the arrangement be made if there are six men and six women, subject to the same restriction?

17 A man stays three days at a hotel and the menu is the same for breakfast each day. He may have any one of three types of egg dish, or two types of fish, or meat. In how many ways can he order his three breakfasts if he does not have egg two days running nor repeat any dish?

18 A boy has five blue marbles, four green marbles and three red marbles. In how many ways can he arrange four of them in a row, if the marbles of any one colour are indistinguishable?

19 I have fifteen books of three different sizes, five of each. In how many ways can I arrange them on my shelf if I keep books of the same size together?

20 Four men and their wives sit on a bench. In how many ways can they be arranged if

(a) there is no restriction,

(b) each man sits next to his wife?

The factorial notation

12.2 There are times when a problem on arrangements leads to an answer involving a product of more factors than it is convenient to write down. The next example shows how this may arise.

Example 4 *In how many ways can the cards of one suit, from a pack of playing cards, be placed in a row?*

Position of card in row	First	Second	...	Twelfth	Thirteenth
Number of possibilities	13	12	...	2	1

The table abbreviates the type of argument used in the last three examples, and it leads to the conclusion that the cards of one suit can be placed in a row in

$$13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \text{ ways}$$

To shorten the answer, the product could be evaluated, giving 6 227 020 800; but it is easier to write

$$13!$$

(which is read, 'factorial thirteen', or by some, 'thirteen shriek'!). Thus,

$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$

and similarly for any other positive integer.

The factorial notation will be used freely in this chapter and Chapter 14, and the reader should become thoroughly used to it before going on to the next section.

Example 5 (a) *Evaluate* $\dfrac{9!}{2!7!}$,

(b) *Write* $40 \times 39 \times 38 \times 37$ *in factorial notation.*

(a) Written in full,

$$\frac{9!}{2!7!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

$$= \frac{9 \times 8}{2 \times 1}$$

$$= 36$$

(b) $$40 \times 39 \times 38 \times 37 = 40 \times 39 \times 38 \times 37 \times \frac{36 \times 35 \times \ldots \times 2 \times 1}{36 \times 35 \times \ldots \times 2 \times 1}$$

$$= \frac{40!}{36!}$$

Exercise 12b

1 Evaluate:

(a) $3!$, (b) $4!$, (c) $5!$, (d) $\frac{10!}{8!}$, (e) $\frac{7!}{4!}$, (f) $\frac{12!}{9!}$,

(g) $\frac{11!}{7!4!}$, (h) $\frac{6!2!}{8!}$, (i) $(2!)^2$, (j) $\frac{6!}{(3!)^2}$, (k) $\frac{10!}{3!7!}$, (l) $\frac{10!}{2!3!5!}$.

2 Express in factorial notation:

(a) $6 \times 5 \times 4$, (b) 10×9, (c) $12 \times 11 \times 10 \times 9$,

(d) $n(n-1)(n-2)$, (e) $(n+2)(n+1)n$, (f) $\frac{10 \times 9}{2 \times 1}$,

(g) $\frac{7 \times 6 \times 5}{3 \times 2 \times 1}$, (h) $\frac{52 \times 51 \times 50}{3 \times 2 \times 1}$, (i) $\frac{n(n-1)}{2 \times 1}$,

(j) $\frac{(n+1)n(n-1)}{3 \times 2 \times 1}$, (k) $\frac{2n(2n-1)}{2 \times 1}$, (l) $n(n-1)\dots(n-r+1)$.

3 Express in factors:

(a) $20! + 21!$, (b) $26! - 25!$, (c) $14! - 2(13!)$,
(d) $15! + 4(14!)$, (e) $(n+1)! + n!$, (f) $(n-1)! - (n-2)!$,
(g) $n! + 2(n-1)!$, (h) $(n+2)! + (n+1)! + n!$.

4 Simplify:

(a) $\frac{15!}{11!4!} + \frac{15!}{12!3!}$, (b) $\frac{21!}{7!14!} + \frac{21!}{8!13!}$,

(c) $\frac{16!}{9!7!} + \frac{2 \times 16!}{10!6!} + \frac{16!}{11!5!}$, (d) $\frac{35!}{16!19!} + \frac{3 \times 35!}{17!18!}$,

(e) $\frac{n!}{(n-r)!r!} + \frac{n!}{(n-r+1)!(r-1)!}$,

(f) $\frac{n!}{(n-r)!r!} + \frac{2 \times n!}{(n-r+1)!(r-1)!} + \frac{n!}{(n-r+2)!(r-2)!}$.

Permutations

12.3 In Example 4, it was found that 13 playing cards could be placed in a row in 13! ways. If we consider n unlike objects placed in a row, using the same method,

Position of object in row	1st	2nd	...	$(n-1)$th	nth
Number of possibilities	n	$n-1$	...	2	1

we find that they may be arranged in $n!$ ways.

The arrangements of the n objects are called **permutations**. Thus

ABC, ACB, BCA, BAC, CAB, CBA,

are the 3! permutations of the three letters A, B, C.

Again, in Example 1, it was found that 3 cards chosen from 5 unlike cards could be arranged in 60 ways. This might be expressed by saying that there are 60 permutations of 3 cards chosen from 5 unlike cards.

A permutation is an arrangement of a number of objects in a particular order. In practice, the order may be in space, such as from left to right in a row; or it may be in time, such as reaching the winning post in a race, or dialling on a telephone.

How many permutations are there of r objects chosen from n unlike objects?

The method is indicated in the table below.

Order of choice of object	1st	2nd	3rd	...	$(r-1)$th	rth
Number of possibilities	n	$(n-1)$	$(n-2)$	...	$(n-r+2)$	$(n-r+1)$

Thus there are

$$n(n-1)(n-2)\dots(n-r+2)(n-r+1)$$

permutations of the objects. But

$$n(n-1)(n-2)\dots(n-r+2)(n-r+1)$$

$$= \frac{n(n-1)(n-2)\dots(n-r+2)(n-r+1) \times (n-r)\dots 2\times 1}{(n-r)\dots 2\times 1}$$

$$= \frac{n!}{(n-r)!}$$

Therefore there are $\boldsymbol{n!/(n-r)!}$ permutations of r objects chosen from n unlike objects, if r is less than n.

(We have already found that there are $n!$ permutations of n unlike objects.)

Example 6 *There are 20 books on a shelf, but the red covers of two of them clash, and they must not be put together. In how many ways can the books be arranged?*

This is best tackled by finding out the number of ways in which the two books *are* together, and subtracting this from the number of ways in which the 20 books can be arranged if there is no restriction.

Suppose the two red books are tied together, then there are 19 objects, which can be arranged in 19! ways. Now if the order of the two red books is reversed, there will again be 19! arrangements; so that there are $2 \times 19!$ ways of arranging the books with the red ones next to each other.

With *no* restriction, 20 books can be arranged in 20! ways; therefore the number of arrangements in which the red books are not together is

$$20! - 2 \times 19! = 18 \times 19!$$

Example 7 *In how many ways can 8 people sit at a round table?*

Since the table is round, the position of people relative to the *table* is of no consequence. Thus, supposing they sit down, and then all move one place to the left, the arrangement is still the same.

Therefore one person may be considered to be fixed, and the other 7 can then be arranged about him or her in 7! ways.

Thus there are 5040 ways in which 8 people can sit at a round table.

Example 8 *In how many ways can the letters of the word* BESIEGE *be arranged?*

First, give the three E's suffixes: $BE_1SIE_2GE_3$. Then, treating the E's as different, the 7 letters may be arranged in 7! ways.

Now, in every distinct arrangement, the 3 E's may be rearranged amongst themselves in 3! ways, without altering the positions of the B, S, I, or G; for instance, SEIBEEG would have been counted 3! times in the 7! arrangements as

$$SE_1IBE_2E_3G,\quad SE_2IBE_3E_1G,\quad SE_3IBE_1E_2G,$$
$$SE_1IBE_3E_2G,\quad SE_2IBE_1E_3G,\quad SE_3IBE_2E_1G.$$

Therefore the number of distinct arrangements of the letters in BESIEGE is $7!/3! = 840$.

In the next exercise there are some examples which are best tackled from first principles, like the next example.

Example 9 *How many even numbers, greater than* 50 000, *can be formed with the digits* 3, 4, 5, 6, 7, 0, *without repetitions?*

Compared with Example 3, §12.1, there are two extra difficulties: the number can have either 5 or 6 digits, and the number cannot begin with 0. Therefore the problem is split up into four parts:

(1) Numbers with 5 digits, the first digit being even.

Position of digit in number	1st	5th	2nd	3rd	4th
Number of possibilities	1	2	4	3	2

This gives $1 \times 2 \times 4 \times 3 \times 2 = 48$ possibilities.

(2) Numbers with 5 digits, the first digit being odd.

Position of digit in number	1st	5th	2nd	3rd	4th
Number of possibilities	2	3	4	3	2

This gives $2 \times 3 \times 4 \times 3 \times 2 = 144$ possibilities.

(3) Numbers with 6 digits, the first digit being even.

Position of digit in number	1st	6th	2nd	3rd	4th	5th
Number of possibilities	2	2	4	3	2	1

This gives $2 \times 2 \times 4 \times 3 \times 2 \times 1 = 96$ possibilities.

(4) Numbers with 6 digits, the first digit being odd.

Position of digit in number	1st	6th	2nd	3rd	4th	5th
Number of possibilities	3	3	4	3	2	1

This gives $3 \times 3 \times 4 \times 3 \times 2 \times 1 = 216$ possibilities.

Therefore the total number of possibilities is $48 + 144 + 96 + 216 = 504$.

Exercise 12c

1 Seven boys and two girls are to sit together on a bench. In how many ways can they arrange themselves so that the girls do not sit next to each other?

2 Eight women and two men are to sit at a round table. In how many ways can they be arranged? If, however, the two men sit directly opposite each other, in how many ways can the ten people be arranged?

3 How many arrangements can be made of the letters in the word TROTTING? In how many of these are the N and the G next to each other?

4 On a bookshelf, four books are bound in leather and sixteen in cloth. If the books are to be arranged so that the leather-bound ones are together, in how many ways can this be done? If, in addition, the cloth-bound books are to be kept together, in how many ways can the shelf be arranged?

5 There is room for ten books on a bedside table, but there are fifteen to choose from. Of these, however, a Bible and a book of ghost stories must go at the ends. In how many ways can the books be arranged?

6 Ten beads of different colours are arranged on a ring. If a salesman claims that no two of his rings are the same, what is the greatest number of rings he could have? (A ring can be turned over.)

7 In his cowhouse, a farmer has seven stalls for cows, and four for calves. If he has ten cows and five calves, in how many ways can he arrange the animals in his cowhouse?

8 At a conference of five powers, each delegation consists of three members. If each delegation sits together, with their leader in the middle, in how many ways can the members be arranged at a round table?

9 How many numbers, divisible by 5, can be made with the digits 2, 3, 4, 5, no digit being used more than once in each number?

10 In a cricket team, the captain has settled the first four places in the batting order, and has decided that the four bowlers will occupy the last four places.

In how many ways can the batting order be made out?

11 How many arrangements can be made of the letters in the word TERRITORY?

12 A man has ten ornaments for his mantelpiece, and of these the clock must go in the centre. If there is only room for seven ornaments altogether, how many arrangements can be made on the mantelpiece?

13 How many odd numbers, greater than 60 000, can be made from the digits 5, 6, 7, 8, 9, 0, if no number contains any digit more than once?

14 A code word consists of three letters, followed by two digits. How many code words can be made, if no letter nor digit is repeated in any code word?

15 How many numbers of five digits can be made from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, when each number contains exactly one even digit and no digit more than once?

16 A bridge player holds five spades, four hearts, two diamonds and two clubs. If he keeps the cards of each suit together, in how many ways can he arrange the cards he holds

(a) if the suits are in the above order,

(b) if the suits may be arranged in any order?

17 Find the number of ways in which the letters of ISOSCELES can be arranged if the two E's are separated.

18 Find how many numbers greater than 400 000 can be made, using all the digits of 416 566.

19 In how many ways can four red beads, three green beads, and five beads of different colours be strung on a circular wire?

20 Six natives and two foreigners are seated in a compartment of a railway carriage with four seats either side. In how many ways can the passengers seat themselves if

(a) the foreigners do not sit opposite each other,

(b) the foreigners do not sit next to each other?

Combinations

12.4 In the last section, attention was given to permutations, where the order of a set of objects was of importance; but in other circumstances, the order of selection is irrelevant. If, for instance, eight tourists find there is only room for five of them at a hotel, they will be chiefly interested in which five of them stay there, rather than in any order of arrangement.

When a selection of objects is made with no regard being paid to order, it is referred to as a **combination**. Thus, ABC, ACB, CBA, are different permutations, but they are the same combination of letters.

Example 10 *In how many ways can* 13 *cards be selected from a pack of* 52 *playing cards?*

First of all, suppose that thirteen cards from the pack are laid on a table in an order from left to right. From the last section, it follows that this can be done in 52!/39! ways.

Now each combination of cards can be arranged in 13! ways, therefore

the number of permutations = 13! × (the number of combinations)

$$\therefore \frac{52!}{39!} = 13! \times (\text{the number of combinations})$$

Therefore the number of combinations of 13 cards chosen from a pack of playing cards is 52!/(39!13!).

In how many ways can r objects be chosen from n unlike objects?

In §12.3 it was shown that there are $n!/(n-r)!$ permutations of r objects chosen from n unlike objects.

Now each combination of r objects can be arranged in $r!$ ways, therefore

the number of permutations = $r!$ × (the number of combinations)

$$\therefore \frac{n!}{(n-r)!} = r! \times (\text{the number of combinations})$$

Hence the number of combinations of r objects chosen from n unlike objects is

$$\frac{n!}{(n-r)!r!}$$

For brevity, the number of combinations of r objects chosen from n unlike objects is written ${}^nC_r$, thus

$$\mathbf{{}^nC_r = \frac{n!}{(n-r)!r!}}$$

${}^nC_r$ is also sometimes written as ${}_nC_r$ and $\binom{n}{r}$ (see §14.5).

Qu. 1 What are the values of (a) ${}^8C_3$, ${}^8C_5$; (b) ${}^{10}C_6$, ${}^{10}C_4$?

Qu. 2 In how many ways can $n-r$ objects be chosen from n unlike objects?

Qu. 3 Show that ${}^nC_r = {}^nC_{n-r}$.

Example 11 *A mixed hockey team containing 5 men and 6 women is to be chosen from 7 men and 9 women. In how many ways can this be done?*

Five men can be selected from 7 men in ${}^7C_5$ ways, and 6 women can be selected from 9 women in ${}^9C_6$ ways.

Now for each of the ${}^7C_5$ ways of selecting the men, there are ${}^9C_6$ ways of selecting the women, therefore there are ${}^7C_5 \times {}^9C_6$ ways of selecting the team.

$$\begin{aligned} {}^7C_5 \times {}^9C_6 &= \frac{7!}{2!5!} \times \frac{9!}{3!6!} \\ &= 21 \times 84 \end{aligned}$$

Therefore the team can be chosen in 1764 ways.

Exercise 12d

1 Evaluate: (a) ${}^{10}C_2$, (b) ${}^{6}C_4$, (c) ${}^{7}C_3$, (d) ${}^{9}C_5$, (e) ${}^{8}C_4$.
Express in factors: (f) ${}^{n}C_2$, (g) ${}^{n}C_3$, (h) ${}^{n}C_{n-2}$, (i) ${}^{n+1}C_2$, (j) ${}^{n+1}C_{n-1}$.

2 In how many ways can a cricket team be selected from thirteen players?

3 There are ten possible players for the VI to represent a tennis club, and of these the captain and the secretary must be in the team. In how many ways can the team be selected?

4 Ten boxes each hold one white ball and one coloured ball, every colour being different. Find the number of ways in which one ball may be taken from each box if half those taken are white.

5 Nine people are going to travel in two taxis. The larger has five seats, and the smaller has four. In how many ways can the party be split up?

6 A girl wants to ask eight friends to tea, but there is only room for four of them. In how many ways can she choose whom to invite if two of them are sisters and must not be separated? (Consider two cases, (a) when both sisters are invited, (b) when neither sister is invited.)

7 In a game of mixed hockey there are ten married couples and two spinsters playing. In how many ways can the two teams be made up, if no husband may play against his wife?

8 A ferry which holds ten people carries a party of thirteen men and seven women across a river. Find the number of ways in which the party may be taken across if all the women go on the first trip.

9 Twelve people each spin a coin. Find the number of ways in which exactly five heads may be obtained.

10 Two punts each hold six people. In how many ways can a party of six boys and six girls divide themselves so that there are equal numbers of boys and girls in each punt?

11 In how many ways can eight white and four black draughtsmen be arranged in a pile?

12 A committee of six is to be formed from nine women and three men. In how many ways can the members be chosen so as to include at least one man?

13 Ten men are present at a club. In how many ways can four be chosen to play bridge if two men refuse to sit at the same table?

14 A man is allowed to take six volumes to a desert island. He is going to choose these from eleven books, one of which contains two volumes, which he will take or leave together. Find the number of ways in which he can make his choice.

15 Four people are to play bridge and four others are to play whist. Find the number of ways in which they may be chosen if eleven people are available.

16 A party of twelve is to dine at three tables at a hotel. In how many ways may they be split up if each table holds four?

17 Twelve people are to travel by three cars, each of which holds four. Find the number of ways in which the party may be divided if two people refuse to travel in the same car.

18 A committee of ten is to be chosen from nine men and six women. In how many ways can it be formed if at least four women are to be on the committee?

19 In how many ways can eleven men be chosen to represent a cricket club if they are selected from seven Englishmen, six Welshmen and five Scots, and if at least one of each nationality must be in the team?

Exercise 12e (Miscellaneous)

1 In the absence of the chairman, a committee of three vice-chairmen and four ordinary members is to sit on a platform. In how many ways can they be arranged if one of the vice-chairmen sits in the middle?

2 In how many ways can a committee of four men and three women be formed from seven men and eight women?

3 Show that the number of ways of choosing six objects from fourteen unlike objects is equal to the number of ways of choosing five objects from fifteen unlike objects.

4 How many arrangements can be made of the letters in THIRTIETH?

5 In how many ways can a committee of eight be arranged at a round table? In how many of these does the chairman sit between the secretary and the treasurer?

6 How many circular rings can be formed from seven differently coloured beads? In how many of these are the red and the blue beads separated?

7 In how many ways can a boy arrange in a row six balls from seven cricket balls, six tennis balls and five squash balls?

8 Find the number of diagonals of a polygon of n sides.

9 How many five-figure numbers can be made from the digits of 10 242?

10 In how many ways can ten books be arranged on a shelf if four of them are kept together?

11 In how many ways can a man who has ten chairs put five in one room, three in a second and two in a third?

12 How many odd numbers, greater than 600 000, can be made from the digits 5, 6, 7, 8, 9, 0,

(a) if repetitions are not allowed,

(b) if repetitions are allowed?

13 How many arrangements can be made with the letters of LEATHERETTE?

14 In how many ways can four mince-pies, three jam tarts, and three cakes be given to ten children if each receives one?

15 In how many ways can a committee of nine be formed from ten men and their wives, if no husband serves on it with his wife?

16 There are six ornaments on my mantelpiece. In how many ways can I put three more on it without changing the order of those already there?

17 How many mixed hockey teams may be made from six married couples, one bachelor and three spinsters, if no wife will play without her husband?

18 A man has ten pieces of clothing to dispose of. In how many ways can he do this if he gives away at least two articles and sells the rest?

19 Eight boys and two girls sit on a bench. If the girls may sit neither at the ends nor together, in how many ways can they be arranged?

20 In how many arrangements of the letters of REVERSE are the V and S separated?

21 In how many ways is it possible to select one or more letters from those in INSIPIDITY?

22 Four men and their wives, four bachelors and four spinsters are travelling in two eight-seat compartments of a train, one of which is a smoking compartment and the other is not. In how many ways can the party be split up if no wife is separated from her husband?

23 A painter has to paint the doors of twelve new council houses and has sufficient paint to do five green and three yellow. If he is given paint of only one colour — blue, green, or yellow — for the remaining doors, in how many ways can the twelve doors be painted?

24 In how many ways can a lift holding eight passengers carry a party of thirteen up a building in two journeys?

25 How many numbers of five or six digits can be formed from the digits 1, 2, 2, 2, 3, 4?

26 In how many ways is it possible to select six letters, including at least one vowel, from the letters of (a) INCOMPUTABLE, (b) FLABELLIFORM?

Chapter 13

Series

Sequences

13.1 The reader should examine the following lists of numbers. Each list is written down in a definite order, and there is a simple rule by which the terms are obtained. Such a list of terms is called a **sequence**.

Qu. 1 Write down the next two terms in each of the following sequences:

(a) $1, 3, 5, 7, \ldots$ (b) $2, 5, 8, 11, \ldots$ (c) $1, 2, 4, 8, \ldots$
(d) $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \ldots$ (e) $1^3, 2^3, 3^3, 4^3, \ldots$ (f) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$
(g) $1, 4, 9, 16, \ldots$ (h) $1, 2, 6, 24, 120, \ldots$ (i) $1, \frac{2}{3}, \frac{3}{9}, \frac{4}{27}, \ldots$
(j) $4, 2, 0, -2, \ldots$ (k) $1, -1, 1, -1, \ldots$ (l) $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$

Suppose one is asked to add up the integers from 1 to 100. This could be done by elementary arithmetic, but it would be very tedious: fortunately there is a short-cut.

First write the numbers down in their natural order:

$$1 + 2 + 3 + \ldots + 98 + 99 + 100$$

Now write the numbers down again in the opposite order, so that we have:

$$\begin{array}{l} 1 + \;\; 2 + \;\; 3 + \ldots + \;98 + \;99 + 100 \\ 100 + \;99 + \;98 + \ldots + \;\;3 + \;\;2 + \;\;1 \\ \hline 101 + 101 + 101 + \ldots + 101 + 101 + 101 \end{array}$$

The numbers in each column have been added together, and, since there are 100 terms in the top line, the total is $100 \times 101 = 10\,100$. But this is twice the sum required, therefore the sum of the integers from 1 to 100 is 5050.

If the terms of a sequence are considered as a sum, for instance

$$1 + 2 + 3 + \ldots + 98 + 99 + 100$$

or

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \ldots$$

the expression is called a **series**. A series may end after a finite number of terms, in which case it is called a **finite series**; or it may be considered not to end, and it is then called an **infinite series**.

Arithmetical progressions

13.2 The method of §13.1, for finding the sum of a series, may only be applied to a certain type, which is usually called an arithmetical progression (often abbreviated to A.P.). For example,

$$1 + 3 + 5 + \dots + 99$$
$$7 + 11 + 15 + \dots + 79$$
$$3 - 2 - 7 - \dots - 42$$
$$1\tfrac{1}{8} + 1\tfrac{1}{4} + 1\tfrac{3}{8} + \dots + 3\tfrac{1}{2}$$
$$-2 - 4 - 6 - \dots - 16$$

are arithmetical progressions. In such a series, any term may be obtained from the previous term by adding a certain number, called the **common difference**. Thus the common differences in the above progressions are $2, 4, -5, \frac{1}{8}, -2$.

Example 1 *Find the third, tenth, twenty-first and* nth *terms of the A.P. with first term* 6 *and common difference* 5.

Position of term	1st	2nd	3rd	4th	10th	21st	nth
Value	6	$6+5$	$6+2\times 5$	$6+3\times 5$	$6+9\times 5$	$6+20\times 5$	$6+(n-1)\times 5$

Note that to find the nth term $n-1$ common differences are added to the first term. (Throughout this chapter it should be assumed that n represents a positive integer.)

The third, tenth, twenty-first, and nth terms are 16, 51, 106, and $5n+1$.

Example 2 *Find the sum of the first twenty terms of the A.P.* $-4 - 1 + 2 + \dots$.

To find the twentieth term, add 19 times the common difference to the first term: $-4 + 19 \times 3 = 53$.

Write S_{20} for the sum of the first twenty terms, then using the method of §13.1,

$$S_{20} = -4 - 1 + 2 + \dots + 53$$

Again,

$$S_{20} = 53 + 50 + 47 + \dots - 4$$

Adding,

$$2S_{20} = 49 + 49 + 49 + \dots + 49 = 20 \times 49$$
$$\therefore S_{20} = 490$$

Therefore the sum of the first twenty terms of the A.P. is 490.

Exercise 13a

1 Which of the following series are arithmetical progressions? Write down the common differences of those that are.

(a) $7 + 8\frac{1}{2} + 10 + 11\frac{1}{2}$, (b) $-2 - 5 - 8 - 11$,
(c) $1 + 1.1 + 1.2 + 1.3$, (d) $1 + 1.1 + 1.11 + 1.111$,
(e) $\frac{1}{2} + \frac{5}{6} + \frac{7}{6} + \frac{3}{2}$, (f) $1^2 + 2^2 + 3^2 + 4^2$,
(g) $n + 2n + 3n + 4n$, (h) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$,
(i) $1\frac{1}{8} + 2\frac{1}{4} + 3\frac{3}{8} + 4\frac{1}{2}$, (j) $19 + 12 + 5 - 2 - 9$,
(k) $1 - 2 + 3 - 4 + 5$, (l) $1 + 0.8 + 0.6 + 0.4$.

2 Write down the terms indicated in each of the following A.P.s:

(a) $3 + 11 + \ldots$, 10th, 19th, (b) $8 + 5 + \ldots$, 15th, 31st,
(c) $\frac{1}{4} + \frac{7}{8} + \ldots$, 12th, nth, (d) $50 + 48 + \ldots$, 100th, nth,
(e) $7 + 6\frac{1}{2} + \ldots$, 42nd, nth, (f) $3 + 7 + \ldots$, 200th, $(n + 1)$th.

3 Find the number of terms in the following A.P.s:

(a) $2 + 4 + 6 + \ldots + 46$, (b) $50 + 47 + 44 + \ldots + 14$,
(c) $2.7 + 3.2 + \ldots + 17.7$, (d) $6\frac{1}{4} + 7\frac{1}{2} + \ldots + 31\frac{1}{4}$,
(e) $407 + 401 + \ldots - 133$, (f) $2 - 9 - \ldots - 130$,
(g) $2 + 4 + \ldots + 4n$, (h) $x + 2x + \ldots + nx$,
(i) $a + (a + d) + \ldots + \{a + (n - 1)d\}$, (j) $a + (a + d) + \ldots + l$.

4 Find the sums of the following A.P.s:

(a) $1 + 3 + 5 + \ldots + 101$, (b) $2 + 7 + 12 + \ldots + 77$,
(c) $-10 - 7 - 4 - \ldots + 50$, (d) $71 + 67 + 63 + \ldots - 53$,
(e) $2.01 + 2.02 + 2.03 + \ldots + 3.00$, (f) $1 + 1\frac{1}{6} + 1\frac{1}{3} + \ldots + 4\frac{1}{2}$,
(g) $x + 3x + 5x + \ldots + 21x$, (h) $a + (a + 1) + \ldots + (a + n - 1)$,
(i) $a + (a + d) + \ldots + \{a + (n - 1)d\}$.

5 Find the sums of the following arithmetical progressions as far as the terms indicated:

(a) $4 + 10 + \ldots$ 12th term, (b) $15 + 13 + \ldots$ 20th term,
(c) $1 + 2 + \ldots$ 200th term, (d) $20 + 13 + \ldots$ 16th term,
(e) $6 + 10 + \ldots$ nth term, (f) $1\frac{1}{4} + 1 + \ldots$ nth term.

6 The second term of an A.P. is 15, and the fifth is 21. Find the common difference, the first term and the sum of the first ten terms.

7 The fourth term of an A.P. is 18, and the common difference is -5. Find the first term and the sum of the first sixteen terms.

8 Find the difference between the sums of the first ten terms of the A.P.s whose first terms are 12 and 8, and whose common differences are respectively 2 and 3.

9 The first term of an A.P. is -12, and the last term is 40. If the sum of the progression is 196, find the number of terms and the common difference.

10 Find the sum of the odd numbers between 100 and 200.

11 Find the sum of the even numbers, divisible by three, lying between 400 and 500.

12 The twenty-first term of an A.P. is $5\frac{1}{2}$, and the sum of the first twenty-one terms is $94\frac{1}{2}$. Find the first term, the common difference and the sum of the first thirty terms.

13 Show that the sum of the integers from 1 to n is $\frac{1}{2}n(n+1)$.
14 The twenty-first term of an A.P. is 37 and the sum of the first twenty terms is 320. What is the sum of the first ten terms?
15 Show that the sum of the first n terms of the A.P. with first term a and common difference d is $\frac{1}{2}n\{2a+(n-1)d\}$.

Geometrical progressions

13.3 Another series of common occurrence is the geometrical progression, for example:

$$1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\ldots+\tfrac{1}{512}$$
$$3+6+12+\ldots+192$$
$$\tfrac{16}{27}-\tfrac{8}{9}+\tfrac{4}{3}-\ldots+\tfrac{27}{4}$$

In such a progression, the ratio of a term to the previous one is a constant, called the **common ratio**. Thus, the common ratios of the above progressions are respectively $\frac{1}{2}$, 2 and $-\frac{3}{2}$.

Qu. 2 Write down the third and fourth terms of the progressions which begin (i) $2+4+\ldots$, (ii) $12+6+\ldots$, (a) if they are A.P.s, (b) if they are G.P.s.

Example 3 *Find the third, tenth, twenty-first and* nth *terms of the G.P. which begins* $3+6+\ldots$.

Position of term	1st	2nd	3rd	4th	10th	21st	nth
Value	3	3×2	3×2^2	3×2^3	3×2^9	3×2^{20}	$3\times 2^{n-1}$

Note that to find the nth term, the first term is multiplied by the $(n-1)$th power of the common ratio.

The third, tenth, twenty-first, and nth terms are 12, 1536, 3 145 728, and $3\times 2^{n-1}$.

Example 4 *Find the sum of the first eight terms of the geometrical progression* $2+6+18+\ldots$.

To find the eighth term, multiply the first term by the seventh power of the common ratio: 2×3^7.

Let S_8 be the sum of the first eight terms of the expression.

$$\therefore S_8 = 2+2\times 3+2\times 3^2+\ldots+2\times 3^7$$

Now multiply both sides by the common ratio and write the terms obtained one place to the right, so that we have

$$S_8 = 2+2\times 3+2\times 3^2+\ldots+2\times 3^7$$
$$3S_8 = \quad 2\times 3+2\times 3^2+\ldots+2\times 3^7+2\times 3^8$$

Subtracting the top line from the lower,

$$2S_8 = -2 + 2 \times 3^8$$
$$\therefore S_8 = 3^8 - 1$$

Therefore the sum of the first eight terms is 6560.

Exercise 13b

1 Which of the following series are geometrical progressions? Write down the common ratios of those that are.
(a) $3 + 9 + 27 + 81$, (b) $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64}$,
(c) $-1 + 2 - 4 + 8$, (d) $1 - 1 + 1 - 1$,
(e) $1 + 1\frac{1}{2} + 1\frac{1}{4} + 1\frac{1}{8}$, (f) $a + a^2 + a^3 + a^4$,
(g) $1 + 1.1 + 1.21 + 1.331$, (h) $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{36}$,
(i) $2 + 4 - 8 - 16$, (j) $\frac{3}{4} + \frac{9}{2} + 27 + 162$.

2 Write down the terms indicated in each of the following geometrical progressions. Do not simplify your answers.
(a) $5 + 10 + \ldots$, 11th, 20th; (b) $10 + 25 + \ldots$, 7th, 19th;
(c) $\frac{2}{3} + \frac{3}{4} + \ldots$, 12th, nth; (d) $3 - 2 + \ldots$, 8th, nth;
(e) $\frac{2}{7} - \frac{3}{7} + \ldots$, 9th, nth; (f) $3 + 1\frac{1}{2} + \ldots$, 19th, $2n$th.

3 Find the number of terms in the following geometrical progressions:
(a) $2 + 4 + 8 + \ldots + 512$, (b) $81 + 27 + 9 + \ldots + \frac{1}{27}$,
(c) $0.03 + 0.06 + 0.12 + \ldots + 1.92$, (d) $\frac{8}{81} - \frac{4}{27} + \frac{2}{9} - \ldots - 1\frac{11}{16}$,
(e) $5 + 10 + 20 + \ldots + 5 \times 2^n$, (f) $a + ar + ar^2 + \ldots + ar^{n-1}$.

4 Find the sums of the geometrical progressions in No. 3. Simplify, but do not evaluate, your answers.

5 Find the sums of the following geometrical progressions as far as the terms indicated. Simplify, but do not evaluate, your answers.
(a) $4 + 12 + 36 + \ldots$, 12th term; (b) $15 + 5 + 1\frac{2}{3} + \ldots$, 20th term;
(c) $1 - 2 + 4 - \ldots$, 50th term; (d) $24 - 12 + 6 - \ldots$, 17th term;
(e) $1.1 + 1.21 + 1.331 + \ldots$, 23rd term; (f) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$, 13th term;
(g) $3 + 6 + 12 + \ldots$, nth term; (h) $1 - \frac{1}{3} + \frac{1}{9} - \ldots$, nth term.

6 The third term of a geometrical progression is 10, and the sixth is 80. Find the common ratio, the first term and the sum of the first six terms.

7 The third term of a geometrical progression is 2, and the fifth is 18. Find two possible values of the common ratio, and the second term in each case.

8 The three numbers, $n - 2$, n, $n + 3$, are consecutive terms of a geometrical progression. Find n, and the term after $n + 3$.

9 A man starts saving on 1st April. He saves 1p the first day, 2p the second, 4p the third, and so on, doubling the amount every day. If he managed to keep on saving under this system until the end of the month (30 days), how much would he have saved? Give your answer in pounds, correct to three significant figures.

10 The first term of a G.P. is 16 and the fifth term is 9. What is the value of the seventh term?

11 Show that the sum of the series $4 + 12 + 36 + 108 + \dots$ to 20 terms is greater than 3×10^9.

12 The numbers $n - 4$, $n + 2$, $3n + 1$ are in geometrical progression. Find the two possible values of the common ratio.

13 What is the common ratio of the G.P. $(\sqrt{2} - 1) + (3 - 2\sqrt{2}) + \dots$? Find the third term of the progression.

14 Find the ratio of the sum of the first 10 terms of the series

$$\log x + \log x^2 + \log x^4 + \log x^8 + \dots$$

to the first term.

Formulae for the sums of A.P.s and G.P.s

13.4 The methods of Examples 2 and 4 will now be applied to general A.P.s and G.P.s to obtain formulae for their sums.

(a) If the first term of an A.P. is a, and the nth term is l, we may find the sum S_n of the first n terms.

We have

$$S_n = a + (a + d) + \dots + (l - d) + l \quad \text{(where there are } n \text{ terms), and again,}$$

$$S_n = l + (l - d) + \dots + (a + d) + a$$

Adding,

$$2S_n = (a + l) + (a + l) + \dots + (a + l) + (a + l)$$

Now there are n terms on the right-hand side,

$$\therefore 2S_n = n(a + l)$$

$$\therefore \mathbf{S_n = \frac{n(a + l)}{2}}$$

(b) If the first term of an A.P. is a, and the common difference is d, the nth term is $a + (n - 1)d$. Substituting $l = a + (n - 1)d$ in the formula above,

$$S_n = \frac{n}{2}\{a + a + (n - 1)d\}$$

$$\therefore \mathbf{S_n = \frac{n}{2}\{2a + (n - 1)d\}}$$

(c) If the first term of a G.P. is a and the common ratio is r, we may find the sum S_n of the first n terms.

The nth term is ar^{n-1}, therefore

$$S_n = a + ar + ar^2 + \dots + ar^{n-1}$$

$$\therefore rS_n = \qquad ar + ar^2 + \dots + ar^{n-1} + ar^n$$

Subtracting,

$$S_n - rS_n = a - ar^n$$
$$\therefore S_n(1 - r) = a(1 - r^n)$$
$$\therefore \boldsymbol{S_n = a\left(\frac{1 - r^n}{1 - r}\right)}$$

An alternative formula for the sum of a G.P. is obtained by multiplying numerator and denominator by -1:

$$S_n = a\left(\frac{r^n - 1}{r - 1}\right)$$

This is more convenient if r is greater than 1.

Example 5 *In an arithmetical progression, the thirteenth term is 27, and the seventh term is three times the second term. Find the first term, the common difference and the sum of the first ten terms.*

[We have two unknowns (the first term and the common difference). We have two pieces of information:
(a) the thirteenth term is 27.
(b) the seventh term is three times the second term.
Thus we can form two equations which will enable us to find the two unknowns.]

Let the first term be a, and let the common difference be d.

Then the thirteenth term is $a + 12d$, therefore

$$a + 12d = 27$$

The seventh term is $a + 6d$, and the second term is $a + d$, therefore

$$a + 6d = 3(a + d)$$
$$\therefore 3d = 2a$$

Substituting in the first equation,

$$a + 8a = 27$$
$$\therefore a = 3$$

and so

$$d = 2$$

Therefore the first term is 3, and the common difference is 2.

To find the sum of the first ten terms, we know that

$$S_n = \frac{n}{2}\{2a + (n - 1)d\}$$
$$\therefore S_{10} = \tfrac{10}{2}(6 + 9 \times 2)$$
$$= 5 \times 24$$

Therefore the sum of the first ten terms is 120.

Example 6 *In a geometrical progression, the sum of the second and third terms is 6, and the sum of the third and fourth terms is* -12. *Find the first term and the common ratio.*

[As in the last example, we have two unknowns (the first term and the common ratio). We have two pieces of information:
(a) the sum of the second and third terms is 6,
(b) the sum of the third and fourth terms is -12.
We may therefore write down two equations and these will enable us to find the two unknowns.]

Let the first term be a, and let the common ratio be r. Then the second term is ar, and the third term is ar^2, therefore

$$ar + ar^2 = 6$$

The third term is ar^2, and the fourth term is ar^3, therefore

$$ar^2 + ar^3 = -12$$

Factorising the left-hand sides of the equations,

$$ar(1 + r) = 6$$
$$ar^2(1 + r) = -12$$

We may eliminate a by dividing:

$$\frac{ar(1 + r)}{ar^2(1 + r)} = -\frac{6}{12}$$

$$\therefore \frac{1}{r} = -\frac{1}{2}$$

$$\therefore r = -2$$

Substituting $r = -2$ in $ar(1 + r) = 6$,

$$a(-2)(-1) = 6$$
$$\therefore a = 3$$

Therefore the first term is 3, and the common ratio is -2.

Example 7 *The sum of a number of consecutive terms of an arithmetical progression is* $-19\frac{1}{2}$, *the first term is* $16\frac{1}{2}$, *and the common difference is* -3. *Find the number of terms.*

With the notation of §13.4,

$$S_n = \frac{n}{2}\{2a + (n - 1)d\}$$

Substituting $S_n = -19\frac{1}{2}$, $a = 16\frac{1}{2}$, $d = -3$:

$$-\frac{39}{2}=\frac{n}{2}\{33-3(n-1)\}$$

$$\therefore\ -39=n(36-3n)$$

$$\therefore\ 3n^2-36n-39=0$$

Dividing through by 3,

$$n^2-12n-13=0$$

$$\therefore\ (n-13)(n+1)=0$$

$$\therefore n=13 \quad \text{or} \quad -1$$

Therefore the number of terms is 13.

Example 8 *What is the smallest number of terms of the geometrical progression,* $8+24+72+\ldots$, *that will give a total greater than* 6 000 000?

With the notation of §13.4,

$$S_n=a\left(\frac{r^n-1}{r-1}\right)$$

Substituting $a=8$ and $r=3$,

$$S_n=8\left(\frac{3^n-1}{3-1}\right)=4(3^n-1)$$

Now if we solve the equation

$$4(3^n-1)=6\,000\,000$$

the first integer greater than the value of n found from this will be the number of terms required.

To solve the equation:

$$3^n-1=1\,500\,000$$
$$\therefore\ 3^n=1\,500\,001$$

Taking logarithms (base 10) of both sides,

$$n\lg 3=\lg 1\,500\,001$$

$$\therefore\ n=\frac{\lg 1\,500\,001}{\lg 3}$$

$$\approx\frac{6.1761^*}{0.4771}$$

$$=12.94, \quad \text{correct to four significant figures}$$

Therefore the number of terms required to make a total exceeding 6 000 000 is 13.

*If a calculator is used it is not necessary, or desirable, to write these figures down.

Arithmetic and geometric means

13.5 If three numbers a, b, c are in arithmetical progression, b is called the **arithmetic mean** of a and c. The common difference of the progression is given by $b-a$ or $c-b$. Therefore

$$b-a=c-b$$
$$\therefore 2b=a+c$$

Therefore the arithmetic mean of a and c is $(a+c)/2$. This is the ordinary 'average' of a and c.

If three numbers a, b, c are in geometrical progression, b is called the **geometric mean** of a and c. The common ratio is given by b/a or c/b. Therefore

$$\frac{b}{a}=\frac{c}{b}$$
$$\therefore b^2=ac$$

Therefore the geometric mean of a and c is $\sqrt{(ac)}$. If a rectangle is drawn with sides a and c, then b is the side of a square whose area is equal to that of the rectangle.

Qu. 3 Find (a) the arithmetic mean, (b) the geometric mean of 4 and 64.
Qu. 4 The reciprocal of the harmonic mean of two numbers is the arithmetic mean of their reciprocals. Find the harmonic mean of 5 and 20. Also find the arithmetic and geometric means of 5 and 20.
Qu. 5 Find an expression for the harmonic mean of a and c.

Exercise 13c

1 Find the sum of the even numbers up to and including 100.
2 How many terms of the series $2-6+18-54+\ldots$ are needed to make a total of $\frac{1}{2}(1-3^8)$?
3 The fifth term of an A.P. is 17 and the third term is 11. Find the sum of the first seven terms.
4 The fourth term of a G.P. is -6 and the seventh term is 48. Write down the first three terms of the progression.
5 Find the sum of the first eight terms of the G.P. $5+15+\ldots$.
6 What is the difference between the sums to ten terms of the A.P. and G.P. whose first terms are $-2+4\ldots$?
7 The sum of the second and fourth terms of an arithmetical progression is 15, and the sum of the fifth and sixth terms is 25. Find the first term and the common difference.
8 The second term of an arithmetical progression is three times the seventh, and the ninth term is 1. Find the first term, the common difference, and which is the first term less than 0.

9 In a geometrical progression, the sum of the second and third terms is 9, and the seventh term is eight times the fourth. Find the first term, the common ratio, and the fifth term.

10 The fourth term of an arithmetical progression is 15, and the sum of the first five terms is 55. Find the first term and the common difference, and write down the first five terms.

11 The sum of the first three terms of an arithmetical progression is 3, and the sum of the first five terms is 20. Find the first five terms of the progression.

12 The sum of the first two terms of a geometrical progression is 3, and the sum of the second and third terms is -6. Find the first term and the common ratio.

13 How many terms of the A.P. $15 + 13 + 11 + \ldots$ are required to make a total of -36?

14 Which is the first term of the geometrical progression $5 + 10 + 20 + \ldots$ to exceed 400 000?

15 Find how many terms of the G.P. $1 + 3 + 9 + \ldots$ are required to make a total of more than a million.

16 The sum of the first six terms of an arithmetical progression is 21, and the seventh term is three times the sum of the third and fourth. Find the first term and the common difference.

17 In an arithmetical progression, the sum of the first five terms is 30, and the third term is equal to the sum of the first two. Write down the first five terms of the progression.

18 Find the difference between the sums of the first ten terms of the geometrical and arithmetical progressions which begin, $6 + 12 + \ldots$.

19 The sum of the first n terms of a certain series is $n^2 + 5n$, for all integral values of n. Find the first three terms and prove that the series is an arithmetical progression.

20 The second, fourth, and eighth terms of an A.P. are in geometrical progression, and the sum of the third and fifth terms is 20. Find the first four terms of the progression.

21 A man pays a premium of £100 at the beginning of every year to an Insurance Company on the understanding that at the end of fifteen years he can receive back the premiums which he has paid with 5% compound interest. What should he receive? Give your answer correct to three significant figures.

22 A man earned in a certain year £2000 from a certain source and his annual earnings from this time continued to increase at the rate of 5%. Find to the nearest £ the whole amount he received from this source in this year and the next seven years. Give your answer correct to three significant figures.

Proof by induction

13.6 It sometimes happens that a result is found by some means which does not provide a proof. For example, consider the following table:

n	1	2	3	4	5
Sum of the integers up to n	1	3	6	10	15
n^3	1	8	27	64	125
Sum of the cubes of the integers up to n	1	9	36	100	225

Here the terms in the fourth row are the squares of the corresponding terms in the second row. Thus it is natural to suppose that

$$1^3 + 2^3 + \dots + n^3 = (1 + 2 + \dots + n)^2$$

Now $1 + 2 + \dots + n$ is an arithmetical progression whose sum is $\frac{1}{2}n(n+1)$. Therefore we should suppose that

$$1^3 + 2^3 + \dots + n^3 = \tfrac{1}{4}n^2(n+1)^2$$

In proof by induction, it is shown that if the result holds for some particular value of n, say k, then it also holds for $n = k + 1$. It is then verified that the result does hold for some value of n, usually 1 or 2.

Example 9 *Prove by induction that* $1^3 + 2^3 + \dots + n^3 = \frac{1}{4}n^2(n+1)^2$.

Suppose the result holds for a particular value of n, say k; that is,

$$1^3 + 2^3 + \dots + k^3 = \tfrac{1}{4}k^2(k+1)^2$$

Then, adding the next term of the series, $(k+1)^3$, to both sides, we obtain

$$\begin{aligned} 1^3 + 2^3 + \dots + k^3 + (k+1)^3 &= \tfrac{1}{4}k^2(k+1)^2 + (k+1)^3 \\ &= (k+1)^2\left(\frac{k^2}{4} + k + 1\right) \\ &= (k+1)^2\left(\frac{k^2 + 4k + 4}{4}\right) \end{aligned}$$

$$\therefore\ 1^3 + 2^3 + \dots + (k+1)^3 = \tfrac{1}{4}(k+1)^2(k+2)^2$$

Now this is the formula with $n = k + 1$. Therefore if the result holds for $n = k$, then it also holds for $n = k + 1$; but if $n = 1$,

$$\text{L.H.S.} = 1^3 = 1 \quad \text{and} \quad \text{R.H.S.} = \tfrac{1}{4} \times 1^2 \times 2^2 = 1$$

Therefore, since the result is true for $n = 1$, it follows, by what has been shown above, that it must also be true for $n = 2$. From this it follows that the result is true for $n = 3$, and so on, for all positive integral values of n.

Exercise 13d

Prove the following results by induction:

1 $1 + 2 + \dots + n = \frac{1}{2}n(n+1)$.
2 $1^2 + 2^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1)$.

3 $1 \times 2 + 2 \times 3 + \ldots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$.

4 $1 \times 3 + 2 \times 4 + \ldots + n(n+2) = \frac{1}{6}n(n+1)(2n+7)$.

5 $3 + 8 + \ldots + (n^2 - 1) = \frac{1}{6}n(n-1)(2n+5)$.

6 $a + ar + \ldots + ar^{n-1} = a\left(\dfrac{1-r^n}{1-r}\right)$.

7 $\dfrac{1}{1 \times 2} + \dfrac{1}{2 \times 3} + \ldots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$.

8 $\dfrac{1}{1 \times 3} + \dfrac{1}{2 \times 4} + \ldots + \dfrac{1}{n(n+2)} = \dfrac{3}{4} - \dfrac{2n+3}{2(n+1)(n+2)}$.

9 $\dfrac{3}{4} + \dfrac{5}{36} + \ldots + \dfrac{2n-1}{n^2(n-1)^2} = 1 - \dfrac{1}{n^2}$.

10 $\dfrac{1}{1 \times 2 \times 3} + \dfrac{1}{2 \times 3 \times 4} + \ldots + \dfrac{1}{n(n+1)(n+2)} = \dfrac{1}{4} - \dfrac{1}{2(n+1)(n+2)}$.

11 $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$. [Use the formula for differentiating a product.]

12 $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{1}{3}n(4n^2 - 1)$.

13 $1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3 = n^2(2n^2 - 1)$.

14 $4^2 + 7^2 + 10^2 + \ldots + (3n+1)^2 = \frac{1}{2}n(6n^2 + 15n + 11)$.

15 Show that $\dbinom{n}{r} + \dbinom{n}{r-1} = \dbinom{n+1}{r}$, and prove by induction that

$$(1+x)^n = 1 + nx + \ldots + \binom{n}{r}x^r + \ldots + x^n$$

where $\dbinom{n}{r} = \dfrac{n!}{(n-r)!r!}$ and r is a positive integer, less than or equal to n.

Further series

13.7 Certain series can be summed by means of the results:

$$1 + 2 + \ldots + n = \tfrac{1}{2}n(n+1)$$
$$1^2 + 2^2 + \ldots + n^2 = \tfrac{1}{6}n(n+1)(2n+1)$$
$$1^3 + 2^3 + \ldots + n^3 = \tfrac{1}{4}n^2(n+1)^2$$

which appear in the last section and exercise.

It should be noted that they may be used to sum the series to more or less than n terms. For instance,

$$\begin{aligned} 1^3 + 2^3 + \ldots + (2n+1)^3 &= \tfrac{1}{4}(2n+1)^2\{(2n+1)+1\}^2 \\ &= \tfrac{1}{4}(2n+1)^2(2n+2)^2 \\ &= \tfrac{1}{4}(2n+1)^2 4(n+1)^2 \\ &= (2n+1)^2(n+1)^2 \end{aligned}$$

Qu. 6 Find the sums of the following series:

(a) $1 + 2 + \dots + 2n$, (b) $1^2 + 2^2 + \dots + (n+1)^2$,

(c) $1^3 + 2^3 + \dots + (n-1)^3$, (d) $1 + 2 + \dots + (2n-1)$,

(e) $1^2 + 2^2 + \dots + (2n)^2$, (f) $1^3 + 2^3 + \dots + (2n-1)^3$.

Example 10 *Find the sum of the series* $1^3 + 3^3 + 5^3 + \dots + (2n+1)^3$

This series can be thought of as $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \dots + (2n+1)^3$ with the even terms missing.

We found above that

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \dots + (2n+1)^3 = (2n+1)^2(n+1)^2$$

and so it remains to find the sum of the series

$$\begin{aligned} 2^3 + 4^3 + 6^3 + \dots + (2n)^3 &= 2^3 \times 1^3 + 2^3 \times 2^3 + 2^3 \times 3^3 + \dots + 2^3 \times n^3 \\ &= 8(1^3 + 2^3 + 3^3 + \dots + n^3) \\ &= 8 \times \tfrac{1}{4}n^2(n+1)^2 = 2n^2(n+1)^2 \end{aligned}$$

$$\begin{aligned} \text{Therefore } 1^3 + 3^3 + 5^3 + \dots + (2n+1)^3 &= (2n+1)^2(n+1)^2 - 2n^2(n+1)^2 \\ &= (n+1)^2\{(2n+1)^2 - 2n^2\} \\ &= (n+1)^2(4n^2 + 4n + 1 - 2n^2) \end{aligned}$$

Therefore the sum is $(n+1)^2(2n^2 + 4n + 1)$.

Example 11 *Find the sum of n terms of the series* $2 \times 3 + 3 \times 4 + 4 \times 5 + \dots$.

The mth term of this series is $(m+1)(m+2)$, or $m^2 + 3m + 2$. Therefore we require the sum of

$$\begin{aligned} &1^2 + 3 \times 1 + 2 \\ +\ &2^2 + 3 \times 2 + 2 \\ +\ &3^2 + 3 \times 3 + 2 \\ +\ &\dots\dots\dots\dots \\ +\ &n^2 + 3 \times n + 2 \end{aligned}$$

Now the sums of the three columns are respectively

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \dots + n^2 &= \tfrac{1}{6}n(n+1)(2n+1) \\ 3(1 + 2 + 3 + \dots + n) &= \tfrac{3}{2}n(n+1) \\ (2 + 2 + 2 + \dots + 2) &= 2n \end{aligned}$$

Therefore the sum of the series is

$$\begin{aligned} \frac{1}{6}n(n+1)(2n+1) + \frac{3}{2}n(n+1) + 2n &= \frac{n}{6}\{(n+1)(2n+1) + 9(n+1) + 12\} \\ &= \frac{n}{6}(2n^2 + 3n + 1 + 9n + 9 + 12) \\ &= \frac{n}{6}(2n^2 + 12n + 22) \end{aligned}$$

$$\frac{1}{6}n(n+1)(2n+1)+\frac{3}{2}n(n+1)+2n=\frac{n}{3}(n^2+6n+11)$$

Therefore the sum of the first n terms of the series $2 \times 3 + 3 \times 4 + 4 \times 5 + \ldots$ is $\frac{1}{3}n(n^2+6n+11)$.

The $\sum$ notation

13.8 It is useful to have a short way of writing expressions like

$$1^2+2^2+\ldots+n^2$$

This is done by writing

$$\sum m^2$$

which means, 'the sum of all the terms like m^2'. For extra precision, however, numbers are placed below and above the $\sum$, to show where the series begins and ends. Thus

$$\sum_1^n m^2 = 1^2+2^2+\ldots+n^2$$

and

$$\sum_2^5 m(m+2) = 2 \times 4 + 3 \times 5 + 4 \times 6 + 5 \times 7$$

Exercise 13e

1 Write in full:

(a) $\sum_1^4 m^3$, (b) $\sum_2^n m^2$, (c) $\sum_1^n (m^2+m)$,

(d) $\sum_1^3 \frac{1}{m(m+1)}$, (e) $\sum_2^5 2^m$, (f) $\sum_1^4 (-1)^m m^2$,

(g) $\sum_1^n m^m$, (h) $\sum_3^6 \frac{(-1)^m}{m}$, (i) $\sum_n^{n+2} m(m-1)$,

(j) $\sum_{n-2}^n \frac{m}{m+1}$.

2 Write in the $\sum$ notation:

(a) $1+2+3+\ldots+n$,

(b) $1^4+2^4+\ldots+n^4+(n+1)^4$,

(c) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$,

(d) $3^2+3^3+3^4+3^5$,

(e) $2 \times 7 + 3 \times 8 + 4 \times 9 + 5 \times 10 + 6 \times 11$,

(f) $1 + \frac{2}{3} + \frac{3}{9} + \frac{4}{27} + \frac{5}{81}$,

(g) $\frac{1 \times 3}{4} + \frac{2 \times 5}{6} + \frac{3 \times 7}{8} + \frac{4 \times 9}{10} + \frac{5 \times 11}{12}$,

(h) $-1 + 2 - 3 + 4 - 5 + 6$,

(i) $1 - 2 + 4 - 8 + 16 - 32$,

(j) $1 \times 3 - 2 \times 5 + 3 \times 7 - 4 \times 9 + 5 \times 11$.

3 Use the results quoted at the beginning of §13.7 to find the sums of the following series:
(a) $1 + 2 + 3 + \ldots + (2n + 1)$,
(b) $1^2 + 2^2 + 3^2 + \ldots + (n - 1)^2$,
(c) $1^3 + 2^3 + 3^3 + \ldots + (2n)^3$,
(d) $3 + 5 + 7 + \ldots + (2n + 1)$,
(e) $2 + 5 + 8 + 11 + \ldots$, to n terms,
(f) $5 + 9 + 13 + 17 + \ldots$, to n terms,
(g) $2 + 5 + 10 + \ldots + (n^2 + 1)$,
(h) $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + \ldots$, to n terms,
(i) $1 \times 3 + 2 \times 4 + 3 \times 5 + 4 \times 6 + \ldots$, to n terms,
(j) $2^2 + 4^2 + 6^2 + \ldots + (2n)^2$,
(k) $1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2$,
(l) $2 + 10 + 30 + \ldots + (n^3 + n)$,
(m) $2 + 12 + 36 + \ldots + (n^3 + n^2)$.

Infinite geometrical progressions

13.9 Consider the geometrical progression

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^{n-1}}$$

The sum of these n terms, obtained by the formula of §13.4, is given by

$$S_n = \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = 2\left(1 - \left(\frac{1}{2}\right)^n\right)$$

Now as n increases, $(\frac{1}{2})^n$ approaches zero; and $(\frac{1}{2})^n$ can be made as close to zero as we like, if n is large enough. Therefore the sum of n terms approaches 2, as closely as we please, as n increases.

This is what is meant by writing that the infinite series

$$1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{m-1}} + \ldots = 2$$

The limit 2 is called its **sum to infinity**.

In general, the sum of the geometrical progression

$$\boldsymbol{a + ar + ar^2 + \dots + ar^{n-1} = a\left(\frac{1-r^n}{1-r}\right)}$$

If r lies between -1 and $+1$, i.e. $|r| < 1$, assuming that r^n approaches zero as n increases, the sum to infinity of the series

$$a + ar + ar^2 + \dots + ar^{m-1} + \dots = \frac{a}{1-r}$$

Example 12 *Express as fractions in their lowest terms:* (a) $0.0\dot{7}$, (b) $0.\dot{4}\dot{5}$.

(a) $0.0\dot{7}$ means 0.0777 ..., which may be written

$$\frac{7}{100} + \frac{7}{1000} + \frac{7}{10\,000} + \dots$$

This is a geometrical progression, and in the notation of §13.4, $a = \frac{7}{100}$ and $r = \frac{1}{10}$. Therefore

$$S_n = \frac{7}{100}\left(\frac{1-(\frac{1}{10})^n}{1-\frac{1}{10}}\right)$$

Therefore the sum to infinity, S_∞, is given by

$$S_\infty = \frac{7}{100}\left(\frac{1}{\frac{9}{10}}\right) = \frac{7}{100} \times \frac{10}{9} = \frac{7}{90}$$

$$\therefore\ 0.0\dot{7} = \frac{7}{90}$$

(b) $0.\dot{4}\dot{5}$ means 0.454 545 ..., which may be written

$$\frac{45}{100} + \frac{45}{10\,000} + \frac{45}{1\,000\,000} + \dots$$

In this geometrical progression, $a = \frac{45}{100}$, and $r = \frac{1}{100}$.

$$\therefore\ S_n = \frac{45}{100}\left(\frac{1-(\frac{1}{100})^n}{1-\frac{1}{100}}\right)$$

$$\therefore\ S_\infty = \frac{45}{100}\left(\frac{1}{\frac{99}{100}}\right) = \frac{45}{100} \times \frac{100}{99} = \frac{5}{11}$$

$$\therefore\ 0.\dot{4}\dot{5} = \frac{5}{11}$$

Using this method, any recurring decimal can be expressed as a rational number. (See §2.3.)

Exercise 13f

1 Write down the sums of the first n terms of the following series, and deduce their sums to infinity:

(a) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots$, (b) $12 + 6 + 3 + 1\frac{1}{2} + \dots$,

(c) $\dfrac{3}{10} + \dfrac{3}{100} + \dfrac{3}{1000} + \dfrac{3}{10\,000} + \dots$, (d) $\dfrac{13}{100} + \dfrac{13}{10\,000} + \dfrac{13}{1\,000\,000} + \dots$,

(e) $0.5 + 0.05 + 0.005 + \dots$, (f) $0.54 + 0.0054 + 0.000\,054 + \dots$,

(g) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$, (h) $54 - 18 + 6 - 2 + \dots$.

2 Express the following recurring decimals as rational numbers:
(a) $0.\dot{8}$, (b) $0.\dot{1}\dot{2}$, (c) $3.\dot{2}$, (d) $2.\dot{6}\dot{9}$, (e) $1.00\dot{4}$, (f) $2.\dot{9}6\dot{0}$.

3 If the sum to infinity of a G.P. is three times the first term, what is the common ratio?

4 The sum to infinity of a G.P. is 4 and the second term is 1. Find the first, third, and fourth terms.

5 The second term of a G.P. is 24 and its sum to infinity is 100. Find the two possible values of the common ratio and the corresponding first terms.

Exercise 13g (Miscellaneous)

1 Find the sum of the integers between 1 and 100 which are divisible by 3.

2 How many terms of the geometrical progression $\frac{1}{16} + \frac{1}{8} + \frac{1}{4} + \dots$ are needed to make a total of $2^{16} - \frac{1}{16}$?

3 Prove by induction that $1 \times 4 + 2 \times 5 + \dots + n(n+3) = \frac{1}{3}n(n+1)(n+5)$.

4 Show that the sums to infinity of the geometrical progressions

$$3 + \tfrac{9}{4} + \tfrac{27}{16} + \dots \quad \text{and} \quad 4 + \tfrac{8}{3} + \tfrac{16}{9} + \dots$$

are equal.

5 How many terms of the arithmetical progression $2 + 3\frac{1}{4} + 4\frac{1}{2} + \dots$ are needed to make a total of 204?

6 An arithmetical progression has thirteen terms whose sum is 143. The third term is 5. Find the first term.

7 The sum of n terms of a certain series in $3n^2 + 10n$ for all values of n. Find the nth term and show that the series is an arithmetical progression.

8 Find the sum of the series $2 + 6 + \dots + (n^2 - n)$.

9 Show that the sum of the first n odd numbers is a perfect square. Show also, that $57^2 - 13^2$ is the sum of certain consecutive odd numbers, and find them.

10 What is the sum of the integers from 1 to 100, inclusive, which are not divisible by 6?

11 Find the sum of the first n terms of the geometrical progression

$$5 + 15 + 45 + \dots$$

What is the smallest number of terms whose total is more than 10^8?

12 The sum to infinity of a geometrical progression with a positive common ratio is 9 and the sum of the first two terms is 5. Find the first four terms of the progression.

13 Show that, if $\log a$, $\log b$, $\log c$ are consecutive terms of an arithmetical progression, then a, b, c are in geometrical progression.

14 The eighth term of an arithmetical progression is twice the third term, and the sum of the first eight terms is 39. Find the first three terms of the progression, and show that its sum to n terms is $\frac{3}{8}n(n+5)$.

15 Find the number n such that the sum of the integers from 1 to $n-1$ is equal to the sum of the integers from $n+1$ to 49.

16 Show that there are two possible geometrical progressions in each of which the first term is 8, and the sum of the first three terms is 14. Find the second term and the sum of the first seven terms in each progression.

17 Prove by induction that

$$\frac{1}{2}+\frac{1}{6}+\ldots+\frac{1}{n(n-1)}=1-\frac{1}{n}$$

18 Find the sum of the series $3+6+11+\ldots+(n^2+2)$.

19 If a and b are the first and last terms of an arithmetical progression of $r+2$ terms, find the second and the $(r+1)$th terms.

20 The sum of n terms of a certain series is 4^n-1 for all values of n. Find the first three terms and the nth term, and show that the series is a geometrical progression.

21 A child wishes to build up a triangular pile of toy bricks so as to have 1 brick in the top row, 2 in the second, 3 in the third and so on. If he has 100 bricks, how many rows can he complete and how many bricks has he left?

22 Show that the sum of the odd numbers from 1 to 55 inclusive is equal to the sum of the odd numbers from 91 to 105 inclusive.

23 The second, fifth, and eleventh terms of an arithmetical progression are in geometrical progression, and the seventh term is 4. Find the first term and the common difference. What is the common ratio of the geometrical progression?

24 A chess board has 64 squares. Show that ten thousand million people each prepared to contribute a million pounds could not bring sufficient money to put 1p on the first square, 2p on the second, 4p on the third, 8p on the fourth, and so on for the 64 squares.

25 Prove that

$$\log a+\log ax+\log ax^2+\ldots \quad \text{to } n \text{ terms} = n\log a+\tfrac{1}{2}n(n-1)\log x$$

26 Given the series $1+2x+3x^2+4x^3+\ldots$,

(a) find the sum of the first n terms when $x=1$,

(b) find, by multiplying by $1-x$, the sum of the first n terms when x is not equal to one.

Chapter 14

The binomial theorem

Pascal's triangle

14.1 It is well known that

$$(a+b)^2 = a^2 + 2ab + b^2$$

and it is the object of this chapter to show how higher powers of $a+b$ can be expanded with little difficulty.

Most readers will not be able to write down similar expressions for $(a+b)^3$ and $(a+b)^4$ without doing some work on paper, and so the long multiplication is given below. The reason for printing the coefficients in heavy type will appear later.

$$\begin{array}{l} \mathbf{1}a^2 + \mathbf{2}ab \;\; + \mathbf{1}b^2 \\ \qquad\quad a + b \\ \hline \mathbf{1}a^3 + \mathbf{2}a^2b + \mathbf{1}ab^2 \\ \qquad\quad \mathbf{1}a^2b + \mathbf{2}ab^2 + \mathbf{1}b^3 \\ \hline \mathbf{1}a^3 + \mathbf{3}a^2b + \mathbf{3}ab^2 + \mathbf{1}b^3 \end{array} \qquad \begin{array}{l} \mathbf{1}a^3 + \mathbf{3}a^2b + \mathbf{3}ab^2 \;\; + \mathbf{1}b^3 \\ \qquad\qquad\qquad\quad a + b \\ \hline \mathbf{1}a^4 + \mathbf{3}a^3b + \mathbf{3}a^2b^2 + \mathbf{1}ab^3 \\ \qquad\quad \mathbf{1}a^3b + \mathbf{3}a^2b^2 + \mathbf{3}ab^3 + \mathbf{1}b^4 \\ \hline \mathbf{1}a^4 + \mathbf{4}a^3b + \mathbf{6}a^2b^2 + \mathbf{4}ab^3 + \mathbf{1}b^4 \end{array}$$

The results so far obtained are summarised below.

$$\begin{aligned} (a+b)^2 &= \phantom{\mathbf{1}a^4 +{}} \mathbf{1}a^2 + \mathbf{2}ab + \mathbf{1}b^2 \\ (a+b)^3 &= \mathbf{1}a^3 + \mathbf{3}a^2b + \mathbf{3}ab^2 + \mathbf{1}b^3 \\ (a+b)^4 &= \mathbf{1}a^4 + \mathbf{4}a^3b + \mathbf{6}a^2b^2 + \mathbf{4}ab^3 + \mathbf{1}b^4 \end{aligned}$$

It is clearer, however, if the coefficients are written alone.

$$\begin{array}{ccccccccc} & & \mathbf{1} & & \mathbf{2} & & \mathbf{1} & & \\ & \mathbf{1} & & \mathbf{3} & & \mathbf{3} & & \mathbf{1} & \\ \mathbf{1} & & \mathbf{4} & & \mathbf{6} & & \mathbf{4} & & \mathbf{1} \end{array}$$

The reader may be able to guess the next line and, more important, may be able to see how the table can be continued, obtaining each line from the previous one.

To show the construction of the table of coefficients, the last three lines of the long multiplications are written overleaf, leaving out the letters.

$$\begin{array}{cccc} 1 & 2 & 1 & \\ & 1 & 2 & 1 \\ \hline 1 & 3 & 3 & 1 \end{array} \qquad \begin{array}{ccccc} 1 & 3 & 3 & 1 & \\ & 1 & 3 & 3 & 1 \\ \hline 1 & 4 & 6 & 4 & 1 \end{array}$$

Thus it may be seen that every coefficient in the table is obtained from the two on either side of it in the row above. In this way the next line can be obtained:

$$\begin{array}{ccccccccccc} & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

For completeness, it may be observed that

$(a+b)^0 = \mathbf{1}$ and $(a+b)^1 = \mathbf{1}a + \mathbf{1}b$

Therefore the table of coefficients may be written in a triangle (known as Pascal's triangle, after the French mathematician and philosopher Blaise Pascal, 1623–1662) as follows:

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 1 & & 1 & & & & & \\ & & & & 1 & & 2 & & 1 & & & & \\ & & & 1 & & 3 & & 3 & & 1 & & & \\ & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

When an expression is written as a series of terms, it is said to be **expanded**, and the series is called its **expansion**. Thus the expansion of $(a+b)^3$ is

$$a^3 + 3a^2b + 3ab^2 + b^3$$

Certain points should be noted about the expansion of $(a+b)^n$. They should be verified for the cases $n = 2, 3, 4$, in the expansions obtained so far.

(a) Reading from either end of each row, the *coefficients* are the same.
(b) There are $(n+1)$ terms.
(c) Each term is of degree n.
(d) The coefficients are obtained from the row in Pascal's triangle beginning 1, n.

Example 1 *Expand $(a+b)^6$ in descending powers of a.*

There will be 7 terms, involving

$$a^6, \quad a^5b, \quad a^4b^2, \quad a^3b^3, \quad a^2b^4, \quad ab^5, \quad b^6,$$

each of which is of degree 6. Their coefficients, obtained from Pascal's triangle, are respectively

$$1, \quad 6, \quad 15, \quad 20, \quad 15, \quad 6, \quad 1.$$

Therefore the expansion of $(a+b)^6$ in descending powers of a is

$$a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

Example 2 *Expand* $(2x+3y)^3$ *in descending powers of* x.

Here $a=2x$, $b=3y$, and so there will be four terms involving

$$(2x)^3,\quad (2x)^2(3y),\quad (2x)(3y)^2,\quad (3y)^3.$$

Their coefficients, obtained from Pascal's triangle are respectively

$$1,\qquad 3,\qquad 3,\qquad 1.$$

Therefore the expansion of $(2x+3y)^3$, in descending powers of x is

$$(2x)^3+3(2x)^2(3y)+3(2x)(3y)^2+(3y)^3$$

which simplifies to

$$8x^3+\ 36x^2y\ +\ 54xy^2\ +27y^3$$

Example 3 *Obtain the expansion of* $(2x-\frac{1}{2})^4$, *in descending powers of* x.

Here $a=2x$ and $b=-\frac{1}{2}$, therefore the five terms of the expansion will involve

$$(2x)^4,\quad (2x)^3(-\tfrac{1}{2}),\quad (2x)^2(-\tfrac{1}{2})^2,\quad (2x)(-\tfrac{1}{2})^3,\quad (-\tfrac{1}{2})^4$$

and their coefficients will be respectively

$$1,\qquad 4,\qquad 6,\qquad 4,\qquad 1.$$

$$\begin{aligned}\therefore (2x-\tfrac{1}{2})^4 &= (2x)^4+4(2x)^3(-\tfrac{1}{2})+6(2x)^2(-\tfrac{1}{2})^2+4(2x)(-\tfrac{1}{2})^3+(-\tfrac{1}{2})^4\\ &= 16x^4+4(8x^3)(-\tfrac{1}{2})+6(4x^2)(\tfrac{1}{4})+4(2x)(-\tfrac{1}{8})+\tfrac{1}{16}\end{aligned}$$

Therefore the expansion of $(2x-\frac{1}{2})^4$, in descending powers of x, is

$$16x^4-16x^3+6x^2-x+\tfrac{1}{16}$$

Note that terms are alternately + and −, according to the even or odd degree of $(-\frac{1}{2})$.

Example 4 *Use Pascal's triangle to obtain the value of* $(1.002)^5$, *correct to six places of decimals.*

1.002 may be written $(1+0.002)$, so that the expansion of $(a+b)^5$ may be used, with $a=1$ and $b=0.002$.

The terms in the expansion will involve

$$1,\quad (0.002),\quad (0.002)^2,\quad (0.002)^3,\quad (0.002)^4,\quad (0.002)^5$$

and their coefficients will be

$$1,\qquad 5,\qquad 10,\qquad 10,\qquad 5,\qquad 1,$$

respectively. Now the last three terms will make no difference to the answer, correct to six places of decimals. Therefore

$$\begin{aligned}(1.002)^5 &\approx 1+5(0.002)+10(0.002)^2\\ &= 1+0.010+0.000\,040\end{aligned}$$

and so $(1.002)^5=1.010\,040$, correct to six places of decimals.

Exercise 14a

This exercise is intended to give the reader some practice in using Pascal's triangle; calculators should not be used in the numerical questions.

1 Expand:

(a) $(a+b)^5$, (b) $(x+y)^3$, (c) $(x+2y)^4$,

(d) $(1-z)^4$, (e) $(2x+3y)^4$, (f) $(4z+1)^3$,

(g) $(a-b)^6$, (h) $(a-2b)^3$, (i) $(3x-y)^4$,

(j) $(2x+\frac{1}{3})^3$, (k) $\left(x-\frac{1}{x}\right)^5$, (l) $\left(\frac{x}{2}+\frac{2}{x}\right)^4$,

(m) $(a+b)^7$, (n) $(a^2-b^2)^5$, (o) $(a-b)^3(a+b)^3$.

2 Simplify, leaving surds in the answers, where appropriate:
(a) $(1+\sqrt{2})^3+(1-\sqrt{2})^3$, (b) $(2+\sqrt{3})^4+(2-\sqrt{3})^4$,
(c) $(1+\sqrt{2})^3-(1-\sqrt{2})^3$, (d) $(2+\sqrt{6})^4-(2-\sqrt{6})^4$,
(e) $(\sqrt{2}+\sqrt{3})^4+(\sqrt{2}-\sqrt{3})^4$, (f) $(\sqrt{6}+\sqrt{2})^3-(\sqrt{6}-\sqrt{2})^3$.

3 Write down the expansion of $(2+x)^5$ in ascending powers of x. Taking the first three terms of the expansion, put $x=0.001$, and find the value of $(2.001)^5$ correct to five places of decimals.

4 Write down the expansion of $(1+\frac{1}{4}x)^4$. Taking the first three terms of the expansion, put $x=0.1$, and find the value of $(1.025)^4$, correct to three places of decimals.

5 Expand $(2-x)^6$ in ascending powers of x. Taking $x=0.002$, and using the first three terms of the expansion, find the value of $(1.998)^6$ as accurately as you can. Examine the fourth term of the expansion to find to how many places of decimals your answer is correct.

Introduction to the binomial theorem

14.2 In the last section it was shown how $(a+b)^n$ could be expanded, for a known value of n, by using Pascal's triangle. If n is large, this may involve a considerable amount of addition, and when (as is often the case) only the first few terms are required, it is much quicker to use a formula that will be obtained in the next section.

The last section began with the expansions of $(a+b)^2$ and $(a+b)^3$. Now, consider the expansions of $(a+b)(c+d)$ and $(a+b)(c+d)(e+f)$.

It is easily seen that

$$(a+b)(c+d)=ac+ad+bc+bd$$

To obtain the expansion of $(a+b)(c+d)(e+f)$, each term of $ac+ad+bc+bd$ is multiplied by e and f, giving

$$ace+ade+bce+bde+acf+adf+bcf+bdf$$

Note that each term contains one factor from each bracket, and that the expansion consists of the sum of all such combinations.

Now the expansion of $(a+b)(c+d)(e+f)(g+h)$ would be obtained by multiplying each term of the expansion by g and by h. So, continuing this method of expansion, it follows that, if the product of n factors is expanded, each term contains one factor from each bracket, and that the expansion consists of the sum of all such combinations.

The expansion of $(a+b)^5$ will be obtained by an argument making use of this fact.

$$(a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b)$$

(a) Choosing an a from each bracket we obtain a^5.
(b) The term in a^4 is obtained by choosing a b from one bracket, and a's from the other four. This can be done in ${}^5C_1$ ways, giving ${}^5C_1a^4b$.
(c) The term in a^3 is obtained by choosing b's from two brackets, and a's from the other three. This can be done in ${}^5C_2$ ways, giving ${}^5C_2a^3b^2$.
(d) Similarly, the terms in a^2 and a are ${}^5C_3a^2b^3$ and ${}^5C_4ab^4$.
(e) Choosing a b from each bracket we obtain b^5.

$$\therefore (a+b)^5 = a^5 + {}^5C_1a^4b + {}^5C_2a^3b^2 + {}^5C_3a^2b^3 + {}^5C_4ab^4 + b^5$$

The binomial theorem

14.3 *If n is a positive integer,*

$$(a+b)^n = a^n + {}^nC_1a^{n-1}b + \ldots + {}^nC_ra^{n-r}b^r + \ldots + b^n$$

where ${}^nC_r = \dfrac{n!}{(n-r)!r!}$.

The expansion of $(a+b)^n$ is obtained as follows.

$$(a+b)^n = (a+b)(a+b)\ldots(a+b), \quad \text{to } n \text{ factors.}$$

(a) Choosing an a from each bracket we obtain a^n.
(b) The term in a^{n-1} is obtained by choosing a b from one bracket, and a's from the other $n-1$. This can be done in ${}^nC_1$ ways, giving ${}^nC_1a^{n-1}b$.
(c) The term in a^{n-2} is obtained by choosing a b from two brackets, and a's from the other $n-2$. This can be done in ${}^nC_2$ ways, giving ${}^nC_2a^{n-2}b^2$.
(d) The term in a^{n-r} is obtained by choosing a b from r brackets, and a's from the other $n-r$. This can be done in ${}^nC_r$ ways, giving ${}^nC_ra^{n-r}b^r$.
(e) Choosing a b from each bracket we obtain b^n.

This proves the theorem.

When only the first few terms of an expansion are required, the theorem is used in the form

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots + b^n$$

This follows immediately, since

$$^nC_1 = n, \qquad ^nC_2 = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2!} \qquad \text{and}$$

$$^nC_3 = \frac{n!}{(n-3)!3!} = \frac{n(n-1)(n-2)}{3!}$$

In case the name of the theorem is not understood, it may be helpful to remark that an expression with one term is called a mononomial, one which has two terms is a binomial, and one with three terms is a trinomial. Thus the theorem about the expansion of a power of two terms is called the binomial theorem.

Example 5 *Find the coefficient of* x^{10} *in the expansion of* $(2x-3)^{14}$.

The term in $(2x)^{10}(-3)^4$ is the only one needed, and by the binomial theorem it is

$$^{14}C_4(2x)^{10}(-3)^4$$

Therefore the coefficient of x^{10} is $\dfrac{14!}{10!4!}2^{10} \times 3^4$.

It is important to note that we could equally well have written the term as

$$^{14}C_{10}(2x)^{10}(-3)^4$$

because $^{14}C_{10} = {^{14}C_4}$. This is clear if they are written in factorial notation:

$$^{14}C_{10} = \frac{14!}{4!10!} \quad \text{and} \quad ^{14}C_4 = \frac{14!}{10!4!}$$

Alternatively, $^{14}C_{10}$ is the number of ways of choosing ten objects from fourteen unlike objects; but if ten are chosen, four are left, and so it must also be the number of ways of choosing four objects from fourteen unlike objects, which is $^{14}C_4$.

Qu. 1 Show that $^nC_{n-r} = {^nC_r}$.

It is useful to note in Example 5 that the numbers whose factorials appear in the coefficient

$$\frac{14!}{10!4!}$$

are all indices. 14 is the index of $2x-3$, 10 is the index of $2x$ and 4 is the index of -3. That this is always the case should be clear if the term in $a^{n-r}b^r$ in the expansion of $(a+b)^n$ is written with factorial notation:

$$\frac{n!}{(n-r)!r!}a^{n-r}b^r$$

Example 6 *Obtain the first four terms of the expansion of $(1+\frac{1}{2}x)^{10}$ in ascending powers of x. Hence find the value of $(1.005)^{10}$, correct to four decimal places.*

Using the second form of the binomial theorem,

$$\left(1+\frac{1}{2}x\right)^{10} = 1 + 10\left(\frac{x}{2}\right) + \frac{10\times 9}{2\times 1}\left(\frac{x}{2}\right)^2 + \frac{10\times 9\times 8}{3\times 2\times 1}\left(\frac{x}{2}\right)^3 + \dots$$

$$= 1 + 5x + \frac{45}{4}x^2 + 15x^3 + \dots$$

Now $\frac{1}{2}x = 0.005$, if $x = 0.01$; so substituting this value of x,

$$(1.005)^{10} \approx 1 + 5(0.01) + 11.25(0.01)^2 + 15(0.01)^3$$
$$= 1 + 0.05 + 0.001\,125 + 0.000\,015$$

Therefore $(1.005)^{10} = 1.0511$, correct to four places of decimals.

Example 7 *Obtain the expansion of $(1+x-2x^2)^8$, as far as the term in x^3.*

$(1+x-2x^2)^8$ may be written $\{1+(x-2x^2)\}^8$, which may then be expanded by the binomial theorem.

$$\{1+(x-2x^2)\}^8$$

$$= 1 + 8(x-2x^2) + \frac{8\times 7}{2!}(x-2x^2)^2 + \frac{8\times 7\times 6}{3!}(x-2x^2)^3 + \dots$$

$$= 1 + 8(x-2x^2) + 28(x^2-4x^3+4x^4) + 56(x^3 + \text{other terms}) + \dots$$

$$= 1 + 8x - 16x^2 + 28x^2 - 112x^3 + 56x^3 + \text{terms in } x^4 \text{ and higher powers}$$

$\therefore (1+x-2x^2)^8 = 1 + 8x + 12x^2 - 56x^3$ as far as the term in x^3.

Exercise 14b

Calculators should not be used in this exercise.

1 Write down the terms indicated, in the expansions of the following, and simplify your answers:
(a) $(x+2)^8$, term in x^5; (b) $(3u-2)^5$, term in u^3;
(c) $(2t-\frac{1}{2})^{12}$, term in t^7; (d) $(2x+y)^{11}$, term in x^3.

2 Write down, and simplify, the terms indicated, in the expansions of the following in ascending powers of x:
(a) $(1+x)^9$, 4th term; (b) $(2-x/2)^{12}$, 4th term;
(c) $(3+x)^7$, 5th term; (d) $(x+1)^{20}$, 3rd term.

3 Write down, and simplify, the coefficients of the terms indicated, in the expansions of the following:
(a) $(\frac{1}{2}t+\frac{1}{2})^{10}$, term in t^4; (b) $(4+\frac{3}{4}x)^6$, term in x^3;
(c) $(2x-3)^7$, term in x^5; (d) $(3+\frac{1}{3}y)^{11}$, term in y^5.

4 Write down the coefficients of the terms indicated, in the expansions of the following in ascending powers of x:
(a) $(1+x)^{16}$, 3rd term; (b) $(2-x)^{20}$, 18th term;
(c) $(3+2x)^6$, 4th term; (d) $(2+\frac{3}{2}x)^8$, 5th term.

5 Write down the terms involving

(a) $x^4\left(\dfrac{1}{x}\right)^2$, (b) $x^3\left(\dfrac{1}{x}\right)^3$, in the expansion of $\left(x+\dfrac{1}{x}\right)^6$.

6 Write down the constant terms in the expansions of

(a) $\left(x-\dfrac{1}{x}\right)^8$, (b) $\left(2x^2-\dfrac{1}{2x}\right)^6$.

7 Find the coefficients of the terms indicated in the expansions of the following:

(a) $\left(x+\dfrac{1}{x}\right)^6$, term in x^4; (b) $\left(2x+\dfrac{1}{x}\right)^7$, term in $\dfrac{1}{x^5}$;

(c) $\left(x-\dfrac{2}{x}\right)^8$, term in x^6.

8 Find the ratio of the term in x^5 to the term in x^6, in the expansion of $(2x+3)^{20}$.

9 Find the ratio of the term in x^7 to the term in x^8 in the expansion of $(3x+\frac{2}{3})^{17}$.

10 Find the ratio of the term in a^r to the term in a^{r+1} in the expansion of $(a+b)^n$.

11 Write down the first four terms of the expansions of the following, in ascending powers of x:
(a) $(1+x)^{10}$, (b) $(1+\frac{1}{2}x)^9$, (c) $(1-x)^{11}$,
(d) $(x+1)^{12}$, (e) $(2+\frac{1}{2}x)^8$, (f) $(2-\frac{1}{2}x)^7$.

12 Use the binomial theorem to find the values of
(a) $(1.01)^{10}$, correct to three places of decimals;
(b) $(2.001)^{10}$, correct to six significant figures;
(c) $(0.997)^{12}$, correct to three places of decimals;
(d) $(1.998)^8$, correct to two places of decimals.

13 Expand the following as far as the terms in x^3:
(a) $(1+x+x^2)^3$, (b) $(1+2x-x^2)^6$,
(c) $(1-x-x^2)^4$, (d) $(2+x+x^2)^5$,
(e) $(1-x+x^2)^8$, (f) $(2+x-2x^2)^7$,
(g) $(3-2x+x^2)^4$, (h) $(3+x+x^3)^4$.

Convergent series

14.4 The series

$$1+x+x^2+\ldots+x^{n-1}$$

is a geometrical progression, with common ratio x, and may be summed by the method of §13.4. In this way

$$1 + x + x^2 + \dots + x^{n-1} = \frac{1 - x^n}{1 - x}$$

If x lies between -1 and $+1$, we will assume that x^n approaches zero as n increases, which makes the right-hand side of the identity approach $1/(1 - x)$.

Thus when we write

$$1 + x + x^2 + \dots + x^r + \dots = \frac{1}{1 - x}$$

we mean that the left-hand side can be made to differ as little as we please from the right-hand side, providing enough terms are taken. It must not be forgotten, however, that we have taken x to lie between -1 and $+1$.

A series of terms, whose sum approaches a finite value as the number of terms is increased indefinitely is called a **convergent** series, and the finite value is called its **sum to infinity**.

Thus $1 + x + x^2 + \dots + x^r + \dots$ is a convergent series, provided x lies between -1 and $+1$, and its sum is $1/(1 - x)$.

To emphasise the necessity for the condition

$$-1 < x < +1 \qquad (\text{i.e. } |x| < 1)$$

the behaviour of the series for other values of x is examined below.

(a) If $x = 1$, $1 + x + x^2 + \dots + x^{n-1} = n$. Therefore as n increases, the value of the series increases indefinitely.

(b) If $x = -1$,

$$1 + x + x^2 + \dots + x^{n-1} = 1 - 1 + 1 - \dots + (-1)^{n-1}$$

which is equal to 1 or 0, according to whether n is odd or even.

(c) If x is greater than 1, x^n is greater than 1, and can be made as large as we like, if n is sufficiently large. Therefore the sum of the series, $(1 - x^n)/(1 - x)$, can be made as large as we like.

(d) When x is less than -1, $1 - x$ is positive and x^n is numerically greater than 1. If n is even, x^n is positive, therefore $1 - x^n$ is negative and so the sum $(1 - x^n)/(1 - x)$ is negative. If n is odd, x^n is negative, therefore $1 - x^n$ is positive and so the sum is positive. Hence the sum is alternately positive and negative.

It is beyond the scope of this book to give tests to discover whether any particular series is convergent, but this section has been written to draw the reader's attention to the fact that series are not always convergent.

The binomial theorem for any index

14.5 It has been shown that

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \dots + b^n$$

where n is a positive integer.

Now it will be *assumed* that

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

(the series being continued indefinitely), for *any* rational value of n *provided* $-1 < x < +1$, **i.e.** $|x| < 1$. The proof is beyond the scope of this book.

It should be remembered that, if n is a positive integer, there will only be a finite number of terms (see §14.3).

For the sake of those who go on to read other books, it should be added that the index, n, is often called the **exponent**.

The coefficient of x^r in the expansion of $(1 + x)^n$ is usually written $\binom{n}{r}$, that is,

$$\binom{n}{r} = \frac{n(n-1)(n-2)\dots(n-r+1)}{1 \times 2 \times 3 \times \dots \times r}$$

(It should be noticed that for each factor in the top line, there is a corresponding factor in the bottom line.)

Unlike ${}^nC_r$, the symbol $\binom{n}{r}$ may be used when n is *not* a positive integer.

Historical note. Pascal's triangle was given by a Chinese author of the early fourteenth century, but Pascal made considerable use of it in connection with problems on probability, and it became associated with his name. From it he obtained the theorem for positive integral indices. The series for fractional and negative indices was given by Newton in 1676.

Example 8 *Use the binomial theorem to expand* $1/(1 - x)$ *in ascending powers of* x, *as far as the term in* x^3.

(This example has been chosen because the result has already been established in §14.4.)

Since $1/(1 - x)$ may be written $(1 - x)^{-1}$, the binomial theorem may be used. Thus

$$(1 - x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + \\ + \frac{(-1)(-2)(-3)}{3!}(-x)^3 + \dots$$

$$\therefore \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \text{ provided } |x| < 1.$$

Example 9 *Obtain the first five terms of the expansion of* $\sqrt{(1 + 2x)}$ *in ascending powers of* x. *State the values of* x *for which the expansion is valid.*

Since $\sqrt{(1 + 2x)} = (1 + 2x)^{1/2}$, the binomial theorem may be used.

$$(1+2x)^{1/2} = 1 + \frac{1}{2}(2x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}(2x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}(2x)^3 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!}(2x)^4 + \ldots$$

$$\therefore \sqrt{(1+2x)} = 1 + x - \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3 - \tfrac{5}{8}x^4 + \ldots$$

For the expansion to be valid, $-1 < 2x < +1$, i.e. $|x| < \frac{1}{2}$.

Example 10 *Expand $1/(2+x)^2$ in ascending powers of x, as far as the term in x^3, and state for what values of x the expansion is valid.*

First it may be observed that $1/(2+x)^2 = (2+x)^{-2}$. However, the binomial theorem has been stated for $(1+x)^n$. Therefore a factor must be taken out, in order to leave the bracket in this form.

$$(2+x)^{-2} = \{2(1+\tfrac{1}{2}x)\}^{-2} = 2^{-2}(1+\tfrac{1}{2}x)^{-2}$$
$$= \tfrac{1}{4}(1+\tfrac{1}{2}x)^{-2}$$

and this may now be expanded.

$$\left[\text{Alternatively: } \frac{1}{(2+x)^2} = \frac{1}{2^2(1+\frac{1}{2}x)^2} = \tfrac{1}{4}(1+\tfrac{1}{2}x)^{-2}.\right]$$

$$\frac{1}{4}\left(1+\frac{1}{2}x\right)^{-2} = \frac{1}{4}\left\{1 + (-2)\left(\frac{x}{2}\right) + \frac{(-2)(-3)}{2!}\left(\frac{x}{2}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{x}{2}\right)^3 + \ldots\right\}$$

$$\therefore \frac{1}{(2+x)^2} = \frac{1}{4}\left(1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3 + \ldots\right)$$

For the expansion to be valid, $-1 < \frac{1}{2}x < +1$, i.e. $|x| < 2$.

Exercise 14c

Calculators should not be used in this exercise.

1 Evaluate the following binomial coefficients:

(a) $\binom{5}{3}$, (b) $\binom{-2}{4}$, (c) $\binom{\frac{1}{2}}{2}$, (d) $\binom{-\frac{1}{4}}{3}$.

2 Expand the following in ascending powers of x, as far as the terms in x^3, and state the values of x for which the expansions are valid.

(a) $(1+x)^{-2}$, (b) $(1+x)^{1/3}$, (c) $(1+x)^{3/2}$,

(d) $(1-2x)^{1/2}$, (e) $\left(1+\dfrac{x}{2}\right)^{-3}$, (f) $(1-3x)^{-1/2}$,

(g) $\dfrac{1}{1+3x}$, (h) $\sqrt{(1-x^2)}$, (i) $\sqrt[3]{(1-x)}$,

(j) $\dfrac{1}{\sqrt{(1+2x)}}$, (k) $\dfrac{1}{(1+x/2)^2}$, (l) $\sqrt{(1-2x)^3}$,

(m) $\dfrac{1}{2+x}$, (n) $\sqrt{(2-x)}$, (o) $\sqrt[3]{(3+x)}$,

(p) $\dfrac{1}{\sqrt{(2+x^2)}}$, (q) $\dfrac{1}{(3-x)^2}$, (r) $\dfrac{3}{\sqrt[3]{(3-x^3)}}$.

3 Use the binomial theorem to find the values of the following:

(a) $\sqrt{(1.001)}$, correct to six places of decimals.

(b) $\dfrac{1}{(1.02)^2}$, correct to four places of decimals.

(c) $\sqrt{(0.998)}$, correct to six places of decimals.

(d) $\sqrt[3]{(1.03)}$, correct to four places of decimals.

(e) $\dfrac{1}{\sqrt{(0.98)}}$, correct to four places of decimals.

4 Find the first four terms of the expansions of the following in ascending powers of x:

(a) $\dfrac{1+x}{1-x}$, (b) $\dfrac{x+2}{(1+x)^2}$, (c) $\dfrac{1-x}{\sqrt{(1+x)}}$,

(d) $\sqrt{\dfrac{1+x}{1-x}}$, [Multiply numerator and denominator by $\sqrt{(1+x)}$.]

(e) $\dfrac{2x-3}{x+2}$, (f) $\sqrt{\dfrac{(1-x)^3}{1+x}}$, (g) $\dfrac{x+3}{\sqrt[3]{(1-3x)}}$.

5 Find the first four terms of the expansion of $(1-8x)^{1/2}$ in ascending powers of x. Substitute $x=\frac{1}{100}$ and obtain the value of $\sqrt{23}$ correct to five significant figures.

6 Expand $(1-x)^{1/3}$ in ascending powers of x as far as the fourth term. By taking the first two terms of the expansion and substituting $x=\frac{1}{1000}$, find the value of $\sqrt[3]{37}$, correct to six significant figures. [Hint: $27\times 37=999$.]

7 Obtain the first four terms of the expansion of $(1-16x)^{1/4}$. Substitute $x=1/10\,000$ and use the first two terms to find $\sqrt[4]{39}$. To how many significant figures is your answer accurate?

Exercise 14d (Miscellaneous)

Calculators should not be used in this exercise.

1 Write down the sixth term of the expansion of $(3x+2y)^{10}$ in ascending powers of x, and evaluate the term when $x=\frac{1}{2}$ and $y=\frac{1}{3}$.

2 (a) Expand $\left(2x+\frac{1}{2x}\right)^5$ in descending powers of x.

(b) Simplify $(\sqrt{2}+\sqrt{3})^4-(\sqrt{2}-\sqrt{3})^4$.

3 Write down the expansion of $(a-b)^5$ and use the result to find the value of $(9\frac{1}{2})^5$ correct to the nearest 100.

4 (a) Expand $(a+b)^{11}$ in descending powers of a as far as the fourth term.

(b) Find the middle term in the expansion of $(6x+\frac{1}{3}y)^{10}$.

(c) Find the constant term in the expansion of $(x^2+2/x)^9$.

5 Expand $(x+2)^5$ and $(x-2)^4$. Obtain the coefficient of x^5 in the product of the expansions.

6 Obtain the expansion of $(x-2)^2(1-x)^6$ in ascending powers of x as far as the term in x^4.

7 (a) Expand $(2+3x)^4$ and simplify the coefficients.

(b) Obtain the first four terms in the expansion of $(1+2x+3x^2)^6$ in ascending powers of x.

8 Find the first four terms in the expansions of

(a) $(1-x+2x^2)^5$, (b) $(1+x)^{-4}$,

in ascending powers of x.

9 (a) Write down the expansion of $(1+x)^{-3}$ as far as the term in x^4, simplifying each term.

(b) Write down the first four terms of the expansion of $(2+\frac{1}{4}x)^{10}$ in ascending powers of x. Hence find the value of 2.025^{10}, correct to the nearest whole number.

10 (a) Find the middle term of the expansion of $(2x+3)^8$, and the value of this term when $x=\frac{1}{12}$.

(b) Find the first four terms in the expansion of $(1-2x)^{-2}$.

11 (a) Find the value of the fifth term in the expansion of $(\sqrt{2}+\sqrt{3})^8$.

(b) Give the expansion of $(1+x)^{1/3}$ up to and including the term in x^2. Hence, by putting $x=\frac{1}{8}$, calculate the cube root of 9, giving your answer correct to three decimal places.

12 Obtain the first four terms of the expansion of $(1+8x)^{1/2}$ in ascending powers of x. By putting $x=\frac{1}{100}$, obtain the value of $\sqrt{3}$, correct to five places of decimals.

13 If x is so small that its fourth and higher powers may be neglected, show that

$$\sqrt[4]{(1+x)}+\sqrt[4]{(1-x)}=a-bx^2$$

and find the numbers a and b.

Hence by putting $x=\frac{1}{16}$ show that the sum of the fourth roots of 17 and of 15 is 3.9985 approximately.

14 Find the first four terms of the expansion of $\frac{x+3}{(1+x)^2}$ in ascending powers of x.

15 Show that, if x is small enough for its cube and higher powers to be neglected,

$$\sqrt{\frac{1-x}{1+x}}=1-x+\frac{x^2}{2}$$

By putting $x=\frac{1}{8}$, show that $\sqrt{7}\approx 2\frac{83}{128}$.

Chapter 15

Vectors

Introduction

15.1 Consider the following sentences:

(a) The temperature is 15°C.
(b) The journey lasted 2 hours.
(c) The plane is flying due East at 800 km/h.
(d) A horizontal force of 2 newtons was applied to the ruler at right-angles to its length.
(e) Shift the piano 10 m to the right.

One does not have to be a scientist to see that the first two sentences differ from the others in one very important respect: the first two are complete when the magnitude of the quantity is given, but in the others it is necessary to define both the magnitude and the direction. A quantity which is completely specified by its magnitude alone is called a **scalar** quantity and one which requires both the magnitude and the direction to be given is called a **vector** quantity. (Strictly speaking, a vector quantity must also obey the triangle law of addition; see §15.6.)

Let us consider sentences (d) and (e) in more detail. The effect of the force applied to the ruler will be determined by the point at which the force is applied; if it is applied to the end of the ruler, the ruler will start to rotate, but if it is applied to the mid-point of the ruler, the ruler will start to slide without rotating. So when we describe a force we shall have to give not only its magnitude and direction, but we shall also have to state its line of action. (This is usually done by describing a point through which the force passes.) Vectors which have a definite line of action are called **localised vectors**.

On the other hand, when the piano in sentence (e) is shifted, then, as we can see in Fig. 15.1, every point in the piano is shifted 10 m to the right. All the line segments AA′, BB′, CC′, DD′ and PP′ are equal in length and they are parallel to one another. Any one of them can be used to describe the shift which has been applied to the piano.

Vectors which do not have a particular line of action are called **free vectors**; all free vectors which have the same magnitude and direction are equivalent to one

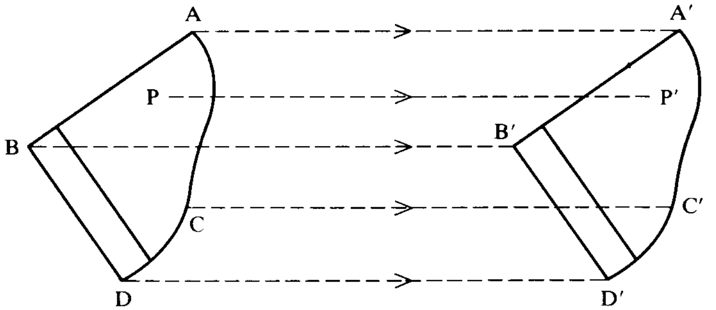


Figure 15.1

another and, in the example above, we write $\overrightarrow{AA'} = \overrightarrow{BB'} = \overrightarrow{CC'} = \overrightarrow{DD'} = \overrightarrow{PP'}$.

In this chapter all the vectors described, with one important exception, will usually be free vectors. The main exception is the position vector (see §15.7), which must always start from the origin.

Displacement vectors

15.2 Looking at a map of England, we see that Cambridge is about eighty miles from Oxford, and it is approximately North East of Oxford. This is an example of a very common type of vector quantity, namely a **displacement**. The displacement of one point from another can be defined, as in the example above, by giving the distance and the direction. Alternatively, when using Cartesian coordinates, the displacement can be defined by giving the increase in the x-coordinate and the increase in the y-coordinate.

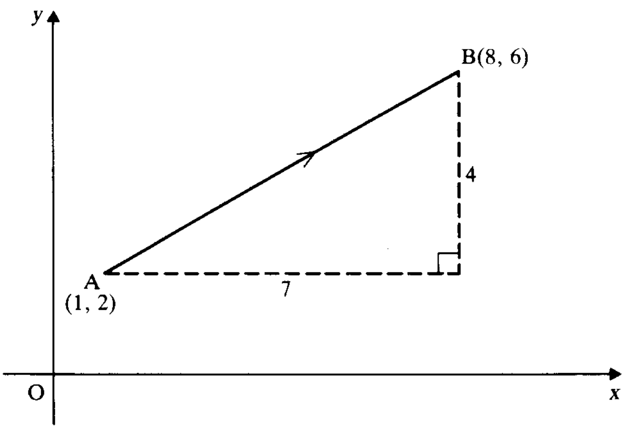


Figure 15.2

In Fig. 15.2, A is the point (1, 2) and B is the point (8, 6), so the displacement from A to B is '7 across and 4 up'. Obviously it is desirable to have a concise way of making statements like this; the normal notation is $\begin{pmatrix} 7 \\ 4 \end{pmatrix}$. The upper number is the increase in the x-coordinate, and the lower one is the increase in the y-coordinate. It is also necessary to make it clear that the displacement goes *from* A *to* B, and so we write

$$\overrightarrow{AB} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

The displacement *from* B *to* A is written $\overrightarrow{BA}$, and, in the case we are considering, this is equal to $\begin{pmatrix} -7 \\ -4 \end{pmatrix}$.

Qu. 1 Write down the displacement vector $\overrightarrow{AB}$ for each of the following pairs of points:
(a) A(3, 5), B(5, 9), (b) A(9, 7), B(12, 4), (c) A(12, 5), B(5, 4),
(d) A(2, 3), B(2, 5), (e) A(5, 1), B(8, 1).

Fig. 15.3 illustrates the fact that the displacement from $A(x_1, y_1)$ to $B(x_2, y_2)$ is

$$\overrightarrow{AB} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}.$$

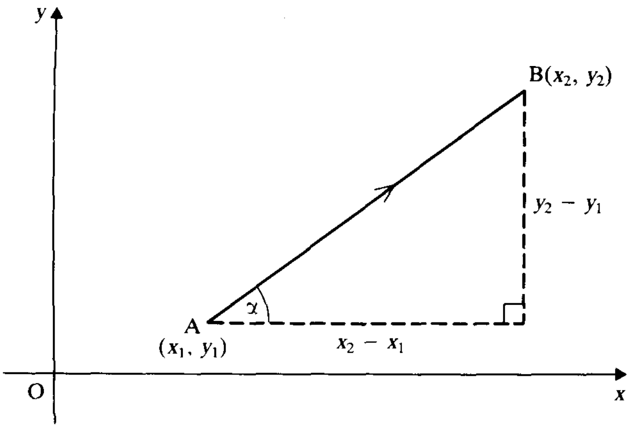


Figure 15.3

Notice that the magnitude of the vector (i.e. its length) is given by

$$AB = \sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2\}}$$

and that its direction is defined by the angle α which it makes with the x-axis, where

$$\tan \alpha = \frac{y_2 - y_1}{x_2 - x_1}$$

(note that this is the gradient of the line AB). The magnitude is never negative and the angle is usually given in the range $-180° \leqslant \alpha \leqslant +180°$; however, angles outside this range may be used, provided the meaning is clear. In the special case when $x_2 = x_1$, $\tan \alpha$ is not defined, because the denominator is zero. However if a diagram is consulted, it is clear that in this case the vector is parallel to the y-axis.

Example 1 *Find the magnitude and direction of the displacement vector* $\overrightarrow{AB}$, *where* A *and* B *are the points* (2, 1) *and* (8, 9) *respectively. Find also the magnitude and direction of* $\overrightarrow{BA}$, *giving the angle correct to the nearest tenth of a degree.*

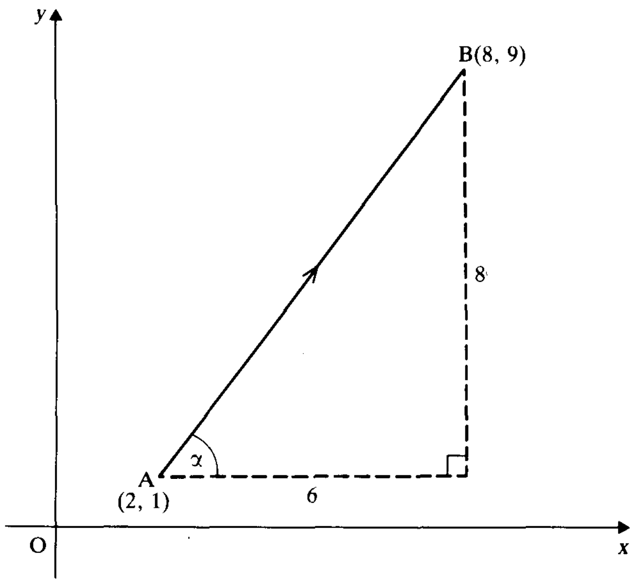


Figure 15.4

From Fig. 15.4, we can see that

$$\overrightarrow{AB} = \begin{pmatrix} 8-2 \\ 9-1 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

$$\therefore \ AB^2 = 6^2 + 8^2 = 36 + 64 = 100$$

$$AB = 10$$

We can also see that the direction is given by

$$\tan \alpha = \frac{8}{6}$$

$$\therefore \ \alpha = 53.1°, \quad \text{correct to the nearest tenth of a degree}$$

Similarly,

$$\overrightarrow{BA} = \begin{pmatrix} 2-8 \\ 1-9 \end{pmatrix} = \begin{pmatrix} -6 \\ -8 \end{pmatrix}$$

$\overrightarrow{BA}$ is inclined at $-126.9°$ to the x-axis

Qu. 2 Find the magnitude and direction of each of the vectors in Qu. 1.

Unit vectors

15.3 Any vector whose magnitude is 1, for example $\begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix}$, is called a **unit vector**. The unit vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are especially important because they are parallel to the x-axis and y-axis respectively; they are called **base vectors**, and the letters **i** and **j** are reserved for them (**i** and **j** are always printed in bold type; in manuscript they should be underlined).

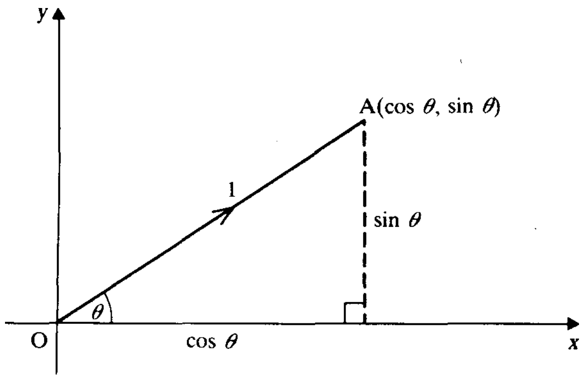


Figure 15.5

The unit vector $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ is very useful as it is inclined at an angle θ to the x-axis (see Fig. 15.5).

Multiplication by a scalar

15.4 In Fig. 15.6, the displacement $\overrightarrow{AB}$ has been enlarged by a factor k, that is $\overrightarrow{AB'} = k\overrightarrow{AB}$. If $\overrightarrow{AB} = \begin{pmatrix} a \\ b \end{pmatrix}$, then AP $= a$ and PB $= b$. Also, since the triangles APB and AP′B′ are similar, AB′ $= k$AB, AP′ $= ka$ and P′B′ $= kb$, and so

$$\overrightarrow{AB'} = \begin{pmatrix} ka \\ kb \end{pmatrix}$$

Thus we can write $\begin{pmatrix} 20 \\ 30 \end{pmatrix} = 10\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 5 \\ 0 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 5\mathbf{i}$, etc.

In general

$$k\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ka \\ kb \end{pmatrix}$$

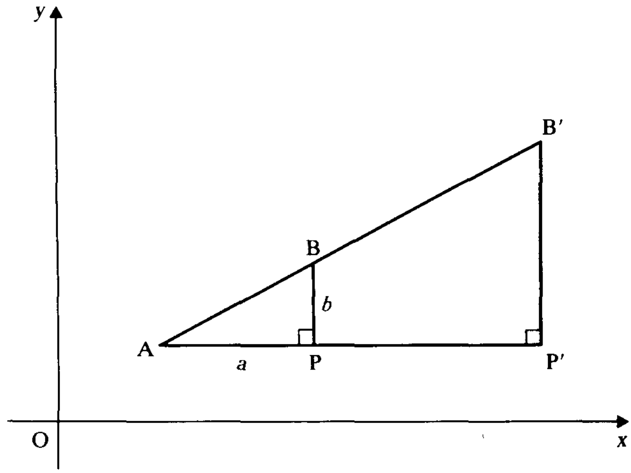


Figure 15.6

Equal vectors

15.5 In Fig. 15.7, $\overrightarrow{AB} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $\overrightarrow{DC} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, that is the displacement from A to B is the same as that from D to C. In this sense we can say that the vectors $\overrightarrow{AB}$

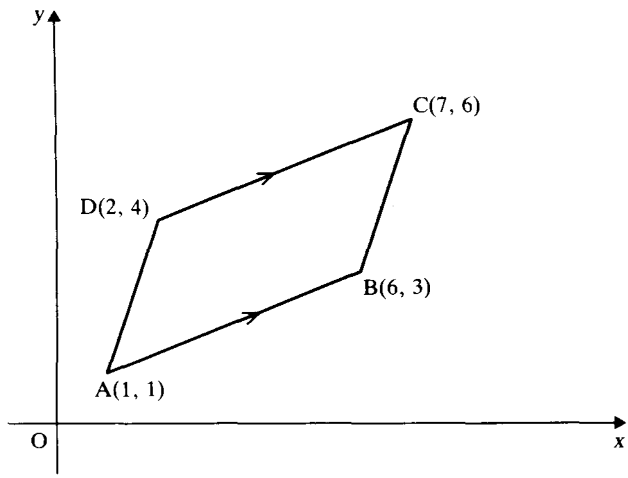


Figure 15.7

and $\overrightarrow{DC}$ are equal. Vectors are equal when they have the same magnitude and direction. Notice that $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are also equal; the figure ABCD is of course a parallelogram.

Example 2 *Given that* A *is the point* (1, 3) *and that* $\overrightarrow{AB}$ *and* $\overrightarrow{AD}$ *are* $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ *and* $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ *respectively, find the coordinates of the vertices* B, C *and* D *of the parallelogram* ABCD (Fig. 15.8).

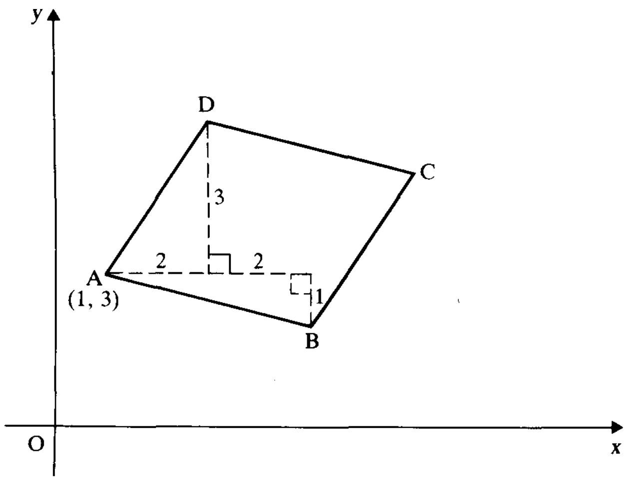


Figure 15.8

B is the point $(1+4, 3-1) = (5, 2)$

D is the point $(1+2, 3+3) = (3, 6)$

$$\overrightarrow{DC} = \overrightarrow{AB} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

hence

C is the point $(3+4, 6-1) = (7, 5)$

Example 3 *Given the points* A(1, 1), B(5, 4), C(8, 9) *and* D(0, 3), *show that* ABCD *is a trapezium* (Fig. 15.9).

$$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad \text{and} \quad \overrightarrow{DC} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$$

$$\therefore 2\,\overrightarrow{AB} = \overrightarrow{DC}$$

Hence $\overrightarrow{DC}$ is parallel to $\overrightarrow{AB}$ (and twice as long). So ABCD is a trapezium.

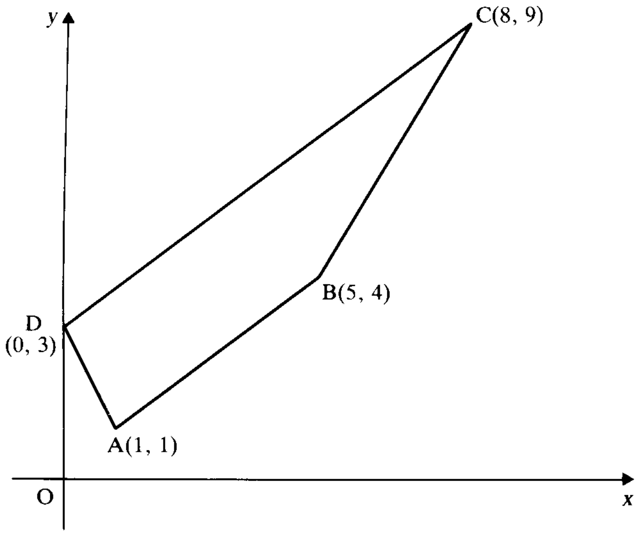


Figure 15.9

Addition and subtraction of vectors

15.6 If we make the displacement $\begin{pmatrix}2\\3\end{pmatrix}$ and follow this with the displacement $\begin{pmatrix}5\\1\end{pmatrix}$, then overall we shall have moved 7 units to the right and 4 units up. We could also achieve the same result by making the displacement $\begin{pmatrix}5\\1\end{pmatrix}$ *first* and the displacement $\begin{pmatrix}2\\3\end{pmatrix}$ *second*. We write

$$\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}5\\1\end{pmatrix}=\begin{pmatrix}5\\1\end{pmatrix}+\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}7\\4\end{pmatrix}$$

and we say that we have 'added' the vectors. In Fig. 15.10, $\overrightarrow{PQ}=\begin{pmatrix}2\\3\end{pmatrix}$, $\overrightarrow{QR}=\begin{pmatrix}5\\1\end{pmatrix}$ and $\overrightarrow{PR}=\begin{pmatrix}7\\4\end{pmatrix}$. Notice that $\overrightarrow{PQ}+\overrightarrow{QR}=\overrightarrow{PR}$ (this is the 'triangle law of addition', which some readers may have met in physics). We could also say that $\overrightarrow{PQ'}=\begin{pmatrix}5\\1\end{pmatrix}$, $\overrightarrow{Q'R}=\begin{pmatrix}2\\3\end{pmatrix}$ and that $\overrightarrow{PQ'}+\overrightarrow{Q'R}=\overrightarrow{PR}$.

In general

$$\begin{pmatrix}a_1\\b_1\end{pmatrix}+\begin{pmatrix}a_2\\b_2\end{pmatrix}=\begin{pmatrix}a_1+a_2\\b_1+b_2\end{pmatrix}$$

If $\overrightarrow{AB}=\begin{pmatrix}h\\k\end{pmatrix}$, then $\overrightarrow{BA}=\begin{pmatrix}-h\\-k\end{pmatrix}$. Notice that $\overrightarrow{BA}=-\overrightarrow{AB}$, and also that

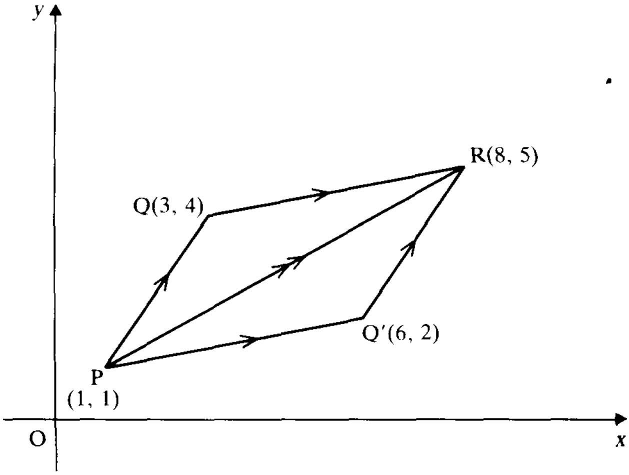


Figure 15.10

$\overrightarrow{AB} + \overrightarrow{BA} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; the vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is called the **zero vector** and is denoted by $\mathbf{0}$.

Any vector $\begin{pmatrix} x \\ y \end{pmatrix}$ can be expressed as $\begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix}$, and this in turn can be written $x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix} = x\mathbf{i} + y\mathbf{j}$.

To subtract vectors, see Fig. 15.11, where C′ is the point on CB produced, such that BC′ = CB.

$$\begin{aligned} \overrightarrow{AB} - \overrightarrow{BC} &= \overrightarrow{AB} + (-\overrightarrow{BC}) \\ &= \overrightarrow{AB} + \overrightarrow{BC'} \\ &= \overrightarrow{AC'} \end{aligned}$$

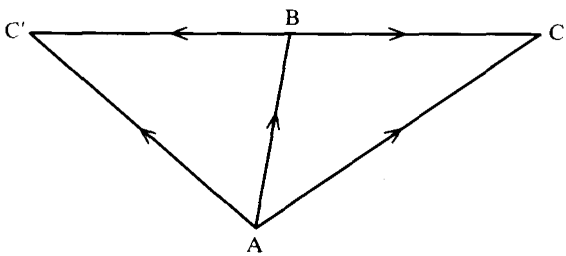


Figure 15.11

Thus

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} - \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 - a_2 \\ b_1 - b_2 \end{pmatrix}$$

It is frequently convenient to use a single letter to represent a vector. When this is done, a lower case letter (i.e. not a capital letter) is always used and it is always printed in bold type (in manuscript it must be underlined). For example we may write

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} \quad \text{and} \quad \mathbf{x} - \mathbf{y} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

In **i**, **j** notation, the statement above could be written

$$\mathbf{x} = 2\mathbf{i} + \mathbf{j} \qquad \mathbf{y} = \mathbf{i} + 5\mathbf{j}$$

$$\mathbf{x} + \mathbf{y} = 3\mathbf{i} + 6\mathbf{j} \quad \text{and} \quad \mathbf{x} - \mathbf{y} = \mathbf{i} - 4\mathbf{j}$$

This is especially useful for labelling diagrams. For example, Fig. 15.12 illustrates the sum, $\mathbf{a} + \mathbf{b}$, and the difference, $\mathbf{a} - \mathbf{b}$, of the vectors **a** and **b**.

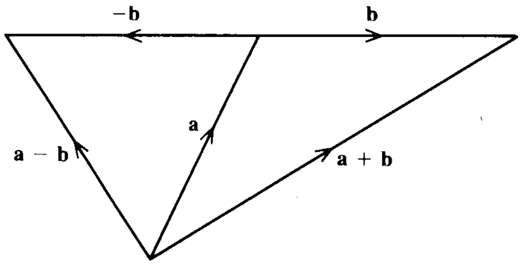


Figure 15.12

When using the single letter notation, an *italic* letter is always used to denote the *magnitude* of the vector which is represented by the same letter in **bold** type, e.g. if $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$, then $a = 5$.

Example 4 *In* Fig. 15.13 *each set of parallel lines is equally spaced and it is given that* $\overrightarrow{OP} = \mathbf{p}$ *and* $\overrightarrow{OU} = \mathbf{u}$. *Express the following vectors in terms of* **p** *and* **u**:
(a) $\overrightarrow{OQ}$, (b) $\overrightarrow{QW}$, (c) $\overrightarrow{OW}$, (d) $\overrightarrow{OM}$, (e) $\overrightarrow{OS}$, (f) $\overrightarrow{OA}$.

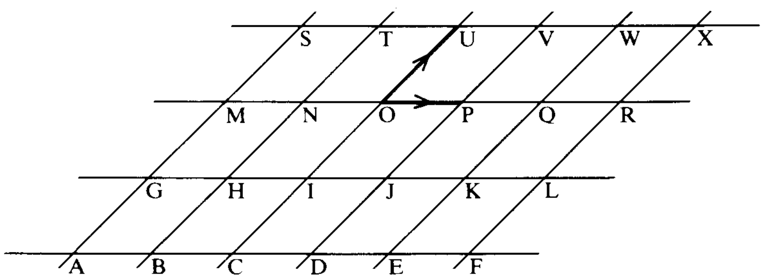


Figure 15.13

(a) $\overrightarrow{OQ} = 2\mathbf{p}$, (b) $\overrightarrow{QW} = \mathbf{u}$,
(c) $\overrightarrow{OW} = 2\mathbf{p} + \mathbf{u}$, (d) $\overrightarrow{OM} = -2\mathbf{p}$,
(e) $\overrightarrow{OS} = \overrightarrow{OM} + \overrightarrow{MS} = -2\mathbf{p} + \mathbf{u}$, (f) $\overrightarrow{OA} = -2\mathbf{p} - 2\mathbf{u}$.

Example 5 *In triangle* OAB (Fig. 15.14), $\overrightarrow{OA} = \mathbf{a}$ *and* $\overrightarrow{OB} = \mathbf{b}$. *Given that* P *and* Q *are the mid-points of* OA *and* OB, *express* $\overrightarrow{PQ}$ *and* $\overrightarrow{AB}$ *in terms of* **a** *and* **b**. *State the geometrical relationship between* $\overrightarrow{PQ}$ *and* $\overrightarrow{AB}$.

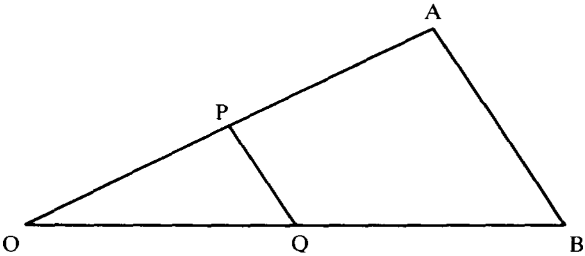


Figure 15.14

Since P and Q are the mid-points of $\overrightarrow{OA}$ and $\overrightarrow{OB}$, we can write

$$\overrightarrow{OP} = \tfrac{1}{2}\mathbf{a} \quad \text{and} \quad \overrightarrow{OQ} = \tfrac{1}{2}\mathbf{b}$$

Now

$$\begin{aligned} \overrightarrow{PQ} &= \overrightarrow{PO} + \overrightarrow{OQ} \\ &= -\tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{b} \\ &= \tfrac{1}{2}(\mathbf{b} - \mathbf{a}) \\ \overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\mathbf{a} + \mathbf{b} \\ \therefore \overrightarrow{AB} &= 2\overrightarrow{PQ} \end{aligned}$$

In other words, $\overrightarrow{AB}$ is parallel to $\overrightarrow{PQ}$ and twice its length.

From a mathematical point of view, the beauty of this kind of argument is that it does not depend upon the actual dimensions of the triangle.

Exercise 15a

1 Given that $\mathbf{x} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$ write down as column vectors:
(a) $2\mathbf{x}$, (b) $3\mathbf{y}$, (c) $-\mathbf{y}$, (d) $\tfrac{1}{2}\mathbf{y}$,
(e) $\mathbf{x} + \mathbf{y}$, (f) $2\mathbf{x} + 3\mathbf{y}$, (g) $\mathbf{x} - \mathbf{y}$, (h) $3\mathbf{x} - 2\mathbf{y}$.

2 Find the magnitude and direction of the vectors:
(a) $3\mathbf{i} + 4\mathbf{j}$, (b) $-5\mathbf{i} + 12\mathbf{j}$, (c) $-10\mathbf{j}$, (d) $\mathbf{i} - \mathbf{j}$.

3 The vector $\overrightarrow{XY}$ has magnitude 10 units and it is inclined at 30° to the x-axis. Express $\overrightarrow{XY}$ as a column vector.

4 The vector $\overrightarrow{PQ}$ has magnitude 5 units and is inclined at 150° to the x-axis. Express $\overrightarrow{PQ}$ in the form $a\mathbf{i} + b\mathbf{j}$, where $a, b \in \mathbb{R}$.

5 A and B are the points (3, 7) and (15, 13) respectively. P is a point on AB such that $\overrightarrow{AP} = s\overrightarrow{AB}$. Write down the vector $\overrightarrow{OP}$ in terms of s. Find the coordinates of P, when

(a) $s = \frac{3}{4}$, (b) $s = \frac{3}{2}$, (c) $s = -2$.

6 In Fig. 15.15, OABC is a quadrilateral and P, Q, R and S are the mid-points of the sides OA, AB, BC and CO, respectively. Given that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$, express the following vectors in terms of **a**, **b** and **c**:

(a) $\overrightarrow{PS}$, (b) $\overrightarrow{AC}$, (c) $\overrightarrow{QR}$.

What can you deduce about the lines PS, AC and QR?

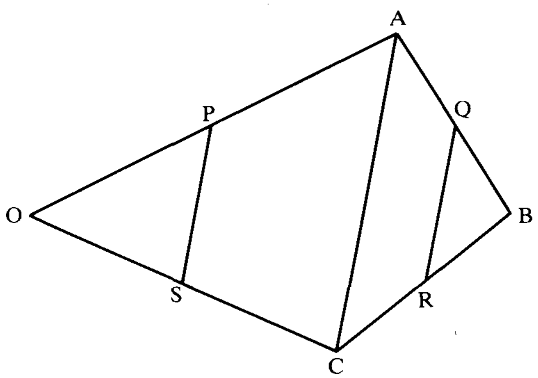


Figure 15.15

7 In No. 6 above, X is the mid-point of PR, and Y is the mid-point of QS. Express $\overrightarrow{OX}$ and $\overrightarrow{OY}$ in terms of **a**, **b** and **c**. State clearly in words the deduction which can be made from these expressions.

Position vectors

15.7 In the preceding exercise, the reader will have noticed that the vector from the origin O to a point P is frequently required. This vector $\overrightarrow{OP}$ is called the **position vector** of the point P; it is always denoted by the single letter **p** (similarly, the position vectors of points A, B, C, ... would be written **a**, **b**, **c**, ...). It is important to notice that position vectors are localised; they *must* start from the origin.

If the coordinates of P are (x, y) then **p** is the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$. Notice that the displacement vector $\overrightarrow{PQ}$ is related to the position vectors of P and Q, as follows:

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = -\mathbf{p} + \mathbf{q} = \mathbf{q} - \mathbf{p}$$

Similarly, $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, $\overrightarrow{XY} = \mathbf{y} - \mathbf{x}$ and so on. Expressions like these are very common in vector geometry and the reader is advised to commit the form of them to memory.

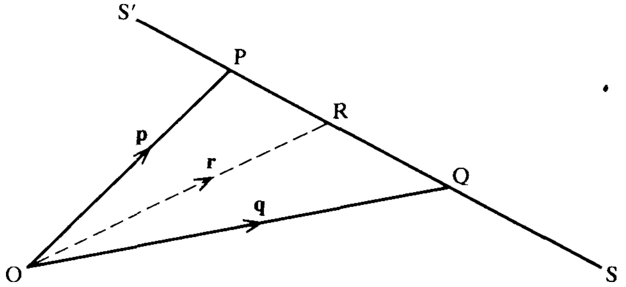


Figure 15.16

In Fig. 15.16, R is a point on the line PQ, such that $\overrightarrow{PR} = t\overrightarrow{PQ}$. Applying the results above to $\overrightarrow{PR}$ and $\overrightarrow{PQ}$, we have

$$\overrightarrow{PR} = t\overrightarrow{PQ}$$

hence

$$\begin{aligned} \mathbf{r} - \mathbf{p} &= t(\mathbf{q} - \mathbf{p}) \\ \therefore \mathbf{r} &= \mathbf{p} + t(\mathbf{q} - \mathbf{p}) \\ &= (1-t)\mathbf{p} + t\mathbf{q} \end{aligned}$$

Since R lies *between* P and Q, $0 < t < 1$. But if $t = 1$, then R will coincide with Q, and if $t = 0$, then R will coincide with P. The position vector of a point such as S, is given by a similar expression, e.g.

$$\mathbf{s} = (1-t)\mathbf{p} + t\mathbf{q}$$

but here the number t is greater than 1. A point such as S′ can be obtained by using a negative value for t.

Example 6 *In* Fig. 15.17, $\overrightarrow{OS} = 2\mathbf{r}$ *and* $\overrightarrow{OQ} = \frac{3}{2}\mathbf{p}$. *Given that* $\overrightarrow{QK} = m\overrightarrow{QR}$ *and* $\overrightarrow{PK} = n\overrightarrow{PS}$, *find two distinct expressions, in terms of* **p**, **r**, *m and n, for* $\overrightarrow{OK}$. *By equating these expressions, find the values of m and n and hence calculate the ratios* QK:KR *and* PK:KS.

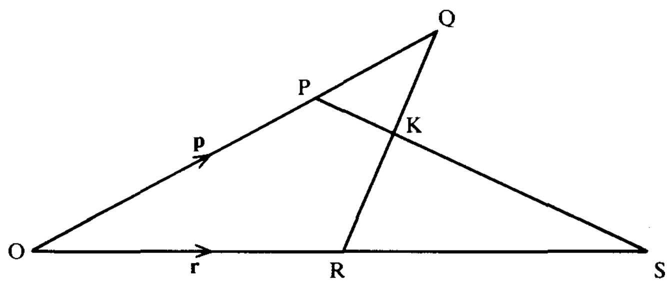


Figure 15.17

$$\begin{aligned} \overrightarrow{QR} &= \overrightarrow{QO} + \overrightarrow{OR} \\ &= -\tfrac{3}{2}\mathbf{p} + \mathbf{r} \end{aligned}$$

One expression for $\overrightarrow{OK}$ is given by

$$\overrightarrow{OK} = \overrightarrow{OQ} + \overrightarrow{QK} = \overrightarrow{OQ} + m\overrightarrow{QR}$$
$$= \tfrac{3}{2}\mathbf{p} + m(\mathbf{r} - \tfrac{3}{2}\mathbf{p})$$

Similarly, $\overrightarrow{OK} = \overrightarrow{OP} + \overrightarrow{PK}$, hence

$$\overrightarrow{OK} = \mathbf{p} + n(-\mathbf{p} + 2\mathbf{r})$$

Equating the two expressions for $\overrightarrow{OK}$, we have

$$\tfrac{3}{2}\mathbf{p} + m(\mathbf{r} - \tfrac{3}{2}\mathbf{p}) = \mathbf{p} + n(-\mathbf{p} + 2\mathbf{r})$$

and rearranging this gives

$$(\tfrac{1}{2} - \tfrac{3}{2}m + n)\mathbf{p} = (2n - m)\mathbf{r}$$

but since **p** and **r** are not parallel, the two sides of this equation cannot be equal unless they are both zero. (The reader should think carefully about this statement, and make sure he or she fully understands it. This argument is very common when vectors are used in geometrical problems.) Hence

$$\tfrac{1}{2} - \tfrac{3}{2}m + n = 0 \tag{1}$$

and $$2n - m = 0 \tag{2}$$

Substituting $m = 2n$ in equation (1), we have

$$\tfrac{1}{2} - \tfrac{3}{2} \times 2n + n = 0$$
$$\tfrac{1}{2} - 3n + n = 0$$
$$2n = \tfrac{1}{2}$$
$$n = \tfrac{1}{4}$$

and hence

$$m = \tfrac{1}{2}$$

But $\overrightarrow{QK} = m\overrightarrow{QR}$ (given) so $\overrightarrow{QK} = \tfrac{1}{2}\overrightarrow{QR}$ and hence $\overrightarrow{QK} = \overrightarrow{KR}$. Therefore

$$QK:KR = 1:1$$

Also $\overrightarrow{PK} = n\overrightarrow{PS}$, so $\overrightarrow{PK} = \tfrac{1}{4}\overrightarrow{PS}$ and hence

$$PK:KS = 1:3$$

Example 7 *At noon, two boats* P *and* Q *are at points whose position vectors are* $4\mathbf{i} + 8\mathbf{j}$ *and* $4\mathbf{i} + 3\mathbf{j}$ *respectively. Both boats are moving with constant velocity; the velocity of* P *is* $4\mathbf{i} + \mathbf{j}$ *and the velocity of* Q *is* $2\mathbf{i} + 5\mathbf{j}$, (*all distances are in kilometres and the time is measured in hours*). *Find the position vectors of* P *and* Q, *and* $\overrightarrow{PQ}$ *after t hours, and hence express the distance* PQ *between the boats in terms of t. Show that the least distance between the boats is* $\sqrt{5}$ km.

After t hours the displacement of P from its starting point is $t(4\mathbf{i}+\mathbf{j})$, hence

$$\begin{aligned}\mathbf{p} &= (4\mathbf{i}+8\mathbf{j})+t(4\mathbf{i}+\mathbf{j})\\ &= (4+4t)\mathbf{i}+(8+t)\mathbf{j}\end{aligned}$$

Similarly

$$\mathbf{q} = (4+2t)\mathbf{i}+(3+5t)\mathbf{j}$$

Hence

$$\begin{aligned}\overrightarrow{PQ} &= \mathbf{q}-\mathbf{p} = -2t\mathbf{i}+(-5+4t)\mathbf{j}\\ \therefore\ PQ^2 &= (-2t)^2+(-5+4t)^2\\ &= 20t^2-40t+25\end{aligned}$$

Hence the distance between the boats is given by

$$PQ = \sqrt{(20t^2-40t+25)}\text{ km}$$

To find the least distance, consider

$$\begin{aligned}PQ^2 &= 20(t^2-2t+1)+5\\ &= 20(t-1)^2+5\end{aligned}$$

Since $(t-1)^2$ cannot be negative, its least value is zero and this is obtained by putting $t=1$. Hence the least value of PQ^2 is 5. (See §10.3.)

$\therefore$ The shortest possible distance between the boats is $\sqrt{5}$ km.

Exercise 15b

1 Given that A is the point (2, 5) and that B is the point (10, −1), find the position vector of a point P on AB, such that
(a) $\overrightarrow{AP}=\overrightarrow{PB}$, (b) $2\overrightarrow{AP}=\overrightarrow{PB}$, (c) $\overrightarrow{AP}=4\overrightarrow{AB}$,
(d) AP:PB = 2:3, (e) AP:PB = 4:1, (f) AP:PB = m:n.

2 Repeat No. 1 for A(−7, 3) and B(−1, −15).

3 A, B, C are three collinear points whose position vectors are **a**, **b** and **c** respectively and $\overrightarrow{AC}=3\overrightarrow{AB}$. Express **c** in the form $\mathbf{c}=m\mathbf{a}+n\mathbf{b}$; find the scalars m and n and verify that $m+n=1$. Show also that if $\mathbf{a}=p\mathbf{b}+q\mathbf{c}$ then $p+q=1$.

4 Repeat No. 3 given that $\overrightarrow{AC}=-2\overrightarrow{AB}$.

5 A stationary observer O observes a ship S at noon, at a point whose coordinates relative to O are (20, 15); the units are kilometres. The ship is moving at a steady 10 km/h on a bearing 150° (a bearing is measured clockwise from North). Express its velocity as a column vector. Write down, in terms of t, its position after t hours. Hence find the value of t when it is due East of O. How far is it from O at this instant?

6 Find numbers m and n such that $m\begin{pmatrix}3\\5\end{pmatrix}+n\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}4\\9\end{pmatrix}$.

7 In Fig. 15.18, $\overrightarrow{OP}=\mathbf{p}$ and $\overrightarrow{OR}=\mathbf{r}$.
P is the mid-point of OQ and PX:XR = 1:3. Express **x** in terms of **p** and **r**. Taking $\overrightarrow{OY}$ to be $h\overrightarrow{OX}$, find $\overrightarrow{QY}$ in terms of **p**, **r** and h and hence find the ratio QY:YR.

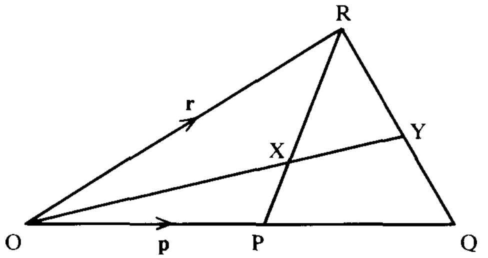


Figure 15.18

8 In Fig. 15.19, OBC is a triangle and the line NL produced meets the line OC produced at M.

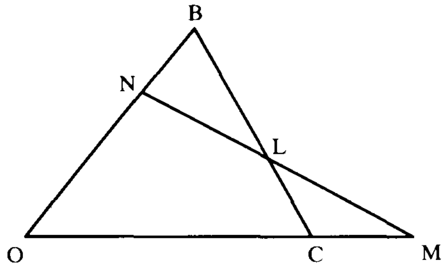


Figure 15.19

Given that $\overrightarrow{ON} = \frac{3}{4}\overrightarrow{OB}$ and $\overrightarrow{BL} = \frac{2}{3}\overrightarrow{BC}$, express the vector $\overrightarrow{NL}$ in terms of **b** and **c**, the position vectors of the points B and C with respect to the origin O. Find an expression for the position vector of any point R on the line NL. Hence express $\overrightarrow{OM}$ as a multiple of $\overrightarrow{OC}$. Find the ratio CM/MO and verify that

$$\frac{ON}{NB} \times \frac{BL}{LC} \times \frac{CM}{MO} = -1$$

9 In a triangle OAB, X is a point on OB such that $\overrightarrow{OX} = 2\overrightarrow{XB}$ and Y is a point on AB such that $2\overrightarrow{BY} = 3\overrightarrow{YA}$. Express **x** and **y** in terms of **a** and **b**. Find the position vector of any point on XY and hence find the position vector of the point Z, where XY produced meets OA produced. Calculate the value of AZ/OZ.

10 Prove that if **a** and **b** are the position vectors of points A and B, then the position vector of a point P on AB, where AP:PB $= m{:}n$ is given by $(m + n)\mathbf{p} = n\mathbf{a} + m\mathbf{b}$.

11 Prove that if $\mathbf{p} = h\mathbf{a} + k\mathbf{b}$ represents the point P on the line AB, then $h + k = 1$.

12 Given that A, B and C are three collinear points whose position vectors satisfy the equation $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = \mathbf{0}$, prove that $\alpha + \beta + \gamma = 0$.

The ratio theorem

15.8 In Fig. 15.20, $\overrightarrow{OA'} = h\mathbf{a}$, $\overrightarrow{OB'} = k\mathbf{b}$ and $\overrightarrow{OC} = h\mathbf{a} + k\mathbf{b}$. We say that $\overrightarrow{OC}$ is a **linear combination** of **a** and **b**. Any point C, whose position vector is a linear combination of **a** and **b**, will be a point in the plane of O, A and B. (So far we have only considered vectors in two dimensions; this last statement becomes very important when we start to consider three dimensions.)

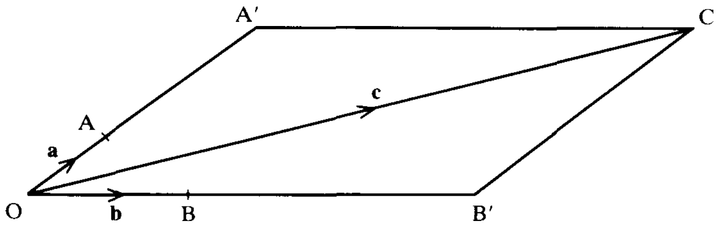


Figure 15.20

However, if $\overrightarrow{OC} = h\mathbf{a} + k\mathbf{b}$ and $h + k = 1$, it can be shown that C lies on the line AB, as follows:

$$\begin{aligned}\mathbf{c} &= h\mathbf{a} + k\mathbf{b}\\ &= (1-k)\mathbf{a} + k\mathbf{b}\\ &= \mathbf{a} + k(\mathbf{b} - \mathbf{a})\end{aligned}$$

Using the double letter notation this last equation becomes

$$\overrightarrow{OC} = \overrightarrow{OA} + k\overrightarrow{AB}$$

hence

$$\overrightarrow{AC} = k\overrightarrow{AB}$$

so C is the point on AB such that $\overrightarrow{AC} = k\overrightarrow{AB}$ (see Fig. 15.21).

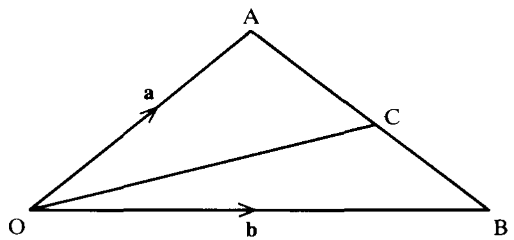


Figure 15.21

Similarly $\overrightarrow{BC} = h\overrightarrow{BA}$. Hence AC:CB $= k{:}h$.

Conversely if we are given that C is a point on the line AB such that AC:CB $= m{:}n$, then we can write

$$\frac{AC}{CB} = \frac{m}{n}$$

i.e. $n\overrightarrow{AC} = m\overrightarrow{CB}$

$\therefore n(\mathbf{c} - \mathbf{a}) = m(\mathbf{b} - \mathbf{c})$

$n\mathbf{c} - n\mathbf{a} = m\mathbf{b} - m\mathbf{c}$

$m\mathbf{c} + n\mathbf{c} = n\mathbf{a} + m\mathbf{b}$

$(m + n)\mathbf{c} = n\mathbf{a} + m\mathbf{b}$

$$\therefore \mathbf{c} = \left(\frac{n}{m+n}\right)\mathbf{a} + \left(\frac{m}{m+n}\right)\mathbf{b}$$

This is usually called the **ratio theorem.** Notice that the sum of the coefficients $n/(m + n)$ and $m/(m + n)$ is 1.

Example 8 *If* $\mathbf{c} = \frac{2}{5}\mathbf{a} + \frac{3}{5}\mathbf{b}$, *show that* C *is a point on* AB *and that* AC:CB = 3:2.

Since $\frac{2}{5} + \frac{3}{5}$ is equal to 1, C lies on the line AB. Also,

$$AC:CB = \tfrac{3}{5}:\tfrac{2}{5} = 3:2$$

The centroid of a triangle

15.9 In Fig. 15.22, ABC is any triangle and P is the mid-point of BC. G is the point on AP such that AG:GP = 2:1. The origin is not shown in the diagram.

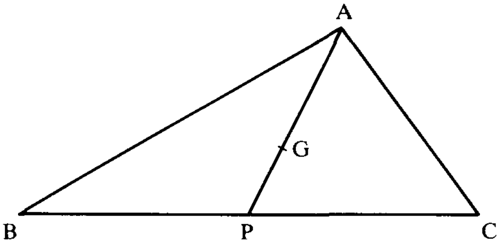


Figure 15.22

Since BP:PC = 1:1,

$$\mathbf{p} = \tfrac{1}{2}\mathbf{b} + \tfrac{1}{2}\mathbf{c}$$

and since AG:GP = 2:1,

$$\begin{aligned}\mathbf{g} &= \tfrac{1}{3}\mathbf{a} + \tfrac{2}{3}\mathbf{p} \\ &= \tfrac{1}{3}\mathbf{a} + \tfrac{2}{3}(\tfrac{1}{2}\mathbf{b} + \tfrac{1}{2}\mathbf{c}) \\ &= \tfrac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})\end{aligned}$$

This last expression is symmetrical in **a**, **b** and **c** (that is, the letters **a**, **b** and **c** can be interchanged without altering **g**), so the same result could be obtained by dividing the median from B to AC (or that from C to AB) in the ratio 2:1. Hence the point G, whose position vector is given by

$$\mathbf{g} = \tfrac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$$

is the point of intersection of the three medians. G is called the **centroid** of the triangle.

Qu. 3 Find the centroid of the triangle whose vertices are A(1, 2), B(3, 7) and C(2, 3).

Menelaus' theorem

15.10 In Fig. 15.23, OAB is any triangle and PQR is a straight line intersecting the sides of the triangle as shown.

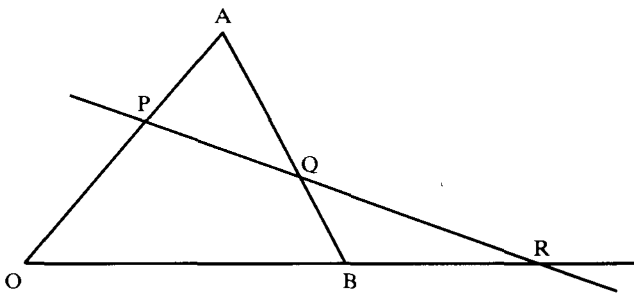


Figure 15.23

Menelaus' theorem can be stated as follows: if $\overrightarrow{OP} = \alpha\overrightarrow{PA}$, $\overrightarrow{AQ} = \beta\overrightarrow{QB}$ and $\overrightarrow{BR} = \gamma\overrightarrow{RO}$, then $\alpha\beta\gamma = -1$. (Notice that since R is on OB produced, γ is a negative number.)

This famous theorem appeared in a treatise published by Menelaus in 100 AD, although it was probably known to Euclid almost 400 years earlier. These great mathematicians would not, of course, have expressed the proof in vector notation.

Menelaus' theorem can be proved by vector methods, as follows:

$\overrightarrow{OP} = \alpha\overrightarrow{PA}$, hence $\mathbf{p} = \alpha(\mathbf{a} - \mathbf{p})$.

$$\therefore (1 + \alpha)\mathbf{p} = \alpha\mathbf{a} \tag{1}$$

$\overrightarrow{AQ} = \beta\overrightarrow{QB}$, hence $\mathbf{q} - \mathbf{a} = \beta(\mathbf{b} - \mathbf{q})$.

$$\therefore (1 + \beta)\mathbf{q} = \mathbf{a} + \beta\mathbf{b} \tag{2}$$

$\overrightarrow{BR} = \gamma\overrightarrow{RO}$, hence $\mathbf{r} - \mathbf{b} = -\gamma\mathbf{r}$.

$$\therefore (1 + \gamma)\mathbf{r} = \mathbf{b} \tag{3}$$

From equation (1) we have

$$\mathbf{a} = \left(\frac{1 + \alpha}{\alpha}\right)\mathbf{p}$$

and from equation (3),

$$\mathbf{b} = (1 + \gamma)\mathbf{r}$$

Substituting these expressions for **a** and **b** in equation (2) gives

$$(1 + \beta)\mathbf{q} = \left(\frac{1 + \alpha}{\alpha}\right)\mathbf{p} + \beta(1 + \gamma)\mathbf{r}$$

$$\therefore \mathbf{q} = \frac{(1 + \alpha)}{\alpha(1 + \beta)}\mathbf{p} + \frac{\beta(1 + \gamma)}{(1 + \beta)}\mathbf{r}$$

However, Q is a point on PR, so, using the ratio theorem (see §15.8),

$$\frac{(1 + \alpha)}{\alpha(1 + \beta)} + \frac{\beta(1 + \gamma)}{(1 + \beta)} = 1$$

$$(1 + \alpha) + \alpha\beta(1 + \gamma) = \alpha(1 + \beta)$$

$$1 + \alpha + \alpha\beta + \alpha\beta\gamma = \alpha + \alpha\beta$$

$$\therefore \alpha\beta\gamma = -1$$

The result looks slightly more elegant, and it is perhaps easier to remember, if the diagram is re-lettered as in Fig. 15.24.

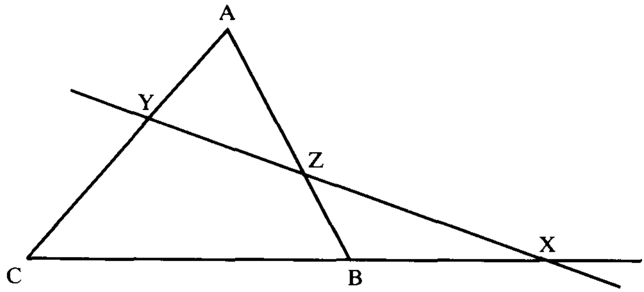


Figure 15.24

Menelaus' theorem can then be expressed

$$\frac{AZ}{ZB} \times \frac{BX}{XC} \times \frac{CY}{YA} = -1$$

Vectors in three dimensions

15.11 So far in this chapter, we have only considered vectors in two dimensions, but the real world is three dimensional, so we must now consider the problems which arise when vectors are used in three dimensions. One of the great attractions of vectors is that the transition from two dimensions to three is very easy. First we will look at Cartesian coordinates in three dimensions. For convenience, the three axes Ox, Oy and Oz will be mutually perpendicular. They cannot be drawn mutually perpendicular on a flat page, so Fig. 15.25 should be viewed with the page held in a vertical plane so that the z-axis is vertical and in the plane of the page, the y-axis is horizontal and in the plane of the page, and the x-axis is imagined to be horizontal, but coming out of the page at right angles to the plane of the page. By convention, the three axes must form a 'right-handed set'. If the thumb, index finger and middle finger of the right hand are stretched out so that they are mutually perpendicular, it should be possible to make the thumb correspond to the x-axis, the index finger to the y-axis and the middle finger to the z-axis. (In a 'left-handed set' the x-axis would go into, instead of come out of, the page.)

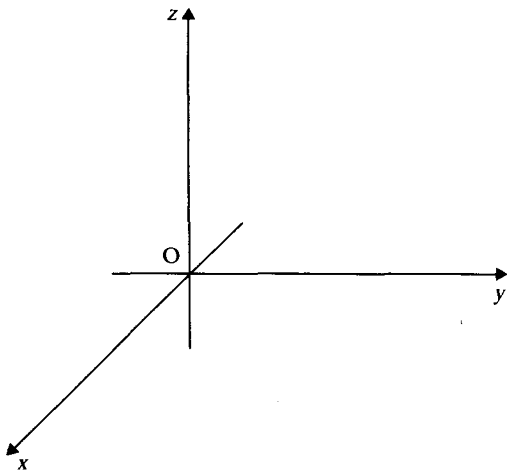


Figure 15.25

A point A(2, 3, 5) is located in the usual way, namely by starting from the origin and moving 2 units along Ox, 3 units parallel to Oy and 5 units parallel to Oz (see Fig. 15.26).

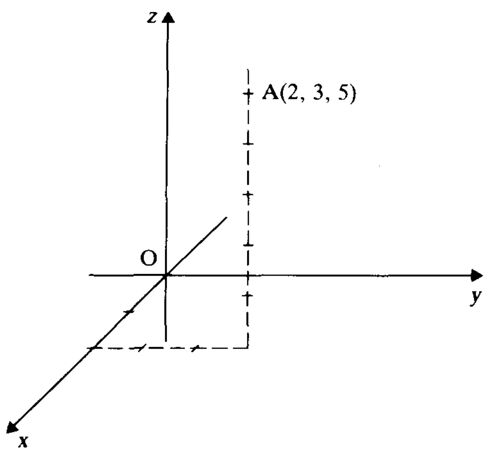


Figure 15.26

The position vector of this point A is written $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$. Similarly the displacement vector from A(2, 3, 5) to B(3, 6, 4) is written $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$.

In general, if A is the point (x_1, y_1, z_1) and B is the point (x_2, y_2, z_2) then we write

$$\mathbf{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{AB} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$$

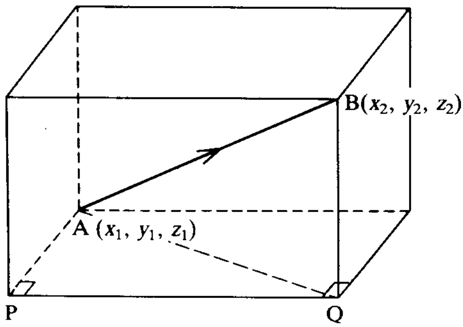


Figure 15.27

Fig. 15.27 represents a cuboid, in which AP is parallel to the x-axis, PQ is parallel to the y-axis and QB is parallel to the z-axis. Hence

$$AP = x_2 - x_1, \quad PQ = y_2 - y_1 \quad \text{and} \quad QB = z_2 - z_1$$

The length of vector $\overrightarrow{AB}$ can be found, using Pythagoras' theorem, as follows. In the right-angled triangle ABQ,

$$AB^2 = AQ^2 + QB^2$$

and, in the right-angled triangle APQ,

$$AQ^2 = AP^2 + PQ^2$$

hence

$$\begin{aligned} AB^2 &= AP^2 + PQ^2 + QB^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \end{aligned}$$

Multiplication of a vector by a scalar in three dimensions is defined by a simple extension of the method used in two dimensions (see §15.4), that is,

$$k \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} kx \\ ky \\ kz \end{pmatrix}$$

Addition and subtraction are also defined by a similar method to that used before, i.e.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

and

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}$$

All the results described so far in this chapter [e.g. the centroid of a triangle ABC is given by $\mathbf{g} = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$] are equally valid in three dimensions.

The letter **k** is always used to represent the unit vector parallel to the z-axis. Consequently in **i**, **j**, **k** notation the vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ becomes $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Example 9 *If* A *and* B *are the points* (1, 1, 1) *and* (13, 4, 5) *respectively, find, in terms of* **i**, **j** *and* **k**, *the displacement vector* $\overrightarrow{AB}$. *Find also the unit vector parallel to* $\overrightarrow{AB}$.

$$\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{b} = 13\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$$

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = 12\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$$

$$\therefore AB^2 = 12^2 + 3^2 + 4^2 = 169$$

$$\therefore AB = 13$$

The magnitude of $\overrightarrow{AB}$ is 13 and so the vector $\frac{1}{13}\overrightarrow{AB}$ is a parallel vector of magnitude 1. Hence the required unit vector is $\frac{12}{13}\mathbf{i} + \frac{3}{13}\mathbf{j} + \frac{4}{13}\mathbf{k}$.

Example 10 *Using the points* A *and* B *in* Example 9, *find the point* P *on* $\overrightarrow{AB}$ *such that* AP:PB = 1:3.

We are given that AP:PB = 1:3, so $\overrightarrow{AP} = \frac{1}{4}\overrightarrow{AB}$, hence

$$\begin{aligned} 4(\mathbf{p} - \mathbf{a}) &= (\mathbf{b} - \mathbf{a}) \\ \therefore 4\mathbf{p} &= 4\mathbf{a} + \mathbf{b} - \mathbf{a} \\ &= 3\mathbf{a} + \mathbf{b} \\ &= 3(\mathbf{i} + \mathbf{j} + \mathbf{k}) + (13\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}) \\ &= 16\mathbf{i} + 7\mathbf{j} + 8\mathbf{k} \\ \therefore \mathbf{p} &= 4\mathbf{i} + \tfrac{7}{4}\mathbf{j} + 2\mathbf{k} \end{aligned}$$

Hence P is the point $(4, \frac{7}{4}, 2)$.

Example 11 *Show that the points* A(1, 2, 3), B(3, 8, 1), C(7, 20, −3) *are collinear.*

$$\begin{aligned} \overrightarrow{AB} &= (3\mathbf{i} + 8\mathbf{j} + \mathbf{k}) - (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \\ &= 2\mathbf{i} + 6\mathbf{j} - 2\mathbf{k} \end{aligned}$$

Similarly

$$\overrightarrow{BC} = 4\mathbf{i} + 12\mathbf{j} - 4\mathbf{k}$$

hence

$$\overrightarrow{BC} = 2\overrightarrow{AB}$$

consequently $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are in the same direction and so ABC is a straight line.

Qu. 4 Find the centroid of the triangle whose vertices are A(1, 2, 3), B(3, 7, 4), C(2, 0, 5).

Qu. 5 Prove that A(1, 2, 1), B(4, 7, 8), C(6, 4, 12) and D(3, −1, 5) are the vertices of a parallelogram.

The vector equation of a line

15.12 Given any two points A and B, with position vectors **a** and **b**, the position vector of any point R on $\overrightarrow{AB}$ can be expressed as follows:

$$\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$$

Let $\overrightarrow{AR} = t\overrightarrow{AB}$, where $t \in \mathbb{R}$, hence

$$\overrightarrow{OR} = \overrightarrow{OA} + t\overrightarrow{AB}$$
$$\therefore \mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$$
$$= (1 - t)\mathbf{a} + t\mathbf{b}$$

The letter t in this equation represents any real number and, for all values of t, **r** is the position vector of a point on $\overrightarrow{AB}$. The equation $\mathbf{r} = (1 - t)\mathbf{a} + t\mathbf{b}$ is called the **vector equation** of the line AB. The number t is called the **parameter**; for any value of the parameter, R is a point on AB.

Example 12 *Find the equation of the line through the points* A(1, 2, 3) *and* B(4, 4, 4) *and find the coordinates of the point where this line meets the plane* $z = 0$.

$$\overrightarrow{AB} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

Let R be any point on AB, so that

$$\overrightarrow{OR} = \overrightarrow{OA} + t\overrightarrow{AB}, \quad \text{where } t \in \mathbb{R}$$
$$\therefore \mathbf{r} = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + t(3\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$
$$= (1 + 3t)\mathbf{i} + (2 + 2t)\mathbf{j} + (3 + t)\mathbf{k}$$

This is the equation of the line.

The line meets the plane $z = 0$, where $(3 + t) = 0$. Thus the parameter at this point is $t = -3$. Substituting this in the equation of the line, we have

$$\mathbf{r} = -8\mathbf{i} - 4\mathbf{j} + 0\mathbf{k}$$

so the line meets the plane at the point (−8, −4, 0).

Any vector equation of the form $\mathbf{r} = \mathbf{a} + t\mathbf{u}$, where **a** and **u** are given vectors, represents the equation of a line passing through the point whose position vector is **a**. The direction of the line is parallel to the vector **u**. If $\mathbf{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$ then

$$\mathbf{r} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t\begin{pmatrix} l \\ m \\ n \end{pmatrix}$$

If the point R has coordinates (x, y, z), then **r** can be written $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and hence the last equation becomes

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 + tl \\ y_1 + tm \\ z_1 + tn \end{pmatrix}$$

Thus the coordinates of R are $x = x_1 + tl$, $y = y_1 + tm$, $z = z_1 + tn$. These three equations are frequently arranged in the form

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = t$$

Example 13 *Given the equation of the line in the form*

$$\frac{x-2}{3} = \frac{y-4}{5} = \frac{z-7}{2}$$

express the equation in the form $\mathbf{r} = \mathbf{a} + t\mathbf{u}$ *and show that the line passes through the point* (8, 14, 11).

Let $\dfrac{x-2}{3} = \dfrac{y-4}{5} - \dfrac{z-7}{2} = t$, then

$$x - 2 = 3t \qquad y - 4 = 5t \qquad z - 7 = 2t$$

hence

$$x = 2 + 3t$$
$$y = 4 + 5t$$
$$z = 7 + 2t$$

that is, in vector form,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 + 3t \\ 4 + 5t \\ 7 + 2t \end{pmatrix}$$

which can be written in the form $\mathbf{r} = \mathbf{a} + t\mathbf{u}$ as follows:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix} + t\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$$

Compare this with the coordinates (8, 14, 11); when $2 + 3t = 8$, $t = 2$. [Now try this value of the parameter on the y- and z-coordinates.] When $t = 2$, $4 + 5t = 14$ and $7 + 2t = 11$. Hence the line passes through the point (8, 14, 11).

Qu. 6 Find the unit vector which is parallel to the line $\dfrac{x-1}{3} = \dfrac{y-2}{4} = \dfrac{z-7}{12}$.

Qu. 7 Show that the equations

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + m\begin{pmatrix} 4 \\ 6 \\ -2 \end{pmatrix} \text{ and } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 15 \\ -3 \end{pmatrix} + n\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$$

represent the same line.

Planes

15.13 If A, B and C are three given points it is always possible to find a plane which contains all three of them. (Imagine the tips of the thumb and first two fingers of the right hand as the three given points. A flat surface, say a book, can then be placed on these three points to represent the plane passing through them.) A fourth point, P, may or may not lie in the same plane. If it does, then, as was shown in §15.8, the vector $\overrightarrow{AP}$ can be expressed as a linear combination of $\overrightarrow{AB}$ and $\overrightarrow{AC}$, that is scalars m and n can be found so that

$$\overrightarrow{AP} = m\overrightarrow{AB} + n\overrightarrow{AC}$$

hence

$$\mathbf{p} - \mathbf{a} = m(\mathbf{b} - \mathbf{a}) + n(\mathbf{c} - \mathbf{a})$$
$$\therefore\ \mathbf{p} = (1 - m - n)\mathbf{a} + m\mathbf{b} + n\mathbf{c}$$

In other words, $\mathbf{p}$ can be expressed as a linear combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$:

$$\mathbf{p} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$$

where $\alpha + \beta + \gamma = 1$ [since $(1 - m - n) + m + n = 1$].

(It is interesting to compare this with the statement 'if R is a point on the line AB then $\mathbf{r}$ can be expressed as a linear combination of $\mathbf{a}$ and $\mathbf{b}$, in other words $\mathbf{r} = \lambda\mathbf{a} + \mu\mathbf{b}$ where $\lambda + \mu = 1$'.)

Example 14 *Given that* A, B, C *are the points* (1, 1, 1), (5, 0, 0) *and* (3, 2, 1) *respectively, find the equation which must be satisfied by the coordinates* (x, y, z) *of any point,* P, *in the plane* ABC.

As P lies in the plane ABC, we may write $\overrightarrow{AP} = m\overrightarrow{AB} + n\overrightarrow{AC}$. Then, since

$$\overrightarrow{AB} = \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \text{ and } \overrightarrow{AP} = \begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix},$$

$$\begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix} = m\begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix} + n\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Thus

$$\left.\begin{aligned} x &= 1 + 4m + 2n \\ y &= 1 - m + n \\ z &= 1 - m \end{aligned}\right\} \qquad (1)$$

Eliminating n,

$$x - 2y = -1 + 6m$$

and eliminating m,

$$x - 2y + 6z = 5$$

This is the equation of the plane ABC.

In the equations (1), the scalars m and n are usually called 'the parameters of the plane'. For any values of m and n the coordinates (x, y, z) resulting from these equations are the coordinates of a point in the plane ABC. In the two-dimensional world of the plane ABC, we have two degrees of freedom; we can choose a value for m and we can choose a value for n. (Compare this with the one-dimensional world of the line, in §15.12, in which there is only one degree of freedom; that is we can choose a value for the parameter t.)

Qu. 8 Find the equation of the plane containing the points (1, 1, 0), (0, 1, 2), (2, 3, −8).

Qu. 9 Find the equation of the plane which passes through the point (1, 2, 3) and which is parallel to the vectors $\begin{pmatrix} 2 \\ 4 \\ -10 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ -4 \\ 2 \end{pmatrix}$.

The intersection of two planes

15.14 Two non-parallel planes will always meet in a straight line. If we are given the equations of two such planes, say, $3x - 5y + z = 8$ and $2x - 3y + z = 3$, then the equation of the line of intersection can be found as follows.

For any point (x, y, z) which lies in *both* planes, the values of x, y and z fit both equations simultaneously. Hence eliminating z from both equations (in this case by subtracting the second equation from the first) we obtain

$$x - 2y = 5$$

There are infinitely many pairs of values of x and y which satisfy this equation, but if we choose a value for x then the value of y is fixed and *vice versa*. (For example, if $x = 7$ then $y = 1$.)

Let $y = t$, then x must be $5 + 2t$ and substituting these expressions for x and y into the first of the original equations, we obtain

$$\begin{aligned} 3(5 + 2t) - 5t + z &= 8 \\ 15 + 6t - 5t + z &= 8 \\ \therefore z &= -7 - t \end{aligned}$$

Thus

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 + 2t \\ t \\ -7 - t \end{pmatrix}, \quad \text{i.e.} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -7 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

The latter is the equation of the line. It is parallel to the vector $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and it passes through the point $(5, 0, -7)$. A typical point on the line can be written $(5 + 2t, t, -7 - t)$ and it can easily be verified that, for all values of t, this point lies in both of the planes. If we substitute its coordinates into the first equation, we obtain

$$\begin{aligned} 3x - 5y + z &= 3(5 + 2t) - 5t + (-7 - t) \\ &= 15 + 6t - 5t - 7 - t \\ &= 8 \end{aligned}$$

and substituting in the second equation gives

$$\begin{aligned} 2x - 3y + z &= 2(5 + 2t) - 3t + (-7 - t) \\ &= 10 + 4t - 3t - 7 - t \\ &= 3 \end{aligned}$$

Exercise 15c

1 Given the points A and B below, write down the displacement vector, $\overrightarrow{AB}$, in each case:
(a) A(1, 0, 2), B(3, 6, 4); (b) A(5, 0, 4), B(3, 0, 4);
(c) A(2, 1, 3), B(6, 4, 3); (d) A(5, 4, 7), B(2, 8, 1);
(e) A(k, $2k$, $3k$), B($3k$, $2k$, k).

2 For each part of No. 1, write down the position vector of the mid-point of AB.

3 For each part of No. 1, write down the position vector of the point P, such that $\overrightarrow{AP} = 5\overrightarrow{AB}$.

4 Find the equation of the plane through the points (1, 2, 0), (1, 1, 1) and (0, 3, 0).

5 Find the equation of the plane through the point (1, 1, 1) parallel to the vectors $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{i} + \mathbf{j}$.

6 Find the coordinates of the point where the line

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

meets the plane $x - 2y + 3z = 26$.

7 Show that the line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ lies in the plane

$$2x + 3y - 5z = -7$$

8 Find the point of intersection of the lines

$$\mathbf{r} = (1 + m)\mathbf{i} + (2 + m)\mathbf{j} + (4 + 2m)\mathbf{k}$$

and

$$\mathbf{r} = (1 + 3n)\mathbf{i} + 5n\mathbf{j} + (3 + 7n)\mathbf{k}$$

9 Show that the lines $\mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} + m\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + n\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ do not meet.

(Non-parallel lines which do not meet are called *skew* lines.)

10 Given four points A, B, C and D, the point G, whose position vector $\mathbf{g}$ is defined by $\mathbf{g} = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d})$, is called the centroid of A, B, C and D. Prove that G lies on the line joining D to M, the centroid of triangle ABC. Find the ratio DG:GM.

11 Find the equation of the line of intersection of the planes

$$4x + 3y + z = 10$$
$$x + y + z = 6$$

12 Show that the three planes whose equations are

$$2x + 3y + z = 8$$
$$x + y + z = 10$$
$$3x + 5y + z = 6$$

contain a common line.

The scalar product of two vectors

15.15 So far we have added and subtracted vectors, and vectors have been multiplied by scalars, but we have not 'multiplied' one vector by another vector. In vector work there are two kinds of 'multiplication'; in one of them, the result is a scalar quantity, so this is called scalar multiplication, while in the other the result is a vector quantity. The latter kind, 'vector multiplication', is beyond the scope of Book 1. (See Book 2, Chapter 21.)

Definition

Given two vectors **a** *and* **b** *(see* Fig. 15.28), *whose magnitudes are a and b respectively, the* **scalar product a.b** *is* $ab \cos \theta$, *where* θ *is the angle between the vectors.*

(The scalar product is always written with a very distinct dot between the **a** and the **b**. It is quite common to call this the 'dot product' of **a** and **b**.)

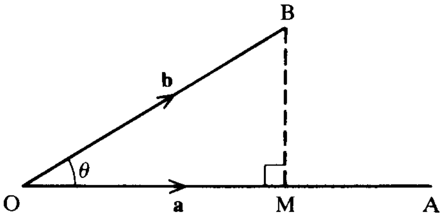


Figure 15.28

At first sight $\mathbf{a}.\mathbf{b} = ab \cos \theta$ might seem a rather odd definition to choose, and one might reasonably ask why it should be this and not, say $ab \tan \theta$, or $ab \sin \theta$. This particular definition, $ab \cos \theta$, is useful because it has many interesting mathematical properties, some of which will appear in the next few sections. Also, applied mathematicians and physicists find it a useful concept; in particular the 'work done' when the point of application of a force **F** (a vector) undergoes a displacement **x** (a vector) is given by **F.x** (a scalar).

Notice that $\mathbf{b}.\mathbf{a} = ba \cos \theta$, which of course is the same as $ab \cos \theta$, so the order of **a** and **b** in the scalar product does not matter, in other words scalar multiplication is *commutative*. (This may seem to be a rather trivial remark, nevertheless it is very important; in contrast vector multiplication is not commutative). We shall frequently require the scalar products of the base vectors **i** and **j**, so the following results should be memorised (bearing in mind that $\cos 0° = 1$ and $\cos 90° = 0$):

$$\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1$$
$$\mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{i} = 0$$

For any vector **a**, the scalar product **a.a** is equal to a^2, and for any pair of perpendicular vectors **a** and **b** the scalar product **a.b** is zero (because cos 90° is zero). Conversely if we know that the scalar product of a pair of vectors is zero, then we can deduce that the vectors are perpendicular (or one of the vectors is zero).

Example 15 *Given that* OA = 6, OB = 4 *and* ∠AOB = 60°, *calculate the value of* $\overrightarrow{OA}.\overrightarrow{OB}$.

$$\begin{aligned}\overrightarrow{OA}.\overrightarrow{OB} &= 6 \times 4 \times \cos 60° \\ &= 6 \times 4 \times 0.5 \\ &= 12\end{aligned}$$

There is an alternative form of this definition. Note that in Fig. 15.28 $b\cos\theta = \text{OM}$; the length OM is often called the projection of $\overrightarrow{\text{OB}}$ onto $\overrightarrow{\text{OA}}$. Consequently we can say that the scalar product, **a.b**, is the product of OA and the projection of $\overrightarrow{\text{OB}}$ onto $\overrightarrow{\text{OA}}$. The A and B in this statement can, of course, be interchanged.

Although **a.b** has been called a 'product' and the process has been called scalar 'multiplication', it is necessary to establish that this 'multiplication' obeys the same rules that we are familiar with, from working with real numbers.

We have already seen that the order of **a** and **b** in the scalar product does not matter, so the *commutative** law, **a.b** = **b.a**, is obeyed.

Since **a.b** is scalar, it is impossible to attach any meaning to a triple product (**a.b**).**c**; consequently there is no question of scalar products obeying the *associative** law. However (**a.b**)**c** could be taken to mean the scalar **a.b** multiplied by the vector **c**, as in §15.4, so great care is needed.

It is, however, very important that we should be able to remove brackets from **a**.(**b** + **c**) and obtain **a.b** + **a.c**. The law

$$\mathbf{a}.(\mathbf{b}+\mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$$

is called the *distributive** law and this is proved in the next section.

The proof of the distributive law

15.16 In Fig. 15.29, $\overrightarrow{\text{OA}} = \mathbf{a}$, $\overrightarrow{\text{OB}} = \mathbf{b}$, $\overrightarrow{\text{OC}} = \mathbf{c}$ and $\overrightarrow{\text{OR}} = \mathbf{b}+\mathbf{c}$.

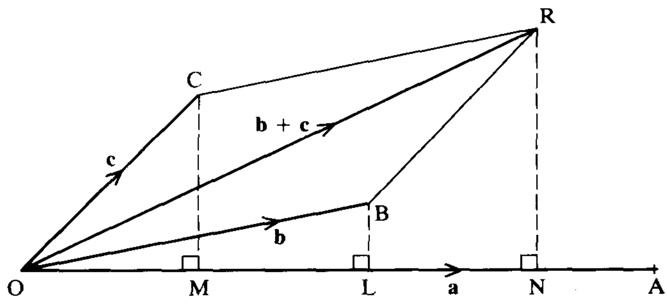


Figure 15.29

$$\mathbf{a}.\mathbf{b} = \text{OA} \times \text{OL} \quad \text{(the product of OA and the projection of } \overrightarrow{\text{OB}} \text{ onto } \overrightarrow{\text{OA}})$$

and similarly

$$\mathbf{a}.\mathbf{c} = \text{OA} \times \text{OM}$$

Adding,

$$\mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c} = \text{OA} \times (\text{OL} + \text{OM})$$

but since OC and BR are opposite sides of a parallelogram, the projection of $\overrightarrow{\text{OC}}$

*The terms commutative, associative, distributive may be new to some readers; they are explained in more detail in Chapter 25.

onto $\overrightarrow{OA}$ is equal to the projection of $\overrightarrow{BR}$ onto $\overrightarrow{OA}$. Hence OM = LN. Thus

$$\begin{aligned}\mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c} &= \text{OA} \times (\text{OL} + \text{LN}) = \text{OA} \times \text{ON} \\ &= \overrightarrow{OA}.\overrightarrow{OR} \\ &= \mathbf{a}.(\mathbf{b} + \mathbf{c})\end{aligned}$$

With this law proved, we may now proceed to remove brackets according to the normal rules of algebra, e.g.

$$(\mathbf{a} + \mathbf{b}).(\mathbf{c} + \mathbf{d}) = \mathbf{a}.(\mathbf{c} + \mathbf{d}) + \mathbf{b}.(\mathbf{c} + \mathbf{d}) = \mathbf{a}.\mathbf{c} + \mathbf{a}.\mathbf{d} + \mathbf{b}.\mathbf{c} + \mathbf{b}.\mathbf{d}$$

In particular, if we wish to form the scalar product of **a** and **b**, where

$$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} \quad \text{and} \quad \mathbf{b} = 5\mathbf{i} + 6\mathbf{j} + 7\mathbf{k}$$

then, bearing in mind that $\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1$ and $\mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{i} = 0$, we have

$$\mathbf{a}.\mathbf{b} = (2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}).(5\mathbf{i} + 6\mathbf{j} + 7\mathbf{k}) = 2 \times 5 + 3 \times 6 + 4 \times 7 = 56$$

In general,

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = (x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}).(x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) = x_1x_2 + y_1y_2 + z_1z_2$$

Example 16 *Given that* $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}$ *and* $\mathbf{b} = 8\mathbf{i} - 6\mathbf{j}$, *find* a^2, b^2 *and* $\mathbf{a}.\mathbf{b}$. *Hence find the angle between the vectors* **a** *and* **b**.

$$\begin{aligned}a^2 = \mathbf{a}.\mathbf{a} &= (4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}).(4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}) \\ &= 16 + 9 + 144 \\ &= 169\end{aligned}$$

Similarly, $b^2 = 100$.

Hence $a = 13$ and $b = 10$.

$$\begin{aligned}\mathbf{a}.\mathbf{b} &= (4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}).(8\mathbf{i} - 6\mathbf{j}) \\ &= 32 - 18 \\ &= 14\end{aligned}$$

However, by definition, $\mathbf{a}.\mathbf{b} = ab \cos \theta$, where θ is the angle between **a** and **b**, consequently

$$14 = 13 \times 10 \cos \theta$$

$$\therefore \cos \theta = \frac{14}{130}$$

$$\therefore \theta = 83.8^\circ$$

The angle between **a** and **b** is 83.8°, correct to the nearest tenth of a degree.

Example 17 *Prove that* $\mathbf{p} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ *is perpendicular to* $\mathbf{q} = 5\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$.

$$\begin{aligned}\mathbf{p}.\mathbf{q} &= (2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}).(5\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}) \\ &= 10 + 6 - 16 \\ &= 0\end{aligned}$$

Since neither **p** nor **q** is zero, we can deduce that

$$\cos\theta = 0$$

where θ is the angle between **p** and **q**, so $\theta = 90°$. Hence **p** is perpendicular to **q**.

Qu. 10 Given that $a = 10$ and $b = 15$ and that the angle between **a** and **b** is 120°, calculate the value of **a.b**.

Qu. 11 Write down the condition for the vectors

$$\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k} \quad \text{and} \quad \mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$$

to be perpendicular.

Qu. 12 Find the angle between the vectors

$$\mathbf{p} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \mathbf{q} = 2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$$

***Qu. 13** The unit vector **u** makes angles α, β and γ with the x-, y- and z-axes respectively. By considering **u.i**, or otherwise, show that

$$\mathbf{u} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$$

and prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

($\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the **direction-cosines** of **u**.)

Qu. 14 Find the direction-cosines of the unit vector parallel to $3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$ and calculate the angles this vector makes with the axes.

It is frequently convenient to have a symbol for the *unit* vector in the direction of a given vector **r**; the normal symbol for this is $\hat{\mathbf{r}}$. So if we use r to represent the magnitude of **r**, the unit vector $\hat{\mathbf{r}}$ is given by

$$\hat{\mathbf{r}} = \frac{1}{r}\mathbf{r}$$

e.g. if we are given that $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j}$, then $r = 5$ and

$$\hat{\mathbf{r}} = \frac{1}{5}\mathbf{r} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

Postscript

15.17 This chapter has been concerned with vectors in two and three dimensions, but there is no reason why we should stop at three! The only problem is that it is rather difficult to draw a four-dimensional vector, especially on a two-dimensional page! However, provided we are prepared to sacrifice the luxury of drawing pictures of our vectors, we may still continue to use the algebraic rules which have been developed for two and three dimensions. Thus if

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 5 \\ 7 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 3 \\ -1 \\ 4 \\ 3 \end{pmatrix}$$

then

$$\mathbf{a}+\mathbf{b}=\begin{pmatrix}4\\1\\9\\10\end{pmatrix},\quad \mathbf{a}-\mathbf{b}=\begin{pmatrix}-2\\3\\1\\4\end{pmatrix}\quad\text{and}\quad 5\mathbf{a}=\begin{pmatrix}5\\10\\25\\35\end{pmatrix}$$

We may even define the 'magnitude' of **a** as $\sqrt{(1^2+2^2+5^2+7^2)}$. Indeed it is possible to define a 'scalar product', although in this context it is usual to call it the 'inner product'. If

$$\mathbf{p}=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\quad\text{and}\quad \mathbf{q}=\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}$$

the inner product of **p** and **q** is $x_1y_1+x_2y_2+x_3y_3+x_4y_4$. Readers who wish to know more should refer to a more advanced book on vector methods.

Exercise 15d (Miscellaneous)

1 In Fig. 15.30, OABC and OPQR are parallelograms; $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OC}=\mathbf{c}$, $\overrightarrow{OP}=\frac{2}{3}\mathbf{a}$, $\overrightarrow{OR}=\frac{1}{2}\mathbf{c}$. Express the following vectors in terms of **a** and **c**:
(a) $\overrightarrow{OB}$, (b) $\overrightarrow{AC}$, (c) $\overrightarrow{OQ}$, (d) $\overrightarrow{PR}$, (e) $\overrightarrow{RC}$,
(f) $\overrightarrow{AQ}$, (g) $\overrightarrow{QC}$, (h) $\overrightarrow{PB}$, (i) $\overrightarrow{PC}$, (j) $\overrightarrow{BQ}$.

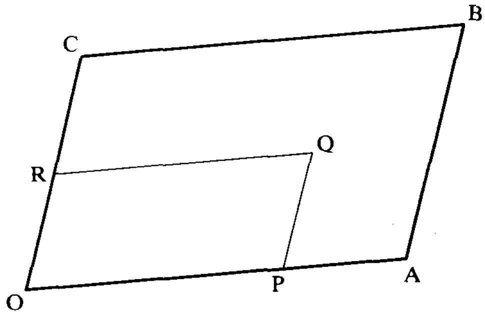


Figure 15.30

2 Find scalars h and k, such that $h\begin{pmatrix}3\\5\end{pmatrix}+k\begin{pmatrix}4\\7\end{pmatrix}=\begin{pmatrix}6\\3\end{pmatrix}$.

3 Given two points A and B, with position vectors **a** and **b**, find, in terms of **a** and **b**, the position vector of the point P, such that
(a) P is the mid-point of AB, (b) B is the mid-point of AP,
(c) $\overrightarrow{AP}:\overrightarrow{PB}=3:7$, (d) $\overrightarrow{AP}=\frac{3}{8}\overrightarrow{AB}$,
(e) $\overrightarrow{PA}=2\overrightarrow{AB}$.

4 (a) Find the scalar product of $\mathbf{a}=\begin{pmatrix}7.2\\9.6\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}12\\-9\end{pmatrix}$,

(b) find the magnitudes of **a** and **b**,

(c) calculate the angle between **a** and **b**.

5 Given that P, Q and R are the points (8, 10), (6, 20) and (16, 16) respectively, calculate the value of the scalar product $\overrightarrow{PQ}.\overrightarrow{PR}$. Hence calculate the size of the angle QPR.

6 The points A, B and C have coordinates (4, −1, 5), (8, 0, 6) and (5, −3, 3) respectively. Prove that the angle BAC is a right angle.

7 In Fig. 15.31, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \frac{4}{3}\mathbf{b}$ and $\overrightarrow{AP} = \frac{2}{3}\overrightarrow{AB}$.

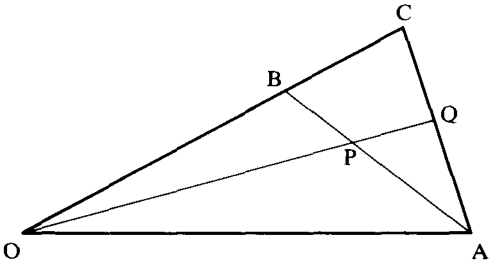


Figure 15.31

Given that $\overrightarrow{AQ} = m\overrightarrow{AC}$ and that $\overrightarrow{OQ} = n\overrightarrow{OP}$, calculate the values of m and n, and the ratio AQ:QC.

8 Find the coordinates of the point P where the line

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

meets the plane $3x + 2y - 2z + 7 = 0$.

9 Find the equation of the line through the points (2, 3, 7) and (3, 1, 4). Find also the equation of the plane perpendicular to this line which passes through the origin.

10 Find the equation of the plane containing the line

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

and passing through the point (1, 0, 3).

11 Given the vectors **a** and **b**, where

$$\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$$

and

$$\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$$

prove that the vector

$$\mathbf{c} = (y_1z_2 - y_2z_1)\mathbf{i} + (z_1x_2 - z_2x_1)\mathbf{j} + (x_1y_2 - x_2y_1)\mathbf{k}$$

is perpendicular to both **a** and **b**.

12 Prove that the planes

$$x - 2y + z = 5$$
$$3x - 3y + z = 3$$
$$5x - 4y + z = 1$$

meet in a common line and find the coordinates of the point where this line meets the plane $z = 0$.

13 A destroyer sights a ship travelling with constant velocity $5\mathbf{j}$, whose position vector at the time of sighting is $2000(3\mathbf{i} + \mathbf{j})$ relative to the destroyer, distances being in m and velocity in m s^{-1}. The destroyer immediately begins to move with velocity $k(4\mathbf{i} + 3\mathbf{j})$, where k is a constant, in order to intercept the ship. Find k and the time to interception.

Find also the distance between the vessels when half the time to interception has elapsed. (O & C)

14 The position vectors, relative to the origin O, of points A and B are respectively $\mathbf{a}$ and $\mathbf{b}$. State, in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vector of the point T which lies on AB and is such that $\overrightarrow{AT} = 2\overrightarrow{TB}$. (Give reasons.)

Find the position vector of the point M on OT produced such that BM and OA are parallel.

If AM is produced to meet OB produced in K, determine the ratio OB:BK. (O & C)

15 The point A has coordinates $(2, 0, -1)$ and the plane π has the equation $x + 2y - 2z = 8$. The line through A parallel to the line $\dfrac{x}{-2} = y = \dfrac{z+1}{2}$ meets π in the point B and the perpendicular from A to π meets π in the point C.

(a) Find the coordinates of B and C.
(b) Show that the length of AC is 4/3.
(c) Find sin $\angle$ABC. (O & C: MEI)

16 Of the following equations, which represent lines and which represent planes?

(a) $\dfrac{x-2}{1} = \dfrac{y-1}{2} = \dfrac{z-3}{-1}$,

(b) $x + 2y - z = 1$,

(c) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$.

Describe, or show in a clear diagram, how these lines and planes are related to each other. (O & C: SMP)

17 Points P, Q and R have position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$. If $\mathbf{p} = (1 - \alpha)\mathbf{q} + \alpha\mathbf{r}$, for some number α, describe the position of P relative to Q and R.

OABC are four non-coplanar points in space. A, B, C have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ relative to O. The position vector of V is $2\mathbf{a} - \mathbf{c}$, and of W is

$-2\mathbf{a} + 3\mathbf{b}$. If VW meets the plane OBC in U, find the position vector of U and show that U is on BC.

Use scalar products to show that if V is in the plane through O perpendicular to OB, and W is in the plane through O perpendicular to OC, then U is in the plane through O perpendicular to OA. (O & C: SMP)

18 The vertices A, B and C of a triangle have position vectors **a**, **b** and **c** respectively relative to an origin O. The point P is on BC such that BP:PC = 3:1; the point Q is on CA such that CQ:QA = 2:3; the point R is on BA produced such that BR:AR = 2:1. The position vectors of P, Q and R are **p**, **q** and **r**, respectively. Show that **q** can be expressed in terms of **p** and **r** and hence show that P, Q and R are collinear. State the ratio of the lengths of the line segments PQ and QR. (JMB)

19 The position vectors of the points A, B and C are given by $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{c} = 11\mathbf{i} + \lambda\mathbf{j} + 14\mathbf{k}$. Find

(a) the unit vector parallel to AB,

(b) the position vector of the point D such that ABCD is a parallelogram,

(c) the value of λ if A, B and C are collinear,

(d) the position vector of the point P on AB if AP:PB = 2:1. (C)

20 A tetrahedron OABC with vertex O at the origin is such that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$. Show that the line segments joining the mid-points of opposite edges bisect one another. Given that two pairs of opposite edges are perpendicular prove that

$$\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{c} = \mathbf{c}.\mathbf{a}$$

and show that the third pair of opposite edges is also perpendicular. Prove also that, in this case,

$$OA^2 + BC^2 = OB^2 + AC^2 \qquad \text{(L)}$$

Chapter 16

The general angle and Pythagoras' theorem

The general angle

16.1 Consider a wheel which is free to rotate about a fixed axis, and suppose that one spoke is marked with a thin line of paint. If the wheel starts from rest and makes one revolution, the marked spoke turns through 360°, and if the wheel makes another revolution the spoke turns through 360° again. Thus we may say that the wheel has turned through a total of 720°, and by using angles greater than 360° the number of revolutions may be specified, as well as the position of the marked spoke.

Now on the x-axis of a graph the positive direction is usually taken to the right and the negative direction is opposite to this. Similarly, if the wheel mentioned above was rotating anti-clockwise, we could take that sense to be positive, and then a clockwise rotation would be considered negative. Angles measured from the x-axis in an anti-clockwise sense are positive, and those measured in a clockwise sense are negative (see Fig. 16.1).

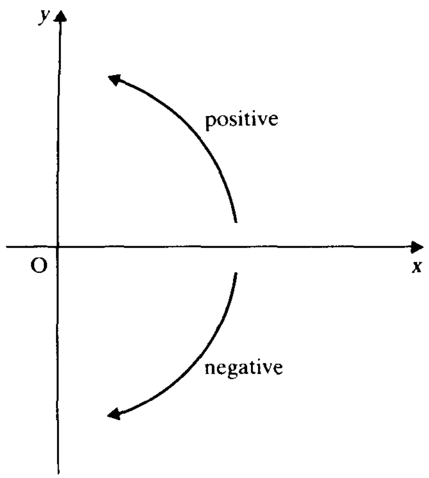


Figure 16.1

Trigonometrical ratios of angles of any magnitude are required in connection with oscillating bodies and rotation about an axis, and in physics they arise in connection with such topics as alternating currents. But as the reader may only have had the six ratios defined for a limited range of angles, we will now give a general definition.

The axes divide the plane into four quadrants, and, as angles are measured in an anti-clockwise direction from the x-axis, the quadrants are numbered as in Fig. 16.2. For the present, a point P(x, y) and its coordinates will be given a suffix corresponding to the quadrant it lies in.

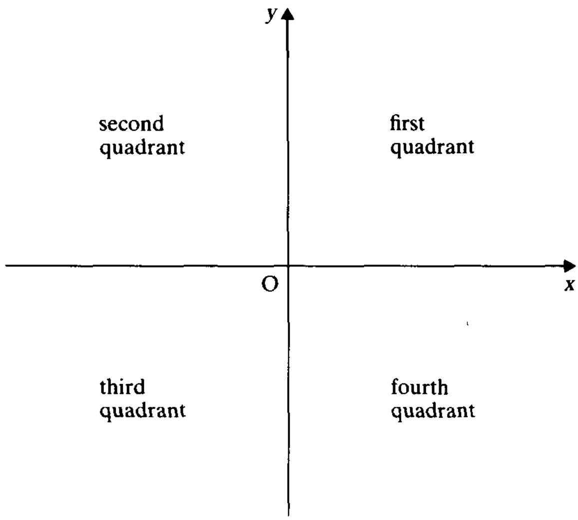


Figure 16.2

For an acute angle θ_1 (see Fig. 16.3),

$$\sin\theta_1 = \frac{y_1}{r}, \qquad \cos\theta_1 = \frac{x_1}{r}, \qquad \tan\theta_1 = \frac{y_1}{x_1}$$

In each case, r is the length of the vector $\overrightarrow{OP}$, and, as in the previous chapter, it should always be taken to be positive. Now

$$\frac{\sin\theta_1}{\cos\theta_1} = \frac{y_1/r}{x_1/r} = \frac{y_1}{x_1} = \tan\theta_1$$

so for an angle θ of *any* magnitude we shall define the six trigonometrical ratios as follows:

$$\sin\theta = \frac{y}{r}, \qquad \cos\theta = \frac{x}{r}, \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}$$

For an angle θ_2 in the second quadrant (see Fig. 16.3), y_2 is positive

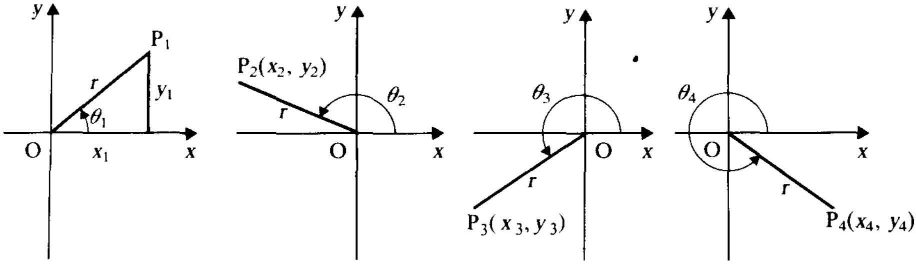


Figure 16.3

(abbreviated +ve) but x_2 is negative (abbreviated −ve), therefore

$\sin\theta_2$ is +ve, $\cos\theta_2$ is −ve, $\tan\theta_2$ is −ve

In the third quadrant, x_3 and y_3 are both negative, hence

$\sin\theta_3$ is −ve, $\cos\theta_3$ is −ve, $\tan\theta_3$ is +ve

For an angle θ_4 in the fourth quadrant, x_4 is positive, and y_4 is negative, hence

$\sin\theta_4$ is −ve, $\cos\theta_4$ is +ve, $\tan\theta_4$ is −ve

These results can be summarised by writing which ratios are positive in each quadrant:

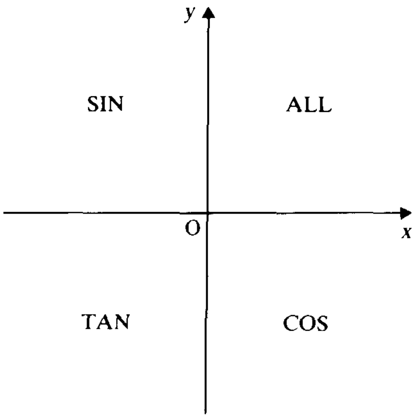


Figure 16.4

The signs of the ratios can be worked out as above quite easily, but for those who like them, there are mnemonics for the first letters in the four quadrants of Fig. 16.4. One such is **A**ll **S**illy **T**om **C**ats. The signs of cosec θ, sec θ, cot θ are, of course, the same as their reciprocals.

A useful point to note is that angles for which OP is equally inclined to the positive or negative x-axis have trigonometrical ratios of the same magnitude, their signs being determined as above. Thus the ratios of 150°, 210°, 330° are *numerically* the same as the ratios of 30°, since in each case the acute angle between OP and the x-axis is 30°, as shown overleaf.

$\sin 150° = +\sin 30°$
$\cos 150° = -\cos 30°$
$\tan 150° = -\tan 30°$

$\sin 210° = -\sin 30°$
$\cos 210° = -\cos 30°$
$\tan 210° = +\tan 30°$

$\sin 330° = -\sin 30°$
$\cos 330° = +\cos 30°$
$\tan 330° = -\tan 30°$

Qu. 1 Express in terms of the trigonometrical ratios of acute angles:
(a) sin 170°, (b) tan 300°, (c) cos 200°,
(d) sin (−50°), (e) cos (−20°), (f) sin 325°,
(g) tan (−140°), (h) cos 164°, (i) cosec 230°,
(j) tan 143°, (k) cos (−130°), (l) sin 250°,
(m) tan (−50°), (n) cot 200°, (o) cos 293°,
(p) sin (−230°), (q) sec 142°, (r) cot 156°,
(s) cosec (−53°), (t) sec (−172°).

Graphs of sin θ, cos θ, tan θ

16.2 It is instructive to draw the graphs of sin θ, cos θ, and tan θ. Fig. 16.5 shows how the graph of sin θ may be drawn from the definition. Construct a circle of unit radius, then sin $\theta = y$. Dotted lines show this for $\theta = 30°, 60°, 90°$, and the rest of the figure is drawn similarly.

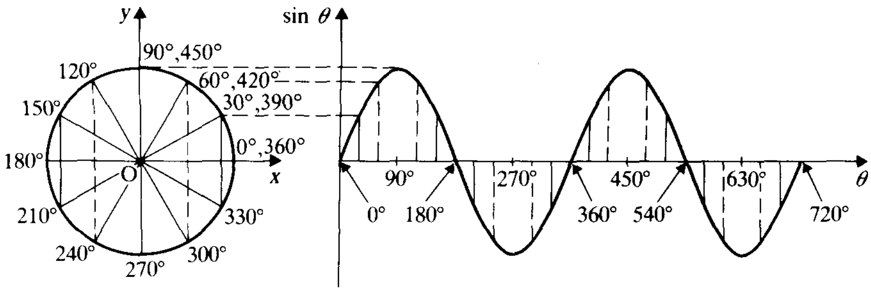


Figure 16.5

It will be seen that the graph of sin θ repeats itself at intervals of 360°. (That this is so should be clear from the way it was drawn, because points on the graph separated by 360° correspond to the same point on the circle.) If a function repeats itself at regular intervals, like sin θ, it is called a **periodic** function, and the interval is called its **period** (see §2.15).

The graph of cos θ may be drawn in a similar way to that of sin θ. In this case, since $\cos \theta = x/r$, the values of x are used instead of y.

The graph of tan θ may also be drawn from a unit circle, but in this case a tangent is drawn at the point (1, 0) (see Fig. 16.6). If P is any point on the circle, and OP meets the tangent at Q, then the y-coordinate of Q is equal to $\tan\theta$.

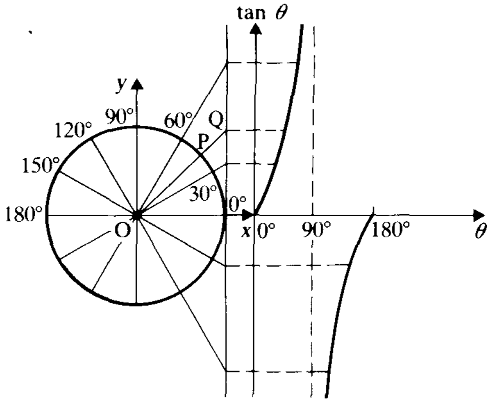


Figure 16.6

Qu. 2 Complete the graph of $\tan\theta$ up to $\theta = 720°$.
Qu. 3 What are the periods of $\cos\theta$ and $\tan\theta$?

Trigonometrical ratios of 30°, 45°, 60°

16.3 The trigonometrical ratios of 30°, 45°, and 60° are frequently needed, and they may be obtained from two figures. Fig. 16.7 represents an equilateral triangle with an altitude constructed. The sides of the triangle are 2 units, and so, by Pythagoras' theorem, the altitude is $\sqrt{3}$ units. The ratios of 30° and 60° may now be read off. Fig. 16.8 represents a right-angled isosceles triangle with two sides of unit length. By Pythagoras' theorem the hypotenuse is $\sqrt{2}$ units, and so the ratios of 45° may be read off.

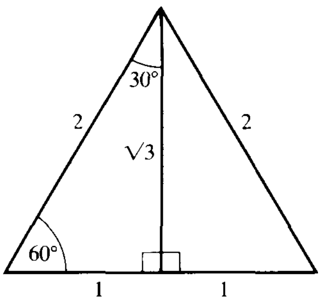


Figure 16.7

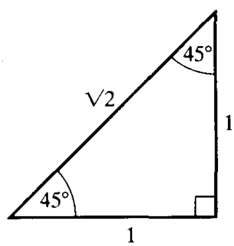


Figure 16.8

Qu. 4 Write down the values of (a) $\sin 30°$, (b) $\cos 30°$, (c) $\cos 45°$, (d) $\tan 30°$, (e) $\sec 60°$, (f) $\operatorname{cosec} 60°$, (g) $\tan 45°$, (h) $\operatorname{cosec} 45°$.

Trigonometrical equations

16.4 Most equations in algebra have a finite number of roots, but in many cases trigonometrical equations have an unlimited number. For instance, the equation $\sin\theta=0$ is satisfied by $\theta=0°$, $\pm180°$, $\pm360°$, $\pm540°$ and so on, indefinitely. In this book it will be specified for what range of values the roots are required.

Example 1 *Solve the equation* $\sin\theta=-\frac{1}{2}$ *for values of* θ *from* 0° *to* 360° *inclusive.*

The acute angle whose sine is $\frac{1}{2}$ is 30° and Fig. 16.9 indicates the angles between 0° and 360° whose sines are $\pm\frac{1}{2}$. But $\sin\theta$ is negative only in the third and fourth quadrants. Therefore the roots of the equation in the required range are 210° and 330°.

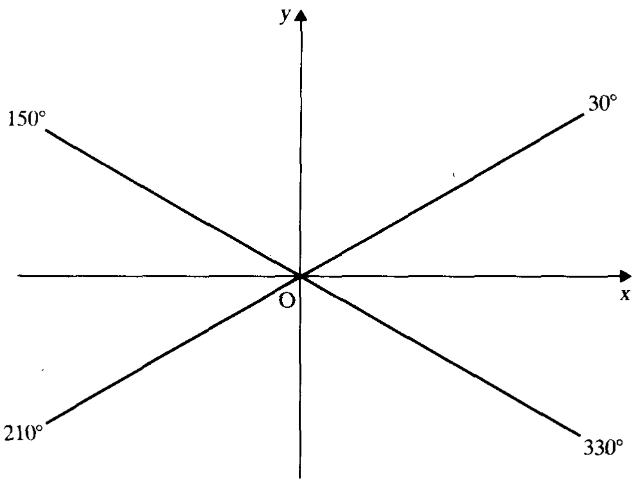


Figure 16.9

Example 2 *Solve the equation* $\cos 2\theta=0.6428$, *for values of* θ *between* $-180°$ *and* $+180°$.

[Note that since θ must lie between $-180°$ and $+180°$, 2θ may lie between $-360°$ and $+360°$.]

From a calculator or tables it can be seen that the acute angle whose cosine is 0.6428 is 50° (see note on accuracy after the Preface), and since $\cos 2\theta$ is positive only in the first and fourth quadrants

$$2\theta=-310°,\quad -50°,\quad 50°,\quad 310°$$
$$\therefore\ \theta=-155°,\quad -25°,\quad 25°,\quad 155°$$

Example 3 *Solve the equation** $2\sin^2\theta=\sin\theta$, *for values of* θ *from* 0° *to* 360° *inclusive.*

*In order to avoid brackets $(\sin\theta)^2$ is written $\sin^2\theta$.

[This equation is a quadratic equation for $\sin\theta$, and may be solved by factorisation.]

$$2\sin^2\theta - \sin\theta = 0$$
$$\therefore \sin\theta\,(2\sin\theta - 1) = 0$$
$$\therefore \sin\theta = 0 \quad \text{or} \quad \sin\theta = \tfrac{1}{2}$$

If $\sin\theta = 0$, $\theta = 0°, 180°, 360°$. If $\sin\theta = \frac{1}{2}$, $\theta = 30°, 150°$.

Therefore the roots of the equation, from 0° to 360° inclusive are 0°, 30°, 150°, 180°, and 360°.

(Note that if we had divided both sides of the equation by $\sin\theta$, giving $2\sin\theta = 1$, we should have lost some of the roots, namely those for which $\sin\theta = 0$.)

Example 4 *Solve the equation* $\tan\theta = 2\sin\theta$, *for values of* θ *from* 0° *to* 360° *inclusive.*

[Equations are often solved by factorisation, so look for a common factor.] Remembering that $\tan\theta = \sin\theta/\cos\theta$ we may write

$$\frac{\sin\theta}{\cos\theta} = 2\sin\theta$$

$$\therefore 2\sin\theta\cos\theta = \sin\theta$$

$$\therefore 2\sin\theta\cos\theta - \sin\theta = 0$$

$$\therefore \sin\theta\,(2\cos\theta - 1) = 0$$

$$\therefore \sin\theta = 0 \quad \text{or} \quad \cos\theta = \tfrac{1}{2}$$

If $\sin\theta = 0$, $\theta = 0°, 180°, 360°$. If $\cos\theta = \frac{1}{2}$, $\theta = 60°, 300°$.

Therefore the required values of θ are 0°, 60°, 180°, 300°, and 360°.

Exercise 16a

1 Write down the values of the following, leaving surds in your answers (calculators should not be used in this question):

(a) cos 270°,	(b) sin 540°,	(c) cos (−180°),
(d) tan 135°,	(e) sin 150°,	(f) cos 210°,
(g) tan 120°,	(h) cos (−30°),	(i) sin (−120°),
(j) sin 405°,	(k) cos (−135°),	(l) sin 225°,
(m) tan (−60°),	(n) sin (−270°),	(o) tan 210°.

2 Sketch the graph of $\sin\theta$, for values of θ from −360° to 360°.

3 Sketch the graph of $\cos\theta$, for values of θ from 0° to 720°, and state its period.

4 Draw the graph of $\tan\theta$, for values of θ from 0° to 720°. (This has been started in Fig. 16.6.) What is the period of $\tan\theta$?

5 Sketch the graphs of (a) $\cos 2\theta$, (b) $\sin\frac{1}{2}\theta$, (c) $\sin\frac{3}{2}\theta$, (d) $\cos(\theta + 60°)$, (e) $\sin(\theta - 45°)$, for values of θ from 0° to 360°, stating the period of each.

6 Find the values of θ from 180° to 360°, inclusive, which satisfy the following equations:

(a) $\cos\theta = -\frac{1}{2}$, (b) $\tan\theta = 1$,
(c) $\operatorname{cosec}\theta = 2$, (d) $\sin\theta = -0.7660$,
(e) $\cos\theta = 0.6$, (f) $\tan\theta = -\sqrt{3}$,
(g) $\cos(\theta + 60°) = 0.5$, (h) $\sin(\theta - 30°) = -\sqrt{3}/2$.

7 Solve the following equations for values of θ from 0° to 360°, inclusive:

(a) $\sin^2\theta = \frac{1}{4}$, (b) $\tan^2\theta = \frac{1}{3}$,
(c) $\sin 2\theta = \frac{1}{2}$, (d) $\tan 2\theta = -1$,
(e) $\cos 3\theta = \sqrt{3}/2$, (f) $\sin 3\theta = -1$,
(g) $\sin^2 2\theta = 1$, (h) $\sec 2\theta = 3$,
(i) $\tan^2 3\theta = 1$, (j) $4\cos 2\theta = 1$,
(k) $\sin(2\theta + 30°) = 0.8$, (l) $\tan(3\theta - 45°) = \frac{1}{2}$.

8 Solve the following equations for values of θ from −180° to +180°, inclusive:

(a) $\tan^2\theta + \tan\theta = 0$, (b) $2\cos^2\theta = \cos\theta$,
(c) $3\sin^2\theta + \sin\theta = 0$, (d) $2\sin^2\theta - \sin\theta - 1 = 0$,
(e) $2\cos^2\theta + 3\cos\theta + 1 = 0$, (f) $4\cos^3\theta = \cos\theta$,
(g) $\tan\theta = \sin\theta$, (h) $\sec\theta = 2\cos\theta$,
(i) $\cot\theta = 5\cos\theta$, (j) $4\sin^2\theta = 3\cos^2\theta$,
(k) $3\cos\theta = 2\cot\theta$, (l) $\tan\theta = 4\cot\theta + 3$,
(m) $5\sin\theta + 6\operatorname{cosec}\theta = 17$, (n) $3\cos\theta + 2\sec\theta + 7 = 0$.

9 Write down the maximum and minimum values of the following expressions, giving the smallest positive or zero value of θ for which they occur:

(a) $\sin\theta$, (b) $3\cos\theta$, (c) $2\cos\frac{1}{2}\theta$,
(d) $-\frac{1}{2}\sin 2\theta$, (e) $1 - 2\sin\theta$, (f) $3 + 2\cos 3\theta$,

(g) $\dfrac{1}{2+\sin\theta}$, (h) $\dfrac{1}{4-3\cos\theta}$, (i) $\sec\frac{3}{2}\theta$,

(j) $\tan^2\theta$, (k) $\dfrac{1}{1+\operatorname{cosec}\theta}$, (l) $\dfrac{2}{3-2\cot\theta}$,

(m) $\dfrac{\cos\theta}{\cos\theta+\sin\theta}$.

10 State, with reasons, which of the following equations have no roots:

(a) $2\sin\theta = 3$, (b) $\sin\theta + \cos\theta = 0$,
(c) $\sin\theta + \cos\theta = 2$, (d) $3\sin\theta + \operatorname{cosec}\theta = 0$,
(e) $4\operatorname{cosec}^2\theta - 1 = 0$, (f) $\operatorname{cosec}\theta = \sin\theta$,
(g) $\sec\theta = \sin\theta$.

11 Sketch on the same axes, for values of θ from −360° to 360°, the graphs of (a) $\sin\theta$, $\operatorname{cosec}\theta$; (b) $\cos\theta$, $\sec\theta$; (c) $\tan\theta$, $\cot\theta$.

12 Sketch the graphs of the following functions and state the period in each case:

(a) $y = \sin 2x$, (b) $y = \cos(x/3)$, (c) $y = \tan 3x$,
(d) $y = \tan(x/2)$, (e) $y = \sin(2x/3)$.

Trigonometrical ratios of $-\theta$, $180 \pm \theta$, $90 \pm \theta$

16.5 The reader who has drawn the graphs of $y = \sin\theta$ and $y = \cos\theta$ may have noticed that they are the same, except for the positions of the y-axes relative to the curves.

Fig. 16.10 suggests that, for any angle α,

$$\cos\alpha = \sin(90° + \alpha)$$

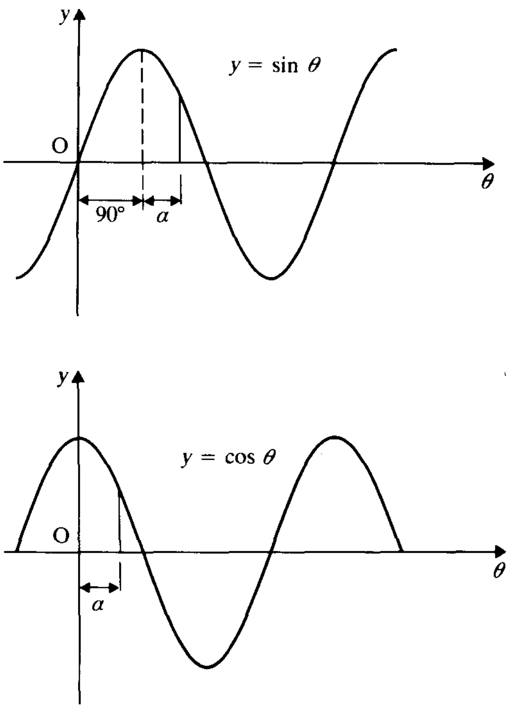


Figure 16.10

and other relationships of this sort may be found from the graphs. Some people find the graphs help them to remember such relationships, but now it will be shown how they may be obtained from first principles.

For any value of θ, in the notation of §16.1 we have by definition

$$\sin\theta = \frac{y}{r}, \qquad \cos\theta = \frac{x}{r}$$

Consider:

(a) ratios of $-\theta$. In Fig. 16.3, p. 327, the angle $-\theta$ is obtained by replacing (x, y) by $(x, -y)$,

$$\therefore \sin(-\theta) = -\frac{y}{r} = -\sin\theta$$

$$\cos(-\theta) = \frac{x}{r} = \cos\theta$$

i.e. $\sin\theta$ is an odd function and $\cos\theta$ is an even function (see §2.14).

(b) ratios of $(180° - \theta)$. Replace (x, y) by $(-x, y)$, hence

$$\sin(180° - \theta) = \frac{y}{r} = \sin\theta$$

$$\cos(180° - \theta) = -\frac{x}{r} = -\cos\theta$$

(c) ratios of $(180° + \theta)$. Replace (x, y) by $(-x, -y)$, hence

$$\sin(180° + \theta) = -\frac{y}{r} = -\sin\theta$$

$$\cos(180° + \theta) = -\frac{x}{r} = -\cos\theta$$

[Note that in all these cases above, OP is inclined at an angle θ to the positive or negative x-axis, the ratios of these angles have the same magnitude as those of θ, and their signs are determined as on page 327 if θ is acute.]

(d) ratios of $(90° - \theta)$. Replace (x, y) by (y, x), hence

$$\sin(90° - \theta) = \frac{x}{r} = \cos\theta$$

$$\cos(90° - \theta) = \frac{y}{r} = \sin\theta$$

(e) ratios of $(90° + \theta)$. Replace (x, y) by $(-y, x)$, hence

$$\sin(90° + \theta) = \frac{x}{r} = \cos\theta$$

$$\cos(90° + \theta) = -\frac{y}{r} = -\sin\theta$$

Qu. 5 Express the following in terms of the trigonometrical ratios of θ:

(a) $\tan(90° - \theta)$,	(b) $\operatorname{cosec}(180° - \theta)$,	(c) $\sec(90° + \theta)$,
(d) $\cot(90° + \theta)$,	(e) $\sec(-\theta)$,	(f) $\operatorname{cosec}(180° + \theta)$,
(g) $\cos(270° - \theta)$,	(h) $\sin(360° + \theta)$,	(i) $\tan(-\theta)$,
(j) $\sin(\theta - 90°)$,	(k) $\cos(\theta - 180°)$,	(l) $\sec(270° + \theta)$.

Pythagoras' theorem

16.6 The reader will be familiar with Pythagoras' theorem, and will have found that it is a very useful one. In trigonometry it retains its importance and provides relations between trigonometrical ratios.

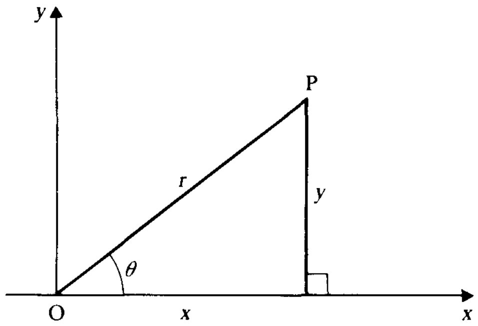


Figure 16.11

In Fig. 16.11, the triangle is right-angled and so, by Pythagoras' theorem,

$$x^2 + y^2 = r^2$$

But $\cos\theta = x/r$ and $\sin\theta = y/r$, so we divide by r^2 obtaining

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$$\therefore \boldsymbol{\cos^2\theta + \sin^2\theta \equiv 1}$$

(If P is not in the first quadrant, OP^2 is still $x^2 + y^2$ by the distance formula of §1.8 and the proof continues as before.)

The $\equiv$ symbol is used to stress that the relationship is an identity, i.e. it holds for *all* values of θ.

Two similar identities can be deduced from this. Dividing through by $\cos^2\theta$,

$$1 + \frac{\sin^2\theta}{\cos^2\theta} \equiv \frac{1}{\cos^2\theta}$$

but $\tan\theta = \sin\theta/\cos\theta$ and $\sec\theta = 1/\cos\theta$, therefore

$$\boldsymbol{1 + \tan^2\theta \equiv \sec^2\theta}$$

Dividing the original identity by $\sin^2\theta$,

$$\frac{\cos^2\theta}{\sin^2\theta} + 1 \equiv \frac{1}{\sin^2\theta}$$

but $\cos\theta/\sin\theta = \cot\theta$ and $1/\sin\theta = \operatorname{cosec}\theta$, therefore

$$\boldsymbol{\cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta}$$

Historical note. The equivalent of the identity

$$\cos^2\theta + \sin^2\theta = 1$$

is found in the *Syntaxis* written during the first century A.D. by Claudius Ptolemy. Instead of sines and cosines, he used chords. (If a chord subtends an angle 2θ at the centre of a circle, the ratio of the chord to the diameter of the circle is $\sin\theta$.)

Example 5 *Solve the equation* $1 + \cos\theta = 2\sin^2\theta$, *for values of* θ *between* 0° *and* 360°.

[The square on the right-hand side indicates that the equation is a quadratic, and to solve it, we must write it in terms of either $\cos\theta$ or $\sin\theta$.] We know that

$$\cos^2\theta + \sin^2\theta = 1$$

hence $$\sin^2\theta = 1 - \cos^2\theta$$

so substituting $2 - 2\cos^2\theta$ for $2\sin^2\theta$, we obtain

$$1 + \cos\theta = 2 - 2\cos^2\theta$$

This quadratic for $\cos\theta$ is solved by factorisation:

$$2\cos^2\theta + \cos\theta - 1 = 0$$
$$\therefore (2\cos\theta - 1)(\cos\theta + 1) = 0$$
$$\therefore \cos\theta = \tfrac{1}{2} \quad \text{or} \quad -1$$

If $\cos\theta = \frac{1}{2}$, $\theta = 60°, 300°$. If $\cos\theta = -1$, $\theta = 180°$.

Therefore the roots of the equation between 0° and 360° are 60°, 180°, and 300°.

Example 6 *Simplify* $1/\sqrt{(x^2 - a^2)}$ *when* $x = a \operatorname{cosec}\theta$.

Substituting $x = a\operatorname{cosec}\theta$, we obtain

$$\frac{1}{\sqrt{(a^2\operatorname{cosec}^2\theta - a^2)}}$$

But the $\operatorname{cosec}^2\theta$ in the denominator suggests the use of the identity

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

With this the expression $(a^2\operatorname{cosec}^2\theta - a^2)$ may be simplified, giving

$$a^2\operatorname{cosec}^2\theta - a^2 = a^2(\cot^2\theta + 1) - a^2 = a^2\cot^2\theta$$

Thus the original expression becomes

$$\frac{1}{\sqrt{(a^2\cot^2\theta)}} = \frac{1}{a\cot\theta} = \frac{1}{a}\tan\theta$$

Example 7 *Eliminate* θ *from the equations* $x = a\sin\theta$, $y = b\tan\theta$.

[Since $\sin\theta$ and $\tan\theta$ are the reciprocals of $\operatorname{cosec}\theta$ and $\cot\theta$ we use the identity $\operatorname{cosec}^2\theta = \cot^2\theta + 1$.]

$$\operatorname{cosec}\theta = \frac{a}{x} \quad \text{and} \quad \cot\theta = \frac{b}{y}$$

Substituting into the identity $\operatorname{cosec}^2\theta = \cot^2\theta + 1$,

$$\frac{a^2}{x^2} = \frac{b^2}{y^2} + 1$$

Exercise 16b

1 If $s = \sin\theta$, simplify:

(a) $\sqrt{(1-s^2)}$, (b) $\dfrac{s}{\sqrt{(1-s^2)}}$, (c) $\dfrac{1-s^2}{s}$.

2 If $c = \cos\theta$, simplify:

(a) $\sqrt{(1-c^2)}$, (b) $\dfrac{\sqrt{(1-c^2)}}{c}$, (c) $\dfrac{c}{1-c^2}$.

3 If $t = \tan\theta$, simplify:

(a) $\sqrt{(1+t^2)}$, (b) $t(1+t^2)$, (c) $\dfrac{t}{\sqrt{(1+t^2)}}$.

4 If $c = \operatorname{cosec}\theta$, simplify:

(a) $\sqrt{(c^2-1)}$, (b) $\dfrac{\sqrt{(c^2-1)}}{c}$, (c) $\dfrac{c}{c^2-1}$.

5 If $x = a\sin\theta$, simplify:

(a) $a^2 - x^2$, (b) $\dfrac{1}{\sqrt{(a^2-x^2)}}$, (c) $\dfrac{a^2-x^2}{x}$.

6 If $y = b\cot\theta$, simplify:

(a) $b^2 + y^2$, (b) $y\sqrt{(b^2+y^2)}$, (c) $\dfrac{y}{b^2+y^2}$.

7 If $z = a\sec\theta$, simplify:

(a) $z^2 - a^2$, (b) $\dfrac{1}{\sqrt{(z^2-a^2)}}$, (c) $\dfrac{\sqrt{(z^2-a^2)}}{z}$.

In Nos. 8–13, solve the equations, giving values of θ from 0° to 360° inclusive.

8 $3 - 3\cos\theta = 2\sin^2\theta$.
9 $\cos^2\theta + \sin\theta + 1 = 0$.
10 $\sec^2\theta = 3\tan\theta - 1$.
11 $\operatorname{cosec}^2\theta = 3 + \cot\theta$.
12 $3\tan^2\theta + 5 = 7\sec\theta$.
13 $2\cot^2\theta + 8 = 7\operatorname{cosec}\theta$.

14 If $\sin\theta = \frac{3}{5}$, find without using tables or calculators, the values of (a) $\cos\theta$, (b) $\tan\theta$.
15 If $\cos\theta = -\frac{8}{17}$, and θ is obtuse, find without using tables or calculators, the values of (a) $\sin\theta$, (b) $\cot\theta$.
16 If $\tan\theta = \frac{7}{24}$ and θ is reflex, find without using tables or calculators, the values of (a) $\sec\theta$, (b) $\sin\theta$.

Prove the following identities:

17 $\tan\theta + \cot\theta = 1/(\sin\theta\cos\theta)$.
18 $\operatorname{cosec}\theta + \tan\theta\sec\theta = \operatorname{cosec}\theta\sec^2\theta$.
19 $\sec^2\theta - \operatorname{cosec}^2\theta = \tan^2\theta - \cot^2\theta$.
20 $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$.
21 $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$.
22 $2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$.
23 $\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta\operatorname{cosec}^2\theta$.

24 $\sec^4\theta - \operatorname{cosec}^4\theta = \dfrac{\sin^2\theta - \cos^2\theta}{\cos^4\theta\sin^4\theta}$.

25 $\dfrac{1}{\tan^2\theta + 1} + \dfrac{1}{\cot^2\theta + 1} = 1$.

26 $(\sec^2\theta - 1)(\operatorname{cosec}^2\theta - 1) = 1$.
27 $\sqrt{(\sec^2\theta - 1)} + \sqrt{(\operatorname{cosec}^2\theta - 1)} = \sec\theta\operatorname{cosec}\theta$.
28 $\sqrt{(\sec^2\theta - \tan^2\theta)} + \sqrt{(\operatorname{cosec}^2\theta - \cot^2\theta)} = 2$.

29 $\dfrac{1 - \cos^2\theta}{\sec^2\theta - 1} = 1 - \sin^2\theta$.

30 $\dfrac{\sec\theta - \operatorname{cosec}\theta}{\tan\theta - \cot\theta} = \dfrac{\tan\theta + \cot\theta}{\sec\theta + \operatorname{cosec}\theta}$.

31 $\dfrac{\cos\theta}{\sqrt{(1 + \tan^2\theta)}} + \dfrac{\sin\theta}{\sqrt{(1 + \cot^2\theta)}} = 1$.

Eliminate θ from the following equations:

32 $x = a\cos\theta$, $y = b\sin\theta$.
33 $x = a\cot\theta$, $y = b\operatorname{cosec}\theta$.
34 $x = a\tan\theta$, $y = b\cos\theta$.
35 $x = 1 - \sin\theta$, $y = 1 + \cos\theta$.
36 $x = a\sec\theta$, $y = b + c\cos\theta$.
37 $x = a\operatorname{cosec}\theta$, $y = b\sec\theta$.
38 $x = 1 + \tan\theta$, $y = \cos\theta$.
39 $x = \sin\theta + \cos\theta$, $y = \sin\theta - \cos\theta$.
40 $x = \sec\theta + \tan\theta$, $y = \sec\theta - \tan\theta$.

Exercise 16c (Miscellaneous)

1 Express in terms of the ratios of acute angles:
(a) $\cos 205°$, (b) $\tan 153°$, (c) $\sec 309°$,
(d) $\sin(-215°)$, (e) $\cot 406°$, (f) $\operatorname{cosec} 684°$.

2 Find the values of the following, leaving surds in your answers:
(a) $\sin 270°$, (b) $\cos 150°$, (c) $\cot 210°$,
(d) $\cos 315°$, (e) $\operatorname{cosec} 240°$, (f) $\sec 585°$,
(g) $\tan(-225°)$, (h) $\sin(-690°)$, (i) $\cos(-300°)$.

3 Solve the following equations for values of θ from 0° to 360° inclusive:
(a) $2 \sin \theta = 1$, (b) $\tan \theta + 1 = 0$, (c) $\cos \theta = 0.8$,
(d) $\tan 2\theta = 1$, (e) $\sec 2\theta = 4$, (f) $\sin \frac{1}{2}\theta = \frac{1}{2}$,
(g) $3 \cos (\theta - 10°) = 1$, (h) $\sin (\theta + 30°) = 0.7$, (i) $\cot \frac{1}{2}\theta = 0.9$.

4 Solve the following equations for values of θ from $-180°$ to $+180°$ inclusive:
(a) $2 \sin^2 \theta + \sin \theta = 0$, (b) $3 \cos^2 \theta = 2 \sin \theta \cos \theta$,
(c) $2 \sin^2 \theta + 1 = 3 \sin \theta$, (d) $3 \cos^2 \theta = 7 \cos \theta + 6$,
(e) $4 \sin \theta + \operatorname{cosec} \theta = 4$, (f) $10 \cos \theta + 1 = 2 \sec \theta$,
(g) $\tan \theta + 2 \cot \theta = 3$, (h) $10 \sin \theta \cos \theta - 5 \sin \theta + 4 \cos \theta = 2$.

5 Find the maximum and minimum values of the following functions of θ. Give the smallest non-negative values of θ for which they occur.

(a) $3 + 2 \sin \theta$, (b) $1 - 3 \cos \theta$, (c) $4 \sin \frac{3}{2}\theta$,

(d) $3 \sin^2 \frac{1}{2}\theta$, (e) $\dfrac{1}{2 + 3 \cos \theta}$, (f) $\dfrac{1}{3 - 2 \sin 2\theta}$.

6 Express in terms of the trigonometrical ratios of θ:
(a) $\cot (90° - \theta)$, (b) $\sin (90° + \theta)$, (c) $\cos (270° + \theta)$,
(d) $\tan (90° + \theta)$, (e) $\operatorname{cosec} (360° - \theta)$, (f) $\sec (180° - \theta)$,
(g) $\sin (\theta - 180°)$, (h) $\tan (-\theta)$, (i) $\cos (450° - \theta)$.

7 If $s = \sin \theta$ and $c = \cos \theta$, simplify:

(a) $\dfrac{1 - s^2}{1 - c^2}$, (b) $\dfrac{sc}{\sqrt{(1 - s^2)}}$, (c) $\dfrac{s}{c^2 - 1}$,

(d) $\dfrac{c^4 - s^4}{c^2 - s^2}$, (e) $\dfrac{s\sqrt{(1 - s^2)}}{c\sqrt{(1 - c^2)}}$, (f) $\dfrac{c}{s} + \dfrac{s}{c}$.

8 Solve the following equations for values of θ from 0° to 360° inclusive:
(a) $2 \cos^2 \theta + \sin \theta = 1$,
(b) $5 \cos \theta = 2(1 + 2 \sin^2 \theta)$,
(c) $2 \tan^2 \theta + \sec \theta = 1$,
(d) $4 \cot^2 \theta + 39 = 24 \operatorname{cosec} \theta$,
(e) $5 \sec \theta - 2 \sec^2 \theta = \tan^2 \theta - 1$,
(f) $\sec \theta + 3 = \cos \theta + \tan \theta (2 + \sin \theta)$.
(g) $3 \sin^2 \theta - \sin \theta \cos \theta - 4 \cos^2 \theta = 0$.

9 Find, without using tables or calculators, the values of
(a) $\sin \theta$, $\tan \theta$, if $\cos \theta = \frac{4}{5}$ and θ is acute.
(b) $\sec \theta$, $\sin \theta$, if $\tan \theta = -\frac{5}{12}$ and θ is obtuse.
(c) $\cos \theta$, $\cot \theta$, if $\sin \theta = \frac{15}{17}$ and θ is acute.
(d) $\sin \theta$, $\sec \theta$, if $\cot \theta = \frac{20}{21}$ and θ is reflex.

Prove the following identities:

10 $\sec \theta + \operatorname{cosec} \theta \cot \theta = \sec \theta \operatorname{cosec}^2 \theta$.

11 $\sin^2 \theta (1 + \sec^2 \theta) = \sec^2 \theta - \cos^2 \theta$.

12 $\dfrac{1 - \cos \theta}{\sin \theta} = \dfrac{1}{\operatorname{cosec} \theta + \cot \theta}$.

13 $\dfrac{\tan\theta+\cot\theta}{\sec\theta+\operatorname{cosec}\theta}=\dfrac{1}{\sin\theta+\cos\theta}$.

14 $\sec^2\theta=\dfrac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta-\sin\theta}$.

15 $\dfrac{1+\sin\theta}{1-\sin\theta}=(\sec\theta+\tan\theta)^2$.

16 $\sec\theta-\sin\theta=\dfrac{\tan^2\theta+\cos^2\theta}{\sec\theta+\sin\theta}$.

17 $\dfrac{1-\sin\theta+\cos\theta}{1-\sin\theta}=\dfrac{1+\sin\theta+\cos\theta}{\cos\theta}$.

Eliminate θ from the following pairs of equations:

18 $x=a\sec\theta$, $y=b\tan\theta$.
19 $x=1-\cos\theta$, $y=1+\sin\theta$.
20 $x=a\cot\theta$, $y=b\sin\theta$.
21 $x=a\sec\theta$, $y=b\cot\theta$.
22 $x=a\tan\theta$, $y=b\sin\theta$.
23 $x=\operatorname{cosec}\theta-\cot\theta$, $y=\operatorname{cosec}\theta+\cot\theta$.
24 $x=\sin\theta+\cos\theta$, $y=\tan\theta$.
25 $x=\cos\theta$, $y=\operatorname{cosec}\theta-\cot\theta$.
26 Plot the graph of $y=\sin x+\cos x$ for values of x from $-180°$ to $180°$ at intervals of 30°. Find from your graph the maximum and minimum values of $\sin x+\cos x$, and the values of x for which they occur.
27 Plot the graph of $y=\sin x+2\cos x$ for values of x from $-180°$ to $180°$ at intervals of 30°. Find from your graph the roots of the equation $\sin x+2\cos x=1$ which lie between $-180°$ and $+180°$.
28 Plot the graphs of $y=\sin 2x$ and $y=\cos 3x$ on the same axes for values of x from 0° to 90°. Find from your graph the root of the equation $\sin 2x=\cos 3x$ which lie between 0° and 90°.
29 Solve the simultaneous equations

$$\sin(x+y)=\frac{1}{\sqrt{2}}$$

$$\cos 2x=-\tfrac{1}{2}$$

for values of x, y from 0° to 360° inclusive.
30 State whether the following functions are odd, even or neither, and state the range of each function. Sketch the graph of each function.
(a) $y=1+\sin x$, (b) $y=2+3\cos x$,
(c) $y=5\sin x+10$, (d) $y=1-\cos x$.

Chapter 17

Trigonometrical identities

The formulae for sin $(A \pm B)$, cos $(A \pm B)$

17.1 Place a rectangular piece of cardboard PQRS in a vertical plane with two edges horizontal, and then turn it through an angle B (see Fig. 17.1). Take the diagonal PR as the unit of length and let angle RPQ be A.

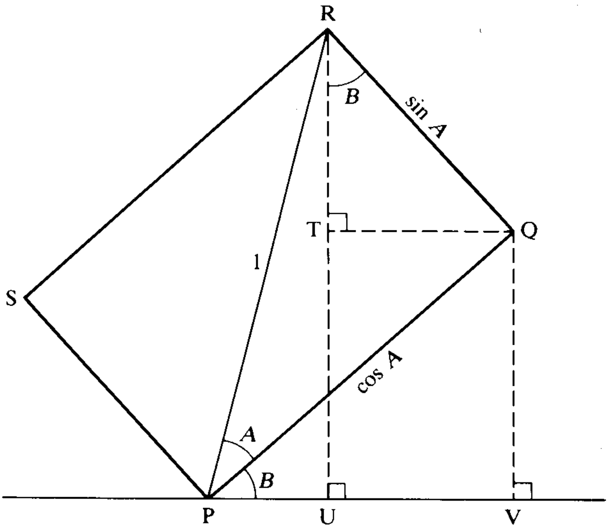


Figure 17.1

What is the height of R *above* P?

One way to find this out is to drop a perpendicular RU from R to the horizontal through P, then from the triangle RPU, $RU = \sin(A + B)$.

Alternatively, since $RQ = \sin A$, $PQ = \cos A$ and angle $QRU = B$, the height of R above P can be found in two parts. First, the height of R above Q, $RT = \sin A \cos B$ (from triangle RTQ). Secondly, the height of Q above P, $QV = \cos A \sin B$ (from triangle PQV). Thus, equating the height of R above P obtained in the two ways,

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

How far to the right of P *is* R?
In triangle RPU, PU = $\cos(A+B)$.
Alternatively, the distance of Q to the right of P, PV = $\cos A \cos B$ (from triangle PQV), and the distance of R to the left of Q, QT = $\sin A \sin B$ (from triangle RTQ). So, equating the distance of R to the right of P obtained in these two ways,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

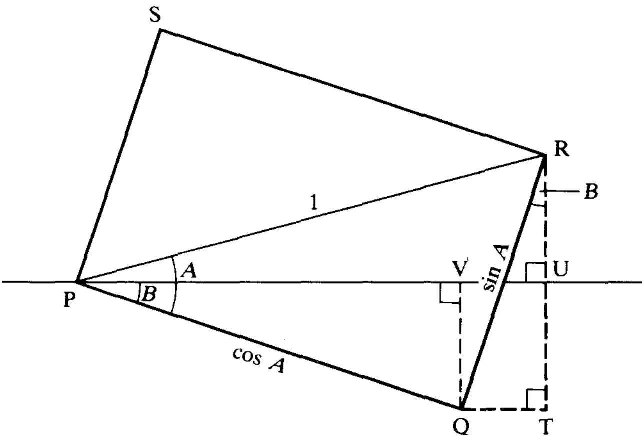


Figure 17.2

Consider now what happens if PQ is tilted through an angle B below the horizontal, as in Fig. 17.2. The height of R above P is now $\sin(A-B)$. R is a distance $\sin A \cos B$ above Q, but Q is a distance $\cos A \sin B$ below P, therefore

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

Further, R is a distance $\cos(A-B)$ to the right of P. Q is a distance $\cos A \cos B$ to the right of P, but R is now a distance $\sin A \sin B$ to the right of Q, therefore

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

The four identities just obtained have many applications apart from their use in trigonometry. They, or identities which will be derived from them, are needed in calculus, coordinate geometry and mechanics. Some applications are found in Chapters 19 and 22.

The outline of a good general proof of the identities (which may be taken on second reading) may be found in §17.6. For the present it will be assumed that they hold for all values of A and B.

Historical note. The equivalents of the identities for $\cos(A+B)$ and $\sin(A-B)$ were known to Ptolemy of Alexandria, almost 2000 years ago.

The formulae for tan $(A \pm B)$

17.2 Two more identities will be deduced from the four just obtained. They give tan $(A+B)$ and tan $(A-B)$ in terms of tan A and tan B.

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

Therefore, using the formulae for sin $(A+B)$ and cos $(A+B)$,

$$\tan(A+B) = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

Dividing numerator and denominator of the right-hand side by cos A cos B,

$$\tan(A+B) = \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}}$$

$$= \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A}{\cos A} \times \dfrac{\sin B}{\cos B}}$$

$$\therefore \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Similarly,

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

For convenience, the six identities are printed together:

$$\mathbf{\cos(A+B) = \cos A \cos B - \sin A \sin B}$$

$$\mathbf{\cos(A-B) = \cos A \cos B + \sin A \sin B}$$

$$\mathbf{\sin(A+B) = \sin A \cos B + \cos A \sin B}$$

$$\mathbf{\sin(A-B) = \sin A \cos B - \cos A \sin B}$$

$$\mathbf{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}}$$

$$\mathbf{\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}}$$

These are usually called the **addition formulae**; when memorising these, note the following:

(a) the formulae for the ratios of $(A-B)$ are the same as those for $(A+B)$, except for the changes in signs,

(b) the signs on the two sides of each of the sine formulae are the same, but in the cosine formulae they are different,

(c) in the tangent formulae, the signs in the numerators are the same as in the corresponding sine formulae, and those in the denominators are the same as in the cosine formulae.

Example 1 *Find, without using tables or calculators, the value of*

$$\sin(120° + 45°)$$

leaving surds in the answer.

Using the formula for $\sin(A + B)$,

$$\sin(120° + 45°) = \sin 120° \cos 45° + \cos 120° \sin 45°$$

Reference to Figs. 16.7 and 16.8 on page 329 should remind the reader how to obtain the ratios of 30°, 45°, and 60°. Thus we have

$$\sin 120° = \sin 60° = \frac{\sqrt{3}}{2}$$

$$\cos 120° = -\cos 60° = -\tfrac{1}{2}$$

$$\cos 45° = \sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\therefore \sin(120° + 45°) = \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} + \left(-\frac{1}{2}\right) \times \frac{\sqrt{2}}{2}$$

$$\therefore \sin(120° + 45°) = \frac{\sqrt{2}}{4}(\sqrt{3} - 1)$$

Example 2 *If* $\sin A = \frac{3}{5}$ *and* $\cos B = \frac{15}{17}$, *where A is obtuse and B is acute, find the exact value of* $\sin(A + B)$.

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

So it is necessary to find the values of $\cos A$ and $\sin B$, and Figs. 17.3 and 17.4 indicate the method. In Fig. 17.3, the third side of the right-angled triangle is 4 (by Pythagoras' theorem), hence the x-coordinate of P is -4, therefore $\cos A = -\frac{4}{5}$. Similarly, in Fig. 17.4, the y-coordinate of P is 8, and therefore $\sin B = \frac{8}{17}$.

$$\therefore \sin(A + B) = \frac{3}{5} \times \frac{15}{17} + \left(-\frac{4}{5}\right) \times \frac{8}{17}$$

$$= \frac{45}{85} - \frac{32}{85}$$

$$\therefore \sin(A + B) = \frac{13}{85}$$

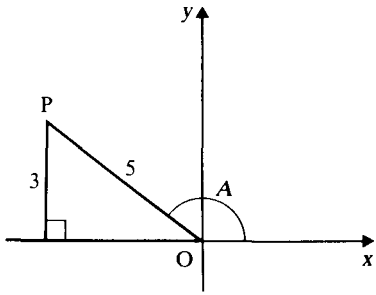


Figure 17.3.

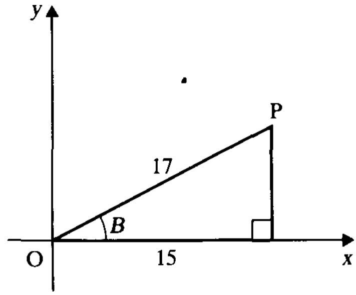


Figure 17.4.

Example 3 *If* $\sin(x+\alpha) = \cos(x-\beta)$, *find* $\tan x$ *in terms of* α *and* β.

Since $\sin(x+\alpha) = \cos(x-\beta)$, we have

$$\sin x \cos\alpha + \cos x \sin\alpha = \cos x \cos\beta + \sin x \sin\beta$$

[Now $\tan x = \sin x/\cos x$, so collect terms in $\sin x$ on one side of the equation, and terms in $\cos x$ on the other.]

Thus

$$\sin x \cos\alpha - \sin x \sin\beta = \cos x \cos\beta - \cos x \sin\alpha$$

$$\therefore \sin x(\cos\alpha - \sin\beta) = \cos x(\cos\beta - \sin\alpha)$$

$$\therefore \frac{\sin x}{\cos x} = \frac{\cos\beta - \sin\alpha}{\cos\alpha - \sin\beta}$$

$$\therefore \tan x = \frac{\cos\beta - \sin\alpha}{\cos\alpha - \sin\beta}$$

Exercise 17a

The questions in this exercise are intended to give the reader practice in using the trigonometrical identities introduced in the preceding section. Do not use a calculator or tables in this exercise; to every question it is possible to give an exact answer. Leave surds in the answers where appropriate.

1 Find the values of the following:
(a) $\cos(45^\circ - 30^\circ)$, (b) $\sin(30^\circ + 45^\circ)$, (c) $\sin(60^\circ + 45^\circ)$,
(d) $\cos 105^\circ$, (e) $\cos(120^\circ + 45^\circ)$, (f) $\sin 165^\circ$,
(g) $\sin 15^\circ$, (h) $\cos 75^\circ$.

2 If $\sin A = \frac{3}{5}$ and $\sin B = \frac{5}{13}$, where A and B are acute angles, find the values of
(a) $\sin(A+B)$, (b) $\cos(A+B)$, (c) $\cot(A+B)$.

3 If $\sin A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$, where A is *obtuse* and B is acute, find the values of
(a) $\sin(A-B)$, (b) $\tan(A-B)$, (c) $\tan(A+B)$.

4 If $\cos A = \frac{3}{5}$ and $\tan B = \frac{12}{5}$, where A and B are both reflex angles, find the values of
(a) $\sin(A - B)$, (b) $\tan(A - B)$, (c) $\cos(A + B)$.
5 If $\tan(x + 45°) = 2$, find the value of $\tan x$.
6 If $\tan(A + B) = \frac{1}{7}$ and $\tan A = 3$, find the value of $\tan B$.
7 If A and B are acute, $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$, find the value of $A + B$.
8 If $\tan A = -\frac{1}{7}$ and $\tan B = \frac{3}{4}$, where A is obtuse and B is acute, find the value of $A - B$.
9 Express as single trigonometrical ratios:

(a) $\dfrac{1}{2}\cos x - \dfrac{\sqrt{3}}{2}\sin x$, (b) $\dfrac{1}{\sqrt{2}}\sin x + \dfrac{1}{\sqrt{2}}\cos x$,

(c) $\dfrac{\sqrt{3} + \tan x}{1 - \sqrt{3}\tan x}$, (d) $\cos 16° \sin 42° - \sin 16° \cos 42°$,

(e) $\dfrac{1}{\cos 24° \cos 15° - \sin 24° \sin 15°}$, (f) $\dfrac{1}{2}\cos 75° + \dfrac{\sqrt{3}}{2}\sin 75°$.

10 Find the values of
(a) $\cos 75° \cos 15° + \sin 75° \sin 15°$, (b) $\sin 50° \cos 20° - \cos 50° \sin 20°$,

(c) $\dfrac{\tan 10° + \tan 20°}{1 - \tan 10° \tan 20°}$, (d) $\cos 70° \cos 20° - \sin 70° \sin 20°$,

(e) $\dfrac{1}{\sqrt{2}}\cos 15° - \dfrac{1}{\sqrt{2}}\sin 15°$, (f) $\dfrac{\sqrt{3}}{2}\cos 15° - \dfrac{1}{2}\sin 15°$,

(g) $\dfrac{1 - \tan 15°}{1 + \tan 15°}$, (h) $\cos 15° + \sin 15°$.

11 Find the value of $\tan A$, when $\tan(A - 45°) = \frac{1}{3}$.
12 Find the value of $\cot B$, when $\cot A = \frac{1}{4}$ and $\cot(A - B) = 8$.
13 From the following equations, find the values of $\tan x$:
(a) $\sin(x + 45°) = 2\cos(x + 45°)$;
(b) $2\sin(x - 45°) = \cos(x + 45°)$;
(c) $\tan(x - A) = \frac{3}{2}$, where $\tan A = 2$;
(d) $\sin(x + 30°) = \cos(x + 30°)$.
14 If $\sin(x + \alpha) = 2\cos(x - \alpha)$, prove that

$$\tan x = \frac{2 - \tan\alpha}{1 - 2\tan\alpha}$$

15 If $\sin(x - \alpha) = \cos(x + \alpha)$, prove that $\tan x = 1$.
16 Solve, for values of x between $0°$ and $360°$, the equations:
(a) $2\sin x = \cos(x + 60°)$,
(b) $\cos(x + 45°) = \cos x$,
(c) $\sin(x - 30°) = \frac{1}{2}\cos x$,
(d) $3\sin(x + 10°) = 4\cos(x - 10°)$.

Prove the following identities:

17 $\sin(A+B)+\sin(A-B)=2\sin A\cos B$.

18 $\cos(A+B)-\cos(A-B)=-2\sin A\sin B$.

19 $\tan A+\tan B=\dfrac{\sin(A+B)}{\cos A\cos B}$.

20 $\tan(A+B+C)=\dfrac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan B\tan C-\tan C\tan A-\tan A\tan B}$.

Hence prove that if A, B, C are angles of a triangle, then

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C$$

The double angle formulae

17.3 The special cases of the identities on page 343, when $A=B$, are even more useful than the identities themselves. For convenience of reference, they are given together, below.

$$\mathbf{\cos 2A=\cos^2 A-\sin^2 A}$$
$$\mathbf{=2\cos^2 A-1}$$
$$\mathbf{=1-2\sin^2 A}$$
$$\mathbf{\sin 2A=2\sin A\cos A}$$
$$\mathbf{\tan 2A=\frac{2\tan A}{1-\tan^2 A}}$$

Further, it is useful to remember that

$$\mathbf{\cos^2 A=\tfrac{1}{2}(1+\cos 2A)}$$
$$\mathbf{\sin^2 A=\tfrac{1}{2}(1-\cos 2A)}$$

To prove the identities concerning $\cos 2A$, we put $B=A$ in the identity

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$

which gives

$$\cos 2A=\cos^2 A-\sin^2 A$$

Now $\cos^2 A+\sin^2 A=1$, so substituting $\sin^2 A=1-\cos^2 A$, we obtain

$$\cos 2A=\cos^2 A-1+\cos^2 A$$
$$\therefore \cos 2A=2\cos^2 A-1$$

If we had substituted $\cos^2 A=1-\sin^2 A$ in the identity

$$\cos 2A=\cos^2 A-\sin^2 A$$

we should have obtained

$$\cos 2A=1-\sin^2 A-\sin^2 A$$
$$\therefore \cos 2A=1-2\sin^2 A$$

The expressions for $\cos^2 A$ and $\sin^2 A$ are obtained by changing the subjects in the formulae

$$\cos 2A = 2\cos^2 A - 1 \quad \text{and} \quad \cos 2A = 1 - 2\sin^2 A.$$

The identities for $\sin 2A$ and $\tan 2A$ are obtained immediately, when the substitution $B = A$ is made in the formulae for $\sin(A+B)$ and $\tan(A+B)$.

Example 4 *Solve the equation* $3\cos 2\theta + \sin\theta = 1$, *for values of* θ *from* $0°$ *to* $360°$ *inclusive.*

[The quadratic equation is liable to occur in various disguises. Here, $\sin\theta$ suggests that the equation may be a quadratic in $\sin\theta$, so we express $\cos 2\theta$ in terms of $\sin\theta$.]

We have $\cos 2\theta = 1 - 2\sin^2\theta$, so, substituting in the equation

$$3\cos 2\theta + \sin\theta = 1$$

it follows that

$$3(1 - 2\sin^2\theta) + \sin\theta = 1$$

This is a quadratic equation for $\sin\theta$, and it is solved by factorisation.

$$\begin{aligned} & 3 - 6\sin^2\theta + \sin\theta = 1 \\ \therefore\ & 6\sin^2\theta - \sin\theta - 2 = 0 \\ \therefore\ & (3\sin\theta - 2)(2\sin\theta + 1) = 0 \\ \therefore\ & \sin\theta = \tfrac{2}{3} \quad \text{or} \quad \sin\theta = -\tfrac{1}{2} \end{aligned}$$

If $\sin\theta = \frac{2}{3}$,

$$\theta = 41.8° \quad \text{or} \quad 180° - 41.8°, \quad \text{correct to one decimal place.}$$

If $\sin\theta = -\frac{1}{2}$,

$$\theta = 180° + 30° \quad \text{or} \quad 360° - 30°$$

Therefore the values of θ between $0°$ and $360°$ which satisfy the equation are $41.8°$, $138.2°$, $210°$, and $330°$.

Example 5 *Prove that* $\sin 3A = 3\sin A - 4\sin^3 A$.

The left-hand side of the identity may be written as $\sin(A + 2A)$, so by using the formula for $\sin(A+B)$ we have

$$\sin(A + 2A) = \sin A\cos 2A + \cos A\sin 2A$$

But the right-hand side of the identity to be proved is in terms of $\sin A$, and this suggests that $\cos 2A$ should be expressed in terms of $\sin A$. (We have only one formula for $\sin 2A$, so it must be used.) Therefore

$$\begin{aligned} \sin 3A &= \sin A(1 - 2\sin^2 A) + \cos A(2\sin A\cos A) \\ &= \sin A - 2\sin^3 A + 2\sin A\cos^2 A \end{aligned}$$

Now $\cos^2 A$ must be expressed in terms of $\sin A$ by means of the identity $\cos^2 A = 1 - \sin^2 A$, therefore

$$\sin 3A = \sin A - 2\sin^3 A + 2\sin A(1 - \sin^2 A)$$
$$= \sin A - 2\sin^3 A + 2\sin A - 2\sin^3 A$$
$$\therefore \mathbf{\sin 3A = 3\sin A - 4\sin^3 A}$$

A formula for $\cos 3A$ in terms of $\cos A$ may be obtained from the expansion of $\cos(2A + A)$. The proof is left as an exercise.

$$\mathbf{\cos 3A = 4\cos^3 A - 3\cos A}$$

Exercise 17b (Oral)

Express more simply:

1 $2\sin 17° \cos 17°$.

2 $\dfrac{2\tan 30°}{1 - \tan^2 30°}$,

3 $2\cos^2 42° - 1$.

4 $2\sin\frac{1}{2}\theta \cos\frac{1}{2}\theta$.

5 $1 - 2\sin^2 22\frac{1}{2}°$.

6 $\dfrac{2\tan\frac{1}{2}\theta}{1 - \tan^2\frac{1}{2}\theta}$.

7 $\cos^2 15° - \sin^2 15°$.

8 $2\sin 2A \cos 2A$.

9 $2\cos^2\frac{1}{2}\theta - 1$.

10 $1 - 2\sin^2 3\theta$.

11 $\dfrac{\tan 2\theta}{1 - \tan^2 2\theta}$.

12 $\sin x \cos x$.

13 $\dfrac{1 - \tan^2 20°}{\tan 20°}$.

14 $\sec\theta \operatorname{cosec}\theta$.

15 $1 - 2\sin^2\frac{1}{2}\theta$.

Exercise 17c

Nos. 1–6 in this exercise are intended to give the reader practice in using the trigonometrical identities introduced in this chapter. Do not use a calculator or tables in these questions; in each case it is possible to give an exact answer. Leave surds in the answers where appropriate.

1 Evaluate:

(a) $2\sin 15° \cos 15°$,

(b) $\dfrac{2\tan 22\frac{1}{2}°}{1 - \tan^2 22\frac{1}{2}°}$,

(c) $2\cos^2 75° - 1$,

(d) $1 - 2\sin^2 67\frac{1}{2}°$,

(e) $\cos^2 22\frac{1}{2}° - \sin^2 22\frac{1}{2}°$,

(f) $\dfrac{1 - \tan^2 15°}{\tan 15°}$,

(g) $\dfrac{1 - 2\cos^2 25°}{1 - 2\sin^2 65°}$,

(h) $\sec 22\frac{1}{2}° \operatorname{cosec} 22\frac{1}{2}°$.

2 Find the values of $\sin 2\theta$ and $\cos 2\theta$ when
(a) $\sin\theta = \frac{3}{5}$, (b) $\cos\theta = \frac{12}{13}$, (c) $\sin\theta = -\sqrt{3}/2$.

3 Find the value of $\tan 2\theta$ when
(a) $\tan\theta = \frac{4}{3}$, (b) $\tan\theta = \frac{8}{15}$, (c) $\cos\theta = -\frac{5}{13}$.

4 Find the values of $\cos x$ and $\sin x$ when $\cos 2x$ is
(a) $\frac{1}{8}$, (b) $\frac{7}{25}$, (c) $-\frac{119}{169}$.

5 Find the values of $\tan\frac{1}{2}\theta$ when $\tan\theta$ is
(a) $\frac{3}{4}$, (b) $\frac{4}{3}$, (c) $-\frac{12}{5}$.

6 If $t = \tan 22\frac{1}{2}°$, use the formula for $\tan 2\theta$ to show that $t^2 + 2t - 1 = 0$. Deduce the value of $\tan 22\frac{1}{2}°$.

Solve the following equations for values of θ from 0° to 360° inclusive:

7 $\cos 2\theta + \cos\theta + 1 = 0$.

8 $\sin 2\theta = \sin\theta$.

9 $\cos 2\theta = \sin\theta$.

10 $3\cos 2\theta - \sin\theta + 2 = 0$.

11 $\sin 2\theta \cos\theta + \sin^2\theta = 1$.

12 $\sin\theta = 6\sin 2\theta$.

13 $2\sin\theta\,(5\cos 2\theta + 1) = 3\sin 2\theta$.

14 $3\tan\theta = \tan 2\theta$.

15 $3\cot 2\theta + \cot\theta = 1$.

16 $4\tan\theta\tan 2\theta = 1$.

17 Eliminate θ from the equations:
(a) $x = \cos\theta$, $y = \cos 2\theta$; (b) $x = 2\sin\theta$, $y = 3\cos 2\theta$;
(c) $x = \tan\theta$, $y = \tan 2\theta$; (d) $x = 2\sec\theta$, $y = \cos 2\theta$.

Prove the following identities:

18 $\dfrac{\cos 2A}{\cos A + \sin A} = \cos A - \sin A$.

19 $\dfrac{\sin A}{\sin B} + \dfrac{\cos A}{\cos B} = \dfrac{2\sin(A+B)}{\sin 2B}$.

20 $\dfrac{\cos A}{\sin B} - \dfrac{\sin A}{\cos B} = \dfrac{2\cos(A+B)}{\sin 2B}$.

21 $\tan A + \cot A = 2\,\text{cosec}\, 2A$.

22 $\cot A - \tan A = 2\cot 2A$.

23 $\dfrac{1}{\cos A + \sin A} + \dfrac{1}{\cos A - \sin A} = \tan 2A\,\text{cosec}\,A$.

24 $\dfrac{\sin 2A}{1 + \cos 2A} = \tan A = \dfrac{1 - \cos 2A}{\sin 2A}$.

25 $\cos 3A = 4\cos^3 A - 3\cos A$.

26 $\text{cosec}\,2x - \cot 2x = \tan x$.

27 $\text{cosec}\,2x + \cot 2x = \cot x$.

28 $\tan x = \sqrt{\dfrac{1 - \cos 2x}{1 + \cos 2x}}$.

29 $\sin 2x = \dfrac{2\tan x}{1 + \tan^2 x}$,

30 $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$.

The *t*-formulae

17.4 In the preceding section the following formulae for $\sin 2x$ and $\cos 2x$ were introduced:

$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$

It is possible to express both $\sin 2x$ and $\cos 2x$ in terms of $\tan x$ and there are many occasions when this is a very useful technique.

In the case of $\sin 2x$ we start by deliberately introducing a factor $\sin x/\cos x$, which is equal to $\tan x$.

$$\begin{aligned}\sin 2x &= 2\sin x\cos x\\ &= 2\frac{\sin x}{\cos x}\cos^2 x\\ &= 2\tan x\cos^2 x\\ &= 2\tan x\times\frac{1}{\sec^2 x}\end{aligned}$$

This last step may seem rather peculiar; its purpose is to enable us to replace $\sec^2 x$ by $1+\tan^2 x$ (see §16.6). Hence

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x}$$

This identity is most frequently used in the form obtained by substituting θ for $2x$, i.e.

$$\sin\theta = \frac{2\tan\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}$$

$$\therefore\ \boldsymbol{\sin\theta = \frac{2t}{1+t^2}}\quad (\text{where } t=\tan\tfrac{1}{2}\theta)$$

This is usually called the ***t*-formula** for $\sin\theta$. The corresponding t-formulae for $\cos\theta$ and $\tan\theta$ are left as exercises for the reader.

Qu. 1 Prove that, in the usual notation, $\boldsymbol{\cos\theta = \dfrac{1-t^2}{1+t^2}}$.

Qu. 2 Prove that $\boldsymbol{\tan\theta = \dfrac{2t}{1-t^2}}$.

Qu. 3 Use the t-formulae to solve the following equations, giving values of θ from 0° to 360° inclusive:

(a) $2\cos\theta + 3\sin\theta - 2 = 0$, (b) $7\cos\theta + \sin\theta - 5 = 0$,
(c) $3\cos\theta - 4\sin\theta + 1 = 0$, (d) $3\cos\theta + 4\sin\theta = 2$.

In due course the reader will find that the t-formulae can be a useful means of tackling certain integrals (see Book 2, §13.3).

The form $a\cos\theta + b\sin\theta$

17.5 Two applications of the identities of §17.2 follow in the next examples.

Example 6 *Solve the equation* $3\cos\theta + 4\sin\theta = 2$, *for values of* θ *from* 0° *to* 360°, *inclusive.*

The solution is obtained by dividing both sides of the equation by some number, so as to leave it in the form

$$\cos\alpha\cos\theta + \sin\alpha\sin\theta = \text{constant}$$

Comparing this with

$$3\cos\theta + \quad 4\sin\theta = 2$$

it follows that

$$\frac{\cos\alpha}{3} = \frac{\sin\alpha}{4}, \quad \text{i.e. } \tan\alpha = \frac{4}{3}$$

From a calculator or tables we find that $\alpha = 53.13°$, and from Fig. 17.5 it follows that $\sin\alpha = \frac{4}{5}$ and $\cos\alpha = \frac{3}{5}$. Therefore we divide the original equation by 5, giving

$$\tfrac{3}{5}\cos\theta + \tfrac{4}{5}\sin\theta = \tfrac{2}{5}$$

$$\therefore \cos\theta\cos\alpha + \sin\theta\sin\alpha = 0.4$$

$$\therefore \cos(\theta - \alpha) = 0.4$$

$$\therefore \theta - 53.13° = 66.42° \quad \text{or} \quad 293.58°$$

Therefore the roots of the equation in the range from 0° to 360° are 119.6° and 346.7°, correct to the nearest tenth of a degree.*

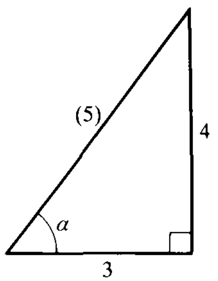


Figure 17.5

Qu. 4 What advantage is there in using the formula for $\cos(A - B)$, rather than that for $\sin(A + B)$ in Example 6?

Example 7 *Find the maximum and minimum values of* $2\sin\theta - 5\cos\theta$, *and the corresponding values of* θ *between* 0° *and* 360°.

This will be solved by writing

$$2\sin\theta - 5\cos\theta = k(\cos\alpha\sin\theta - \sin\alpha\cos\theta)$$

*The figure in the second decimal place should be included in the intermediate working, in order to avoid errors due to premature approximation.

where k and α are to be found. Comparing the two forms of the expression,

$$\frac{\sin\alpha}{\cos\alpha}=\frac{5}{2}, \quad \text{i.e. } \tan\alpha = 2.5$$

From a calculator or tables it is found that $\alpha = 68.20°$; and from Fig. 17.6, it follows that $\cos\alpha = 2/\sqrt{29}$, and $\sin\alpha = 5/\sqrt{29}$. So we may write

$$\begin{aligned} 2\sin\theta - 5\cos\theta &= \sqrt{29}\left(\frac{2}{\sqrt{29}}\sin\theta - \frac{5}{\sqrt{29}}\cos\theta\right) \\ &= \sqrt{29}(\sin\theta\cos\alpha - \cos\theta\sin\alpha) \\ &= \sqrt{29}\sin(\theta-\alpha) \end{aligned}$$

Now the greatest value of $\sin x$ is 1, and this occurs when $x = 90°$, and the least value of $\sin x$ is -1, when $x = 270°$. (Values of x less than 0° or greater than 360° have been ignored.)

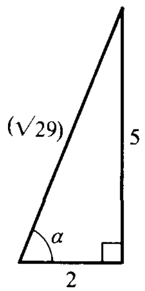


Figure 17.6

Therefore $\sqrt{29}\sin(\theta-\alpha)$ has a maximum value of $\sqrt{29}$ when $\theta - \alpha = 90°$; and it has a minimum value of $-\sqrt{29}$ when $\theta - \alpha = 270°$.

Therefore the maximum and minimum values of

$$2\sin\theta - 5\cos\theta$$

are $\sqrt{29}$ and $-\sqrt{29}$, and are given by

$$\theta = 90° + \alpha = 158.2° \quad \text{and} \quad \theta = 270° + \alpha = 338.2° \text{ respectively.}$$

Exercise 17d

Solve the following equations for values of θ from 0° to 360° inclusive.

1 $\sqrt{3}\cos\theta + \sin\theta = 1$.
2 $5\sin\theta - 12\cos\theta = 6$.
3 $\sin\theta + \cos\theta = \frac{1}{2}$.
4 $\cos\theta - 7\sin\theta = 2$.
5 $2\sin\theta + 7\cos\theta = 4$.
6 $3\tan\theta - 2\sec\theta = 4$.
7 $4\cos\theta\sin\theta + 15\cos 2\theta = 10$.
8 $\cos\theta + \sin\theta = \sec\theta$.
9 Prove that $\cos\theta - \sin\theta = \sqrt{2}\cos(\theta + 45°) = -\sqrt{2}\sin(\theta - 45°)$.

10 Show that $\sqrt{3}\cos\theta - \sin\theta$ may be written as

$$2\cos(\theta + 30°) \quad \text{or} \quad 2\sin(60° - \theta)$$

Find the maximum and minimum values of the expression, and state the values of θ between $0°$ and $360°$ for which they occur.

11 Show that $3\cos\theta + 2\sin\theta$ may be written in the form $\sqrt{13}\cos(\theta - \alpha)$, where $\tan\alpha = \frac{2}{3}$. Hence find the maximum and minimum values of the function, giving the corresponding values of θ from $-180°$ to $+180°$.

12 Show that $3\cos\theta + 4\sin\theta$ may be expressed in the form $R\cos(\theta - \alpha)$, where α is acute. Find the values of R and α.

13 By expressing $\cos\theta + 2\sin\theta$ in the form $R\sin(\theta + \alpha)$, where α is acute, find the maximum and minimum values of the expression, giving the values of θ between $-180°$ and $180°$ for which they occur.

Find the maximum and minimum values of the following expressions, stating the values of θ, from $0°$ to $360°$ inclusive, for which they occur.

14 $\cos\theta + \sin\theta$.

15 $4\sin\theta - 3\cos\theta$.

16 $\sqrt{3}\sin\theta + \cos\theta$.

17 $8\cos\theta - 15\sin\theta$.

18 $\sin\theta - 6\cos\theta$.

19 $\cos(\theta + 60°) - \cos\theta$.

20 $3\sqrt{2}\cos(\theta + 45°) + 7\sin\theta$.

Proof of the addition formulae, using vectors

17.6 In this section we shall use vectors, and in particular the scalar product of vectors (see §15.15), to give a more general proof of the formula

$$\cos(A - B) = \cos A\cos B + \sin A\sin B$$

The diagram (Fig. 17.7) shows A and B as acute angles, but the subsequent working is valid for angles of *any* magnitude.

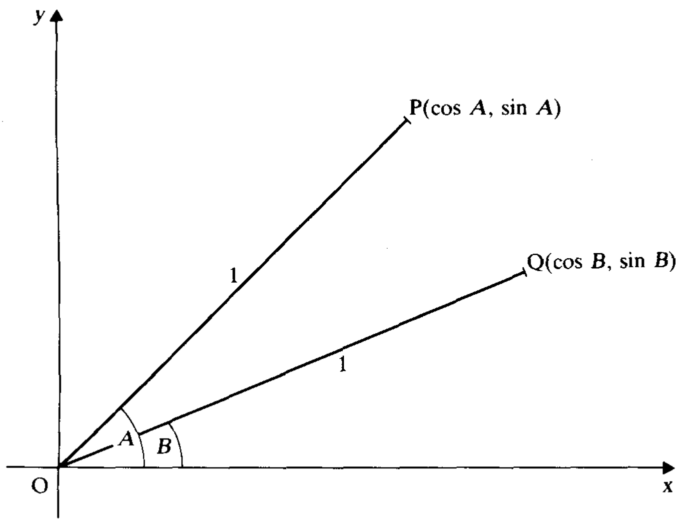


Figure 17.7

In the diagram, $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are unit vectors, i.e. they are vectors whose length is one unit. $\overrightarrow{OP}$ is inclined at an angle A to the x-axis, and $\overrightarrow{OQ}$ is inclined at an angle B to the x-axis. Consequently the coordinates of the points P and Q are $(\cos A, \sin A)$ and $(\cos B, \sin B)$ respectively and the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ can be written

$$\overrightarrow{OP} = \cos A\,\mathbf{i} + \sin A\,\mathbf{j} \quad \text{and} \quad \overrightarrow{OQ} = \cos B\,\mathbf{i} + \sin B\,\mathbf{j}$$

Taking the scalar product of these vectors, we have

$$\begin{aligned}\overrightarrow{OP}.\overrightarrow{OQ} &= (\cos A\,\mathbf{i} + \sin A\,\mathbf{j}).(\cos B\,\mathbf{i} + \sin B\,\mathbf{j})\\ &= \cos A \cos B + \sin A \sin B\end{aligned}$$

But, from the basic definition of the scalar product, we know that

$$\overrightarrow{OP}.\overrightarrow{OQ} = \text{OP} \times \text{OQ} \cos \angle \text{POQ} = 1 \times 1 \times \cos(A-B) = \cos(A-B)$$

Equating these two expressions for $\overrightarrow{OP}.\overrightarrow{OQ}$, we obtain

$$\cos(A-B) = \cos A \cos B + \sin A \sin B \tag{1}$$

To obtain the corresponding identity for $\cos(A+B)$, it is only necessary to replace B by $-B$, giving

$$\cos(A-(-B)) = \cos A \cos(-B) + \sin A \sin(-B)$$

and hence

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

(Alternatively the proof above could be repeated with the angle B drawn in the fourth quadrant.)

The formulae for $\sin(A+B)$ can be obtained by replacing A in identity (1) by $90° - A$, which gives

$$\cos\{(90° - A) - B\} = \cos(90° - A) \cos B + \sin(90° - A) \sin B$$

$$\therefore \cos\{90° - (A+B)\} = \cos(90° - A) \cos B + \sin(90° - A) \sin B$$

But $\sin(90° - \theta) = \cos\theta$ and $\cos(90° - \theta) = \sin\theta$, so

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

The corresponding identity for $\sin(A-B)$ can then be obtained by replacing B by $-B$.

Introduction to the factor formulae

17.7 Factors are very useful, in algebra, for solving equations and simplifying expressions, and when dealing with trigonometrical ratios, it is often convenient to be able to factorise a sum of two terms. On the other hand, it is sometimes useful to express a product as a sum or difference of two terms, and it is to this that we turn first.

In §17.2 it was shown that

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

Adding,

$$\cos(A+B) + \cos(A-B) = 2\cos A\cos B$$

and subtracting,

$$\cos(A+B) - \cos(A-B) = -2\sin A\sin B$$

Now, keeping the formulae for $\cos(A+B)$ and $\cos(A-B)$ in mind, work through the next exercise.

Exercise 17e (Oral)

Express as a sum or difference of two cosines:

1 $-2\sin x\sin y$.
2 $2\cos x\cos y$.
3 $2\cos 3\theta\cos\theta$.
4 $-2\sin(S+T)\sin(S-T)$.
5 $2\sin 5x\sin 3x$.
6 $2\cos(x+y)\cos(x-y)$.
7 $2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$.
8 $-2\sin\dfrac{B+C}{2}\sin\dfrac{B-C}{2}$.
9 $-2\sin(x+45°)\sin(x-45°)$.
10 $2\cos(2x+30°)\cos(2x-30°)$.

Following the same method as before, we have

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

Adding,

$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$$

and subtracting,

$$\sin(A+B) - \sin(A-B) = 2\cos A\sin B$$

Again, keeping the formulae for $\sin(A+B)$ and $\sin(A-B)$ in mind, work through the next exercise.

Exercise 17f (Oral)

Express as a sum or difference of two sines:

1 $2\sin x\cos y$.
2 $2\cos x\sin y$.
3 $2\sin 3\theta\cos\theta$.
4 $2\sin(S+T)\cos(S-T)$.
5 $2\cos 5x\sin 3x$.
6 $2\cos(x+y)\sin(x-y)$.

7 $-2\cos 4x \sin 2x$.

8 $2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$.

9 $2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$.

10 $2\sin\frac{R-S}{2}\cos\frac{R+S}{2}$.

The factor formulae

17.8 We may now proceed to the question of factorising a sum or difference of two cosines or sines. The last section has indicated the method, for it was shown that

$$\cos(A+B)+\cos(A-B)=2\cos A\cos B$$
$$\cos(A+B)-\cos(A-B)=-2\sin A\sin B$$
$$\sin(A+B)+\sin(A-B)=2\sin A\cos B$$
$$\sin(A+B)-\sin(A-B)=2\cos A\sin B$$

Here, the right-hand sides of the identities are in factors, but it would be more convenient if the left-hand sides were in the form $\cos P + \cos Q$, etc. Therefore let

$$P=A+B \quad \text{and} \quad Q=A-B$$

Adding,

$$P+Q=2A \quad \therefore A=\frac{P+Q}{2}$$

Subtracting,

$$P-Q=2B \quad \therefore B=\frac{P-Q}{2}$$

Substituting into the four identities above,

$$\mathbf{\cos P+\cos Q=2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}}$$

$$\mathbf{\cos P-\cos Q=-2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}}$$

$$\mathbf{\sin P+\sin Q=2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}}$$

$$\mathbf{\sin P-\sin Q=2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}}$$

Remember how these identities were obtained: this will make it easier to remember them. Many people find it helpful to remember them in the form,

'cos plus cos, equals two cos semi-sum, cos semi-diff.'

Example 8 *Solve the equation* $\sin 3x + \sin x = 0$, *for values of x from* $-180°$ *to* $+180°$, *inclusive.*

$$\sin 3x+\sin x=0$$

therefore, using the formula for $\sin P + \sin Q$,

$$2 \sin 2x \cos x = 0$$
$$\therefore \sin 2x = 0 \quad \text{or} \quad \cos x = 0$$

Now x may lie in the range from $-180°$ to $180°$, therefore $2x$ lies in the range from $-360°$ to $360°$.

If $\sin 2x = 0$,

$$2x = -360°, -180°, 0°, 180°, 360°$$
$$\therefore x = -180°, \ -90°, 0°, \ 90°, 180°$$

If $\cos x = 0$, $x = -90°, 90°$.

Therefore the roots of the equation between $-180°$ and $+180°$, inclusive, are $-180°$, $-90°$, $0°$, $90°$ and $180°$.

Example 9 *Solve the equation* $\cos(x + 20°) - \cos(x + 80°) = 0.5$, *for* $0° \leqslant x \leqslant 360°$.

(The difference of the two cosines suggests using one of the above identities.)

$$\cos(x + 20°) - \cos(x + 80°) = 0.5$$
$$-2 \sin(x + 50°) \sin(-30°) = 0.5$$

But $\sin(-30°) = -\sin 30° = -\frac{1}{2}$.

$$\therefore \sin(x + 50°) = 0.5$$
$$x + 50° = 30°, 150°, 390°, 510°, \ldots$$
$$x = -20°, 100°, 340°, \ldots$$

Therefore the roots of the equation between $0°$ and $360°$ are $100°$ and $340°$.

Example 10 *Solve the equation* $\sin(x + 15°)\cos(x - 15°) = 0.5$, *for values of* x *from* $0°$ *to* $360°$ *inclusive.*

(The product of a sine and a cosine suggests that the left-hand side may be expressed as the sum of two sines.)

$$\sin(x + 15°)\cos(x - 15°) = 0.5$$
$$\therefore 2 \sin(x + 15°)\cos(x - 15°) = 1$$
$$\therefore \sin 2x + \sin 30° = 1$$
$$\therefore \sin 2x = 1 - \sin 30°$$
$$= 0.5$$
$$\therefore 2x = 30°, 150°, 390°, 510°, \ldots$$

Hence the values of x required are $15°$, $75°$, $195°$, $255°$.

Example 11 *Prove the identity*

$$\cos^2 A - \cos^2 B = \sin(A + B)\sin(B - A)$$

[A neat method is to use $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$, $\cos^2 B = \frac{1}{2}(1 + \cos 2B)$.]

$$\cos^2 A - \cos^2 B = \tfrac{1}{2}(\cos 2A - \cos 2B)$$
$$= \tfrac{1}{2}\{-2 \sin (A + B) \sin (A - B)\}$$
$$\therefore \cos^2 A - \cos^2 B = \sin (A + B) \sin (B - A)$$

Exercise 17g (Oral)

Express the following in factors:

1 $\cos x + \cos y$.
2 $\sin 3x + \sin 5x$.
3 $\sin 2y - \sin 2z$.
4 $\cos 5x + \cos 7x$.
5 $\cos 2A - \cos A$.
6 $\sin 4x - \sin 2x$.
7 $\cos 3A - \cos 5A$.
8 $\sin 5\theta + \sin 7\theta$.
9 $\sin (x + 30°) + \sin (x - 30°)$.
10 $\cos (y + 10°) + \cos (y - 80°)$.
11 $\sin 3\theta - \sin 5\theta$.
12 $\cos (x + 30°) - \cos (x - 30°)$.
13 $\cos \dfrac{3x}{2} - \cos \dfrac{x}{2}$.
14 $\sin 2(x + 40°) + \sin 2(x - 40°)$.
15 $\cos (90° - x) + \cos y$.
16 $\sin A + \cos B$.
17 $\sin 3x + \sin 90°$.
18 $1 + \sin 2x$.
19 $\cos A - \sin B$.
20 $\frac{1}{2} + \cos 2\theta$.

Further identities and equations

17.9 Example 12 *Solve the equation* $\cos 6x + \cos 4x + \cos 2x = 0$, *for values of* x *from* 0° *to* 180° *inclusive.*

[Remember that equations are very often solved by factorisation, so look to see whether any of the three terms is a factor of the sum of the other pair. Note that $\cos 4x$ is a factor of $\cos 6x + \cos 2x$, so group $\cos 6x$ and $\cos 2x$ together.]

$$\cos 4x + \cos 6x + \cos 2x = 0$$
$$\therefore \cos 4x + 2 \cos 4x \cos 2x = 0$$
$$\therefore \cos 4x(1 + 2 \cos 2x) = 0$$
$$\therefore \cos 4x = 0 \quad \text{or} \quad \cos 2x = -\tfrac{1}{2}$$

If $\cos 4x = 0$,

$$4x = 90°, 270°, 450°, 630°$$
$$\therefore x = 22\tfrac{1}{2}°, 67\tfrac{1}{2}°, 112\tfrac{1}{2}°, 157\tfrac{1}{2}°$$

If $\cos 2x = -\frac{1}{2}$,

$$2x = 120°, 240°$$
$$\therefore x = 60°, 120°$$

Therefore the roots of the equation in the range 0° to 180° are $22\frac{1}{2}°$, 60°, $67\frac{1}{2}°$, $112\frac{1}{2}°$, 120°, $157\frac{1}{2}°$.

Example 13 *If A, B, C are the angles of a triangle, prove that*

$$\cos A + \cos B + \cos C - 1 = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

Split the left-hand side into two pairs of terms. Now,

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

But since $A + B = 180° - C$,

$$\frac{A+B}{2} = 90° - \frac{C}{2}$$

$$\therefore \cos \frac{A+B}{2} = \sin \frac{C}{2}$$

Seeing this factor $\sin (C/2)$ on the right-hand side, write

$$\cos C - 1 = -2 \sin^2 \frac{C}{2}$$

Therefore

$$\cos A + \cos B + \cos C - 1 = 2 \sin \frac{C}{2} \cos \frac{A-B}{2} - 2 \sin^2 \frac{C}{2}$$

$$= 2 \sin \frac{C}{2} \left(\cos \frac{A-B}{2} - \sin \frac{C}{2} \right)$$

On the right-hand side of the identity to be proved, $\sin (C/2)$ is multiplied by a function of A and B, so in the last bracket we must express $\sin (C/2)$ in terms of A and B. This has been done above.

$$\therefore \cos A + \cos B + \cos C - 1 = 2 \sin \frac{C}{2} \left(\cos \frac{A-B}{2} - \cos \frac{A+B}{2} \right)$$

$$= -2 \left(\cos \frac{A+B}{2} - \cos \frac{A-B}{2} \right) \sin \frac{C}{2}$$

$$= -2 \left(-2 \sin \frac{A}{2} \sin \frac{B}{2} \right) \sin \frac{C}{2}$$

$$\therefore \cos A + \cos B + \cos C - 1 = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

Exercise 17h

Prove the following identities:

1 $\dfrac{\cos B + \cos C}{\sin B - \sin C} = \cot \dfrac{B-C}{2}$.

2 $\dfrac{\cos B - \cos C}{\sin B + \sin C} = -\tan \dfrac{B-C}{2}$.

3 $\dfrac{\sin B+\sin C}{\cos B+\cos C}=\tan\dfrac{B+C}{2}$. **4** $\dfrac{\sin B-\sin C}{\sin B+\sin C}=\cot\dfrac{B+C}{2}\tan\dfrac{B-C}{2}$.

5 $\sin x+\sin 2x+\sin 3x=\sin 2x\,(2\cos x+1)$.
6 $\cos x+\sin 2x-\cos 3x=\sin 2x\,(2\sin x+1)$.
7 $\cos 3\theta+\cos 5\theta+\cos 7\theta=\cos 5\theta\,(2\cos 2\theta+1)$.
8 $\cos\theta+2\cos 3\theta+\cos 5\theta=4\cos^2\theta\cos 3\theta$.
9 $1+2\cos 2\theta+\cos 4\theta=4\cos^2\theta\cos 2\theta$.
10 $\sin\theta-2\sin 3\theta+\sin 5\theta=2\sin\theta\,(\cos 4\theta-\cos 2\theta)$.
11 $\cos\theta-2\cos 3\theta+\cos 5\theta=2\sin\theta\,(\sin 2\theta-\sin 4\theta)$.
12 $\sin x-\sin(x+60°)+\sin(x+120°)=0$.
13 $\cos x+\cos(x+120°)+\cos(x+240°)=0$.

Solve the following equations, for values of x from 0° to 360° inclusive:

14 $\cos x+\cos 5x=0$. **15** $\cos 4x-\cos x=0$.
16 $\sin 3x-\sin x=0$. **17** $\sin 2x+\sin 3x=0$.
18 $\sin(x+10°)+\sin x=0$.
19 $\cos(2x+10°)+\cos(2x-10°)=0$.
20 $\cos(x+20°)-\cos(x-70°)=0$.

Exercise 17i (Miscellaneous)

Do not use a calculator or tables in Nos. 1–6.

1 If $\sin A=\frac{5}{13}$, $\sin B=\frac{8}{17}$, where A and B are acute, find the values of
(a) $\cos(A+B)$, (b) $\sin(A-B)$, (c) $\tan(A+B)$.
2 If $\cos A=\frac{15}{17}$, $\sin B=\frac{20}{29}$, where A is reflex and B is obtuse, find the values of
(a) $\sin(A+B)$, (b) $\cos(A-B)$, (c) $\cot(A-B)$.
3 Find the values of

(a) $\cos 80°\cos 20°+\sin 80°\sin 20°$,

(b) $\dfrac{\tan 15°+\tan 30°}{1-\tan 15°\tan 30°}$,

(c) $\sin 40°\cos 50°+\sin 50°\cos 40°$.

4 Find the values of $\sin x$ and $\cos x$ when $\cos 2x$ is (a) $\frac{1}{9}$, (b) $\frac{49}{81}$.
5 Find the value of $\tan\theta$ when $\tan 2\theta$ is (a) $-\frac{20}{21}$, (b) $\frac{36}{77}$.
6 If $\sin\theta=\frac{35}{37}$, where θ is acute, find the values of (a) $\sin 2\theta$, (b) $\cos 2\theta$.

Solve the following equations, giving values of θ from 0° to 360° inclusive:

7 $\cos 2\theta+5\cos\theta=2$. **8** $2\sin 2\theta=3\sin\theta$.
9 $\tan 2\theta+\tan\theta=0$. **10** $4\cos\theta-3\sin\theta=1$.
11 $3\cos\theta+2\sin\theta=2.5$.

Eliminate θ from the following equations:

12 $x=2\cos 2\theta$, $y=3\cos\theta$. **13** $x=2\tan\theta$, $y=\tan 2\theta$.

In Nos. 14 and 15, using $t = \tan \frac{1}{2}\theta$, express in terms of t:

14 $3 \cos \theta + 4 \sin \theta + 5$. **15** $\sqrt{\left(\dfrac{1 + \sin \theta}{1 - \sin \theta}\right)}$.

Find the maximum and minimum values of the following, giving the values of θ between 0° and 360° for which they occur:

16 $5 \cos \theta - 12 \sin \theta$.
17 $12 \cos \theta + 35 \sin \theta$.
18 $48 \cos \theta - 55 \sin \theta$.

19 Prove that $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$.

20 If $2A + B = 45°$, show that $\tan B = \dfrac{1 - 2 \tan A - \tan^2 A}{1 + 2 \tan A - \tan^2 A}$.

Solve the following equations for values of θ from 0° to 180° inclusive:

21 $\cos \theta + \cos 3\theta + \cos 5\theta = 0$. **22** $\sin 2\theta + \sin 4\theta + \sin 6\theta = 0$.
23 $\sin \theta - 2 \sin 2\theta + \sin 3\theta = 0$. **24** $\cos \frac{1}{2}\theta + 2 \cos \frac{3}{2}\theta + \cos \frac{5}{2}\theta = 0$.
25 $\sin \theta + \cos 2\theta - \sin 3\theta = 0$.

Prove the following identities. A, B, C are to be taken as the angles of a triangle.

26 $\sin A + \sin (B - C) = 2 \sin B \cos C$.
27 $\cos A - \cos (B - C) = -2 \cos B \cos C$.

28 $\sin A + \sin B + \sin C = 4 \cos \dfrac{A}{2} \cos \dfrac{B}{2} \cos \dfrac{C}{2}$.

29 $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$.
30 $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

Chapter 18

Further topics in trigonometry

Introduction

18.1 One of Euler's many contributions to mathematics is the invention of a standard notation for labelling triangles. In this notation the vertices are always labelled with capital letters, say A, B and C, and the same symbols are used to represent the sizes of the angles at these vertices. The corresponding lower case letters, a, b, c, are then used to represent the lengths of the sides opposite the vertices, i.e. the letter a is used to represent the length of the side BC (see Fig. 18.1).

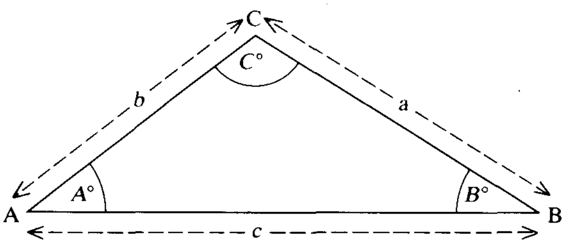


Figure 18.1

The traditional unit of measurement for angles is the degree (but it is not the only one, see §18.5); the degree has been used for over 2000 years. The traditional sub-unit is the minute, which is 1/60th of a degree, and the standard symbol for it is a small dash. So 35° 12′ is equal to $35\frac{12}{60}°$; in decimals this becomes 35.2°. For more awkward numbers a calculator can be used to convert the number of minutes into a decimal fraction of a degree.

In the next two sections Euler's notation will be used to introduce two important rules, the **sine rule** and the **cosine rule**. These rules are used to 'solve' triangles; that is, given sufficient data to define a unique triangle, the sine and cosine rules can be used to calculate the sizes of the remaining sides and angles.

The sine rule

18.2 In the triangle in Fig. 18.2, CP is perpendicular to AB.

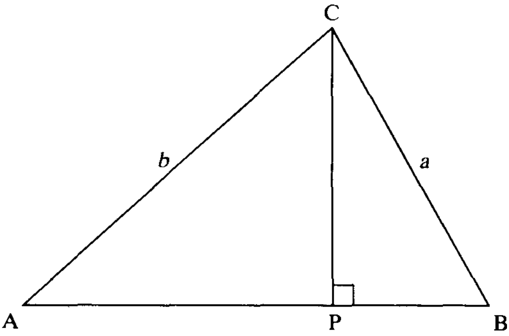


Figure 18.2

By elementary trigonometry the length of the altitude CP is equal to $b \sin A$ (from triangle APC) and it is also equal to $a \sin B$ (from triangle BPC). Equating these expressions, we have

$$a \sin B = b \sin A$$

and hence

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Applying the same argument to the line from A, perpendicular to BC, we could obtain

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Putting these expressions together, we have,

$$\boldsymbol{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$$

This expression, which, by virtue of its symmetrical appearance, is easy to remember, is called the **sine rule**.

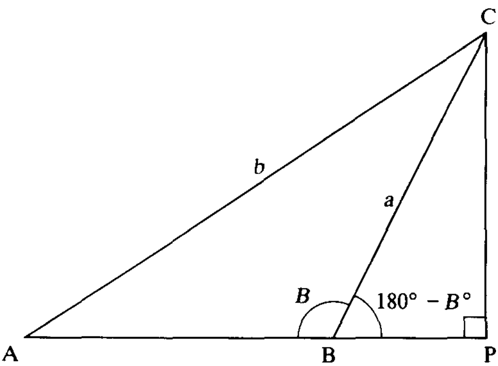


Figure 18.3

However, in drawing Fig. 18.1, we have assumed that all the angles are acute; if one of them is obtuse, the proof must be modified. Suppose that B is the obtuse angle as shown in Fig. 18.3.

In *this* diagram, CP is the perpendicular line from C to AB *produced.* By elementary trigonometry $\text{CP} = a \sin \angle \text{CBP} = a \sin(180° - B)$. However $\sin(180° - B)$ is equal to $\sin B$ and so we can write

$$\text{CP} = a \sin B = b \sin A$$

and proceed with the proof as before.

Example 1 *In triangle* PQR, $r = 5.75$ *and the sizes of angles* P *and* Q *are* 42° *and* 65° *respectively. Calculate the length of* PR.

With these letters (see Fig. 18.4), the sine rule becomes

$$\frac{p}{\sin P} = \frac{q}{\sin Q} = \frac{r}{\sin R}$$

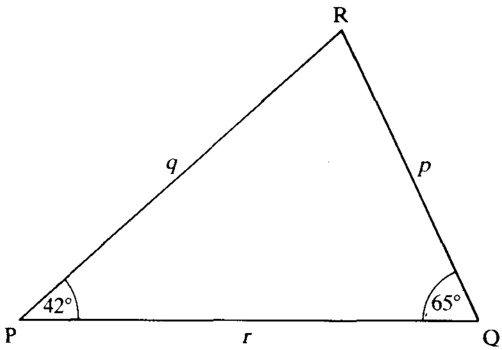


Figure 18.4

Notice that when two angles are given, the remaining angle can be calculated from the fact that the sum of the three angles of a triangle is 180°, so $R = 73°$. Substituting the data, and this value of R, we obtain

$$\frac{p}{\sin 42°} = \frac{q}{\sin 65°} = \frac{5.75}{\sin 73°}$$

In this example, the length of PR, i.e. q, is required. Making q the subject of the formula above, we obtain

$$q = \frac{5.75}{\sin 73°} \times \sin 65°$$

$$= 5.45, \quad \text{correct to three significant figures}$$

Example 2 *In triangle* ABC, $a = 4.73$, $c = 3.58$ *and* $C = 42°\ 12'$. *Calculate the size of angle* A.

Firstly, we note that $42° \, 12' = 42\frac{12}{60}° = 42.2°$, and secondly, from Fig. 18.5, we can see that two triangles can be drawn with these data. (It is very important that a sketch should be drawn, so that this sort of difficulty can be anticipated.)

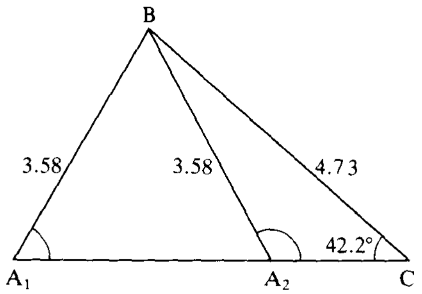


Figure 18.5

By the sine rule,

$$\frac{4.73}{\sin A} = \frac{b}{\sin B} = \frac{3.58}{\sin 42.2}$$

In this case, the middle term is superfluous; the other two terms give

$$\frac{\sin A}{4.73} = \frac{\sin 42.2°}{3.58}$$

$$\therefore \sin A = \frac{\sin 42.2°}{3.58} \times 4.73$$

$$(= 0.8875)^*$$

$$\therefore A = 62.560° \quad or \quad 117.440°$$

$$= 62.6° \quad or \quad 117.4°, \quad \text{correct to the nearest tenth of a degree}$$

There are two points to note here.

(1) The step marked with the asterisk indicates the figures which appear on a calculator at this stage; it is not necessary to write them down. (Indeed, to write them down, correct to four significant figures, and then to use the *corrected* figures to find A is poor calculator technique.)

(2) The alternative value of A, namely, $A = 117.4°$, follows from the fact that $\sin \theta = \sin (180° - \theta)$, i.e. in this case, $\sin 62.6° = \sin 117.4°$. If we inspect the diagram, we can see that both answers are perfectly reasonable, because the triangle A_1BA_2 is isosceles.

A case like this one, where there are two possible answers, is called *the ambiguous case.*

The sine rule can be used when two angles are given (as in Example 1) or when one of the given sides is opposite the given angle (as in Example 2), but, as the reader should be able to see with a little experimentation, it is useless when the lengths of the three sides are given, or when two sides and the *included* angle (i.e. the angle between them) are given. In these circumstances we must turn to the cosine rule. [Some readers may prefer to work Exercise 18a, Nos. 1–3, first.]

The cosine rule

18.3 There are several possible proofs of the cosine rule; this one uses the idea of the scalar product (see §15.15).

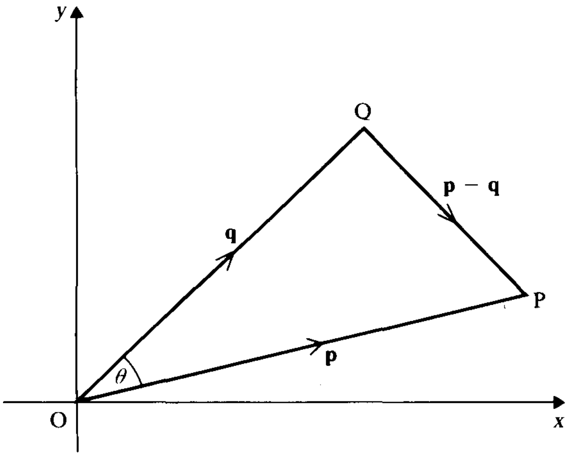


Figure 18.6

In the triangle OPQ (Fig. 18.6), the angle POQ is equal to θ and $\overrightarrow{QP} = \mathbf{p} - \mathbf{q}$. Consider the scalar product $\overrightarrow{QP}.\overrightarrow{QP}$:

$$\begin{aligned}\overrightarrow{QP}.\overrightarrow{QP} &= (\mathbf{p} - \mathbf{q}).(\mathbf{p} - \mathbf{q}) \\ &= \mathbf{p}.\mathbf{p} + \mathbf{q}.\mathbf{q} - 2\mathbf{p}.\mathbf{q} \\ &= p^2 + q^2 - 2pq \cos\theta\end{aligned}$$

But $\overrightarrow{QP}.\overrightarrow{QP}$ is equal to QP^2,

$$\therefore \; QP^2 = p^2 + q^2 - 2pq \cos\theta$$

So, if we are given the values of p and q, and the size of the included angle θ, we can calculate the length of QP.

The formula looks neater, and it is easier to remember, if Euler's notation is used. If the triangle is re-lettered ABC, as in Fig. 18.7, the cosine rule becomes

$$\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A}$$

The letters a, b and c can be permuted to give the following alternative forms:

$$\begin{aligned}b^2 &= c^2 + a^2 - 2ca \cos B \\ c^2 &= a^2 + b^2 - 2ab \cos C\end{aligned}$$

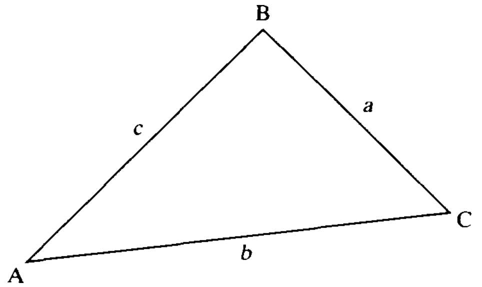


Figure 18.7

Example 3 *In triangle* PQR, $p = 14.3$, $r = 17.5$ *and* $Q = 25° 36'$. *Calculate the length of side* PR.

In this question we are given the lengths of two sides and the size of the included angle, so the cosine rule is appropriate. With these letters it takes the form

$$q^2 = r^2 + p^2 - 2rp \cos Q$$

Substituting the data gives

$$q^2 = 17.5^2 + 14.3^2 - 2 \times 17.5 \times 14.3 \cos 25.6°$$

Hence

$$q = 7.71, \quad \text{correct to three significant figures}$$

(On most calculators it should be possible to do the whole calculation without having to write down any of the intermediate working. If this is possible, it should be done, because mistakes are easily made when figures are transferred from the calculator to paper and *vice versa*. In case of difficulty, consult the calculator's instruction booklet.)

Example 4 *In triangle* XYZ, XY = 3.5, YZ = 4.5 *and* ZX = 6.5. *Calculate the size of angle* Y.

In this case the lengths of the three sides are given. The cosine rule can be used, but first it must be rearranged to make $\cos Y$ the subject.

$$y^2 = z^2 + x^2 - 2zx \cos Y$$

$$\therefore\ 2zx \cos Y = z^2 + x^2 - y^2$$

and hence

$$\cos Y = \frac{z^2 + x^2 - y^2}{2zx}$$

Substituting the data,

$$\cos Y = \frac{3.5^2 + 4.5^2 - 6.5^2}{2 \times 3.5 \times 4.5}$$

$\therefore\ Y = 108.0°$, correct to the nearest tenth of a degree

Once again, if you are using a calculator, the entire calculation should be done without writing down the intermediate steps. Be careful to press the 'equals' key when you have completed the top line (the calculator should display -9.75 at this stage), and, on most calculators, it is essential to enclose the bottom line in brackets, i.e. $(2 \times 3.5 \times 4.5)$.

The area of a triangle

18.4 It is assumed that the reader is familiar with the elementary formula for $\triangle$, the area of a triangle, namely,

$$\triangle = \tfrac{1}{2}bh$$

where b is the length of the base and h is the height of the triangle.

If we are given the lengths b and c and the size of the included angle A (see Fig. 18.8), then the height, h, can be expressed as

$$h = c \sin A$$

and the formula for the area can be written

$$\boldsymbol{\triangle = \tfrac{1}{2}bc \sin A}$$

(The reader should note that this formula can be used for both acute and obtuse angles.)

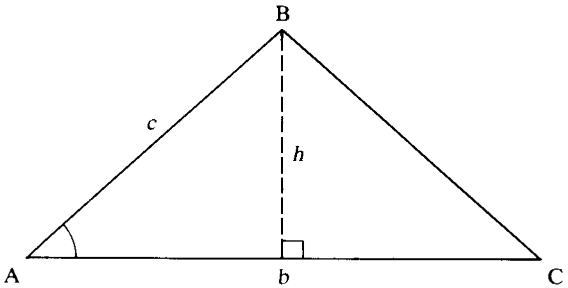


Figure 18.8

Example 5 *In triangle* PQR, QR = 3.5, RP = 4 *and* PQ = 5. *Calculate the size of angle P and hence find the area of the triangle.*

Rearranging the cosine rule (see Example 4),

$$\cos P = \frac{q^2 + r^2 - p^2}{2qr}$$

and substituting the data, i.e. $p = 3.5$, $q = 4$ and $r = 5$, we have

$$\cos P = \frac{16 + 25 - 12.25}{2 \times 4 \times 5}$$

$$= \frac{28.75}{40}$$

$\therefore\ P = 44.0°$, correct to the nearest tenth of a degree

The area of the triangle is given by

$$\begin{aligned}\triangle &= \tfrac{1}{2}qr \sin P\\ &= \tfrac{1}{2} \times 4 \times 5 \times \sin P\\ &= 6.95, \quad \text{correct to three significant figures}\end{aligned}$$

Note. When no units have been explicitly stated, as in the example above, it is assumed that the same units have been used consistently throughout the question, e.g. if the lengths QR, RP and PQ are all given in cm, then the area of PQR is measured in cm^2.

Historical note. The problem of calculating the area of a triangle when the lengths of the three sides are given is a very ancient one. The area can be calculated from the formula

$$\triangle = \sqrt{\{s(s-a)(s-b)(s-c)\}}$$

where $s = \frac{1}{2}(a + b + c)$. This formula is usually known as Heron's formula, after Heron of Alexandria, who lived over two thousand years ago. However the formula was known even before Heron's time. (See Exercise 18f, No. 19.)

Qu. 1 Calculate the area of the triangle in Example 3.
Qu. 2 Use Heron's formula (see *Historical note* above) to calculate the area of the triangle in Example 5.
Qu. 3 Calculate the areas of the triangles in which
(a) $A = 60°$, $b = 3$, $c = 5$;
(b) $C = 110°$, $a = 14$, $b = 11$;
(c) $B = 90°$, $c = 8.6$, $b = 11.4$;
(d) $a = 8$, $b = 11$, $c = 13$;
(e) $a = 12.3$, $b = 14.1$, $c = 13.6$;
(f) $a = 17.6$, $b = 16.9$, $c = 16.1$;
(g) $a = 209$, $b = 313$, $c = 390$.

Exercise 18a

Solve the following triangles:

1 (Sine formula, acute angled)
(a) $a = 12$, $B = 59°$, $C = 73°$;
(b) $A = 75.6°$, $b = 5.6$, $C = 48.3°$;
(c) $A = 73.2°$, $B = 61.7°$, $c = 171$.

2 (Sine formula, obtuse angled)
(a) $A = 36°$, $b = 2.37$, $C = 49°$;
(b) $A = 123.2°$, $a = 11.5$, $C = 37.1°$;
(c) $a = 136$, $B = 104.2°$, $C = 43.1°$.

3 (Sine formula, ambiguous case)
(a) $b = 17.6$, $C = 48° 15'$, $c = 15.3$;
(b) $B = 129°$, $b = 7.89$, $c = 4.56$;
(c) $A = 28° 15'$, $a = 8.5$, $b = 14.8$.

4 (Cosine formula, acute angled)
(a) $a = 5$, $b = 8$, $c = 7$;
(b) $a = 10$, $b = 12$, $c = 9$;
(c) $a = 17$, $b = 13$, $c = 18$.

5 (Cosine formula, acute angled)
(a) $A = 60°$, $b = 8$, $c = 15$;
(b) $a = 14$, $B = 53°$, $c = 12$;
(c) $a = 11$, $b = 9$, $C = 43.2°$.

6 (Cosine formula, obtuse angled)
(a) $a = 8$, $b = 10$, $c = 15$;
(b) $a = 11$, $b = 31$, $c = 24$;
(c) $a = 27$, $b = 35$, $c = 46$.

7 (Cosine formula, obtuse angled)
(a) $a = 17$, $B = 120°$, $c = 63$;
(b) $A = 104° 15'$, $b = 10$, $c = 12$;
(c) $a = 31$, $b = 42$, $C = 104° 10'$.

8 Two points A and B on a straight coastline are 1 km apart, B being due East of A. If a ship is observed on bearings 167° and 205° from A and B respectively, what is its distance from the coastline?

9 A boat is sailing directly towards a cliff. The angle of elevation of a point on the top of the cliff and straight ahead of the boat increases from 10° to 15° as the ship sails a distance of 50 m. What is the height of the cliff?

10 A triangle is taken with sides 10, 11, 15 cm. By how much does its largest angle differ from a right angle?

11 A ship rounds a headland by sailing first 4 nautical miles on a course of 069° then 5 nautical miles on a course of 295°. Calculate the distance and bearing of its new position from its original position.

12 A man travelling along a straight level road in the direction 053° observes a pylon on a bearing of 037°. 800 m further along the road the bearing of the pylon is 296°. Calculate the distance of the pylon from the road.

Radians

18.5 The fact that there are 90 degrees in a right angle has been familiar to the reader since he or she began geometry; but it may not have been realised that the number is an arbitrary one which has come down to us from the Babylonian civilisation. Indeed, an attempt to introduce 100 degrees to the right angle was made after the French Revolution, but it was later dropped, and in 1938 a similar attempt was made by the Germans. The following example also illustrates the arbitrary nature of the number of degrees in a right angle.

Example 6 *An arc* AB *of a circle, centre* O, *subtends an angle of* $x°$ *at* O. *Find expressions in terms of* x *and the radius,* r, *for* (a) *the length of the arc* AB, (b) *the area of the sector* OAB (see Fig. 18.9).

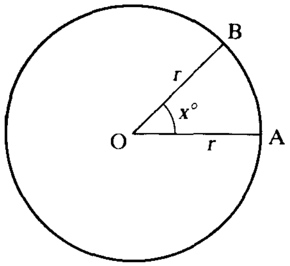


Figure 18.9

(a) The length of an arc of a given circle is proportional to the angle it subtends at the centre. But an angle of 360° is subtended by an arc of length $2\pi r$, therefore an angle of $x°$ is subtended by an arc of length

$$\frac{x}{360} \times 2\pi r$$

Therefore the length of arc AB is $(\pi/180)xr$.

(b) The area of a sector of a given circle is proportional to the angle at the centre. But a sector containing an angle of 360° is the whole circle, which has an area of πr^2, therefore a sector containing an angle of $x°$ has an area of

$$\frac{x}{360} \times \pi r^2$$

Therefore the area of the sector OAB is $\frac{1}{2}(\pi/180)xr^2$.

Thus, in both the length of an arc and the area of a sector, there appears a factor of $\pi/180$, which is due to the unit of measurement of the angle OAB. This suggests a new unit for measuring angles, which is called a **radian**, such that an

$$\text{angle in radians} = \frac{\pi}{180}\,(\text{angle in degrees}) \qquad (1)$$

If we let θ radians equal x degrees, then, referring to Fig. 18.9,

the **length of arc AB** $= r\theta$

and

the **area of sector OAB** $= \frac{1}{2}r^2\theta$

If, then, we construct an angle of 1 radian, the arc AB will be of length r, and so *an arc of a circle equal to the radius subtends at the centre an angle of* 1 *radian.* Radians are sometimes termed circular measure, and are denoted by rad. It follows from the relation (1) above, by putting the angle in degrees equal to 180, that

$$\pi \textbf{ rad} = \textbf{180}^\circ$$

Hence 1 radian = 57.296 degrees and 1 degree = 0.017 453 radians, both correct to five significant figures.

The use of radians extends far beyond finding lengths of arcs and areas of sectors. In later sections it is shown how they have applications in mechanics and calculus.

Exercise 18b (Oral)

1 Convert to degrees:

(a) $\frac{\pi}{2}$ rad, (b) $\frac{\pi}{4}$ rad, (c) $\frac{\pi}{3}$ rad, (d) $\frac{2\pi}{3}$ rad,

(e) $\frac{\pi}{6}$ rad, (f) $\frac{3\pi}{2}$ rad, (g) $\frac{5\pi}{2}$ rad, (h) 4π rad,

(i) 5π rad, (j) $\frac{4\pi}{3}$ rad, (k) $\frac{7\pi}{2}$ rad, (l) $\frac{3\pi}{4}$ rad.

2 Convert to radians, leaving π in your answer:
(a) 360°, (b) 90°, (c) 45°, (d) 15°,
(e) 60°, (f) 120°, (g) 300°, (h) 270°,
(i) 540°, (j) 30°, (k) 150°, (l) 450°.

3 What is the length of an arc which subtends an angle of 0.8 rad at the centre of a circle of radius 10 cm?

4 An arc of a circle subtends an angle of 1.2 rad at any point on the remaining part of the circumference. Find the length of the arc, if the radius of the circle is 4 cm.

5 An arc of a circle subtends an angle of 0.5 rad at the centre. Find the radius of the circle, if the length of the arc is 3 cm.

6 Find, in radians, the angle subtended at the centre of a circle of radius 2.5 cm by an arc 2 cm long.

7 What is the area of a sector containing an angle of 1.5 rad, in a circle of radius 2 cm?

8 The radius of a circle is 3 cm. What is the angle contained by a sector of area 18 cm^2?

9 An arc subtends an angle of 1 rad at the centre of a circle, and a sector of area 72 cm^2 is bounded by this arc and two radii. What is the radius of the circle?

10 The arc of a sector in a circle, radius 2 cm, is 4 cm long. What is the area of the sector?

Exercise 18c

1 Express in radians, leaving π in your answers:
(a) $22\frac{1}{2}°$, (b) $1080°$, (c) $12'$, (d) $37°\ 30'$.

2 Express in degrees:

(a) $\dfrac{2\pi}{5}$ rad, (b) $\dfrac{\pi}{36}$ rad, (c) $\dfrac{7\pi}{12}$ rad, (d) $\dfrac{7\pi}{2}$ rad.

3 Find the length of an arc of a circle, which subtends an angle of 31° at the centre, if the radius of the circle is 5 cm.

4 The chord AB of a circle subtends an angle of 60° at the centre. What is the ratio of chord AB to arc AB?

5 An arc of a circle, radius 2.5 cm, is 3 cm long. What is the angle subtended by the arc at the centre
(a) in radians, (b) in degrees?

6 A segment is cut off a circle of radius 5 cm by a chord AB, 6 cm long. What is the length of the minor arc AB?

7 What is the area of a sector containing an angle of 1.4 rad in a circle whose radius is 2.4 cm?

8 A chord AB subtends an angle of 120° at O, the centre of a circle with radius 12 cm. Find the area of
(a) sector AOB, (b) triangle AOB, (c) the minor segment AB.

9 An arc AB of a circle with radius 6 cm subtends an angle of 40° at the centre. Find the area bounded by the diameter BC, CA and the arc AB.

10 Two equal circles of radius 5 cm are situated with their centres 6 cm apart. Calculate what area lies within both circles.

11 A chord PQ of a circle with radius r, subtends an angle θ at the centre. Show that the area of the minor segment PQ is $\frac{1}{2}r^2(\theta - \sin\theta)$, and write down the area of the major segment PQ in terms of r and θ.

12 A circle of radius r is drawn with its centre on the circumference of another circle of radius r. Show that the area common to both circles is $2r^2(\pi/3 - \sqrt{3}/4)$.

Angular velocity

18.6 A man who buys an electric motor is usually interested in the rate at which it goes, and he may be told that it does 12 000 revolutions per minute (rev/min). On the other hand the drum of a barograph turns at the rate of 49 degrees per day. In either case the rate of turning, which is called average angular velocity, is given by

$$\text{average angular velocity} = \frac{\text{angle turned}}{\text{time taken}}$$

Qu. 4 Find the average angular velocity of the second hand of a watch
(a) in degrees per second (deg/s), (b) in rev/min.

Qu. 5 Convert
(a) 500 rev/min into deg/s, (b) 1 rev/week into deg/h.

In many cases of turning, however, the angular velocity is not constant, so consider the average angular velocity in a small interval of time δt. If the angle turned through in this time is $\delta\theta$ radians,

$$\text{average angular velocity} = \frac{\delta\theta}{\delta t}\ \text{rad/s}$$

But as $\delta t \to 0$,

$$\frac{\delta\theta}{\delta t} \to \frac{d\theta}{dt}$$

$$\therefore \text{average angular velocity} \to \frac{d\theta}{dt}$$

$\dfrac{d\theta}{dt}$ is called **angular velocity** and is denoted by ω (the Greek letter omega). Therefore

$$\boldsymbol{\omega = \frac{d\theta}{dt}}$$

[In motion in a straight line average velocity $= \dfrac{\text{distance}}{\text{time}}$ and if a distance δs is travelled in a time δt, average velocity $= \dfrac{\delta s}{\delta t}$. But $\dfrac{\delta s}{\delta t} \to \dfrac{ds}{dt}$ as $\delta t \to 0$ and so the velocity at an instant is given by $v = \dfrac{ds}{dt}$. In this way there is a parallel between linear motion and angular motion.]

If a particle moves in a circle of radius r with speed v and angular velocity ω about the centre, the relation between r, v, ω can be obtained from one of the results obtained in §18.5. If s is the distance of the particle measured along the circumference of the circle from a fixed point,

$$s = r\theta$$

Differentiating with respect to time (remember r is constant),

$$\frac{ds}{dt} = r\frac{d\theta}{dt}$$

$$\therefore v = r\omega$$

Remember that ω must be measured in radians/unit time. Three sets of possible units for v, r, ω are shown in the table below:

v	r	ω
m/s	m	rad/s
km/h	km	rad/h
cm/min	cm	rad/min

Example 7 *A belt runs round a pulley attached to the shaft of a motor. If the belt runs at 0.75 m/s and the radius of the pulley is 6 cm, find the angular velocity of the pulley* (a) *in rad/s,* (b) *in rev/min.*

(a) Using the result $v = r\omega$,

$$\omega = \frac{75}{6} = 12.5 \text{ rad/s}$$

(b) $$12.5 \text{ rad/s} = \frac{12.5}{2\pi} \text{ rev/s}$$

$$= \frac{12.5}{2\pi} \times 60 \text{ rev/min}$$

$$\approx 120 \text{ rev/min}$$

(The sign $\approx$ means 'is approximately equal to'.) Therefore the angular velocity is 12.5 rad/s or approximately 120 rev/min.

Exercise 18d

Use the result $v = r\omega$ where you can.

1 Express the angular velocity of the minute hand of a clock in
(a) rev/min, (b) deg/s, (c) rad/s.

2 A wheel is turning at 200 rev/min. Express this angular velocity in
(a) deg/s, (b) rad/s.

3 A cook can rotate the handle of her egg whisk 32 times in 5 seconds. Each time the handle rotates, the paddles rotate four times. At what speed are the paddles rotating in
(a) rev/min, (b) rad/s?

4 The Earth rotates on its axis approximately $365\frac{1}{4}$ times in a year. Calculate its angular velocity in rad/h, correct to three significant figures.

5 The cutters of a well-known electric shaver rotate about 3000 times a minute, and the distance from the axis to the tip of the cutter is 0.65 cm. Find
(a) the angular velocity of the cutter in rad/s,
(b) the speed of the tip of a cutter in cm/s.

6 When I dial 0 on the telephone, the dial rotates through 334° in $1\frac{1}{2}$ s approximately. What is the average angular speed of the dial in rad/s, and what is the speed of a point on the circumference of the dial if its diameter is 8 cm?

7 A motor runs at 1200 rev/min. What is its angular velocity in rad/s? If the shaft of the motor is 2.5 cm in diameter, at what speed is a point on the circumference of the shaft moving?

8 A point on the rim of a wheel of diameter 2.5 m is moving at a speed of 44 m/s relative to the axis. At what rate in (a) rad/s, (b) rev/min, is the wheel turning?

9 If a cotton reel drops 1.76 m in 0.7 s, the end of the cotton being held still, at what average angular velocity, in rev/min, is the reel turning, if its diameter is 3 cm?

10 A belt runs round two pulleys of diameters 26.25 cm and 15 cm. If the larger rotates 700 times in a minute, find the angular velocity of the smaller in rad/s.

11 The Earth moves round the sun approximately in a circle of radius 150 000 000 km. Find its angular speed in rad/s, and obtain its speed along its orbit in km/s.

12 Taking the Earth to be a sphere of radius 6300 km which rotates about its axis once in 23.93 hours what error will be made in calculating the velocity of a point on the equator, if it is assumed that the Earth rotates once in 24 hours? Express your answer in km/h, correct to two significant figures.

Inverse trigonometrical functions

18.7 Can you find an angle $x°$, such that $\sin x° = 0.5$? This sort of problem arises frequently in mathematics; indeed we have already met it earlier in this chapter. An answer can be easily obtained from tables or from a calculator. In this particular case, the angle $x°$ is an angle in one of the 'standard' triangles described in §16.3, i.e. 30°. But this is not the complete solution; we can see from the graph of $y = \sin x$ (Fig. 18.10), that 150° is also a possibility and, since $\sin x$

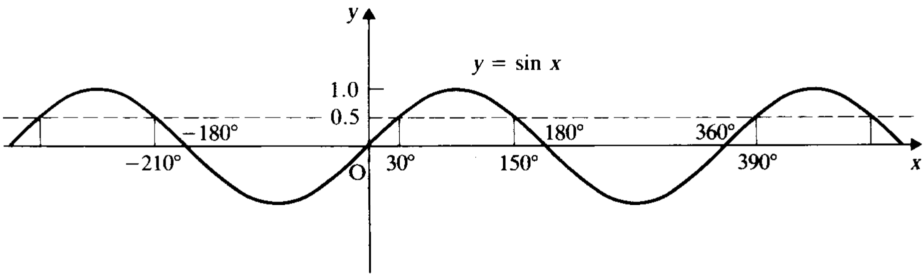


Figure 18.10

has a period of 360° (it repeats itself every 360°), we can add any multiple of 360° to these two angles. Hence there are infinitely many values of x which satisfy the equation $\sin x = 0.5$; they can be expressed in the form

$$x = 30° + n360° \quad \text{or} \quad x = 150° + n360°$$

where n is any integer, positive or negative. If we were working in radians, this general solution would take the form

$$x = \frac{\pi}{6} + 2n\pi \quad \text{or} \quad x = \frac{5\pi}{6} + 2n\pi$$

Qu. 6 Write down the general solution, in degrees, of the equation $\cos x° = -0.5$.

Qu. 7 Write down, in radians, the general solution of the equation $\tan x = 1$.

In advanced trigonometry, it is useful to have a standard abbreviation for the phrase 'the angle whose sine is x', etc. The usual abbreviation for this is arcsin x; and arccos x, arctan x are used for the inverses of the cos and tan functions. This is the standard notation on all microcomputers and it is also found on many pocket calculators, but the notation $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$, is also used.

However, the fact that there are infinitely many angles whose sine is x, causes some problems. For instance, if you were designing a pocket calculator, which of the infinitely many possible answers would you choose to show on the display? (Try finding arcsin, arccos and arctan of ± 0.2, ± 0.4, ± 0.8, etc., on your pocket calculator. Can you discover the principle which the manufacturer of your calculator is using to select the angle shown on the display?)

Another serious problem is that if we are intending to describe arcsin x, arccos x and arctan x, as *functions*, then we must ensure that the function has *exactly one value*, for any given value of x (see §2.8). Consequently we must define these functions rather more carefully than we have done so far.

Definitions

(a) **arcsin *x*** *is the angle (in radians) between* $-\frac{1}{2}\pi$ *and* $+\frac{1}{2}\pi$, *inclusive, whose sine is* x.
(b) **arccos *x*** *is the angle (in radians) between* 0 *and* π, *inclusive, whose cosine is* x.
(c) **arctan *x*** *is the angle (in radians) between* $-\frac{1}{2}\pi$ *and* $+\frac{1}{2}\pi$, *whose tangent is* x.
(The angles within these ranges are often called the *principal values*.)

If desired, these definitions may be expressed in degrees, but for advanced work in trigonometry, radians are more common than degrees.

Qu. 8 Why is the range $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$ unsuitable for arccos x?

Notice that, since there is no angle whose sine is greater than 1, an expression such as arcsin 2 is meaningless. The function arcsin x only makes sense if x is numerically smaller than (or equal to) 1; in other words, *the domain* of the function arcsin x is $\{x: -1 \leqslant x \leqslant +1\}$. The function arccos x has the same domain, but in the function arctan x, the variable x can take any (real) value, i.e. the domain of arctan x is $\mathbb{R}$ (see Fig. 18.11).

Like all inverse functions, the graphs of arcsin x, arccos x and arctan x are the reflections of the graphs of the corresponding functions in the line $y = x$.

In diagrams (i) and (ii), the solid parts of the graphs represent the principal values of arcsin x and arccos x respectively; the broken parts of the graphs represent the other values.

Exercise 18e

All the questions in this exercise use the angles in the 'standard' triangles (see §16.3). *Do not use a calculator.*

Write down the general solutions of the following equations (in degrees):

1 $\sin x^\circ = 1/\sqrt{2}$. **2** $\cos x^\circ = 1$. **3** $\tan x^\circ = \sqrt{3}$.
4 $\sin x^\circ = -1$. **5** $\cos x^\circ = -1/2$. **6** $\tan x^\circ = -1/\sqrt{3}$.

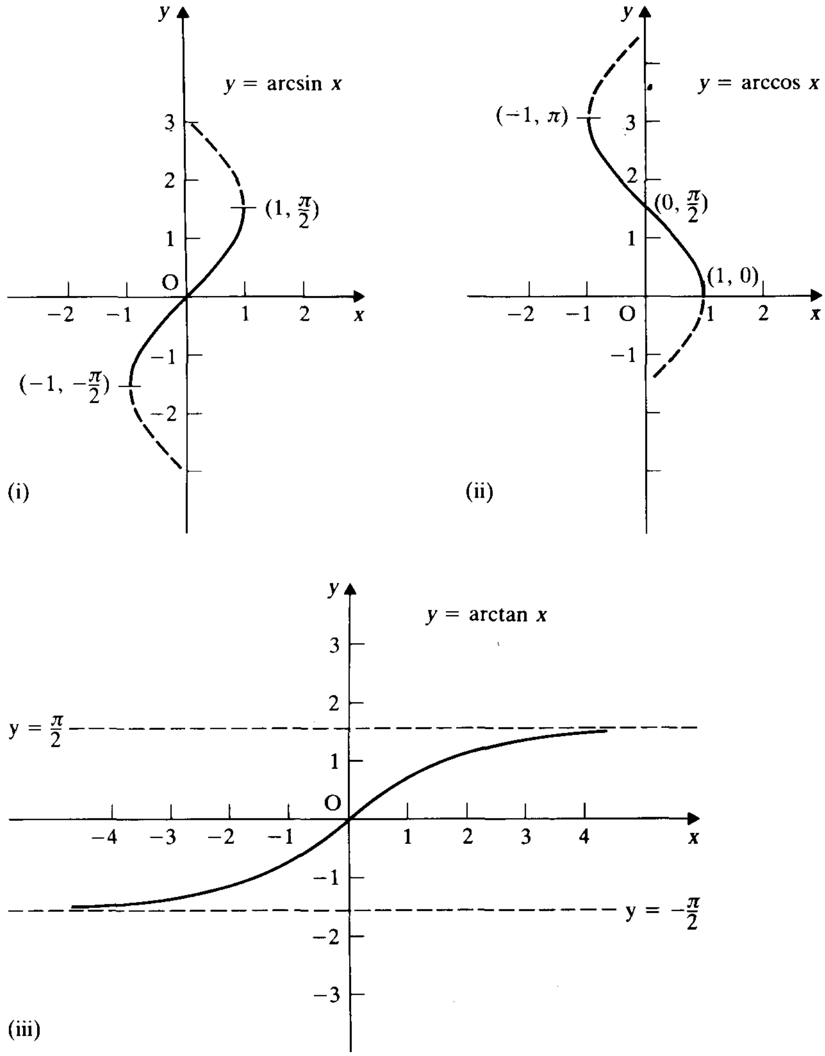


Figure 18.11

Write down, *in radians*, the general solutions of the following equations:

7 $\cos x = \frac{1}{2}$.
8 $\tan x = -1$.
9 $\sin 2x = \frac{1}{2}$.
10 $\cos^2 x = \frac{3}{4}$.

Write down, *in radians*, the values of

11 $\arcsin(\sqrt{3}/2)$.
12 $\arccos(1/\sqrt{2})$.
13 $\arctan 1$.
14 $\arcsin(-\frac{1}{2})$.
15 $\arccos(-\sqrt{3}/2)$.
16 $\arctan(-1)$.
17 $\arcsin(-1)$.
18 $\arccos(-1)$.
19 $\arctan 0$.
20 $\arccos 0$.

Exercise 18f (Miscellaneous)

1 Solve the following triangles:
(a) $A = 60°$, $b = 8$, $c = 15$;
(b) $a = 14$, $B = 53°$, $c = 12$;
(c) $a = 11$, $b = 9$, $C = 43.2°$.

2 Solve the triangles:
(a) $a = 17$, $B = 120°$, $c = 63$;
(b) $A = 104° \, 15'$, $b = 10$, $c = 12$;
(c) $a = 31$, $b = 42$, $C = 104°$.

3 Solve the triangles:
(a) $c = 11.6$, $A = 54.2°$, $B = 26.4°$;
(b) $a = 4.96$, $b = 6.01$, $A = 31.2°$;
(c) $A = 20°$, $a = 15$, $c = 10$.

4 Calculate the areas of the following triangles:
(a) $x = 5$, $y = 8$, $Z = 35°$;
(b) $x = 4$, $y = 5$, $z = 6$;
(c) $x = 25$, $y = 35$, $z = 9$.

5 Convert to degrees:

(a) $\frac{2\pi}{5}$, (b) $\frac{5\pi}{6}$, (c) $\frac{3\pi}{8}$, (d) $\frac{7\pi}{12}$.

6 Convert to radians, leaving π in your answers:
(a) 330°, (b) 50°, (c) 75°, (d) 24°.

7 The area of a sector of a circle, diameter 7 cm, is 18.375 cm^2. What is the length of the arc of the sector?

8 A radar scanner rotates at a speed of 30 rev/min. Express this angular velocity in rad/s.

9 What is the angular velocity of the hour hand of a clock in
(a) rev/min, (b) rad/s?

10 Two cog-wheels have radii 10 cm and 15 cm. If the larger wheel is turning with an angular velocity of 50 rad/s, what is the angular velocity of the smaller one when the teeth of the cog-wheels are engaged?

11 A circular coin is placed on a flat horizontal surface and held stationary while an identical coin, also placed on the horizontal surface, rolls around its perimeter, without slipping. Through how many radians does the second coin turn?

12 Investigate the effect on the cosine rule if, in the usual notation, a, b and c are given, and $c > a + b$.

13 Investigate the effect on the sine rule if, in the usual notation, a, b and A, are given, and
(a) $a < b \sin A$, (b) $b \sin A < a < b$, (c) $b < a$.

14 The lengths of the sides of a triangle are 10, x and $(x - 2)$. The side of length $(x - 2)$ is opposite an angle of 60°. Find the value of x.

15 In the triangle XYZ, $x = 29$, $y = 21$ and $z = 20$. Calculate:
(a) the area of the triangle,

(b) the length of the perpendicular from Z to XY.

16 The points A and B lie on a circle, radius 1 cm, centre O, the origin. The radii OA and OB are inclined at angles α and β, respectively, to the x-axis. Write down the coordinates of A and B in terms of α and β. By applying the cosine rule to triangle OAB prove that

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

17 In the cosine rule, substitute $\cos A = 2\cos^2\frac{A}{2} - 1$, and hence prove that

$$\cos\frac{A}{2} = \sqrt{\left\{\frac{s(s-a)}{bc}\right\}}$$

where $s = \frac{1}{2}(a + b + c)$.

18 In the cosine rule, substitute $\cos A = 1 - 2\sin^2\frac{A}{2}$, and hence prove that

$$\sin\frac{A}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}}$$

19 Use the results of Nos. 18 and 19 to prove Heron's formula for the area of a triangle,

$$\triangle = \sqrt{\{s(s-a)(s-b)(s-c)\}}$$

20 Prove Heron's formula by eliminating A from the formulae

$$a^2 = b^2 + c^2 - 2bc\cos A \qquad \text{and} \qquad \triangle = \tfrac{1}{2}bc\sin A$$

[Hint: use $\cos^2 A + \sin^2 A = 1$.]

Chapter 19

Derivatives of trigonometrical functions

Small angles

19.1 A glance at Fig. 19.1 will show the reader that, for small acute angles, tan θ, θ and sin θ are practically equal.

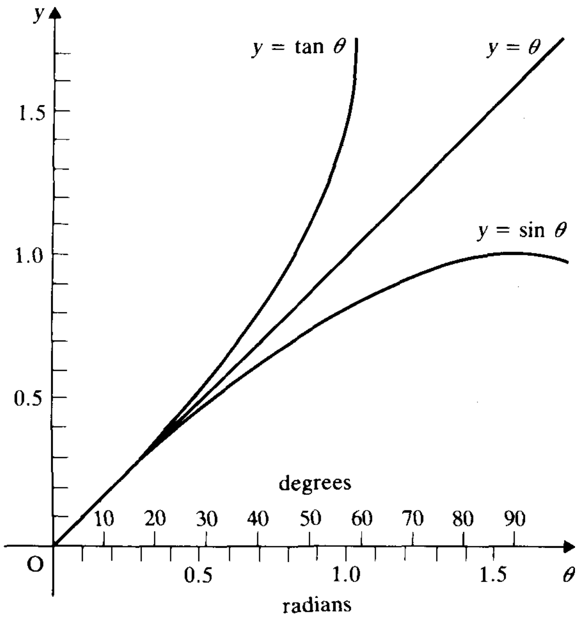


Figure 19.1

This is borne out by seven-figure tables:

Angle in degrees	10°	5°	1°
θ (radians)	0.174 532 9	0.087 266 5	0.017 453 3
$\tan\theta$	0.176 327 0	0.087 488 7	0.017 455 1
$\sin\theta$	0.173 648 2	0.087 155 7	0.017 452 4

We shall now consider this geometrically.

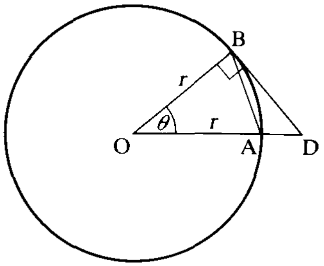


Figure 19.2

In Fig. 19.2, the chord AB subtends an angle θ at the centre of a circle of radius r, and the tangent at B meets OA at D. Consider the three areas: triangle AOB, sector AOB, triangle DOB.

(a) In triangle AOB, two sides of length r include an angle θ, therefore its area is $\frac{1}{2}r^2 \sin\theta$ (see §18.4).
(b) From §18.5, the area of sector AOB is $\frac{1}{2}r^2\theta$.
(c) In triangle DOB, B is a right angle, therefore $BD = r\tan\theta$ and so its area is $\frac{1}{2}r^2 \tan\theta$.

From the figure it can be seen that

triangle AOB < sector AOB < triangle DOB

$$\therefore \tfrac{1}{2}r^2 \sin\theta < \tfrac{1}{2}r^2\theta < \tfrac{1}{2}r^2 \tan\theta$$

But if we divide each term by $\frac{1}{2}r^2$ the order of magnitude is unchanged, therefore

$$\sin\theta < \theta < \tan\theta$$

providing θ is acute, as the figure requires. Again, if we divide each term by $\sin\theta$, the order of magnitude is unchanged, therefore

$$\frac{\sin\theta}{\sin\theta} < \frac{\theta}{\sin\theta} < \frac{\tan\theta}{\sin\theta}$$

But $\tan\theta = \sin\theta/\cos\theta$, therefore

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

Now as $\theta \to 0$, $\cos\theta \to 1$,

$$\therefore \frac{1}{\cos\theta} \to 1$$

Thus $\theta/\sin\theta$ lies between 1 and a function which approaches 1 as $\theta \to 0$.

$$\therefore \frac{\boldsymbol{\theta}}{\mathbf{sin}\,\boldsymbol{\theta}} \to \mathbf{1} \quad \text{as } \theta \to 0$$

(See Chapter 2, Example 17 and Qu. 11.)

This limit (or, more strictly, $(\sin\theta)/\theta \to 1$ as $\theta \to 0$) is required in the next section for the differentiation of $\sin x$.

Another way of expressing the statement that $\theta/\sin\theta \to 1$ as $\theta \to 0$, is to say that, for small values of θ,

$$\mathbf{sin}\,\boldsymbol{\theta} \approx \boldsymbol{\theta}$$

An approximation for $\cos\theta$ is obtained from the identity

$$\cos\theta = 1 - 2\sin^2 \tfrac{1}{2}\theta$$

If θ is small, $\sin\frac{1}{2}\theta \approx \frac{1}{2}\theta$, therefore

$$\cos\theta \approx 1 - 2(\tfrac{1}{2}\theta)^2$$

Therefore, for small values of θ,

$$\mathbf{cos}\,\boldsymbol{\theta} \approx \mathbf{1} - \tfrac{1}{2}\boldsymbol{\theta}^2$$

Example 1 *Find the approximate value of* $\dfrac{1-\cos 2\theta}{\theta\tan\theta}$ *when* θ *is small.*

We cannot put $\theta = 0$, as the numerator and denominator would both be zero. Since $\cos\theta \approx 1 - \frac{1}{2}\theta^2$,

$$\cos 2\theta \approx 1 - \tfrac{1}{2}(2\theta)^2 = 1 - 2\theta^2$$

Therefore the numerator $\approx 2\theta^2$. But the denominator $\approx \theta^2$, since $\tan\theta \approx \theta$. Therefore, when θ is small,

$$\frac{1-\cos 2\theta}{\theta\tan\theta} \approx \frac{2\theta^2}{\theta^2}$$

$$\therefore \frac{1-\cos 2\theta}{\theta\tan\theta} \approx 2 \quad \text{when } \theta \text{ is small}$$

Qu. 1 Find approximations for the following functions when θ is small:

(a) $\dfrac{\sin 3\theta}{2\theta}$, (b) $\dfrac{\sin 4\theta}{\sin 2\theta}$, (c) $\dfrac{1-\cos\theta}{\theta^2}$,

(d) $\dfrac{\theta\sin\theta}{1-\cos 2\theta}$, (e) $\dfrac{\sin(\alpha+\theta)\sin\theta}{\theta}$, (f) $\dfrac{\sin(\alpha+\theta)-\sin\alpha}{\theta}$,

(g) $\dfrac{\sin\theta\tan\theta}{1-\cos 3\theta}$, (h) $\sin\theta\operatorname{cosec}\tfrac{1}{2}\theta$, (i) $\dfrac{\tan(\alpha+\theta)-\tan\alpha}{\theta}$.

Derivatives of sin *x* and cos *x*

19.2 The graph of sin x may be sketched, as shown in Fig. 19.3, and from it may be obtained a rough graph of its gradient. The gradient is zero at B, D, F, positive from A to B and from D to F, and negative from B to D, giving a graph like the one in Fig. 19.4.

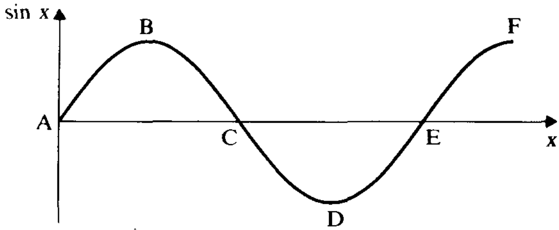


Figure 19.3

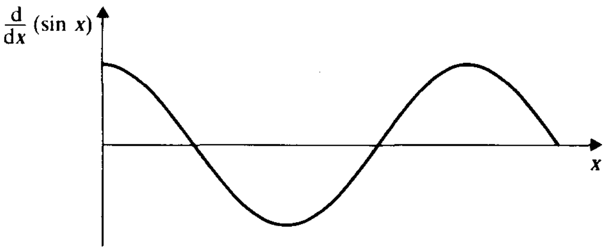


Figure 19.4

Qu. 2 Does Fig. 19.4 resemble any graph you have met so far?
Qu. 3 Express $\sin A - \sin B$ in factors. (See §17.8.)

We shall now find the derivative of sin x from first principles, using the definition in §3.8, that is

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

(The reader is advised to review §3.8 before proceeding further.)

In this case, $f(x) = \sin x$, and so

$$f(x+h) - f(x) = \sin(x+h) - \sin x$$

Using the factor formula (see Qu. 3 above), this can be written

$$f(x+h) - f(x) = 2\cos\frac{x+h+x}{2}\sin\frac{h}{2}$$

$$\therefore \frac{f(x+h)-f(x)}{h} = \frac{2\cos\frac{2x+h}{2}\sin\frac{h}{2}}{h}$$

$$= \frac{\cos(x+\frac{1}{2}h)\sin\frac{1}{2}h}{\frac{1}{2}h} \qquad (1)$$

But we know that when $h \to 0$,

$$\cos(x+\tfrac{1}{2}h) \to \cos x \quad \text{and} \quad \frac{\sin\frac{1}{2}h}{\frac{1}{2}h} \to 1$$

Therefore, when $h \to 0$, the right-hand side of equation (1) tends to $\cos x$. So, for this function,

$$\mathbf{f'(x) = \cos x}$$

In Leibnitz notation, this is written

$$y = \sin x$$

$$\frac{dy}{dx} = \cos x$$

Or, more concisely,

$$\frac{\mathbf{d}}{\mathbf{d}x}(\sin x) = \cos x$$

Qu. 4 At what stage in the above is it necessary to have x in radians?

Qu. 5 Prove from first principles that

$$\frac{\mathbf{d}}{\mathbf{d}x}(\cos x) = -\sin x$$

Remember that these results hold only if x is in radians.

Example 2 *Differentiate* (a) $\sin(2x+3)$, (b) $\cos^2 x$, (c) $\sin x°$.

(a) Let $y = \sin(2x+3)$, $t = 2x+3$, then $y = \sin t$.

$$\therefore \frac{dy}{dt} = \cos t \quad \text{and} \quad \frac{dt}{dx} = 2$$

But $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx} = (\cos t)2$,

$$\therefore \frac{d}{dx}\{\sin(2x+3)\} = 2\cos(2x+3)$$

(b) Let $y = \cos^2 x$, $t = \cos x$, then $y = t^2$.

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2t(-\sin x) = -2\cos x \sin x$$

$$\therefore \frac{d}{dx}(\cos^2 x) = -\sin 2x$$

(c) Let $y = \sin x°$. Now $x° = (\pi/180)x$ radians,

$$\therefore y = \sin \frac{\pi}{180} x$$

Put $t = (\pi/180)x$, then $y = \sin t$.

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = (\cos t)\frac{\pi}{180} = \frac{\pi}{180}\cos\frac{\pi}{180}x = \frac{\pi}{180}\cos x°$$

$$\therefore \frac{d}{dx}(\sin x°) = \frac{\pi}{180}\cos x°$$

Qu. 6 Differentiate (a) $\cos 3x$, (b) $\sin^2 x$, (c) $2 \sin 2x$, (d) $\cos^3 x$.

Example 3 *Integrate* (a) $\cos 2x$, (b) $3 \sin \frac{1}{2}x$.

The method used here is to change cos to sin, or sin to cos, and to determine the coefficient by differentiation:

(a) $\dfrac{d}{dx}(\sin 2x) = 2 \cos 2x.$

$$\therefore \frac{d}{dx}(\tfrac{1}{2}\sin 2x) = \cos 2x$$

$$\therefore \int \cos 2x \, dx = \tfrac{1}{2}\sin 2x + c$$

(b) $\dfrac{d}{dx}(3 \cos \tfrac{1}{2}x) = -\tfrac{3}{2}\sin \tfrac{1}{2}x.$

$$\therefore \frac{d}{dx}(-2 \times 3 \cos \tfrac{1}{2}x) = 3 \sin \tfrac{1}{2}x$$

$$\therefore \int 3 \sin \tfrac{1}{2}x \, dx = -6 \cos \tfrac{1}{2}x + c$$

Exercise 19a

1 Differentiate:
(a) $\cos 2x$, (b) $\sin 6x$, (c) $\cos(3x - 1)$,
(d) $\sin(2x - 3)$, (e) $-3 \cos 5x$, (f) $2 \sin 4x$,
(g) $-4 \sin \frac{3}{2}x$, (h) $2 \sin \frac{1}{2}(x + 1)$, (i) $\sin x^2$.

2 Integrate:
(a) $\sin 3x$, (b) $\cos 3x$, (c) $2 \sin 4x$,
(d) $2 \cos 2x$, (e) $-\frac{1}{2}\sin 6x$, (f) $6 \cos 4x$,
(g) $\sin(2x + 1)$, (h) $3 \cos(2x - 1)$, (i) $\frac{2}{3}\sin \frac{1}{2}x$.

3 Differentiate:
(a) $\sin^2 x$, (b) $4 \cos^2 x$, (c) $\cos^3 x$,

(d) $2\sin^3 x$, (e) $3\cos^4 x$, (f) $\sqrt{(\sin x)}$,
(g) $\sqrt{(\cos x)}$, (h) $\cos^2 3x$, (i) $\sin^2 2x$,
(j) $-2\sin^3 3x$, (k) $3\sin^4 2x$, (l) $\sqrt{(\sin 2x)}$.

4 Differentiate:
(a) $x\cos x$, (b) $x\sin 2x$, (c) $x^2\sin x$,

(d) $\sin x\cos x$, (e) $\dfrac{\sin x}{x}$, (f) $\dfrac{\cos 2x}{x}$,

(g) $\dfrac{x}{\sin x}$, (h) $\dfrac{x^2}{\cos x}$, (i) $\dfrac{\sin x}{\cos x}$,

(j) $\cot x$, (k) $\dfrac{1}{\cos x}$, (l) $\operatorname{cosec} x$.

5 A particle moves in a straight line such that its velocity in m/s, t s after passing through a fixed point O, is $3\cos t - 2\sin t$. Find
(a) its distance from O after $\frac{1}{2}\pi$ s,
(b) its acceleration after π s,
(c) the time when its velocity is first zero.

6 A particle is moving in a straight line in such a way that its distance from a fixed point O, t s after the motion begins, is $\cos t + \cos 2t$ cm. Find
(a) the time when the particle first passes through O,
(b) the velocity of the particle at this instant,
(c) the acceleration when the velocity is zero.

7 The distance of a particle from a fixed point O is given by

$$s = 3\cos 2t + 4\sin 2t$$

Show that the velocity v and the acceleration a are given by $v^2 + 4s^2 = 100$, $a + 4s = 0$. Hence find
(a) the greatest distance of the particle from O,
(b) the acceleration at this instant.

8 The velocity at time t of a particle moving in a straight line is $6\cos 2t + \cos t$, and when $t = 0$, the particle is at O. Find
(a) the time when v is first zero,
(b) the distance from O at this instant,
(c) the acceleration at the same instant.

9 Find the area between the curve $y = \sin 3x$ and the x-axis between $x = 0$ and $x = \frac{1}{3}\pi$.

10 Sketch the curve $y = 1 + \cos x$ from $x = -\pi$ to $x = \pi$, and find the area enclosed by the curve and the x-axis between these limits.

11 Find the maximum value of $y = x + \sin 2x$ which is given by a value of x between 0 and $\frac{1}{2}\pi$. Sketch the graph of y for $0 \leqslant x \leqslant \frac{1}{2}\pi$ and find the area bounded by the curve, the x-axis and the line $x = \frac{1}{2}\pi$.

12 Find the maximum value of $y = 2\sin x - x$ which is given by a value of x between 0 and $\frac{1}{2}\pi$. Sketch the graph of y for values of x from 0 to π, and find the area between the curve, the x-axis and the line $x = \frac{1}{2}\pi$.

13 Show that $\frac{d}{dx}(\frac{1}{2}x - \frac{1}{4}\sin 2x) = \sin^2 x$ and deduce that

$$\int_0^\pi \sin^2 x \, dx = \tfrac{1}{2}\pi$$

14 Express $\cos^2 x$ in terms of $\cos 2x$, and hence show that

$$\int \cos^2 x \, dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + c$$

15 Show that $\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$, and deduce that

$$\int \cos^3 x \, dx = \tfrac{1}{12}\sin 3x + \tfrac{3}{4}\sin x + c = \sin x - \tfrac{1}{3}\sin^3 x + c$$

16 By expressing $\sin^3 x$ in terms of $\sin x$ and $\sin 3x$, show that

$$\int \sin^3 x \, dx = \tfrac{1}{12}\cos 3x - \tfrac{3}{4}\cos x + c = \tfrac{1}{3}\cos^3 x - \cos x + c$$

17 Express $2\cos 5x\cos 3x$ as a sum of two cosines and hence evaluate

$$\int_0^{\pi/4} 2\cos 5x \cos 3x \, dx$$

Derivatives of tan *x*, cot *x*, sec *x*, cosec *x*

19.3 Using the derivatives of $\sin x$ and $\cos x$, those of the four other trigonometrical ratios can be obtained by writing

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x} \qquad \operatorname{cosec} x = \frac{1}{\sin x}$$

This is left as an exercise for the reader, if he or she has not already done No. 4 (i)–(l) of Exercise 19a. The results are

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

Note:

(a) the similarity of the pair of formulae on each line.

(b) the associations between $\tan x$ and $\sec x$, and between $\cot x$ and $\operatorname{cosec} x$. The same associations occur in the identities $1 + \tan^2 x = \sec^2 x$, $\cot^2 x + 1 = \operatorname{cosec}^2 x$.

(c) that the derivatives of ratios beginning with 'co', i.e. $\cos x$, $\cot x$, $\operatorname{cosec} x$, all have a negative sign.

Exercise 19b

1 Differentiate:

(a) $\tan 2x$, (b) $\cot 3x$, (c) $3 \sec 2x$,
(d) $2 \operatorname{cosec} \frac{1}{2}x$, (e) $-\tan(2x+1)$, (f) $\frac{1}{3}\sec(3x-2)$,
(g) $-2\cot(3x+2)$, (h) $\cot x^2$, (i) $\tan \sqrt{x}$.

2 Differentiate:

(a) $\tan^2 x$, (b) $\sec^2 x$, (c) $2\cot^3 x$,
(d) $3 \operatorname{cosec}^2 x$, (e) $-\tan^2 2x$,* (f) $\frac{1}{2}\cot^2 3x$,
(g) $\frac{1}{6}\sec^3 2x$, (h) $-2\operatorname{cosec}^4 x$, (i) $\sqrt{(\tan x)}$.

3 Differentiate:

(a) $x \tan x$, (b) $\sec x \tan x$, (c) $x^2 \cot x$,

(d) $3x \operatorname{cosec} x$, (e) $\operatorname{cosec} x \cot x$, (f) $\dfrac{\tan x}{x}$,

(g) $\dfrac{\sec x}{x^2}$, (h) $\sin x - x\cos x$, (i) $x\sec^2 x - \tan x$.

4 Integrate:

(a) $\sec^2 2x$, (b) $3\sec x \tan x$, (c) $-\operatorname{cosec}^2 \frac{1}{2}x$,

(d) $\frac{1}{3}\operatorname{cosec} 3x \cot 3x$, (e) $2\sec^2 x \tan x$, (f) $\dfrac{1}{\cos^2 x}$,

(g) $\dfrac{\sin x}{\cos^2 x}$, (h) $\dfrac{1}{\sin^2 2x}$, (i) $\dfrac{\cos 2x}{\sin^2 2x}$.

5 Sketch the graph of the curve $y = \sec^2 x - 1$ between $x = -\frac{1}{2}\pi$ and $x = \frac{1}{2}\pi$. Calculate the area enclosed by the curve, the x-axis and the line $x = \frac{1}{4}\pi$.

6 Find the volume generated by revolving the area bounded by the x-axis, the lines $x = \pm\frac{1}{4}\pi$ and the curve $y = \sec x$ about the x-axis.

7 Find the minimum values of the following functions which are given by values of x between 0 and $\frac{1}{2}\pi$:

(a) $\tan x + 3\cot x$, (b) $\sec x + 8\operatorname{cosec} x$, (c) $6\sec x + \cot x$.

8 By expressing $\tan^2 x$ in terms of $\sec^2 x$, show that

$$\int \tan^2 x \, dx = \tan x - x + c$$

9 Express $\cot^2 x$ in terms of $\operatorname{cosec}^2 x$ and hence integrate $\cot^2 x$.

Exercise 19c (Miscellaneous)

1 Convert to degrees:

(a) $\dfrac{2\pi}{5}$, (b) $\dfrac{5\pi}{6}$, (c) $\dfrac{3\pi}{8}$, (d) $\dfrac{7\pi}{12}$.

*The following method of working often overcomes the initial difficulty some students find with the chain rule:

$$\frac{d}{dx}(3\sin^4 5x) = \frac{d}{dx}\{3(\sin 5x)^4\} = 3 \times 4(\sin 5x)^3 \times \cos 5x \times 5 = 60\sin^3 5x \cos 5x.$$

2 Convert to radians, leaving π in your answers:
(a) 330°, (b) 50°, (c) 75°, (d) 24°.

3 Use tables or a calculator to find the values of
(a) sin 2 rad, (b) sec 0.5 rad, (c) tan 1.32 rad, (d) cos 2.98 rad.

4 The area of a sector of a circle, diameter 7 cm, is 18.375 cm². What is the length of the arc of the sector?

5 A sector with an area of $\frac{2}{3}$ cm² is bounded by an arc of length $\frac{5}{6}$ cm. What is the radius of the circle? Also find the angle contained by the sector, giving your answer in degrees.

6 A chord AB subtends a right angle at the centre of a circle of radius r. BC is a chord in the minor segment, inclined at 15° to BA. Show that the area bounded by the two chords and the arc AC is $\frac{1}{2}r^2(\frac{1}{6}\pi + \frac{1}{2}\sqrt{3} - 1)$.

7 The common chord of two circles of radii 13 cm and 37 cm is 24 cm long. Calculate the area common to both circles.

8 Draw, on the same diagram, the graphs of $y = x - 1$ and $y = \sin x$, where x is in radians, and $-\pi \leqslant x \leqslant +\pi$. Hence show that the equation

$$x = 1 + \sin x$$

has one root only. Estimate this root from your graph.

9 Draw the graph of $\cos 2\theta$ for values of θ from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$. Use your graph to solve the equation $\cos^2 \theta = \frac{1}{2}(1 + \theta)$.

10 Draw the graph of $\cos 3\theta + \cos \theta$ for values of θ from 0 to π, and find the roots of the equation

$$2 \cos 3\theta + 2 \cos \theta + 1 = 0$$

in this range.

11 A radar scanner rotates at a speed of 30 rev/min. Express this angular velocity in rad/s.

12 What is the angular velocity of the hour hand of a clock
(a) in rev/min, (b) in rad/s?

13 A wheel of diameter 3 m is rotating with an angular velocity of 420 rev/min. Find, taking π as 22/7,
(a) the angular velocity of the wheel in rad/s,
(b) the velocity of a point on the circumference in km/h.

14 A lift goes down a distance of 6 m in $3\frac{1}{2}$ s, and a cable to the counter-weight passes over a pulley of diameter 0.5 m. What is the average angular velocity of the pulley while the lift is in motion?

15 In order to investigate the effect of acceleration on the human body, a man is placed in a cabin which is made to travel in a circle of radius 10 m. If the speed of the cabin reaches 160 km/h, what is its angular velocity in rev/min at that instant?

16 Find approximations for the following when θ is small:

(a) $\dfrac{\sin \theta \tan \theta}{\theta^2}$, (b) $\dfrac{1 - \cos 2\theta}{\theta \sin 3\theta}$, (c) $\dfrac{\cos(\theta + \alpha) - \cos \alpha}{\theta}$.

17 Show that, if θ is small,

(a) $\sin(\frac{1}{6}\pi + \theta) \approx \frac{1}{2} + \frac{1}{2}\sqrt{3}\theta - \frac{1}{4}\theta^2$, (b) $\cos(\frac{1}{4}\pi + \theta) \approx \frac{1}{2}\sqrt{2}(1 - \theta - \frac{1}{2}\theta^2)$.

18 Differentiate:

(a) $\sin 3x$, (b) $\tan \frac{1}{2}x$, (c) $\cos x^2$,
(d) $\sqrt{(\cos x)}$, (e) $2 \operatorname{cosec}^3 x$, (f) $4 \sin^2 \frac{1}{2}x$,
(g) $-3 \sec^3 2x$, (h) $\sqrt{(\sin 2x)}$, (i) $3 \tan^2 2x$.

19 Integrate:

(a) $\cos 2x$, (b) $\sin(2x - 1)$, (c) $3 \cos \frac{1}{2}x$,

(d) $\sec^2 \frac{1}{2}x$, (e) $\operatorname{cosec} x \cot x$, (f) $\sec 2x \tan 2x$,

(g) $\dfrac{\cos x}{\sin^2 x}$, (h) $\dfrac{1}{\cos^2 2x}$, (i) $x \sin x^2$.

20 Differentiate:

(a) $x \sin x$, (b) $\sin x \cos 2x$, (c) $x^2 \tan^2 x$,

(d) $\dfrac{\sec x}{x}$, (e) $\dfrac{\cos 2x}{\sin 3x}$, (f) $\sin x \tan 2x$,

(g) $\dfrac{\sin x}{x^2}$, (h) $2 \cos x + 2x \sin x - x^2 \cos x$.

21 If $x = a \sec \theta$, $y = b \tan \theta$, show that

$$\frac{dy}{dx} = \frac{b}{a} \operatorname{cosec} \theta \quad \text{and} \quad \frac{d^2y}{dx^2} = -\frac{b}{a^2} \cot^3 \theta$$

22 If $x = a \cos \theta$, $y = b \sin \theta$, show that

$$\frac{d^2y}{dx^2} = -\frac{b}{a^2} \operatorname{cosec}^3 \theta$$

23 A particle travels in a straight line in such a way that its distance from a fixed point O after time t is $3 \cos 2t + 4 \sin 2t$. Find

(a) the distance of the particle from O when it first comes to rest instantaneously,
(b) its acceleration at this instant,
(c) its maximum velocity.

24 A particle is moving in a straight line with velocity $\sin 2t + 7 \sin t$ cm/s, t s after passing through a fixed point O. Find

(a) the maximum velocity of the particle,
(b) the greatest distance of the particle from O.

25 Evaluate:

(a) $\displaystyle\int_0^{\pi/2} \sin 2x \, dx$, (b) $\displaystyle\int_{-\pi/3}^{\pi/6} \sec^2 x \, dx$,

(c) $\displaystyle\int_0^{\pi} \sin^2 x \, dx$, (d) $\displaystyle\int_0^{\pi/4} \cos 3x \sin 5x \, dx$.

Chapter 20

Loci

Introduction

20.1 'Percy the goat is tethered to a fixed point O by a rope which is 6 m long. If Percy moves so that the rope is always taut, describe his path.' Readers will have little difficulty deciding that the goat moves around a circle, centre O, radius 6 m. (A scale drawing could be made, using a piece of string 6 cm long, fixed at one end by a drawing-pin and with a pencil at the other end; as the pencil moves, keeping the string taut, a circle can be drawn.)

Now consider this problem: 'Percy the goat is tethered by means of a ring which can slide freely on a rope which is 6 m long. The ends of the rope are attached to two fixed points A and B which are 4 m apart. Describe the goat's path.' In this case there are probably few readers who could give the path a name. However, a scale drawing could be made, using a piece of string 6 cm long with its ends attached to two drawing pins which are fixed, 4 cm apart. Use a pencil to trace the goat's path, being careful to keep the string taut. The diagram should look something like Fig. 20.1. Note that at all points on the path, $AP + PB = 6$.

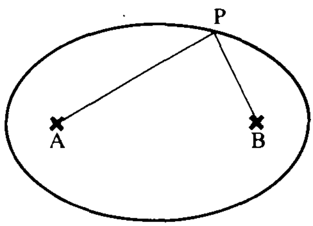


Figure 20.1

In this chapter, we shall use the techniques introduced earlier in the book to investigate problems like this. In particular, the moving point P will be represented by a point in the Cartesian plane with coordinates (x, y) (we shall only consider two-dimensional problems) and we shall endeavour to find an equation which expresses, algebraically, the conditions governing the motion of

P. (It is customary, in this context, to use P to represent the *moving* point; any *fixed* points are usually represented by A, B or C, although in many cases the origin O will be used as a fixed point.) The path traced out by the point P, as it moves according to the given conditions, is called the **locus** of P, and the equation satisfied by the coordinates of P is called the equation of the locus.

The equation of a locus

20.2 In the first of the introductory problems above, the given condition is $OP = 6$, so, if O is the origin, the equation of the locus can be obtained by applying Pythagoras' theorem in Fig. 20.2.

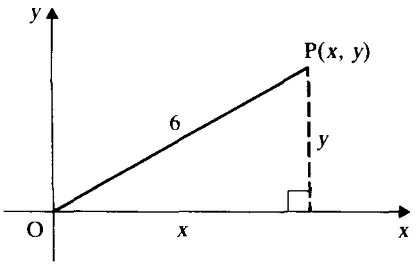


Figure 20.2

i.e. the equation of the locus is

$$x^2 + y^2 = 36$$

In the second problem, we shall take the two fixed points to be $(-2, 0)$ and $(2, 0)$, respectively (Fig. 20.3).

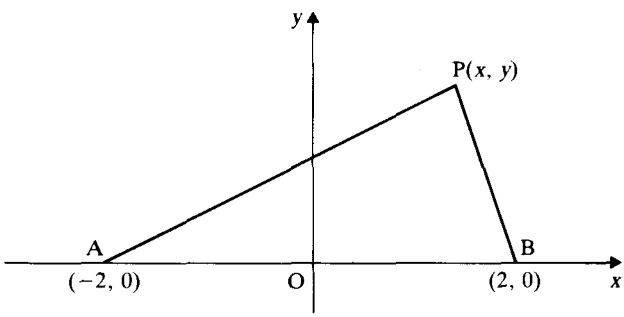


Figure 20.3

Applying the usual formula for the distance between two points we obtain

$$AP = \sqrt{\{(x+2)^2 + y^2\}} \quad \text{and} \quad BP = \sqrt{\{(x-2)^2 + y^2\}}$$

The condition which governs the movement of the point P is $AP + PB = 6$, so the equation of the locus is

$$\sqrt{\{(x+2)^2+y^2\}}+\sqrt{\{(x-2)^2+y^2\}}=6$$

Qu. 1 Show that when the equation above is simplified it can be expressed as

$$\frac{x^2}{9}+\frac{y^2}{5}=1$$

(We shall see in Book 2 that this is the equation of an ellipse.)

Example 1 *Find the equation of the locus of a point* P *which moves so that it is equidistant from two fixed points* A *and* B *whose coordinates are* (3, 2) *and* (5, − 1) *respectively*

Let P be the point (x, y). Expressed geometrically, the condition satisfied by P is

$$PA = PB$$

However, since we shall use Pythagoras' theorem to express the lengths of PA and PB in terms of x and y, it is neater to square this equation, obtaining

$$PA^2 = PB^2$$

Now

$$PA^2 = (x-3)^2 + (y-2)^2$$
$$PB^2 = (x-5)^2 + (y+1)^2$$

therefore the equation which must be satisfied by the coordinates of P is

$$(x-3)^2 + (y-2)^2 = (x-5)^2 + (y+1)^2$$

i.e. $x^2 - 6x + 9 + y^2 - 4y + 4 = x^2 - 10x + 25 + y^2 + 2y + 1$

Therefore the equation of the locus of points equidistant from (3, 2) and (5, − 1) is $4x - 6y - 13 = 0$.

The locus is actually the perpendicular bisector (or mediator) of AB. Because of the close connection between the locus and the equation connecting the points lying on the locus, the equation itself is often referred to as the locus.

Qu. 2 Find the equation of the locus in Example 1 by using the fact that it is the perpendicular bisector of AB.

Note. When drawing graphs it is often useful to take different scales on the two axes, but in coordinate geometry the scales must be the same or the figures will be distorted.

Example 2 *Find the locus of a point* P, *whose distance from the point* A(− 1, 2) *is twice its distance from the origin.*

Let $P(x, y)$ be a point on the locus (Fig. 20.4), then

$$PA = 2PO$$

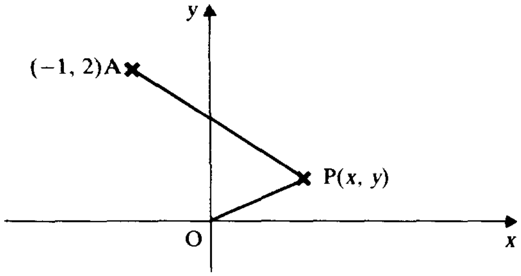


Figure 20.4

The lengths of PA and PO may be written down by the method of §1.2, but as both expressions involve a square root, it is neater to square first, giving

$$PA^2 = 4PO^2$$
$$\therefore (x+1)^2 + (y-2)^2 = 4(x^2 + y^2)$$
$$\therefore x^2 + 2x + 1 + y^2 - 4y + 4 = 4x^2 + 4y^2$$

Therefore the locus of P is $3x^2 + 3y^2 - 2x + 4y - 5 = 0$.

Exercise 20a

1 Find the equation of a circle with centre at the origin and radius 5 units.

2 What is the locus of a point which moves so that its distance from the point (3, 1) is 2 units?

3 What is the locus of a point which is equidistant from the origin and the point $(-2, 5)$?

4 What is the locus of a point which moves so that its distance from the point $(-2, 1)$ is equal to its distance from the point $(3, -2)$?

5 What is the distance of the point (x, y) from the line $x = -1$? Find the locus of a point which is equidistant from the origin and the line $x = -1$.

6 Find the locus of a point which is equidistant from the point (0, 1) and the line $y = -1$.

7 Find the locus of a point which moves so that its distance from the point $A(-2, 0)$ is three times its distance from the origin.

8 A point P moves so that its distance from A(2, 1) is twice its distance from $B(-4, 5)$. What is the locus of P?

9 Find the locus of a point which moves so that its distance from the point (8, 0) is twice its distance from the line $x = 2$.

10 Find the locus of a point which moves so that its distance from the point (2, 0) is half its distance from the line $x = 8$.

11 Find the locus of a point which moves so that the sum of the squares of its distances from the points $(-2, 0)$ and (2, 0) is 26 units.

12 Find the locus of a point which moves so that it is equidistant from the point $(a, 0)$ and the line $x = -a$.

13 A is the point (1, 0), and B is the point $(-1, 0)$. Find the locus of a point P which moves so that $PA + PB = 4$.

14 A is the point $(1, 0)$, and B is the point $(-1, 0)$. Find the locus of a point P which moves so that $PA - PB = 2$.

15 A rectangle is formed by the axes and the lines $x = 4$ and $y = 6$. Find the locus of a point which moves so that the sum of the squares of its distances from the axes is equal to the sum of the squares of its distances from the other two sides.

Further examples

20.3 Example 3 *Show that the equation of the circle on the line segment joining* A$(3, -5)$ *and* B$(2, 6)$ *as diameter is* $(x-3)(x-2)+(y+5)(y-6)=0$.

Let $P(x, y)$ be any point on the circle. The vector $\overrightarrow{AP}$ is perpendicular to the vector $\overrightarrow{BP}$, and hence the scalar product $\overrightarrow{AP}.\overrightarrow{BP}$ is zero (see §15.16).

Now, $\overrightarrow{AP} = (x-3)\mathbf{i} + (y+5)\mathbf{j}$ and $\overrightarrow{BP} = (x-2)\mathbf{i} + (y-6)\mathbf{j}$, so

$$\begin{aligned}\overrightarrow{AP}.\overrightarrow{BP} &= \{(x-3)\mathbf{i} + (y+5)\mathbf{j}\}.\{(x-2)\mathbf{i} + (y-6)\mathbf{j}\}\\ &= (x-3)(x-2) + (y+5)(y-6)\end{aligned}$$

But this scalar product is zero, so the equation of the circle is

$$(x-3)(x-2) + (y+5)(y-6) = 0$$

Qu. 3 Show that the equation in Example 3 may also be found by using the result that the product of the gradients of two perpendicular lines is -1.

Example 4 *A variable point* P *moves on the curve* $y^2 = 4x$ *and* A *is the point* $(1, 0)$. *Find the locus of the mid-point of* AP.

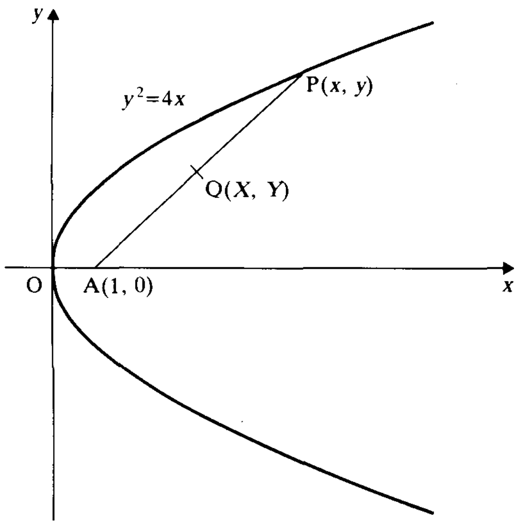


Figure 20.5

Let P be the point (x, y), and let $Q(X, Y)$ be the mid-point of AP (Fig. 20.5). Then the coordinates of Q are given by

$$X = \frac{x+1}{2} \quad \text{and} \quad Y = \frac{y}{2}$$

Since P lies on the given curve, we have

$$y^2 = 4x$$

but $x = 2X - 1$ and $y = 2Y$, therefore

$$4Y^2 = 4(2X - 1)$$

Therefore the locus of the mid-point of AP is $y^2 = 2x - 1$.

Example 5 *A straight line* AB *of length* 10 *units is free to move with its ends on the axes. Find the locus of a point* P *on the line at a distance of* 3 *units from the end on the x-axis.*

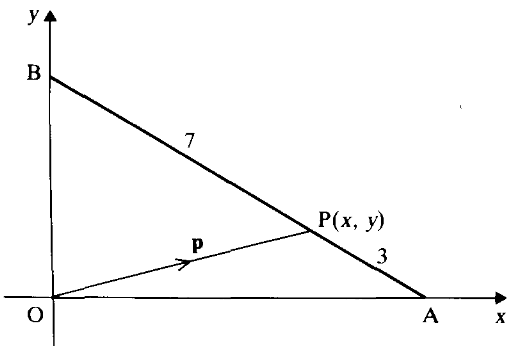


Figure 20.6

Let A be the point $(X, 0)$ and B the point $(0, Y)$ and note that, by Pythagoras' theorem (Fig. 20.6),

$$OA^2 + OB^2 = AB^2$$

and therefore

$$X^2 + Y^2 = 100 \tag{1}$$

Also, let the coordinates of the point P be (x, y). We are given that BP:PA = 7:3, and so **p**, the position vector of the point P, is given by

$$\mathbf{p} = \tfrac{7}{10}\mathbf{a} + \tfrac{3}{10}\mathbf{b}$$

where **a** and **b** are the position vectors of the points A and B (see the ratio theorem, §15.8), hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{7}{10}\begin{pmatrix} X \\ 0 \end{pmatrix} + \frac{3}{10}\begin{pmatrix} 0 \\ Y \end{pmatrix}$$

so, $x=\frac{7}{10}X$ and $y=\frac{3}{10}Y$. From these equations we see that

$$X=\frac{10}{7}x \quad \text{and} \quad Y=\frac{10}{3}y$$

Substituting these into equation (1) we obtain

$$\frac{100}{49}x^2+\frac{100}{9}y^2=100$$

and hence the equation we require is

$$\frac{x^2}{49}+\frac{y^2}{9}=1$$

(We shall see in Book 2 that this is the equation of an ellipse.)

Exercise 20b

1 Find the equations of the circles on the diameters whose ends are
(a) $(-3, 2)$ and $(4, -5)$; (b) $(\frac{1}{2}, 1)$ and $(-\frac{3}{2}, 4)$;
(c) $(0, a)$ and $(a, 0)$; (d) (x_1, y_1) and (x_2, y_2).

2 P is a point on a line of length 12 units, which moves so that its ends lie on the axes. Find the locus of P when it is
(a) the mid-point of the line,
(b) the point of trisection of the line nearer to the y-axis.

3 L and M are the feet of perpendiculars from a point P on to the axes. Find the locus of P when it moves so that LM is of length 4 units.

4 A variable line through the point (3, 4) cuts the axes at Q and R, and the perpendiculars to the axes at Q and R intersect at P. What is the locus of the point P?

5 A variable point P lies on the curve $xy=12$. Q is the mid-point of the line joining P to theoigin. Find the locus of Q.

6 P is a variable point on the curve $y=2x^2+3$, and O is the origin. Q is the point of trisection of OP nearer the origin. Find the locus of Q.

7 A line parallel to the x-axis cuts the curve $y^2=4x$ at P and the line $x=-1$ at Q. Find the locus of the mid-point of PQ.

8 Variable lines through the points O(0, 0) and A(2, 0) intersect at right angles at the point P. Show that the locus of the mid-point of OP is $y^2+x(x-1)=0$.

9 Find the locus of a point which moves so that the sum of the squares of its distances from the lines $x+y=0$ and $x-y=0$ is 4.

10 A is the point (1, 0), B is the point (2, 0) and O is the origin. A point P moves so that angle BPO is a right angle, and Q is the mid-point of AP. What is the locus of Q?

11 A line parallel to the y-axis meets the curve $y=x^2$ at P and the line $y=x+2$ at Q. Find the locus of the mid-point of PQ.

12 M is a variable point on the x-axis, and A is the point (2, 3). A line through A, perpendicular to AM, meets the y-axis at N. Perpendiculars to the axes at M and N meet at P. Find the locus of the point P.

13 M and N are points on the axes, and the line MN passes through the point (3, 2). P is a variable point which moves so that the mid-point of the line joining P to the origin is the mid-point of MN. Find the locus of the point P.

14 A straight line LM, of length 4 units, moves with L on the line $y = x$ and M on the x-axis. Find the locus of the mid-point of LM.

15 A straight line LM meets the x-axis in M and the line $y = x$ in L, and passes through the point (6, 4). What is the locus of the mid-point of LM?

Tangents and normals

20.4 If a tangent touches a curve at the point P, the line through P perpendicular to the tangent is called a **normal**. (See §3.9.)

Example 6 *Find the equations of the tangent and normal to the curve* $y = 3x^2 - 8x + 5$, *at the point where* $x = 2$.

[The equation of a line can be found from its gradient and the coordinates of a point through which it passes. Therefore we begin by finding these.]

$$y = 3x^2 - 8x + 5$$

Therefore the gradient of the tangent, $\dfrac{dy}{dx}$, is given by

$$\frac{dy}{dx} = 6x - 8$$

At the point of contact $x = 2$, and so

$$\frac{dy}{dx} = 6 \times 2 - 8 = 4$$

The y-coordinate of the point of contact may be found by substituting $x = 2$ in the equation of the curve:

$$y = 3 \times 2^2 - 8 \times 2 + 5 = 1$$

Therefore the coordinates of the point of contact are (2, 1).

Using the equation of a line in the form

$$y - y_1 = m(x - x_1)$$

the equation of the tangent is

$$y - 1 = 4(x - 2)$$

i.e. $4x - y - 7 = 0$

The normal is perpendicular to the tangent, and so its gradient is $-\frac{1}{4}$. Therefore its equation may be written

$$y - 1 = -\tfrac{1}{4}(x - 2)$$

i.e. $x + 4y - 6 = 0$

Thus the equations of the tangent and normal to the curve $y = 3x^2 - 8x + 5$ at the point (2, 1) are respectively $4x - y - 7 = 0$ and $x + 4y - 6 = 0$.

Note. It should be emphasised that, when the equation of the tangent was found, the gradient of the curve *at* (2, 1) was used. If we had taken the gradient to be $6x - 8$, the equation $y - 1 = (6x - 8)(x - 2)$ would not have represented a straight line.

Example 7 *Find the equations of the tangents to the curve $xy = 6$ which are parallel to the line $2y + 3x = 0$.*

The gradient of the line $2y + 3x = 0$ is $-\frac{3}{2}$. Therefore we must find at what points on the curve $xy = 6$ the gradient is $-\frac{3}{2}$.

$$y = \frac{6}{x}$$

$$\therefore \frac{dy}{dx} = -\frac{6}{x^2}$$

If $\dfrac{dy}{dx} = -3/2$,

$$-\frac{6}{x^2} = -\frac{3}{2}$$

$$\therefore 3x^2 = 12, \quad \text{and so} \quad x^2 = 4$$

$$\therefore x = \pm 2$$

When $x = 2$, $y = \frac{6}{2} = 3$; and when $x = -2$, $y = -\frac{6}{2} = -3$. Thus the gradient of the curve is $-\frac{3}{2}$ at the points (2, 3) and $(-2, -3)$.

The equations of the tangents may be found from the form $y - y_1 = m(x - x_1)$:

$$y - 3 = -\tfrac{3}{2}(x - 2) \quad \text{and} \quad y + 3 = -\tfrac{3}{2}(x + 2)$$

Therefore the equations of the tangents to the curve $xy = 6$ which are parallel to the line $2y + 3x = 0$ are $3x + 2y - 12 = 0$ and $3x + 2y + 12 = 0$.

Sometimes questions about tangents may be solved without using the calculus. Fig. 20.7 shows a curve with a chord PQ passing through a fixed point P and a variable point Q. When P and Q are distinct, we must obtain distinct roots when the equations of the curve and PQ are solved simultaneously; and when P and Q coincide, producing a tangent, there will be a repeated root.

Example 8 *Show that if the line $y = mx + c$ is a tangent to the curve $4x^2 + 3y^2 = 12$, then $c^2 = 3m^2 + 4$.*

[If the line $y = mx + c$ is a tangent, then the point of contact must be given by an equation with a repeated root.]

Substituting $y = mx + c$ in the equation $4x^2 + 3y^2 = 12$, we obtain

$$4x^2 + 3(mx + c)^2 = 12$$

$$\therefore 4x^2 + 3m^2x^2 + 6mxc + 3c^2 = 12$$

$$\therefore (4 + 3m^2)x^2 + 6mcx + 3c^2 - 12 = 0$$

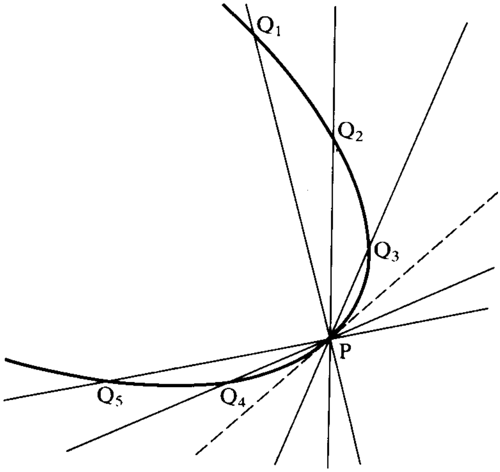


Figure 20.7

Now if the equation $ax^2 + bx + c = 0$ has equal roots then $b^2 = 4ac$ (see §10.2). Therefore if $y = mx + c$ is a tangent,

$$36m^2c^2 = 4(4 + 3m^2)(3c^2 - 12)$$
$$\therefore 9m^2c^2 = 12c^2 - 48 + 9m^2c^2 - 36m^2$$
$$\therefore 12c^2 = 36m^2 + 48$$

Therefore if $y = mx + c$ is a tangent to the curve $4x^2 + 3y^2 = 12$, then $c^2 = 3m^2 + 4$.

This means that the line $y = mx \pm \sqrt{(3m^2 + 4)}$ will touch the curve for all values of m. Hence we may find the tangents parallel to $y = 2x$ by substituting $m = 2$, which gives $y = 2x \pm 4$.

Qu. 4 Find the equations of the tangents to the curve $4x^2 + 3y^2 = 12$ which are (a) parallel to $y = x$, (b) inclined at 60° to the x-axis.

Qu. 5 Solve the following pairs of simultaneous equations:
(a) $y = x$, $y^2 = x^3 + x^2$; (b) $y = 2x$, $y^2 = x^3 + x^2$.
What is the significance of the repeated root in each case?

Exercise 20c

1 Find the equations of the tangents and normals to the following curves at the points indicated:
(a) $y = x^2$, $(2, 4)$;
(b) $y = 3x^2 - 2x + 1$, where $x = 1$;
(c) $y = x + 1/x$, $(-1, -2)$;
(d) $y^2 = 4x$, $(1, -2)$;
(e) $y = x^2 - 2x$, where $x = -2$;
(f) $xy = 4$, where $y = 2$;
(g) $y^3 = x^2$, $(1, 1)$.

2 Show that the following lines touch the given curves and find the coordinates of the points of contact:
(a) $y^2 = 8x$, $y - 2x - 1 = 0$;
(b) $x^2 + y^2 = 8$, $x - y - 4 = 0$;

(c) $xy = 4$, $x + 9y - 12 = 0$; (d) $9x^2 - 4y^2 = 36$, $5x - 2y + 8 = 0$.

3 At what points does the parabola $y = x^2 - 4x + 3$ cut the x-axis? Find the equations of the tangents and normals at these points.

4 Find the equations of the tangents at the points of intersection of the line $y = x + 1$ and the parabola $y = x^2 - x - 2$.

5 Find the equations of the normals to the curve $y = x^2 - 1$ at the points where it cuts the x-axis. What are the coordinates of the point of intersection of these normals?

6 Find the coordinates of the points of intersection of the parabolas $y^2 = x$ and $x^2 = y$. What are the equations of the tangents to the curves at these points?

7 What is the equation of the normal to the curve $y = x^2 - 4x - 12$ at the point where it cuts the y-axis? Where does this normal meet the x-axis?

8 Find the equations of the tangents to the curve $y = x^3 - 3x^2$ which are parallel to the line $y = 9x$.

9 Find the equations of the tangents to the hyperbola $xy = 4$, which are inclined at 135° to the x-axis.

10 Show that the equation of the tangent to the parabola $y = x^2$ at the point (h, k) may be written $y - 2hx + h^2 = 0$. Find the values of h for which the tangent passes through the point $(1, 0)$, and obtain the equations of these tangents.

11 Show that the equation of the tangent to the rectangular hyperbola $xy = c^2$ at the point (h, k) may be written $xk + yh - 2c^2 = 0$. Find the equation of the tangent which passes through the point $(0, c)$.

12 Show that, if the line $y = x + c$ is a tangent to the circle $x^2 + y^2 = 4$, then $c^2 = 8$.

13 Prove that the condition that the line $y = mx + c$ should touch the ellipse $x^2 + 4y^2 = 4$ is $c^2 = 4m^2 + 1$. Hence find the equations of the tangents to the ellipse which are parallel to the line $3x - 8y = 0$.

14 Show that the line $y = mx + c$ touches the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ if $c^2 = a^2m^2 - b^2$. Hence find the equations of the tangents to the hyperbola $9x^2 - 25y^2 = 225$ which are parallel to the line $x - y = 0$.

15 Find the condition that the line $lx + my + n = 0$ should touch the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

Exercise 20d (Miscellaneous)

1 Find the locus of a point which is equidistant from the points $(4, -1)$ and $(3, 7)$.

2 Find the locus of a point which is equidistant from the y-axis and the point $(4, 0)$.

3 A point P moves so that its distance from the point $(5, 0)$ is half its distance from the line $x - 8 = 0$. Find the locus of P.

4 Find the locus of a point which moves so that its distance from the origin is three times its distance from the line $x = a$.

5 Find the locus of a point which moves so that its distance from $(2, 0)$ is twice its distance from $(-1, 0)$. Show that a point P, which moves so that the sum of the squares of the distances from P to the origin and the point $(-4, 0)$ is 16, describes the same locus.

6 If A is the point $(2, 0)$ and B is $(-3, 0)$, find the locus of a point P which moves so that $AP^2 + 2BP^2 = 22$.

7 Find the equation of the circle on the line joining (a, b) to (c, d) as diameter.

8 A straight line of length 24 units moves with its ends on the axes. Find the locus of a point on the line which is
(a) 12 units from the end on the x-axis,
(b) 6 units from the end on the x-axis.

9 A straight line of length 6 units moves with its ends A and B on the axes. Perpendiculars to the axes, erected at the points A and B, meet at P. Find the locus of P.

10 A and B are points on the x- and y-axes, and P is the mid-point of AB. Find the locus of P if the area of triangle AOB is 8 units.

11 A variable line through the point (a, b) cuts the axes at L and M, and the perpendiculars to the axes at L and M meet at P. What is the locus of P?

12 P is a variable point on the curve $4x^2 + y^2 = 36$ and A is the point $(1, 0)$. Find the locus of the mid-point of AP.

13 Find the gradient of the curve $y = 9x - x^2$ at the point where $x = 1$. Find the equation of the tangent to the curve at this point. Where does this tangent meet the line $x = y$?

14 Find the equation of the normal to the parabola $y = \frac{1}{4}x^2$ at the point $(4, 4)$. Find also the coordinates of the point at which this normal meets the parabola again, and show that the length of the chord so formed is $5\sqrt{5}$.

15 Find the equations of the tangents to the rectangular hyperbola $xy = 4$ at the points $(2, 2)$, $(6, \frac{2}{3})$. Show that they intersect on the line $3y = x$.

16 Find the gradient of the curve $y = 4x^2 - 7x + 5$ at each of the points where it is cut by the line $y = 2$. Find the equations of the tangents at these points and show that they meet on the line $15x = 7y$.

17 Find the equation of the normal to the parabola $y = \frac{1}{4}x^2$ which is parallel to $y = 3x$, and find the coordinates of the point on the parabola at which it is the normal.

18 Prove that the line $y = mx + a/m$ touches the parabola $y^2 = 4ax$. Find the equation of the tangent to the parabola $y^2 = 2x$ which is perpendicular to the straight line $2y + 7x = 4$.

19 The gradient of a curve at the point (x, y) is $1 - 2/x^2$. Find the equation of the curve if it passes through the point $(2, 4)$.

Find the point of contact of the tangent which is parallel to the tangent at $(2, 4)$; also find the equations of both these tangents.

20 Show that the line $y = mx + c$ touches the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

if $c^2 = a^2m^2 + b^2$.

Find the equations of the tangents to the ellipse $4x^2 + 9y^2 = 1$ which are perpendicular to $y = 2x + 3$.

Chapter 21

The circle

The equation of a circle

21.1 The work of previous chapters will now be applied to the circle, and we begin by obtaining the equation of a circle, radius r, with its centre at the origin.

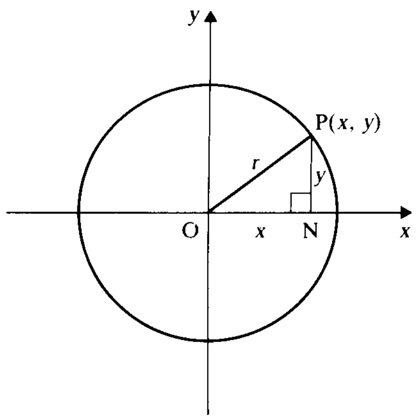


Figure 21.1

We require an equation connecting the coordinates (x, y) of any point P on the circle (see Fig. 21.1). Let N be the foot of the perpendicular from P to the x-axis, so that $ON = x$ and $NP = y$.

Then by Pythagoras' theorem,

$$ON^2 + NP^2 = r^2$$
$$\therefore x^2 + y^2 = r^2$$

Therefore the equation of the circle, radius r, with its centre at the origin is

$$\boldsymbol{x^2 + y^2 = r^2}$$

This is the simplest form in which the equation of a circle can be written, but now, to be quite general, consider the circle, radius r, whose centre is at the point $C(a, b)$.

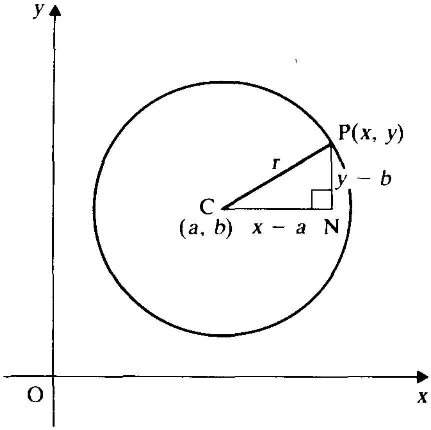


Figure 21.2

Let P(x, y) be any point on the circle, and draw CN and NP parallel to the x- and y-axes, as shown in Fig. 21.2.

Now CN $= x - a$ and NP $= y - b$; but by Pythagoras' theorem in triangle CNP,

$$\mathrm{CN}^2 + \mathrm{NP}^2 = \mathrm{CP}^2$$
$$\therefore (x - a)^2 + (y - b)^2 = r^2$$

Therefore the equation of the circle, radius r, whose centre is at (a, b) is

$$\boldsymbol{(x - a)^2 + (y - b)^2 = r^2}$$

Using this result, the equation of the circle with centre at (4, −1) and radius 2 may be written

$$(x - 4)^2 + (y + 1)^2 = 2^2$$

Expanding the squares:

$$x^2 - 8x + 16 + y^2 + 2y + 1 = 4$$

Collecting the terms:

$$x^2 + y^2 - 8x + 2y + 13 = 0$$

The equation of a circle is usually given in this form. Note that
(a) the coefficients of x^2 and y^2 are equal,
(b) the only other terms are linear (such as may occur in the equation of a straight line).

Qu. 1 Express the equation $(x - a)^2 + (y - b)^2 = r^2$ in the form

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Write down g, f, c, in terms of a, b, r.

Example 1 *Find the radius and the coordinates of the centre of the circle* $2x^2 + 2y^2 - 8x + 5y + 10 = 0$.

[We may find the centre and radius if the equation is expressed in the form $(x-a)^2 + (y-b)^2 = r^2$.]

Divide both sides of the equation of the circle

$$2x^2 + 2y^2 - 8x + 5y + 10 = 0$$

by 2, in order to make the coefficients of x^2 and y^2 equal to 1:

$$x^2 + y^2 - 4x + \tfrac{5}{2}y + 5 = 0$$

Rearrange the terms, grouping those in x and y:

$$x^2 - 4x \quad + \quad y^2 + \tfrac{5}{2}y \qquad = -5$$

Complete the squares (see Appendix, Exercise 5):

$$x^2 - 4x + 4 + y^2 + \tfrac{5}{2}y + (\tfrac{5}{4})^2 = -5 + 4 + \tfrac{25}{16}$$
$$\therefore (x-2)^2 + (y+\tfrac{5}{4})^2 = \tfrac{9}{16}$$
$$\therefore (x-2)^2 + (y+\tfrac{5}{4})^2 = (\tfrac{3}{4})^2$$

Comparing this with the equation of the circle, radius r, centre (a, b):

$$(x-a)^2 + (y-b)^2 = r^2$$

we obtain

$$a = 2, \quad b = -\tfrac{5}{4}, \quad r = \tfrac{3}{4}$$

Therefore the radius is $\tfrac{3}{4}$ and the centre is at the point $(2, -\tfrac{5}{4})$.

Example 2 *Find the equations of the circles which pass through the points* A(0, 2) *and* B(0, 8), *and which touch the x-axis.*

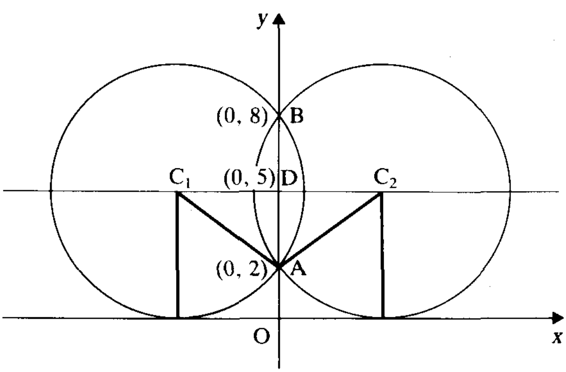


Figure 21.3

Fig. 21.3 suggests a method. The centre of the circle must lie on the perpendicular bisector of the chord AB, i.e. on the line $y = 5$.

Now the circle touches the x-axis, therefore the radius is 5.

If D is the point (0, 5) and C is the centre of either circle, then triangle ADC is right-angled and DC = 4 by Pythagoras' theorem. Therefore the centres of the circles are $(-4, 5)$ and $(4, 5)$ and so their equations are

$$(x \pm 4)^2 + (y - 5)^2 = 5^2$$

Therefore the equations of the circles are $x^2 + y^2 \pm 8x - 10y + 16 = 0$.

Exercise 21a

1 Find the equations of the circles with the following centres and radii:
(a) centre (2, 3), radius 1; (b) centre $(-3, 4)$, radius 5;
(c) centre $(\frac{2}{3}, -\frac{1}{3})$, radius $\frac{2}{3}$; (d) centre $(0, -5)$, radius 5;
(e) centre (3, 0), radius $\sqrt{2}$; (f) centre $(-\frac{1}{4}, \frac{1}{3})$, radius $\frac{1}{2}\sqrt{2}$.

2 Find the radii and the coordinates of the centres of the following circles:
(a) $x^2 + y^2 + 4x - 6y + 12 = 0$, (b) $x^2 + y^2 - 2x - 4y + 1 = 0$,
(c) $x^2 + y^2 - 3x = 0$, (d) $x^2 + y^2 + 3x - 4y - 6 = 0$,
(e) $2x^2 + 2y^2 + x + y = 0$, (f) $36x^2 + 36y^2 - 24x - 36y - 23 = 0$,
(g) $x^2 + y^2 - 2ax - 2by = 0$, (h) $x^2 + y^2 + 2gx + 2fy + c = 0$.

3 Which of the following equations represent circles?
(a) $x^2 + y^2 - 5 = 0$, (b) $x^2 + y^2 + 10 = 0$,
(c) $3x^2 + 2y^2 + 6x - 8y + 100 = 0$, (d) $ax^2 + ay^2 = 1$,
(e) $x^2 + y^2 + 8x + xy + 4 = 0$, (f) $x^2 + y^2 + bxy = 1$,
(g) $x^2 + y^2 + c = 0$, (h) $x^2 + dy^2 - 8x + 10y + 50 = 0$.
Which of them can represent circles if suitable values are given to the constants a, b, c, d?

4 Find the equation of the circle whose centre is at the point (2, 1) and which passes through the point $(4, -3)$.

5 The points (8, 4) and (2, 2) are the ends of a diameter of a circle. Find the coordinates of the centre, and the radius. Deduce the equation of the circle.

6 What is the equation of the circle, centre $(2, -3)$, which touches the x-axis?

7 Find the radii of the two circles, with centres at the origin, which touch the circle $x^2 + y^2 - 8x - 6y + 24 = 0$.

8 Show that the distance of the centre of the circle $x^2 + y^2 - 6x - 4y + 4 = 0$ from the y-axis is equal to the radius. What does this prove about the y-axis and the circle?

9 Find the equations of the circles which touch the x-axis, have radius 5, and pass through the point (0, 8).

10 What is the equation of the circle whose centre lies on the line $x - 2y + 2 = 0$, and which touches the positive axes?

11 A circle passes through the points $A(-5, 2)$, $B(-3, -4)$, $C(1, 8)$. Find the point of intersection of the perpendicular bisectors of AB and BC. What is the equation of the circle?

12 The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ passes through the points $A(-1, -2)$, $B(1, 2)$, $C(2, 3)$. Write down three equations which must be satisfied by g, f, c. Solve these equations and write down the equation of the circle ABC.

13 Find the equations of the circles which pass through
(a) the origin, $(-1, 3)$, $(-4, 2)$;
(b) $(3, 1)$, $(8, 2)$, $(2, 6)$;
(c) $(6, -5)$, $(2, -7)$, $(-6, -1)$.

14 A point moves so that its distance from the origin is twice its distance from the point $(3, 0)$. Show that the locus is a circle, and find its centre and radius.

15 A is the point $(3, -1)$, and B is the point $(5, 3)$. Show that the locus of a point P, which moves so that $PA^2 + PB^2 = 28$, is a circle. Find its centre and radius.

Tangents to a circle

21.2 Elementary geometry will frequently help to simplify working in co-ordinate geometry, as the reader may have found in the last exercise. It provides a simple way of obtaining the equation of a tangent at a given point on a given circle, using the fact that a tangent is perpendicular to the radius through the point of contact. This method will be employed in the next example.

Example 3 *Verify that the point* $(3, 2)$ *lies on the circle* $x^2+y^2-8x+2y+7=0$, *and find the equation of the tangent at this point.*

Substituting the coordinates $(3, 2)$ into the equation $x^2+y^2-8x+2y+7=0$,

$$\text{L.H.S.} = 9 + 4 - 24 + 4 + 7 = 0 = \text{R.H.S.}$$

Therefore $(3, 2)$ lies on the circle.

[The gradient of the tangent can be found from the gradient of the radius through $(3, 2)$; and, in order to find this, we obtain the coordinates of the centre of the circle.]

The equation of the circle may be written

$$x^2 - 8x \qquad + y^2 + 2y \qquad = -7$$
$$\therefore x^2 - 8x + 16 + y^2 + 2y + 1 = -7 + 16 + 1$$
$$\therefore (x-4)^2 + (y+1)^2 = 10$$

Therefore the centre of the circle is $(4, -1)$. Hence the gradient of the radius through $(3, 2)$ is $(-1-2)/(4-3) = -3$.

Therefore the gradient of the tangent is $\frac{1}{3}$. Using the formula $y - y_1 = m(x - x_1)$, the equation of the tangent at $(3, 2)$ is

$$y - 2 = \tfrac{1}{3}(x-3)$$
$$\therefore 3y - 6 = x - 3$$

Therefore the equation of the tangent to the circle at $(3, 2)$ is $x - 3y + 3 = 0$.

Example 4 *Find the length of the tangents from the point* $(5, 7)$ *to the circle* $x^2 + y^2 - 4x - 6y + 9 = 0$.

[Fig. 21.4 suggests a method. The tangent is perpendicular to the radius through the point of contact, so t can be found by Pythagoras' theorem if d and r are known.]

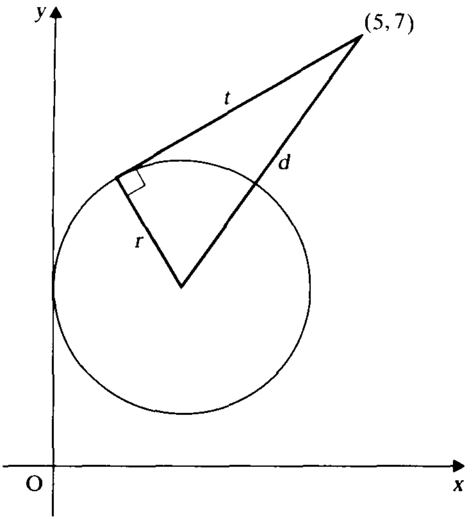


Figure 21.4

In Fig. 21.4, the radius, length r, is perpendicular to the tangent, length t, from the point (5, 7). If the distance of (5, 7) from the centre of the circle is d, then by Pythagoras' theorem $d^2 = t^2 + r^2$, or

$$t^2 = d^2 - r^2$$

To find the coordinates of the centre of the circle $x^2 + y^2 - 4x - 6y + 9 = 0$:

$$x^2 - 4x + 4 + y^2 - 6y + 9 = 4$$
$$\therefore (x-2)^2 + (y-3)^2 = 2^2$$

Therefore the centre is (2, 3) and the radius is 2.

Now, by Pythagoras' theorem,

$$d^2 = (5-2)^2 + (7-3)^2 = 9 + 16 = 25$$

But $r^2 = 4$,

$$\therefore t^2 = 25 - 4 = 21$$

Therefore the length of the tangents from (5, 7) to the circle is $\sqrt{21}$.

Qu. 2 Calculate the lengths of the tangents to the circle in Example 4 from (a) (4, 3), (b) (2, 2). What do you conclude from these results? If in doubt, mark these points in a figure containing the circle.

Exercise 21b

1 Verify that the given points lie on the following circles and find the equations of the tangents to the circles at these points:
(a) $x^2 + y^2 + 6x - 2y = 0$, (0, 0);
(b) $x^2 + y^2 - 8x - 2y = 0$, (3, 5);

(c) $x^2 + y^2 + 2x + 4y - 12 = 0$, $(3, -1)$;
(d) $x^2 + y^2 + 2x - 2y - 8 = 0$, $(2, 2)$;
(e) $2x^2 + 2y^2 - 8x - 5y - 1 = 0$, $(1, -1)$.

2 Find the lengths of the tangents from the given points to the following circles:
(a) $x^2 + y^2 + 4x - 6y + 10 = 0$, $(0, 0)$;
(b) $x^2 + y^2 - 4x - 8y - 5 = 0$, $(8, 2)$;
(c) $x^2 + y^2 + 6x + 10y - 2 = 0$, $(-2, 3)$;
(d) $x^2 + y^2 - 10x + 8y + 5 = 0$, $(5, 4)$;
(e) $x^2 + y^2 = a^2$, (x_1, y_1);
(f) $x^2 + y^2 + 2gx + 2fy + c = 0$, $(0, 0)$.

3 The tangent to the circle $x^2 + y^2 - 2x - 6y + 5 = 0$ at the point (3, 4) meets the x-axis at M. Find the distance of M from the centre of the circle.

4 Find the equations of the tangents to the circle $x^2 + y^2 - 6x + 4y + 5 = 0$ at the points where it meets the x-axis.

5 The tangent to the circle $x^2 + y^2 - 4x + 6y - 77 = 0$ at the point (5, 6) meets the axes at A and B. Find the coordinates of A and B. Deduce the area of triangle AOB.

6 Find the length of the tangents from the origin to the circle

$$x^2 + y^2 - 10x + 2y + 13 = 0$$

Use this answer to show that these two tangents and the radii through the points of contact form a square.

7 Find the length of the tangents to the circle $x^2 + y^2 - 4 = 0$ from the point P(X, Y); and deduce the equation of the locus of P, when it moves so that the length of the tangents to the circle is equal to the distance of P from the point (1, 0).

8 Show that the length of the tangents to the circle $x^2 + y^2 - 4x - 6y + 12 = 0$ from the point P(X, Y) is $\sqrt{(X^2 + Y^2 - 4X - 6Y + 12)}$. Find the locus of P when it moves so that the length of the tangents to the circle is equal to its distance from the origin.

9 Show that the point (x_1, y_1) is outside, on or inside the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

according as to whether ${x_1}^2 + {y_1}^2 + 2gx_1 + 2fy_1 + c$ is positive, zero or negative.

10 Prove that the line $x - y - 3 = 0$ is a common tangent to the circles

$$x^2 + y^2 - 2x - 4y - 3 = 0 \quad \text{and} \quad x^2 + y^2 + 4x - 2y - 13 = 0$$

What are the coordinates of the point in which it meets the other common tangent?

The intersection of two circles

21.3 Example 5 *Find the equation of the common chord of the circles* $x^2 + y^2 - 4x - 2y + 1 = 0$ *and* $x^2 + y^2 + 4x - 6y - 10 = 0$.

The coordinates of the points of intersection A and B of the circles satisfy the two equations

$$x^2 + y^2 - 4x - 2y + 1 = 0$$
$$x^2 + y^2 + 4x - 6y - 10 = 0$$

Therefore, by subtraction, the coordinates of A and B satisfy the equation

$$-8x + 4y + 11 = 0$$

But this equation represents a straight line, and it is satisfied by the coordinates of A and B, therefore it is the equation of the common chord.

Two circles may not intersect but, by subtracting one equation from the other, the equation of a line may still be obtained. What then does the line represent? Qu. 3 suggests an answer.

Qu. 3 What are the squares of the lengths of the tangents from the point P(X, Y) to the circles $x^2 + y^2 - 1 = 0$, $x^2 + y^2 - 6x - 8y + 21 = 0$? What is the locus of P such that the lengths of the tangents from P to the circles are equal?
Qu. 4 Write down the equation of the line joining the origin to the point of intersection of the lines $17x - 15y + 7 = 0$ and $19x - 13y + 7 = 0$.

Orthogonal circles

21.4 If the tangents to two circles at their points of intersection are perpendicular, the circles are said to be **orthogonal**. Since the radius through a point of contact is perpendicular to the tangent, it follows that the tangent to one circle is a radius of the other. Thus if the centres of two orthogonal circles of radii R and r are a distance d apart (Fig. 21.5), it follows by Pythagoras' theorem that

$$d^2 = R^2 + r^2$$

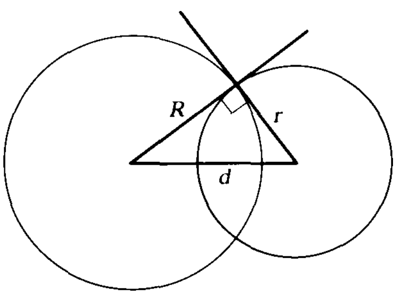


Figure 21.5

Example 6 *Show that the circles*

$$x^2 + y^2 - 6x + 4y + 2 = 0 \quad and \quad x^2 + y^2 + 8x + 2y - 22 = 0$$

are orthogonal.

The centres of the circles are $(3, -2)$ and $(-4, -1)$, and their radii are $\sqrt{11}$ and $\sqrt{39}$.

The sum of the squares of the radii is 50, and the square of the distance between the centres is $7^2 + 1^2 = 50$, therefore the circles are orthogonal.

Exercise 21c (Miscellaneous)

1 Show that the common chord of the circles

$$x^2 + y^2 = 4 \quad \text{and} \quad x^2 + y^2 - 4x - 2y - 4 = 0$$

passes through the origin.

2 Find the coordinates of the point where the common chord of the circles $x^2 + y^2 - 4x - 8y - 5 = 0$ and $x^2 + y^2 - 2x - 4y - 5 = 0$ meets the line joining their centres.

3 Show that the following pairs of circles are orthogonal:
(a) $x^2 + y^2 - 6x - 8y + 9 = 0, \quad x^2 + y^2 = 9$;
(b) $x^2 + y^2 - 4x + 2 = 0, \quad x^2 + y^2 + 6y - 2 = 0$;
(c) $x^2 + y^2 - 6y + 8 = 0, \quad x^2 + y^2 - 4x + 2y - 14 = 0$;
(d) $x^2 + y^2 + 10x - 4y - 3 = 0, \quad x^2 + y^2 - 2x - 6y + 5 = 0$.

4 Prove that the line $y = 2x$ is a tangent to the circle $x^2 + y^2 - 8x - y + 5 = 0$ and find the coordinates of the point of contact.

5 Show that the line $x - 2y + 12 = 0$ touches the circle $x^2 + y^2 - x - 31 = 0$ and find the coordinates of the point of contact.

6 The line $2x + 2y - 3 = 0$ touches the circle $4x^2 + 4y^2 + 8x + 4y - 13 = 0$ at A. Find the equation of the line joining A to the origin.

7 Find the equation of the circle whose centre is at the point (5, 4) and which touches the line joining the points (0, 5) and (4, 1).

8 Find the equation of the tangent to the circle $x^2 + y^2 - 2x + y - 5 = 0$ at the point $(3, -2)$. If this tangent cuts the axes at A and B, find the area of triangle OAB.

9 Find the length of the tangents to the circle $x^2 + y^2 - 2x + 4y - 3 = 0$ from the centre of the circle $x^2 + y^2 + 6x + 8y - 1 = 0$.

10 A tangent is drawn from the point $(-a, 0)$ to a variable circle, centre $(a, 0)$. What is the locus of the point of contact?

11 Prove that the circles $x^2 + y^2 + 3x + y = 0$ and $x^2 + y^2 - 6x - 2y = 0$ touch each other. Find the coordinates of the point of contact and the equation of their common tangent at that point.

12 Show that the line $y = x + 1$ touches the circle $x^2 + y^2 - 8x - 2y + 9 = 0$. What is the equation of the other tangent to the circle from the point (0, 1)?

13 A circle passing through the point (4, 0) is orthogonal to the circle $x^2 + y^2 = 4$. Find the locus of the centre of the variable circle.

14 The circle $x^2 + y^2 - 2x - 4y - 5 = 0$ has centre C, and is cut by the line $y = 2x + 5$ at A and B. Show that BC is perpendicular to AC and find the area of the triangle ABC.

15 Find the equation of the circle which passes through the points (0, 2), $(8, -2)$, (9, 5). Verify that it also passes through the point (2, 6).

16 Find the coordinates of the points A and B at which the line $x - 3y = 0$ meets the circle $x^2 + y^2 - 10x - 5y + 25 = 0$. Find also the coordinates of the point T where the circle touches the axis OX and verify that $OA \times OB = OT^2$.

17 A triangle has vertices (0, 6), (4, 0), (6, 0). Find the equation of the circle through the mid-points of the sides and show that it passes through the origin.

18 Two circles have their centres on the line $y + 3 = 0$ and touch the line $3y - 2x = 0$. If the radii of the circles are $\sqrt{13}$, find the coordinates of their centres and also their equations. [Hint: use similar triangles.]

19 A and B have coordinates $(-3, 0)$ and $(3, 0)$. Show that the locus of a point P which moves such that $PB = 2PA$ is a circle with centre $(-5, 0)$ and radius 4.

20 Show that the line $y = mx + c$ touches the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$.

Chapter 22

Further topics in coordinate geometry

The equation of a straight line

22.1 Straight lines occur so often in coordinate geometry that it is worth while learning to write down their equations by a quick method. Example 9 in Chapter 1 was done by two methods, and what follows is an extension of the second.

Example 1 *Find the equation of the line with gradient* $-\frac{2}{3}$, *which passes through the point* $(1, -4)$.

[Think: the line has equation $y = -\frac{2}{3}x + c$, therefore it may be written

$$3y + 2x = \text{constant}$$

Now since the line passes through $(1, -4)$, the constant may be found by substituting these coordinates in the left-hand side.]

The equation of the line is

$$\mathbf{3}y + 2x = -\mathbf{12} + 2$$

i.e. $2x + 3y + 10 = 0$

Note. Check that the line (a) has gradient $-\frac{2}{3}$, (b) passes through $(1, -4)$.

Given the equation of a line, it is easy to write down the equation of a perpendicular line through a given point. For example, if we require the equation of the line perpendicular to $4x + 5y + 7 = 0$ which passes through $(6, -5)$, we interchange the coefficients of x and y, changing one of the signs, and balance the equation as before. Thus the perpendicular is $5x - 4y = 50$.

Example 2 *Find the equation of the line joining the points* $(a, 0)$, $(0, b)$.

The gradient of the line is $-b/a$. Therefore, using the method of Example 1, its equation is $ay + bx = ab$.

Dividing through by ab, the equation becomes

$$\frac{x}{a}+\frac{y}{b}=1$$

which is known as the **intercept** form of the equation of a line.

Exercise 22a

1 Write down the equations of the lines with the given gradients which pass through the given points:
(a) gradient 1, through (3, 2); (b) gradient -2, through $(1, -3)$;
(c) gradient $\frac{1}{2}$, through $(0, -6)$; (d) gradient $-\frac{1}{3}$, through $(-2, 5)$;
(e) gradient $-\frac{7}{5}$, through $(3, -6)$; (f) gradient $\frac{3}{4}$, through $(-1, 1)$;
(g) gradient $-\frac{5}{6}$, through $(-3, -4)$; (h) gradient $\frac{4}{3}$, through $(-2, 5)$;
(i) gradient $1/t$, through $(at^2, 2at)$ (j) gradient $-t$, through $(at^2, 2at)$;
(k) gradient $-\cot\theta$, through $(a\cos\theta, a\sin\theta)$;
(l) gradient $-1/t^2$, through $(ct, c/t)$.

2 Write down the equations of the perpendiculars to
(a) $3x+2y-1=0$, through (2, 2);
(b) $4x-3y+7=0$, through the origin;
(c) $5x+6y+11=0$, through $(-3, 5)$;
(d) $3x-2y-7=0$, through $(-1, 3)$; (e) $ty-x=at^2$, through (h, k);
(f) $ax+by+c=0$, through (x_1, y_1); (g) $t^2y+x=2ct$, through $(ct, c/t)$.

3 Write down the equations of the lines which make the following intercepts on the x- and y-axes respectively:
(a) 3, 2; (b) -1, 2; (c) $\frac{1}{2}, \frac{1}{5}$; (d) $-\frac{1}{3}, \frac{1}{4}$.

4 Write down the equations of the lines joining the following pairs of points:
(a) (0, 2), (3, 0); (b) $(-1, 0)$, (0, 5); (c) $(-\frac{1}{2}, 0)$, $(0, \frac{2}{3})$.

5 The perpendicular from the origin to a straight line is of length p and makes an angle α with the x-axis (see Fig. 22.1). What intercepts does the line make on the axes? Write down the equation of the line.

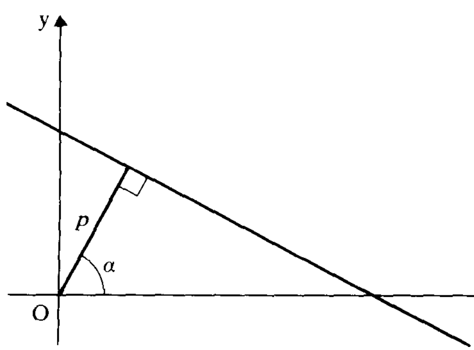


Figure 22.1

6 Use the method of §22.1 to find the equations of the following straight lines:
(a) with gradient 3, through (4, 3);
(b) with gradient $-\frac{1}{2}$, through (2, − 1);
(c) with gradient $\frac{2}{5}$, through (1, 1);
(d) with gradient $-\frac{3}{4}$, through (0, − 3);
(e) joining (3, 2) and (2, −4);
(f) joining (1, 3) and (−3, −6);
(g) joining (− 1, 2) to the mid-point of (3, 5) and (5, − 1);
(h) through (2, 1), perpendicular to $2x - y = 0$;
(i) through (− 1, 3) perpendicular to $3x + 4y - 2 = 0$;
(j) the altitude through A of the triangle A(1, 3), B(2, − 1), C(3, 5);
(k) the altitude through B of the triangle in (j);
(l) through (h, k), perpendicular to $t^2y + x = 2ct$.

7 Show that the rectangular hyperbolas $xy = 1$ and $x^2 - y^2 = 1$ are orthogonal.

8 Show that the ellipse $16x^2 + 25y^2 = 400$ and the hyperbola $4x^2 - 5y^2 = 20$ are orthogonal.

Polar coordinates

22.2 If someone asks me at Harrow to tell him where Enfield is, I may reply that it is about 19 km East and 9 km North, or I might tell him that it is roughly 21 km away on a bearing N 60° E. These two descriptions of the position of Enfield correspond to the two systems of coordinates used in this book. The first is the basis of Cartesian coordinates and we have already met the second in the chapter on vectors. (See also §10.9.)

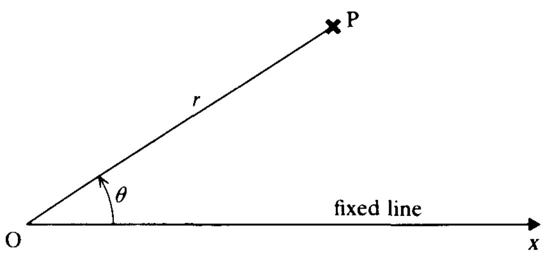


Figure 22.2

Let P be any point and let OP = r, where O is the origin (see Fig. 22.2) and let OP make an angle θ with the x-axis, then r and θ are called the **polar coordinates** of the point P, and the coordinates may be written (r, θ). The x-axis is sometimes called the initial line.

It should be noticed that, while a bearing is usually measured in a clockwise sense from North, in mathematics the polar coordinate θ is normally represented in an anti-clockwise sense.

Thus in Fig. 22.3, the coordinates of A are (2, 30°) and those of B are (3, 90°). A point may be described in different ways, for instance C may be written as

(2, 210°), (2, −150°), (−2, 30°) and so on. If, for any reason, a unique way of referring to each point is required, r may be taken to be positive and θ to lie in the range $-180° < \theta \leqslant 180°$.

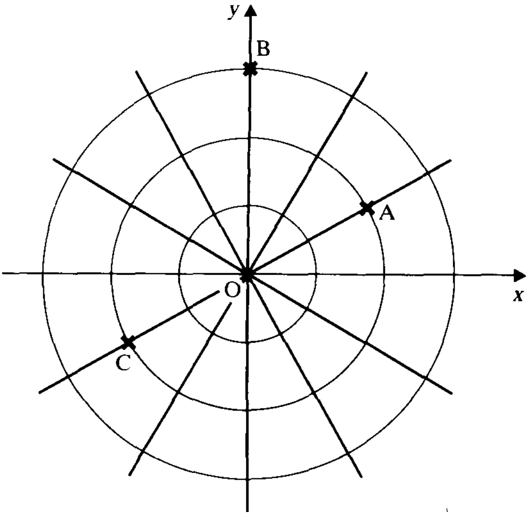


Figure 22.3

Example 3 *Sketch the curve whose polar equation is* $r = a(1 + 2\cos\theta)$.

Take values of θ, and calculate $1 + 2\cos\theta$, as below.

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°
$2\cos\theta$	2	1.732	1.414	1	0	−1	−1.414	−1.732	−2
$1+2\cos\theta$	3	2.732	2.414	2	1	0	−0.414	−0.732	−1

Plot these values (see Fig. 22.4). Now if α is any angle, $\cos(-\alpha) = \cos\alpha$, therefore the same values of r will be obtained for negative values of θ. Thus the curve may be completed.

Example 4 *Find the polar equation of a line such that the perpendicular to it from the origin is of length p and makes an angle α with the x-axis.*

In Fig. 22.5, N is the foot of the perpendicular from the origin to the line, and let P be any point (r, θ) on the line.

In the triangle ONP, N is a right angle and angle PON $= \theta - \alpha$ (or $\alpha - \theta$).

$\therefore r\cos(\theta - \alpha) = p \quad (\text{or } r\cos(\alpha - \theta) = p)$

Therefore, in either case, the polar equation of the line is $r\cos(\theta - \alpha) = p$.

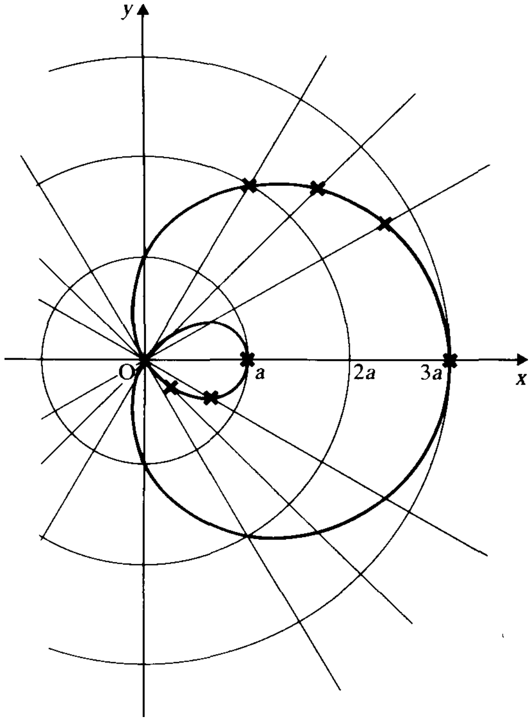


Figure 22.4

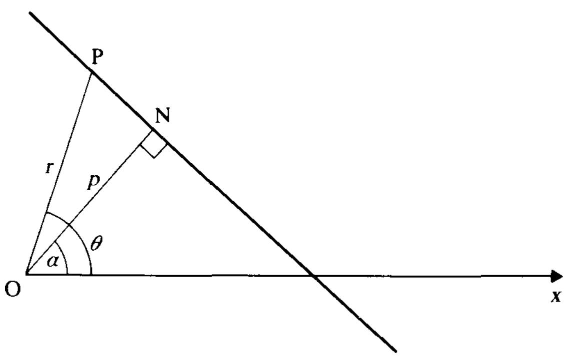


Figure 22.5

Relations between polar and Cartesian coordinates

22.3 In Fig. 22.6, P is the point (x, y) in Cartesian coordinates and (r, θ) in polar coordinates, and PM is an ordinate.

Now, by the definitions of cosine and sine given in §16.1,

$$\cos\theta = \frac{x}{r} \quad \text{and} \quad \sin\theta = \frac{y}{r}$$

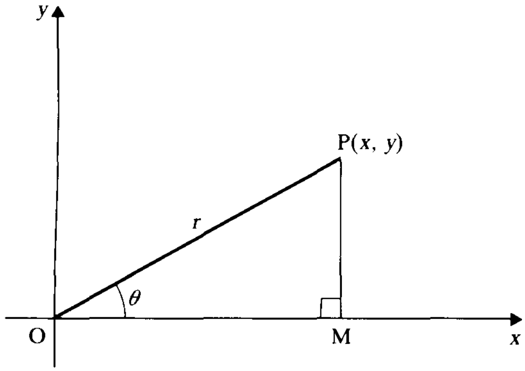


Figure 22.6

Therefore x and y are given in terms of r and θ by the equations

$$\boldsymbol{x = r\cos\theta} \quad \text{and} \quad \boldsymbol{y = r\sin\theta}$$

If, on the other hand, we are given the values of x and y, we can, by inspecting the diagram in Fig. 22.6, write down the values of r and θ.

By Pythagoras' theorem,

$$r^2 = x^2 + y^2$$
$$\therefore \boldsymbol{r = \pm\sqrt{(x^2 + y^2)}}$$

In most cases the positive square root should be taken, but on some occasions it may be necessary to use the negative one. (For instance, in Example 3 above, at the point $x = a$, $y = 0$, r is equal to $-a$.)

The angle θ can be found by elementary trigonometry. In Fig. 22.6, θ is given by

$$\boldsymbol{\tan\theta = \frac{y}{x}}$$

[Here again, care must be taken. For example, the point $x = -1$, $y = -1$, gives $\tan\theta = +1$, but in this case θ is equal to $-135°$, not $45°$; if in doubt, consult the diagram. Compare this with the modulus and argument of a complex number (see §10.9).]

Example 5 *Find the Cartesian equations of*
(a) $r = a(1 + 2\cos\theta)$, (b) $r\cos(\theta - \alpha) = p$.

(a) $r = a(1 + 2\cos\theta)$

[The $\cos\theta$ suggests the relation $x = r\cos\theta$, so multiply through by r.]

$$\therefore r^2 = a(r + 2r\cos\theta)$$
$$\therefore x^2 + y^2 = a\{\sqrt{(x^2 + y^2)} + 2x\}$$
$$\therefore x^2 + y^2 - 2ax = a\sqrt{(x^2 + y^2)}$$

Therefore the Cartesian equation of $r = a(1 + 2\cos\theta)$ is

$$(x^2 + y^2 - 2ax)^2 = a^2(x^2 + y^2)$$

(b) $r\cos(\theta - \alpha) = p$

$\cos(\theta - \alpha)$ may be expanded (see §17.2),

$$\therefore r\cos\theta\cos\alpha + r\sin\theta\sin\alpha = p$$

Therefore the Cartesian equation of $r\cos(\theta - \alpha) = p$ is

$$x\cos\alpha + y\sin\alpha = p$$

Note. The perpendicular from the origin to this line is of length p and makes an angle α with the x-axis. This form of the equation of a straight line is known as the **normal** or **perpendicular** form.

Example 6 *Find the polar equation of the circle whose Cartesian equation is* $x^2 + y^2 = 4x$.

$$x^2 + y^2 = 4x$$

Put $x = r\cos\theta$, $y = r\sin\theta$, then

$$r^2\cos^2\theta + r^2\sin^2\theta = 4r\cos\theta$$
$$\therefore r^2 = 4r\cos\theta$$

Therefore the polar equation of the circle is $r = 4\cos\theta$.

Exercise 22b

1 Sketch the curves given by the following polar equations:
(a) $r = a(1 + \cos\theta)$, (b) $r = a\cos 2\theta$, (c) $r = a(1 - \sin\theta)$,
(d) $r = a\sin 3\theta$, (e) $r = a\sec\theta$, (f) $r = a\tan\theta$,
(g) $r = a\cos\frac{1}{2}\theta$, (h) $r = a(1 + \sin 2\theta)$.

2 Find the polar equations of the following loci:
(a) a circle, centre at the origin, radius a;
(b) a straight line through the origin, inclined at an angle α to the initial line;
(c) a straight line perpendicular to the initial line, at a distance a from the origin;
(d) a straight line parallel to the initial line at a distance a;
(e) a circle on the line joining the origin to $(a, 0)$ as a diameter;
(f) a circle, radius a, touching the initial line at the origin and lying above it;
(g) a circle, radius a, centre on the initial line at a distance c from the origin;
(h) a point which moves so that its distance from the origin is equal to its distance from the straight line $x = 2a$.

3 P_1 is the point (r_1, θ_1), P_2 is (r_2, θ_2) and $\theta_2 > \theta_1$. Show that the area of the triangle OP_1P_2 is $\frac{1}{2}r_1r_2\sin(\theta_2 - \theta_1)$. Deduce that if the Cartesian coordinates of P_1 and P_2 are (x_1, y_1) and (x_2, y_2), then the area of OP_1P_2 is $\frac{1}{2}(x_1y_2 - x_2y_1)$.

4 Deduce from the result of No. 3, that the area of the triangle $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ is

$$\tfrac{1}{2}\{(x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1)\}$$

[If new axes are drawn at (x_3, y_3), the coordinates of P_1 and P_2 referred to them are $(x_1 - x_3, y_1 - y_3)$ and $(x_2 - x_3, y_2 - y_3)$.]

5 Obtain the polar equations of the following loci:

(a) $x^2 + y^2 = a^2$, (b) $x^2 - y^2 = a^2$, (c) $y = 0$,
(d) $y^2 = 4a(a - x)$, (e) $x^2 + y^2 - 2y = 0$, (f) $xy = c^2$.

6 Obtain the Cartesian equations of the following loci:

(a) $r = 2$, (b) $r = a(1 + \cos\theta)$, (c) $r = a\cos\theta$,
(d) $r = a\tan\theta$, (e) $r = 2a(1 + \sin 2\theta)$, (f) $2r^2 \sin 2\theta = c^2$,
(g) $l/r = 1 + e\cos\theta$, (h) $r = 4a\cot\theta\operatorname{cosec}\theta$.

7 Express the following straight lines in the form $x\cos\alpha + y\sin\alpha = p$. State the distance of each line from the origin and give the angle which the perpendicular from the origin makes with the x-axis.

(a) $x + \sqrt{3}y = 2$, (b) $x - y = 4$, (c) $3x + 4y - 10 = 0$,
(d) $5x - 12y + 26 = 0$, (e) $x + 3y - 2 = 0$, (f) $ax + by + c = 0$.

The distance of a point from a line

22.4 Given a point $P_1(x_1, y_1)$ and the line

$$ax + by + c = 0$$

we shall first find the distance, r, of P_1 from a point P_2 on the line, such that $\overrightarrow{P_1P_2}$ makes an angle α with the x-axis (see Fig. 22.7).

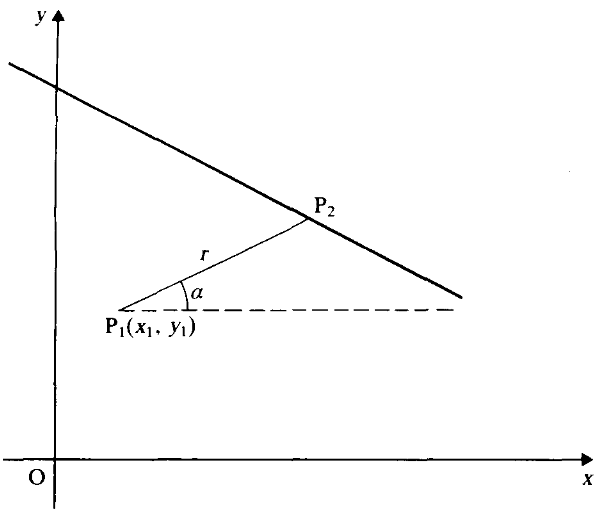


Figure 22.7

P_2 has coordinates $(x_1 + r\cos\alpha,\ y_1 + r\sin\alpha)$, but, as P_2 lies on the line $ax + by + c = 0$, its coordinates satisfy the equation, therefore

$$a(x_1 + r\cos\alpha) + b(y_1 + r\sin\alpha) + c = 0$$

$$\therefore r(a\cos\alpha + b\sin\alpha) = -(ax_1 + by_1 + c)$$

$$\therefore r = -\frac{ax_1 + by_1 + c}{a\cos\alpha + b\sin\alpha} \tag{1}$$

Now take the case when P_1P_2 is perpendicular to the line $ax + by + c = 0$. The gradient of $ax + by + c = 0$ is $-a/b$, therefore the gradient of P_1P_2 is b/a.

$$\therefore \tan\alpha = \frac{b}{a}$$

$$\therefore \sec^2\alpha = 1 + \frac{b^2}{a^2} = \frac{a^2 + b^2}{a^2}$$

$$\therefore \cos\alpha = \pm\frac{a}{\sqrt{(a^2 + b^2)}}$$

and, since $\tan\alpha = b/a$,

$$\sin\alpha = \pm\frac{b}{\sqrt{(a^2 + b^2)}}$$

so in the denominator of equation (1)

$$a\cos\alpha + b\sin\alpha = \pm\left(\frac{a^2}{\sqrt{(a^2 + b^2)}} + \frac{b^2}{\sqrt{(a^2 + b^2)}}\right)$$

$$= \pm\sqrt{(a^2 + b^2)}$$

Therefore the perpendicular distance of (x_1, y_1) from the line $ax + by + c = 0$ is

$$\pm\frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}}$$

The plus or minus sign should be chosen so that this is a positive quantity, in other words, the perpendicular distance is

$$\left|\frac{\boldsymbol{ax_1 + by_1 + c}}{\boldsymbol{\sqrt{(a^2 + b^2)}}}\right|$$

Example 7 *Find the distances of the points* (a) (1, 3), (b) (−3, 4), (c) (4, −2) *from the line* $2x + 3y - 6 = 0$.

The distance of (x_1, y_1) from the line $ax + by + c = 0$ is

$$\left|\frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}}\right|$$

Therefore the distances of (1, 3), (−3, 4), (4, −2) from $2x + 3y - 6 = 0$ are respectively

(a) $\left|\dfrac{2\times 1+3\times 3-6}{\sqrt{(2^2+3^2)}}\right|=\dfrac{5}{\sqrt{13}}$,

(b) $\left|\dfrac{2\times(-3)+3\times 4-6}{\sqrt{(2^2+3^2)}}\right|=0$,

(c) $\left|\dfrac{2\times 4+3\times(-2)-6}{\sqrt{(2^2+3^2)}}\right|=\dfrac{4}{\sqrt{13}}$,

The formula is more easily remembered if two points are noticed: (1) the numerator is obtained by substituting the coordinates of the point into the equation of the line (remember that the perpendicular distance is zero if the point lies on the line), (2) the denominator is the square root of the sum of the squares of the coefficients.

Qu. 1 Find the distances of the given points from the following lines:
(a) $(3, 2)$, $3x-4y+4=0$; (b) $(2, -1)$, $5x+12y=0$;
(c) $(0, -3)$, $x+5y+2=0$; (d) $(2, 5)$, $x+y-1=0$;
(e) $(-4, 2)$, $3y=5x-6$; (f) $(2, 1)$, $y=\frac{2}{3}x+\frac{1}{3}$;
(g) $(0, a)$, $3y=4x$; (h) (p, q), $3x+4y-3p=0$;
(i) (X, Y), $12x-5y+7=0$; (j) (x_1, y_1), $8x=15y$.

Example 8 *Find the equations of the bisectors of the angles between the lines* $4x+3y-12=0$ *and* $y=3x$.

[The angle bisectors are the locus of a point which is equidistant from the two lines, and this provides a method of finding their equations.]

Let $P(X, Y)$ be a point on the locus, then the distances of P from the lines $4x+3y-12=0$ and $y-3x=0$ are

$$\pm\frac{4X+3Y-12}{\sqrt{(4^2+3^2)}} \quad\text{and}\quad \pm\frac{Y-3X}{\sqrt{(3^2+1^2)}}$$

But P is equidistant from the two lines, therefore

$$\frac{4X+3Y-12}{5}=\pm\frac{Y-3X}{\sqrt{10}}$$

[One $\pm$ sign has been dropped, since there are only two distinct equations: one given by the same sign each side, the other by different signs.]

Simplifying these equations we obtain

$$4\sqrt{10}X+3\sqrt{10}Y-12\sqrt{10}=5Y-15X$$

and

$$4\sqrt{10}X+3\sqrt{10}Y-12\sqrt{10}=-5Y+15X$$

Therefore the equations of the angle bisectors of the lines are

$$(4\sqrt{10}+15)x+(3\sqrt{10}-5)y-12\sqrt{10}=0$$

and

$$(4\sqrt{10}-15)x+(3\sqrt{10}+5)y-12\sqrt{10}=0$$

Example 9 *Find the equations of the tangents to the circle*

$$x^2+y^2-4x-2y-8=0$$

which are parallel to the line $3x+2y=0$.

[This will be done by using the result that the perpendicular distance of a tangent from the centre of a circle is equal to the radius.]

The required tangents are parallel to the line $3x+2y=0$, therefore their equations may be written in the form

$$3x+2y+c=0$$

where c is a constant to be determined for each tangent.

To find the centre and radius of the circle

$$x^2+y^2-4x-2y-8=0$$

$$\therefore\ x^2-4x+4+y^2-2y+1=8+4+1$$

$$\therefore\ (x-2)^2+(y-1)^2=13$$

Therefore the centre is (2, 1) and the radius is $\sqrt{13}$.

Now the distance of the point (x_1, y_1) from the line $ax+by+c=0$ is $|(ax_1+by_1+c)/\sqrt{(a^2+b^2)}|$, therefore the distance of the centre of the circle (2, 1) from the line $3x+2y+c=0$ is

$$\left|\frac{3\times 2+2\times 1+c}{\sqrt{(3^2+2^2)}}\right|$$

But if the line is a tangent, this distance is equal to the radius, therefore

$$\left|\frac{8+c}{\sqrt{13}}\right|=\sqrt{13}$$

$$\therefore\ \pm(8+c)=13$$

Taking the positive sign, $8+c=13$, and so $c=5$. With the negative sign, $-8-c=13$, and so $c=-21$.

Therefore the equations of the tangents parallel to $3x+2y=0$ are $3x+2y+5=0$ and $3x+2y-21=0$.

Exercise 22c

1 Write down the distances of the given points from the following lines:

(a) (2, 5), $4x+3y-2=0$;
(b) $(-1, 3)$, $12x-5y=0$;
(c) $(-2, 0)$, $4x+y-2=0$;
(d) (3, 5), $x-y+2=0$;
(e) $(-1, 7)$, $2x=5y+1$;
(f) (0, 0), $3x=4y+6$;
(g) (2, 3), $y=\frac{4}{5}x+\frac{1}{5}$;
(h) (1, 4), $\frac{1}{2}x+\frac{1}{3}y=1$;
(i) (0, 0), $x\cos\alpha+y\sin\alpha=p$;
(j) (X, Y), $5x-12y+1=0$;
(k) $(c, 2c)$, $8x=15y$;
(l) (x_1, y_1), $y=\frac{3}{4}x-\frac{1}{2}$.

2 Find the equations of the bisectors of the angles between
(a) $3x+4y-7=0$, $y-1=0$; (b) $4x-3y+1=0$, $3x-4y+3=0$;
(c) $5x+12y=0$, $12x+5y-4=0$; (d) $x+y-1=0$, the x-axis.

3 Find the equations of the bisectors of the acute angles between
(a) $3x-4y+2=0$, $x+3=0$; (b) $5x+12y+9=0$, $5x-12y+6=0$;
(c) $x+y+1=0$, $x=7y$.
[Draw figures to determine which equations give the required lines.]

4 What is the locus of a point which moves so that it is equidistant from the point $(2, -3)$ and the line $x+2y=0$?

5 Find the locus of a point which is equidistant from the line $3x-4y+7=0$ and the point $(3, 4)$ on the line.

6 What is the locus of a point which moves so that its distance from $(2, 2)$ is half its distance from $x+y+4=0$?

7 Find the equations of the tangents to the circle $x^2+y^2+4x+8y-5=0$ which are parallel to the line $4y-3x=0$.

8 Show that the line $3x+2y=0$ touches the circle $x^2+y^2+6x+4y=0$, and find the equations of the perpendicular tangents.

9 Find the equation of the circle in the first quadrant with radius 2 which touches the y-axis and the line $3y-4x-3=0$.

10 Prove that the line $y=mx+c$ touches the circle $x^2+y^2=a^2$ if $c^2=a^2(1+m^2)$. Also find the condition that the line $lx+my+n=0$ should touch the circle.

Parameters

22.5 Consider a circle, radius a, centre at the origin (see Fig. 22.8). Let $P(x, y)$ be any point on the circle, and let the angle between PO and the x-axis be θ, then

$$x=a\cos\theta \quad \text{and} \quad y=a\sin\theta$$

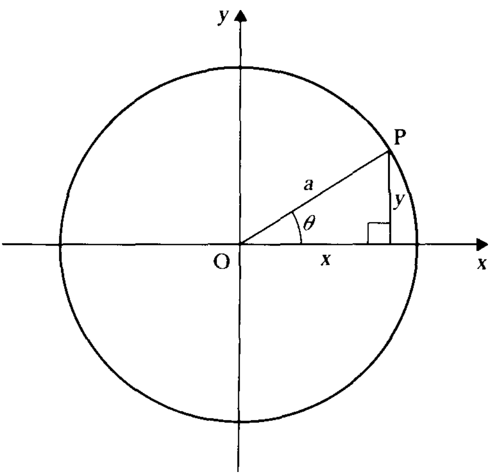


Figure 22.8

These equations, which give the coordinates of any point on the curve in terms of θ, are called **parametric equations**, and θ is called a **parameter**.

If we wish to refer to a particular point on the curve, a single number, the corresponding value of θ, will determine it. Thus $\theta = 60^\circ$ gives the point $(a/2, \sqrt{3}a/2)$. On the other hand, if we were given a value of x, say $\frac{1}{2}a$, there are two corresponding points: $(a/2, \sqrt{3}a/2)$ and $(a/2, -\sqrt{3}a/2)$. Another advantage of parameters is that we may write down the coordinates of a general point on the curve $(a \cos\theta, a \sin\theta)$. If we wrote (x_1, y_1), we should also have to bear in mind the equation $x_1{}^2 + y_1{}^2 = a^2$.

Another example of parameters was used in §22.4. The point

$$(x_1 + r\cos\alpha,\ y_1 + r\sin\alpha)$$

lies on the straight line through (x_1, y_1) with gradient $\tan\alpha$, and in this case the parameter, r, is a distance. However, it is not always possible to give an easy interpretation of a parameter in terms of angles or distances.

Example 10 *Plot the graph of the curve given parametrically by the equations* $x = t^2 - 4$, $y = t^3 - 4t$, *for values of* t *from* -3 *to* $+3$.

A table of values is shown below.

t	-3	-2	-1	0	1	2	3
$x = t^2 - 4$	5	0	-3	-4	-3	0	5
$y = t^3 - 4t$	-15	0	3	0	-3	0	15

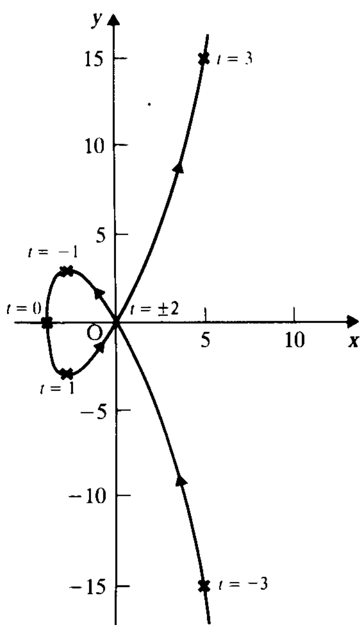


Figure 22.9

The graph has been plotted in Fig. 22.9, and the values of the parameter, t, have been written against the corresponding points. The arrows indicate the direction of motion of a point on the curve as t increases from -3 to $+3$.

Example 11 *Sketch the curve given parametrically by* $x = \sin\theta$, $y = \sin 2\theta$.

A few values of θ will give all the points we need.

θ	0	45°	90°	135°	180°
$x = \sin\theta$	0	0.7071	1	0.7071	0
$y = \sin 2\theta$	0	1	0	-1	0

Plotting these points and joining them by a curve we obtain the part of the curve in Fig. 22.10 which lies to the right of the y-axis.

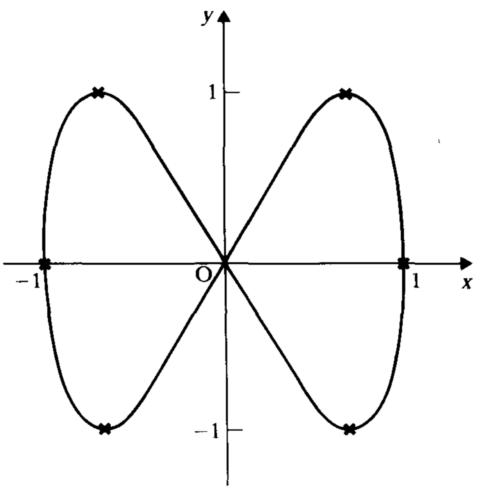


Figure 22.10

Now $\sin(-\alpha) = -\sin\alpha$, so that negative values of θ change the signs of x and y. Therefore the rest of the curve may be drawn in symmetrically.

Qu. 2 Sketch the locus given by $x = t^2$, $y = 1 - t^2$, for real values of t. Is it the line $x + y = 1$?

The graph of the curve given parametrically by the equations $x = t^2 - 4$, $y = t^3 - 4t$ was plotted for values of t from -3 to $+3$ in Example 10. The question may well have risen in the reader's mind, 'What is the equation connecting x and y?' This can be found by eliminating t from the equations

$$x = t^2 - 4 \qquad y = t^3 - 4t$$

Notice that $y = tx$. Therefore we may substitute $t = y/x$ in either of the equations above. Choosing the simpler,

$$x = \frac{y^2}{x^2} - 4$$

$$\therefore x^3 = y^2 - 4x^2$$

Therefore the Cartesian equation of the locus is $y^2 = x^2(x + 4)$.

Example 12 *Find the Cartesian equation of the locus given parametrically by the equations* $x = \sin\theta$, $y = \sin 2\theta$ (*see* Example 11).

$y = \sin 2\theta$, but $\sin 2\theta = 2\sin\theta\cos\theta$, therefore

$$y = 2\sin\theta\cos\theta$$
$$\therefore y^2 = 4\sin^2\theta\cos^2\theta$$

Now $x = \sin\theta$, therefore $1 - x^2 = \cos^2\theta$, and so the Cartesian equation of the locus is $y^2 = 4x^2(1 - x^2)$.

The process of obtaining parametric equations from a given Cartesian equation is not so easy as the reverse, but one method is illustrated in the next example.

Example 13 *Obtain parametric equations for the locus* $y^2 = x^3 - x^2$.

Put $y = tx$ in the equation $y^2 = x^3 - x^2$, then

$$t^2x^2 = x^3 - x^2$$
$$\therefore t^2 = x - 1$$
$$\therefore x = t^2 + 1$$

Therefore the locus may be represented by the parametric equations $x = t^2 + 1$, $y = t^3 + t$.

Note. This method is not suitable for all equations, but it works well when the terms are of degree n and $n - 1$.

Exercise 22d

1 Plot the curves given parametrically by the equations:
(a) $x = t^2 + 1$, $y = t + 2$; from $t = -3$ to $t = +3$.
(b) $x = t^2$, $y = t^3$; from $t = -3$ to $t = +3$.
(c) $x = t$, $y = 1/t$; taking $t = \pm 4, \pm 3, \pm 2, \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}$.
(d) $x = 1 + t$, $y = 3 - 2t$;
(e) $x = at^2$, $y = 2at$;
(f) $x = t^2$, $y = 1/t$;
(g) $x = \dfrac{2 + 3t}{1 + t}$, $y = \dfrac{3 - 2t}{1 + t}$;

(h) $x = 3(t + 1/t)$, $y = 2(t - 1/t)$;
(i) $x = 3\cos\theta$, $y = 2\sin\theta$;
(j) $x = 4\sec\theta$, $y = 3\tan\theta$.

2 Find the values of the parameters and the other coordinates of the given points on the following curves:
(a) $x = t$, $y = 2/t$; where $y = 1\frac{1}{2}$.
(b) $x = at^2$, $y = 2at$; where $x = \frac{9}{4}a$.
(c) $x = \dfrac{1+t}{1-t}$, $y = \dfrac{2+3t}{1-t}$; where $y = -\frac{4}{3}$.
(d) $x = a\cos\theta$, $y = b\sin\theta$; where $x = \frac{1}{2}a$.

3 Find the Cartesian equations of the loci in No. 1.

4 By substituting $y = tx$, find parametric equations for the loci whose Cartesian equations are
(a) $y^4 = x^5$, (b) $y = x^2 + 2x$, (c) $y^2 = x^2 + 2x$,
(d) $x^2 = x^3 - y^3$, (e) $x^3 + y^3 = 3xy$.

5 Show that the parametric equations
(a) $x = 1 + 2t$, $y = 2 + 3t$, (b) $x = 1/(2t - 3)$, $y = t/(2t - 3)$,
both represent the same straight line, and find its Cartesian equation.

6 Show that the line given parametrically by the equations $x = \dfrac{2-t}{1+2t}$, $y = \dfrac{3+t}{1+2t}$ passes through the points (6, 7) and $(-2, -1)$. Find the values of t corresponding to these points.

7 P is the variable point $(t^2, 3t)$ and O is the origin. Find the coordinates of Q, the mid-point of OP, and hence obtain the locus of Q as P varies.

8 P is the variable point $(at^2, 2at)$ on the parabola $y^2 = 4ax$, and Q is the foot of the perpendicular from P to the y-axis. Find the locus of the mid-point of PQ.

9 The line joining the origin to the variable point P$(t, 1/t)$ meets the line $x = 1$ at Q. Find the locus of the mid-point of PQ.

10 Find the coordinates of the points nearest to the origin on the curve $x = t$, $y = 1/t$. What is their distance from the origin?

11 Find the coordinates of the points on the curve $x = at^2$, $y = 2at$ where the distance from the point $(5a, -2a)$ is stationary. Distinguish between maxima, minima and points of inflexion.

12 Find the equations of the chords joining the points with parameters p and q on the following curves:
(a) $x = t^2$, $y = 2t$; (b) $x = t$, $y = -1/t$;
(c) $x = t^3$, $y = t$; (d) $x = t + 1/t$, $y = 2t$.

13 Determine the point on the parabola $x = at^2$, $y = 2at$ where the distance to the line $x - y + 4a = 0$ is least and find the least distance.

14 Find the values of t at the points of intersection of the line $2x - y - 4 = 0$ with the parabola $x = t^2$, $y = 2t$ and give the coordinates of these points.

15 Find the points of intersection of the parabola $x = t^2$, $y = 2t$ with the circle $x^2 + y^2 - 9x + 4 = 0$.

Example 14 *Find the equation of the tangent to the rectangular hyperbola* $xy = c^2$ *at the point* $\mathrm{P}(ct, c/t)$, *and show that, if this tangent meets the axes at* Q *and* R, *then* P *is the mid-point of* QR.

The gradient of the curve is given by

$$\frac{dy}{dx} = \frac{dy}{dt} \bigg/ \frac{dx}{dt}$$

But $y = c/t$,

$$\therefore \frac{dy}{dt} = -\frac{c}{t^2}$$

and $x = ct$,

$$\therefore \frac{dx}{dt} = c$$

$$\therefore \frac{dy}{dx} = \frac{-c/t^2}{c} = -\frac{1}{t^2}$$

Therefore the equation of the tangent at P is

$$yt^2 + x = 2ct$$

This tangent meets the axes at $\mathrm{Q}(2ct, 0)$ and $\mathrm{R}(0, 2c/t)$ therefore $\mathrm{P}(ct, c/t)$ is the mid-point of QR.

Example 15 *Find the coordinates of the points where the line* $4x - 5y + 6a = 0$ *cuts the curve given parametrically by* $(at^2, 2at)$.

If the line $4x - 5y + 6a = 0$ meets the curve at the point $(at^2, 2at)$, then its coordinates must satisfy the equation of the line. Therefore

$$4at^2 - 10at + 6a = 0$$
$$\therefore 2t^2 - 5t + 3 = 0$$
$$\therefore (2t - 3)(t - 1) = 0$$
$$\therefore t = \tfrac{3}{2} \quad \text{or} \quad 1$$

Therefore the coordinates of the points of intersection are $(\frac{9}{4}a, 3a)$ and $(a, 2a)$.

Exercise 22e

1 Find the equations of the tangents and normals to the following curves at the given points:
(a) $x = t^2$, $y = t^3$, $(1, -1)$;
(b) $x = t^2$, $y = 1/t$, $(\frac{1}{4}, 2)$;
(c) $x = at^2$, $y = 2at$, $(a, -2a)$;
(d) $x = ct$, $y = c/t$, $(-c, -c)$;
(e) $x = t^2 - 4$, $y = t^3 - 4t$, $(-3, -3)$;
(f) $x = 3\cos\theta$, $y = 2\sin\theta$, $(\frac{3}{2}, \sqrt{3})$.

2 Find the equations of the tangents and normals to the following curves at the point whose parameter is t:
(a) $x = t^3$, $y = 3t^2$;
(b) $x = at^2$, $y = 2at$;
(c) $x = 4t^3$, $y = 3t^4$;
(d) $x = ct$, $y = c/t$;
(e) $x = a\cos t$, $y = b\sin t$;
(f) $x = a\sec t$, $y = b\tan t$.

3 Find the equations of the chords joining the points whose parameters are p and q on the following curves. Deduce the equations of the tangents at the points p by finding the limiting equations of the chords as q approaches p.
(a) $x = t^2, y = 2t$; (b) $x = 1/t, y = t^2$;
(c) $x = ct, y = c/t$; (d) $x = a\cos t, y = b\sin t$.
[Hint: cancel a factor of $p - q$ in the gradients.]

4 Find the equation of the normal to the parabola $x = at^2, y = 2at$ at the point $(4a, 4a)$. Find also the coordinates of the point where the normal meets the curve again.

5 Find the coordinates of the point where the normal to the rectangular hyperbola $x = ct$, $y = c/t$ at $(2c, \frac{1}{2}c)$ meets the curve again.

6 Find the coordinates of the point where the tangent to the curve $x = 1/t$, $y = t^2$ at $(1, 1)$ meets the curve again.

7 Find the equation of the tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$. For what values of t does the tangent pass through the point $(8a, 6a)$? Write down the equations of the tangents to the parabola from $(8a, 6a)$.

8 Find the equations of the tangents to the hyperbola $x = ct$, $y = c/t$ from the point $(\frac{3}{2}c, \frac{1}{2}c)$.

9 Find the equations of the normals to the parabola $x = at^2$, $y = 2at$ from the point $(14a, -16a)$.

10 The normal to the hyperbola $x = ct$, $y = c/t$ at the point P with parameter p meets the curve again at Q. Find the coordinates of Q.

11 Show that, if a tangent to the curve $x = 1/t$, $y = t^2$ meets the axes in A and B, then PB = 2AP.

12 Show that the tangent at the point t on the astroid $x = a\cos^3 t$, $y = a\sin^3 t$ is the line $y\cos t + x\sin t = a\sin t\cos t$. Show that the tangent meets the axes in points whose distance apart is a.

The parabola

22.6 As no new method is required, work on the parabola is given in the form of exercises. It is intended that any result proved may be used in later questions.

Definition

The locus of a point equidistant from a given point and a given line is called a **parabola**. *The given point is the* **focus** *and the given line the* **directrix**.

Exercise 22f

1 Use compasses and graph paper to plot a parabola from the definition.

2 Given a parabola, take axes with the x-axis through the focus, perpendicular to the directrix, and the origin where the x-axis meets the curve. Let the focus be $(a, 0)$ and show that the equation of the parabola is $y^2 = 4ax$. [It follows from the definition that the equation of the directrix is $x = -a$.]

3 Verify that the point $(at^2, 2at)$ lies on the parabola $y^2 = 4ax$ for all values of t, and that every point on the parabola is given thus.

4 Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

In Fig. 22.11, the tangent and normal at the point P on the parabola $y^2 = 4ax$ meet the x-axis at T and G, and the y-axis at T′ and G′. PN is parallel to the y-axis. S is the focus. LD is the directrix and L is the foot of the perpendicular from P to the directrix.

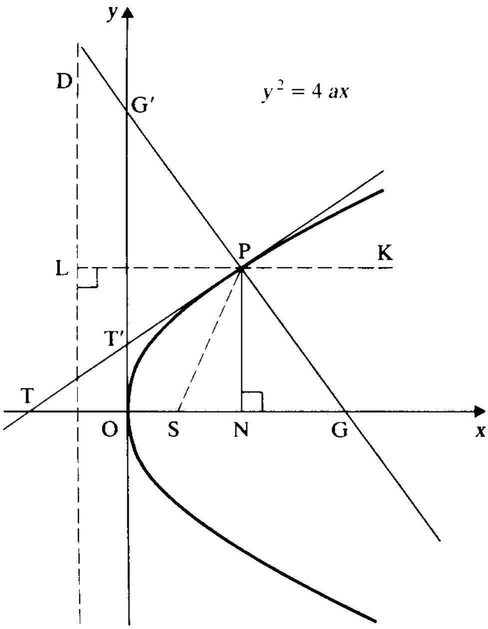


Figure 22.11

5 Show that ST′ = T′L and deduce that
(a) ∠LPT′ = ∠SPT′ [use the definition of the curve],
(b) ∠SPG = ∠KPG.
[This proves the optical property of the parabola, i.e. that light from a point source at the focus is reflected in rays parallel to the axis.]

6 Show that L, T′, S are collinear (i.e. lie on a straight line), and that LS is perpendicular to PT.

7 Show that TS = SP = SG.

8 Show that LPST is a rhombus and that LPGS is a parallelogram.

9 Show that NG = $2a$.

10 If the parameters of the points P and Q are p and q, show that the tangents to the parabola meet at the point $(apq, ap + aq)$.

11 If PQ passes through the focus prove that, with the notation of No. 10, $pq = -1$.

12 Show that the tangents at the ends of a focal chord meet on the directrix.

13 Show that if the tangents at the ends of a focal chord meet the tangent at the vertex at U and V, then $\angle$USV is a right angle.

14 Show that the locus of the mid-point of a focal chord is $y^2 = 2a(x - a)$.

15 Show that if the tangents to the parabola at P and Q meet on the line $x = ah$, then the locus of the mid-point of the chord PQ is $y^2 = 2a(x + ah)$.

16 If the tangents to the parabola at P and Q intersect on the line $y = k$, find the locus of the mid-point of PQ.

17 Find the values of t for which the normal at $(at^2, 2at)$ passes through the point $(5a, 2a)$. Hence find the equations of the normals to the parabola from $(5a, 2a)$.

18 Find the equations of the tangents to the parabola from the point $(4a, 5a)$.

Exercise 22g (Miscellaneous)

1 Show that the ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $2x^2 - 3y^2 = 6$ are orthogonal.

2 Sketch the curve whose polar equation is $r = a \cos 3\theta$.

3 Sketch the curve whose polar equation is $r = a(1 + \sin \theta)$ and from this obtain a sketch of the curve $r(1 + \sin \theta) = a$.

4 Find the polar equation of a parabola, taking the focus as the origin and the axis as the initial line.

5 Calculate the area of the triangle A(2, 5), B(3, $-$1), C(4, 6).

6 Find the polar equation of $(x^2 + y^2 + ax)^2 = a^2(x^2 + y^2)$ and the Cartesian equation of $r(1 + \sin \theta) = a$.

7 Express the equation $7x - 24y - 10 = 0$ in perpendicular form and state the distance of the line from the origin.

8 Find the equations of the bisectors of the angles between
(a) $6x - 7y + 11 = 0$, $2x + 9y - 3 = 0$; (b) $7x - y = 3$, $x + y = 2$.

9 Find the locus of a point which moves so that its distance from the line $y + x - 2 = 0$ is equal to its distance from the point $(-1, -1)$.

10 Find the equations of the tangents to the circle

$$x^2 + y^2 - 12x - 14y + 75 = 0$$

which are parallel to the line $3y - x = 0$.

11 A straight line through the point (1, 1) and the variable point $P(t, 1/t)$ meets the y-axis at Q. Find the locus of the mid-point of PQ.

12 The chord PQ of the hyperbola $x = ct$, $y = c/t$ meets the axes at A and B. Show that the mid-point of PQ is also the mid-point of AB.

13 Find the equations of the tangent and normal to the curve $x = t^3 - t^2$, $y = t^2 - 1$ at the point (4, 3).

14 Find the equation of the tangent at $P(t^2, 1/t)$ to the curve $xy^2 = 1$. If the tangent meets the x-axis at Q, find the locus of the mid-point of PQ.

15 The tangent at $P(t^2, 1/t)$ to the curve $xy^2 = 1$ meets the y-axis at A, the x-axis at B and the curve again at Q. Show that AP:PB:BQ = 1:2:1.

16 Find the equations of the tangents to the parabola $x = at^2$, $y = 2at$ from the point $(5a, 6a)$.

17 Find the coordinates of the point where the normal to the parabola $x = at^2$, $y = 2at$ at $(9a, 6a)$ meets the curve again.

18 Show that if the tangent at $P(t, t^3)$ to the curve $y = x^3$ meets the curve again at Q, then the y-axis divides PQ in the ratio 1:2.

19 A tangent to the rectangular hyperbola $x = ct$, $y = c/t$ meets the axes at A and B. Show that the area of triangle AOB is constant.

20 Show that if the tangent to a parabola at P meets the axis at T, and N is the foot of the perpendicular from P to the axis, then TN is bisected by the vertex.

Chapter 23

Variation and experimental laws

Variation

23.1 'Variation' in its mathematical sense is concerned with certain ways in which one variable depends on one or more others. The idea is bound up with ratio and proportion which the reader will have met in elementary arithmetic. Some readers may need to revise these ideas and to appreciate their power for the first time.

Proportion arises in arithmetic in a number of ways. For instance the circumference C of a circle is proportional to its radius r: this is usually expressed in the form of an equation,

$$C = 2\pi r$$

Sometimes a graph shows us that two variables are in proportion; for example Fig. 23.1 shows the 'travel graph' of a car moving at a steady speed of 50 km/h along a road. Note that: (1) the gradient of the graph is uniform, (2) the straight line passes through the origin.

Yet another aspect of proportion, and indeed the most basic, is used in arithmetic when we use ratios.

To summarise, if y is proportional to x, then

(a) $y = kx$, where k is some constant,

(b) the graph of y against x is a straight line through the origin,

(c) if x_1, y_1 and x_2, y_2 are corresponding values of x and y, then

$$\frac{y_1}{y_2} = \frac{x_1}{x_2}$$

Note that any one of these statements follows from either of the others. The equivalence of (a) and (b) is familiar from the work of Chapter 1. The equivalence of (b) and (c) can be seen by writing (c) in the form

$$\frac{y_1}{x_1} = \frac{y_2}{x_2}$$

which shows that (x_1, y_1) and (x_2, y_2) lie on the same straight line through the origin.

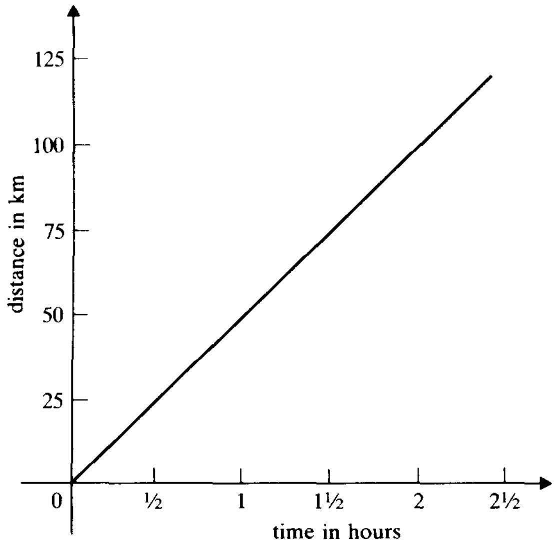


Figure 23.1

In the last paragraph we used the phrase,

'y is proportional to x'

Sometimes another phrase with exactly the same meaning is used instead, namely,

'y varies as x'

Other examples of variation will already be familiar to the reader. For instance, the area A of a circle is given in terms of its radius r by the equation

$$A = \pi r^2$$

Here A is *not* proportional to r, but it *is* proportional to r^2 and we express this by saying that 'A *varies* as the square of r'.

Another example is the volume V of a sphere in terms of its radius r. The equation connecting V, r is

$$V = \tfrac{4}{3}\pi r^3$$

Again, V is *not* proportional to r, but it *is* proportional to r^3 and we express this by saying that 'V varies as the cube of r'.

Kinematics provides another example. If a distance of 60 km is travelled at a constant speed u km/h, the time t h is given by the equation

$$t = 60/u$$

We may say that t and u are inversely proportional, or we may express this by saying 't varies inversely as u'.

The 'inverse square law' may be familiar to the reader: one example of this is the force F exerted by the Earth on a given satellite at distance r from the centre of the Earth. The equation connecting F and r is

$$F = \frac{k}{r^2} \qquad \text{(where } k \text{ is a constant)}$$

This may also be expressed by saying that 'F varies inversely as the square of r'.

Qu. 1 Express the following equations as statements involving the word 'varies':

(a) $s = 16t^2$, (b) $V = \pi r^3$, (c) $y = \frac{10}{x^2}$,

(d) $T = \frac{\pi}{2\sqrt{2}}\sqrt{l}$, (e) $p = \frac{200}{V}$, (f) $T^2 = d^3$.

Suppose that a number of spheres are made out of wood of uniform density. Then, unless we know the density of the wood, we cannot calculate the weight W of one of these spheres from its diameter d. We can, however, say that

W varies on the cube of d

or write

$$W = kd^3 \qquad \text{(where } k \text{ is a constant)}$$

Further, if W_1, d_1 and W_2, d_2 are the weights and diameters of two of the spheres,

$$W_1 = kd_1^{\,3}$$
$$W_2 = kd_2^{\,3}$$

so that, by division,

$$\frac{W_1}{W_2} = \frac{d_1^{\,3}}{d_2^{\,3}}$$

Now, if we know the weight and radius of one of the spheres, this last equation provides us with a very convenient way of calculating the weight of any other when its diameter is known.

Example 1 *A number of spheres are made out of wood of uniform density. A sphere with diameter* 7 cm *weighs* 0.11 kg. *How much will a sphere of diameter* 9 cm *weigh?*

As we have seen above, W varies as d^3. Hence if W_1, d_1 and W_2, d_2 are the weights and diameters of the two spheres,

$$\frac{W_1}{W_2} = \frac{d_1^{\,3}}{d_2^{\,3}}$$

It is often helpful to tabulate the data and it is worth noting that the algebra of the question is simplified if we place the *quantity to be found* in the line labelled (1):

	weight (kg)	diameter (cm)
(1)	W_1	9
(2)	0.11	7

Then substituting into the equation $W_1/W_2 = d_1^3/d_2^3$,

$$\frac{W_1}{0.11} = \frac{9^3}{7^3}$$

$$\therefore W_1 = 0.11 \times \frac{9^3}{7^3}$$

$$= 0.23 \text{ to two significant figures}$$

Therefore the sphere of diameter 9 cm weighs 0.23 kg, correct to 2 significant figures.

[Example 1 illustrates the power of the method: an alternative way of tackling this question would have been to find the density of the wood from the data numbered (2) in the table.]

Qu. 2 In Example 1, what is the effect on W of (a) doubling d, (b) trebling d?

We saw on page 438 (just above Example 1), that from the statement,

'W varies as the cube of d'

could be deduced the equation

$$\frac{W_1}{W_2} = \frac{d_1^{\ 3}}{d_2^{\ 3}}$$

which connects corresponding values W_1, d_1 and W_2, d_2. It is important for the following work to be able to convert a statement to an equation quickly and easily, so some more examples of this process follow.

If we are given that

'y varies as the square of x'

this is simply another way of saying

'y is proportional to x^2'

From this it follows immediately (see page 436) that
(a) $y = kx^2$, where k is some constant,
(b) the graph of y against x^2 is a straight line through the origin,
(c) if x_1, y_1 and x_2, y_2 are corresponding values of x and y, then

$$\frac{y_1}{y_2} = \frac{x_1^{\ 2}}{x_2^{\ 2}}$$

On the other hand, if we are given that

'y varies *inversely* as the square of x'

this is simply another way of saying

'y is proportional to $\dfrac{1}{x^2}$'

from which it follows immediately that
(a) $y = k/x^2$, where k is some constant,
(b) the graph of y against $1/x^2$ is a straight line through the origin,
(c) if x_1, y_1 and x_2, y_2 are corresponding values of x and y, then

$$\frac{y_1}{y_2} = \frac{1/x_1^2}{1/x_2^2}$$

or, multiplying numerator and denominator of the right-hand side by $x_1^2x_2^2$,

$$\frac{y_1}{y_2} = \frac{x_2^2}{x_1^2}$$

Note that, in this case of *inverse* variation, the x's are upside down compared with the y's. 'Inverse' comes from the same root as 'invert', one meaning of which is to 'turn upside down'.

Qu. 3 Write down equations (i) with k's, (ii) with suffixes, similar to those in the last three paragraphs for the following statements:
(a) p varies as q,
(b) p varies inversely as v,
(c) v varies as the cube of x,
(d) u varies as the square root of l,
(e) F varies as the square of c,
(f) H varies inversely as the square of d,
(g) T varies inversely as the square root of g,
(h) A varies as the nth power of s,
(i) the cube of A varies as the square of v.

Example 2 *The length l of a simple pendulum varies as the square of the period T (time to swing to and fro). A pendulum* 0.994 m *long has a period of approximately* 2 s, *find* (a) *the length of a pendulum whose period is* 3 s, (b) *an equation connecting l and T.*

(a) Tabulating the data:

	length (m)	period (s)
(1)	l	3
(2)	0.994	2

l varies as T^2.

$$\therefore \frac{l_1}{l_2} = \frac{T_1^2}{T_2^2}$$

$$\therefore \frac{l}{0.994} = \frac{3^2}{2^2}$$

$$\therefore l = 0.994 \times \frac{9}{4}$$

$$= 2.236$$

Therefore the length of a pendulum whose period is 3 s is 2.24 m.

(b) Tabulating the data again, we enter l and T in the row numbered (1):

	length (m)	period (s)
(1)	l	T
(2)	0.994	2

Substituting in the same equation as before,

$$\frac{l}{0.994} = \frac{T^2}{2^2}$$

$$\therefore l = 0.2485T^2$$

Therefore the equation connecting l, T is $l \approx 0.25T^2$.

Qu. 4 In Example 2, find the effect on l of (a) doubling T, (b) trebling T. What is the effect on T of doubling l?
Qu. 5 Find the period of a pendulum whose length is 0.3 m from the data of Example 2. Time ten swings to and fro of such a pendulum and compare this with your answer.

Example 3 *The weight w* N* *of an astronaut varies inversely as the square of his distance d from the centre of the Earth. If an astronaut's weight on Earth is* 792 N, *what will his weight be at a height of* 230 km *above the Earth? Take the radius of the Earth to be* 6370 km.

We tabulate the data:

	weight (N)	distance from the centre of the Earth (km)
(1)	w	$6370 + 230 = 6600$
(2)	792	6370

*The newton (N) is the absolute unit of force in SI units. The magnitude of 1 kg wt varies with the value of g, since 1 kg wt gives to a mass of 1 kg an acceleration of g m/s^2. In contrast, 1 N gives to the same mass a fixed acceleration of 1 m/s^2, by definition. Hence in a context of varying gravitational pull we use this constant, or absolute, unit of force the newton.

Now w varies inversely as d^2, so if w_1, d_1 and w_2, d_2 are corresponding values,

$$\frac{w_1}{w_2} = \frac{{d_2}^2}{{d_1}^2}$$

$$\therefore \frac{w}{792} = \frac{6370^2}{6600^2}$$

$$\therefore w = 792 \times \frac{6370^2}{6600^2}$$

$$= 737.7$$

Therefore the astronaut's weight would be 738 N.

To find the height above the Earth at which his weight would be halved, we again tabulate the data:

	weight (N)	distance from the centre of the Earth (km)
(1)	396	d
(2)	792	6370

Again using $w_1/w_2 = {d_2}^2/{d_1}^2$, for the new w_1, d_1,

$$\frac{396}{792} = \frac{6370^2}{d^2}$$

$$\therefore d^2 = 2 \times 6370^2$$

$$\therefore d = \sqrt{2} \times 6370$$

$$= 9010$$

Therefore the height above the Earth at which his weight would be halved is $9010 - 6370$ km $= 2640$ km.

Qu. 6 Find an equation in the form $w = k/d^2$ connecting the weight of the astronaut in Example 3 and his distance from the centre of the Earth.

Qu. 7 With the equation of Qu. 6, find the effect on w of (a) doubling d, (b) trebling d.

Qu. 8 Discuss whether the first of the following pairs of variables varies as some power of the second and, if so, state what power:

(a) the cost c of 100 copies of a book and the price p of one,
(b) the cost C of a square of plywood and its side a,
(c) the weight w of a spherical lead shot and its radius r,
(d) the length l of a rectangle of given area and its breadth b,
(e) the surface area S of a scale model and its length l,
(f) the area A of an equilateral triangle and its side a,

(g) the side a of an equilateral triangle and its area A,
(h) the volume V of a regular tetrahedron and its side a,
(i) the side a of a regular tetrahedron and its volume V.

Exercise 23a

1 The area of a circular sector containing a given angle varies as the square of the radius of the circle. If the area of the sector is 2 cm^2 when the radius is 1.6 cm, find the area of the sector containing the same angle when the radius of the circle is 2.7 cm.

2 The distance of the horizon d km varies as the square root of the height h m of the observer above sea level. An observer at a height of 100 m above sea level sees the horizon at a distance of 35.7 km. Find the distance of the horizon from an observer 70 m above sea level.

Also find an equation connecting d and h.

3 The length l cm of a simple pendulum varies as the square of its period T s. A pendulum with period 2 s is 99.4 cm long; find the length of a pendulum whose period is 2.5 s.

What equation connects l and T?

4 Assuming that the length of paper in a roll of given dimensions varies inversely as the thickness of the paper, find the increase in length when the thickness of paper in a 100 m roll is decreased from 0.25 mm to 0.20 mm.

5 A certain type of hollow plastic sphere is designed in such a way that the mass varies as the square of the diameter. Three spheres of this type are made: one has mass 0.10 kg and diameter 9 cm; a second has diameter 14 cm; and a third has mass 0.15 kg. Find the mass of the second, the diameter of the third, and an equation connecting the mass m kg and the diameter d cm of spheres of this type.

6 The circumference C inches of a circle of radius r inches is given by the formula $C = 2\pi r$; if C_1, r_1 and C_2, r_2 are corresponding values of C, r,

$$\frac{C_1}{C_2} = \frac{r_1}{r_2} \qquad (1)$$

(a) What formula gives the circumference C cm of a circle of radius r m? Does equation (1) still hold?

(b) Given that 1 inch = 2.54 cm, what equation gives the circumference C cm of a circle of radius r inches? Does equation (1) still hold?

7 Boyle's law states that, under certain conditions, the pressure exerted by a given mass of gas is inversely proportional to the volume occupied by it. The gas inside a cylinder is compressed by a piston in such a way that Boyle's law may legitimately be applied. When this happens, the volume is decreased from 200 cm^3 to 70 cm^3. If the original pressure of the gas is 9.8×10^4 N/m^2, find the final pressure of the gas.

8 The number of square carpet tiles needed to surface the floor of a hall varies inversely as the square of the length of a side of the tile used. If 2016 tiles of side 0.4 m would be needed to surface the floor of a certain hall, how many tiles of side 0.3 m would be required?

9 If the volume of a model 10 cm long is 72 cm^3, what is the volume of a similar model 6 cm long? What is the length of a similar model with volume 100 cm^3?

10 The maximum speed of yachts of normal dimensions varies as the square root of their length. If a yacht of 20 m can maintain a maximum speed of 12 k, find the maximum speed of a yacht 15 m long. Obtain an equation connecting a yacht's maximum speed v k and its length l m.

11 For similar printing type, the number of characters on a given size of page varies inversely as the square of the height of the type. On a certain page 2200 characters of height 6 mm could be printed. How many characters of similar type of height 5 mm could be printed on the page? When 7000 characters have to be printed on the page with similar type, what height would the type be if the height is a multiple of 0.1 mm?

12 (a) If y varies as x^3 and x varies as t^2, does y vary as any power of t? [Hint: write the statements y varies as x^3, x varies as t^2 as equations with constants k, K.]

(b) p varies inversely as q; q varies as the square of r. Does p vary as any power of r?

13 When I drive round a certain corner at 18 km/h, the sideways frictional force between the tyres of my car and the road is 1050 N. The sideways frictional force F N varies as the square of the speed v km/h. Find an equation connecting F, v and use it to find

(a) the total sideways frictional force at 27 km/h,

(b) the speed at which the sideways frictional force is equal to 6170 N which is half the weight of the loaded car.

14 Assuming that the power H kW developed by a certain car travelling on a level road varies as the cube of the speed v km/h, find an equation connecting H, v for this car, given that it develops 50 kW at 65 km/h. Find the power developed by it at 30 km/h along a level road.

15 The speed of a certain point on a high-speed centrifuge varies as the angular velocity of the centrifuge, and the acceleration of this point varies as the square of the angular velocity. Find the percentage changes in the speed and acceleration of the point when the angular velocity is increased from 56 000 rev/min to 60 000 rev/min.

16 The cube of the surface area of a regular icosahedron varies as the square of its volume. By what factor will the surface area of a regular icosahedron be increased if its volume is doubled?

17 The period T s of a given pendulum varies inversely as the square root of the acceleration due to gravity g m/s^2 at the location of the pendulum. Find the percentage change in the period of a pendulum moved from Greenwich, where $g = 9.812$ m/s^2, to New York where $g = 9.802$ m/s^2. [Hint: use the first two terms of the expansion of $(1+x)^{1/2}$.]

18 The volume and areas of similar solids vary respectively as the cubes and squares of their linear dimensions. Some similar solids are placed in an upward current of air. Assuming that the upthrust of the air current varies as the surface area of the solid and that the weight of the solid varies as its volume, show that some of the solids will rise if their linear dimensions are small enough.

19 The square of the period (time to go round its orbit) of an Earth satellite varies as the cube of its mean distance from the centre of the Earth. The period of the Moon is 28 days and its mean distance from the centre of the Earth is 380 000 km. Find the period, to the nearest minute, of an Earth satellite whose mean distance from the surface of the Earth is 470 km, given that the radius of the Earth is 6370 km.

Also find an equation giving the period of an Earth satellite T hours in terms of its mean distance d km from the centre of the Earth.

20 Like and unlike poles of two bar magnets repel and attract each other respectively with a force which varies inversely as the square of the distance between the poles. The poles of each of two bar magnets are at a distance $2d$ apart. The magnets are placed in line with two unlike poles of the magnets at a distance d apart. They are then placed in line with two unlike poles at a distance $2d$ apart. By what factor is the attractive force between the magnets decreased?

Joint variation

23.2 So far we have only considered examples of variation where one variable, say y, varies as some power of another variable, say x. But there are many examples in science, engineering and everyday life when one variable depends on two or more others. For example, the volume V of a right circular cylinder is given in terms of its radius r and height h by the formula

$$V = \pi r^2 h$$

If we consider a metal rod of uniform circular cross-section which can be cut into lengths, we have a case of this law in which the radius is constant and so

the volume varies as the length

or, using the symbol '$\propto$' as an abbreviation for 'varies as',

$$V \propto h$$

On the other hand, if circular discs are cut out of sheet metal or plywood, h will be constant and so

the volume varies as the square of the radius

or $V \propto r^2$

To summarise, for a right circular cylinder,

if r is constant, $V \propto h$
if h is constant, $V \propto r^2$

In experimental work, if one variable depends on two or more others, it is most convenient to see how the first depends on each of the others in turn while the remainder are held constant. As an illustration of this, consider the discharge of water through a circular hole. The volume of water V will depend in some way on

(a) the radius r of the hole,
(b) the velocity v of the water,
(c) the time t over which the discharge takes place.

It is found that

(1) if v, t are constant, $V \propto r^2$,
(2) if t, r are constant, $V \propto v$,
(3) if r, v are constant, $V \propto t$.

It will be seen that the equation

$$V = kr^2vt \qquad (k \text{ constant})$$

satisfies the conditions (1), (2), (3) and hence it is natural to write

$$V \propto r^2vt$$

Qu. 9 Express the statement 'If z is constant, y varies as x; if x is constant, y varies as the cube of z', as a single equation.

Qu. 10 Write the statement, 'If h, t are constant, W varies as the square of r; if r, t are constant, W varies as h; if r, h are constant, W varies inversely as t', as a single statement using the sign '$\propto$'.

When one variable varies as two or more others, the word *jointly* is sometimes used. For example, with the data of the last paragraph, we might say that V varies jointly as v, t and the square of r.

Qu. 11 'The kinetic energy T of a flywheel varies jointly as its mass m and as the square of its radius r.' Express this statement (a) as an equation with a constant k, (b) as a statement using the sign '$\propto$'.

Qu. 12 'F varies jointly as m and the square of v, and inversely as r.' Express this statement as an equation.

For purposes of calculation, we can rewrite statements in the form

$$A = k\frac{x^3}{t} \qquad \text{(where } k \text{ is some constant)}$$

in terms of the ratios of corresponding values A_1, x_1, t_1 and A_2, x_2, t_2 of the variables. We have

$$A_1 = k\frac{{x_1}^3}{t_1}$$

$$A_2 = k\frac{{x_2}^3}{t_2}$$

$$\therefore \frac{A_1}{A_2} = \frac{{x_1}^3/t_1}{{x_2}^3/t_2}$$

Multiplying numerator and denominator by $t_1 t_2$,

$$\frac{A_1}{A_2} = \frac{x_1^3 t_2}{x_2^3 t_1}$$

Note that A varies inversely as t, and that the ratio t_1/t_2 is 'upside down'.

Qu. 13 If x_1, y_1, z_1 and x_2, y_2, z_2 are corresponding values of x, y, z, write down equations connecting x_1, y_1, z_1 and x_2, y_2, z_2 when
(a) z varies jointly as x and the square of y,
(b) z varies as y and inversely as the square of x,
(c) z varies as the cube of x and as the square of y,
(d) z varies as x when y is constant and z varies as y when x is constant,
(e) z varies as the square of x when y is constant and z varies as the square of y when x is constant,
(f) z varies as the square root of x when y is constant and inversely as y when x is constant.

Example 4 *The total sideways force experienced by a given car rounding a circular bend at a constant speed varies as the square of the speed of the car and inversely as the radius of the circle. A certain car goes round a bend of radius* 50 m *at* 72 km/h *and experiences a total sideways force of* 12 kN. *What sideways force will it experience on going round a bend of radius* 30 m *at* 54 km/h?

Let the sideways force be F kN, the speed be v km/h, and the radius r m, then

$$F \propto \frac{v^2}{r}$$

Therefore, if F_1, v_1, r_1 and F_2, v_2, r_2 are corresponding values of F, v, r,

$$\frac{F_1}{F_2} = \frac{v_1^2/r_1}{v_2^2/r_2} = \frac{v_1^2 r_2}{v_2^2 r_1}$$

	F (kN)	v (km/h)	r (m)
(1)	F	54	30
(2)	12	72	50

$$\therefore \frac{F}{12} = \frac{54^2 \times 50}{72^2 \times 30}$$

$$\therefore F = \frac{12 \times 3^2 \times 5}{4^2 \times 3}$$

$$= \frac{45}{4} = 11.25$$

Therefore the sideways force on the car will be approximately 11 kN.

Variation in parts*

23.3 As an example of variation in parts, consider the cost of having a floor covered with lino tiles. First of all, a man and some materials have to be transported to the site. Here the cost of the man's time and the cost of the running of a van may be taken to vary as the distance s km from the firm's premises and so we may write this part of the cost as £ks, where k is some constant to be found. Second, there is the cost of materials and the man's time doing the job, which may be taken to vary as the area A m^2 of the floor, and so this part of the cost may be written £KA, where K is another constant to be determined. Hence, if the total cost is £C,

$$C = ks + KA$$

Let us suppose that the cost of two contracts is as given in the following table. How much would it cost to lay 40 m^2 of lino tiles at a distance of 75 km from the firm's premises?

cost £C	distance s km	area A m^2
C	75	40
265	45	50
155	60	27

Substituting from the bottom two lines of the table into

$$C = ks + KA$$

we get

$$265 = 45k + 50K \quad (1)$$
$$155 = 60k + 27K \quad (2)$$

$4 \times (1) - 3 \times (2)$:

$$1060 - 465 = (200 - 81)K$$

$$\therefore K = \frac{595}{119} = 5$$

From (2),

$$155 = 60k + 135$$

$$\therefore 20 = 60k$$

$$\therefore k = \tfrac{1}{3}$$

Substituting $K = 5$, $k = \frac{1}{3}$,

$$C = \tfrac{1}{3}s + 5A$$

*The reader is advised to delay reading this section until he has worked at least some of Exercise 23b Nos. 1–12.

When $s = 75$, $A = 40$,

$$C = \tfrac{1}{3} \times 75 + 5 \times 40 = 225$$

Therefore the cost of laying 40 m^2 of lino tiles at a distance of 75 km would be £225.

Qu. 14 The cost £C of manufacturing a certain number of wooden cubes for children is made up of two parts, one of which is constant and the other of which varies as the cube of the side x cm of a brick.

(a) Express the above statement in symbols.

(b) Find the cost of making 1000 $1\frac{1}{4}$ cm cubes if the same number of 2 cm and 1 cm cubes cost respectively £18 and £11.

Exercise 23b

1 The area of a sector of a circle varies jointly as the angle at the centre and the square of the radius. Given that the area of a sector containing an angle of 36° in a circle of radius 10 cm is 31.4 cm^2, find the area of a sector containing an angle of 72° in a circle of radius 5 cm.

2 The number of revolutions per minute of a bicycle wheel varies as the speed of the bicycle and inversely as the diameter of the wheel. A wheel of diameter 63 cm makes 151.5 revolutions per minute when the bicycle is moving at 18 km/h. Another bicycle has wheels of 35 cm diameter; how many revolutions per minute will one of its wheels make when the bicycle is moving at 30 km/h?

3 The flow of water through a circular orifice varies as the square of the diameter of the orifice and as the square root of the head of water. Given that 200 litres of water per second flow through an orifice of diameter 25 mm when the head of water is 4 m, find the flow of water through an orifice of diameter 10 mm when the head of water is 9 m.

4 The kinetic energy of a car (including passengers) varies jointly as the total mass and the square of the speed. A car of total mass 1000 kg travelling at 72 km/h has a kinetic energy of 200 kJ. What is the kinetic energy of a car of total mass 1500 kg travelling at 108 km/h?

5 The volume of a given mass of gas varies directly as its absolute temperature and inversely as its pressure. At an absolute temperature of 283 K and a pressure of 73 cm of mercury, a certain mass of gas has volume 200 cm^3. What will its volume be at standard temperature and pressure, i.e. absolute temperature 273 K and pressure 76 cm of mercury? Also find an equation which expresses the volume V cm^3 of the gas in terms of its absolute temperature T K and its pressure p cm of mercury.

6 The rate at which an electric fire gives out heat varies as the square of the voltage and inversely as the resistance. If a fire with resistance 57.6 ohms gives out approximately 1 kW when the voltage is 240, at what rate will heat be given out by an electric fire with resistance 69 ohms when the voltage is

220? Also find an expression which gives (approximately) the output in kW of an electric fire of resistance R ohms when the voltage is V.

7 The frequency of the note emitted by a plucked wire of a certain type varies as the square root of the tension of the wire and inversely as its length. A wire of length 0.61 m under a tension of 31 N emits a note of frequency 130 s^{-1}. What will be the frequency of the note emitted by a similar wire of length 0.25 m under a tension of 100 N? Find an equation which gives the number of oscillations per second f in terms of the length l m and the tension F N.

8 When a note is produced by blowing across the top of a bottle with a circular mouth, the frequency of the note varies as the internal diameter of the mouth and inversely as the square root of the volume of the bottle. Blowing across a certain bottle, I obtain a note whose frequency is approximately 203 s^{-1}. What is the frequency of the note I should obtain by blowing across the top of a bottle with four times the capacity, and with three-quarters the mouth diameter of the first?

9 The period of a simple pendulum varies as the square root of its length and inversely as the square root of the acceleration due to gravity. On the Earth, the period of a pendulum 99.4 cm long is 2 s. Assuming that the acceleration due to gravity on the surface of the Moon is one-sixth of that on the Earth, what would be the period of a pendulum 1 m long on the Moon?

10 The effectiveness of a spin drier is measured by the central acceleration at a point on the internal surface of the rotating drum. This acceleration varies as the internal diameter of the drum and as the square of its angular speed. Which would be the more effective: a spin drier with internal diameter 0.5 m running at an angular speed of 1600 rev/min, or one with internal diameter 0.3 m running at 2000 rev/min?

11 The rate at which heat is conducted through a metal plate varies jointly as the area of the plate and the temperature difference between the two sides, and inversely as the thickness of the metal. For quick heating of the contents, which saucepan would be better; one with a diameter 15 cm and thickness 2 mm, or another with diameter 20 cm and thickness 3 mm?

12 The light received at a point varies as the power of the source and inversely as the square of its distance from the point. Assuming that each bulb converts an equal proportion of its power into light, which gives better illumination: a 60 W bulb at $1\frac{1}{2}$ m, or a 100 W bulb at 2 m?

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13 The annual cost of running a certain car is made up of two parts, one of which is fixed and the other of which varies as the distance run by the car in the year. In one year the car ran 6000 km at a total cost of £900; in the next year it ran 7200 km at a total cost of £950. How much would it cost to run the car in a year during which it ran 12 000 km? To what extent is the assumption about the cost justified?

14 The cost of printing a circular on octavo paper is partly fixed and partly varies as the number of copies printed. If 100 and 500 copies cost £8.25 and £14.25 respectively, how much will 200 copies cost? Find an equation which gives the cost £C of n copies.

15 When a body is being uniformly accelerated, the distance travelled is the sum of two parts: one part varies as the time, the other varies as the square of the time. The distances travelled by a body in 2 s and 3 s from its original position are respectively 32 m and 57 m. How far will it travel from its original position in 4 s? Find an equation which gives the distance s m in terms of the time t s from its original position.

16 In good road conditions, the driver of a car moving at 30 km/h can stop the car in 11.4 m, and if the car is moving at 60 km/h it can be stopped in 33.6 m. This stopping distance is made up of two parts, one of which varies as the speed of the car, and the other of which varies as the square of the speed. In what distance can the driver stop the car if it is moving at 80 km/h? Find an equation which gives the stopping distance s m in terms of the speed v km/h.

If the car can just be stopped in 25 m, how fast is it moving?

17 Basic slag is advertised in 5 kg packs at £1.25, 10 kg packs at £2.25 and 20 kg packs at £4.25. It is suggested that the cost £C of these packs is partly constant and partly varies as the mass m kg of basic slag. If this is so, what is the equation which gives C in terms of m?

18 The price of a ticket to a dance is made up of two parts, one of which is fixed and the other of which varies inversely as the number of people expected at the dance. For a certain dance, it is found that the price of a ticket would need to be £3 if 100 people were to attend, but if 150 people attended the price of a ticket would need to be £2.50 in order to cover the cost. What would be the price of a ticket in order to cover the cost if only 75 people attended? If the price of a ticket was fixed at £2.70, how many people would have to buy tickets for the cost to be covered?

19 When a certain volume of wax is cast into a square prism, the surface area of the prism may be expressed as the sum of two parts, one of which varies as the square of the side of the cross-section and the other of which varies inversely as the side of the cross-section. If the side of the cross-section is 2 cm, the surface area of the prism is 28 cm^2. When the side of the cross-section is 1 cm, the surface area of the prism is 42 cm^2. What will be the surface area of the prism when the side of the cross-section is $2\frac{1}{2}$ cm?

Also find a formula which gives the surface area S cm^2 of the prism in terms of the side x cm of the cross-section.

20 The volume of a cap of height h cut off from a sphere of radius r (by a plane at distance $r - h$ from the centre) is the sum of two parts, one of which varies as the square of h and the other of which varies as the cube of h. Use the formulae for the volumes of a hemisphere and a sphere (i.e. the volume of the cap when $h = r$ and when $h = 2r$) to find a formula for the volume V of the cap in terms of h, r.

Graphical determination of laws

23.4 A simple experiment is performed to investigate the relationship between the tension in an elastic band and its extension, by fixing the upper end and suspending bodies of different masses in turn from the lower end. The tension

(y N) in the band (given by the weight of each body) is tabulated against the corresponding extension (x cm) measured to the nearest mm.

x	0	1	1.8	2.5	3.3	4.3	5.3
y	0	1	2	3	4	5	6

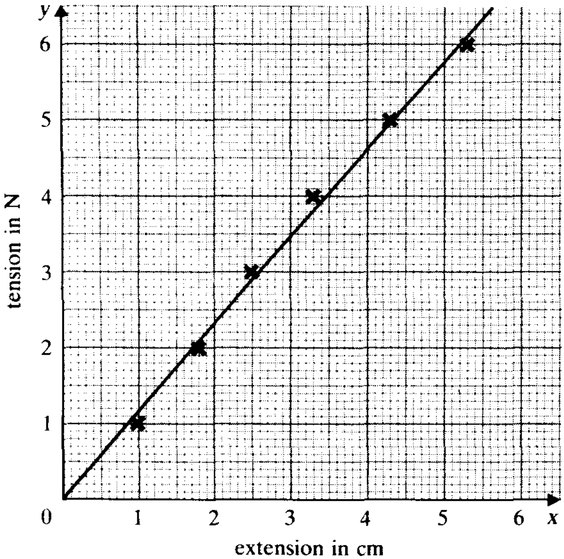

Figure 23.2

When these results are illustrated graphically (see Fig. 23.2) we see that it is possible to draw a straight line about which the points are closely scattered; such a line is then drawn, and we make it pass through the origin since we know that $y=0$ when $x=0$.

A straight line through the origin of gradient m has the equation

$$y = mx$$

and, allowing for experimental error and the limited accuracy in measuring x, we may reasonably deduce this to be the relationship between the x and y of our experiment. Referring to the straight line drawn, when $x=4$, $y \approx 4.6$, and its gradient $m \approx 4.6/4 = 1.2$ correct to 2 significant figures.

So by this experiment we have determined that the law connecting the tension in the given band (y N) and its extension (x cm) is

$$y \approx 1.2x$$

Qu. 15 A trolley accelerates down a slope from rest to v km/h in t s as shown by the following table. Determine graphically the law giving v in terms of t.

v	0	10	20	30	40	50	60
t	0	2.5	4.7	7.1	9.7	11.9	14.5

Example 5 *The following estimate is received for printing copies of a pamphlet.*

No. of copies	50	100	200	500
Cost in £	11.50	12.50	14.50	20.50

(a) *Obtain a law giving the cost, £y, of x copies.*
(b) *Estimate the cost of* 350 *copies.*

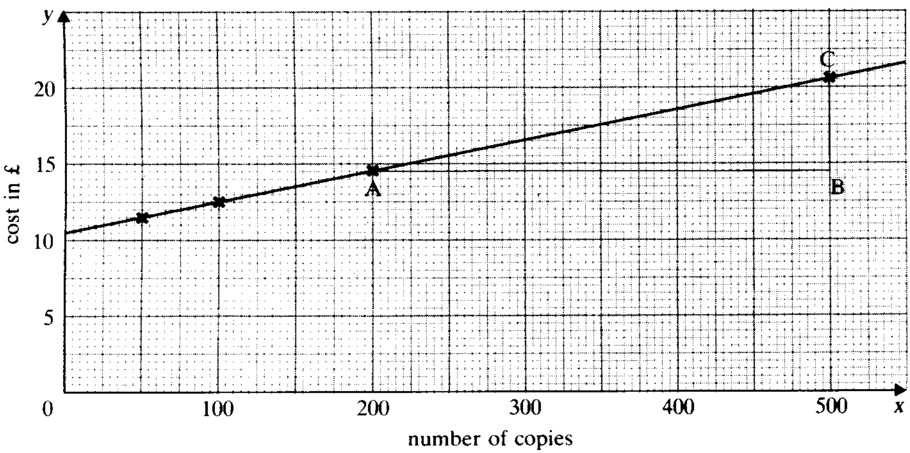

Figure 23.3

(a) Fig. 23.3 shows a straight-line graph, so we assume that the printer has used a *linear* law connecting x and y to make his estimate, i.e. there is an equation connecting the variables of the form

$$y = mx + c$$

Now c is the intercept on the y-axis (see §1.7) and so we can refer to the graph to find that $c = 10.5$, and (from the triangle ABC) that the gradient

$$m = \frac{20.50 - 14.50}{500 - 200} = \frac{6}{300} = 0.02$$

Therefore the law is

$$y = 0.02x + 10.5$$

(b) When $x = 350$,

$$\begin{aligned} y &= 0.02 \times 350 + 10.5 \\ &= 7.0 + 10.5 \\ &= 17.5 \end{aligned}$$

Therefore the cost of 350 copies is £17.50.

Qu. 16 From the solution of Example 5 (a), when $x = 0$, $y = 10.5$. What interpretation may be given to this result?

Note that in Fig. 23.3 we have included the origin of the coordinates (that is to say each axis is calibrated *from zero*) and thus we were able to utilize the y-intercept to find c. This advantage must often be sacrificed in favour of the increased accuracy obtainable by using a larger scale; Example 6 demonstrates how the equation of a straight line is determined in these circumstances.

Example 6 *Find the equation of the line* $y = mx + c$ *in Fig.* 23.4.

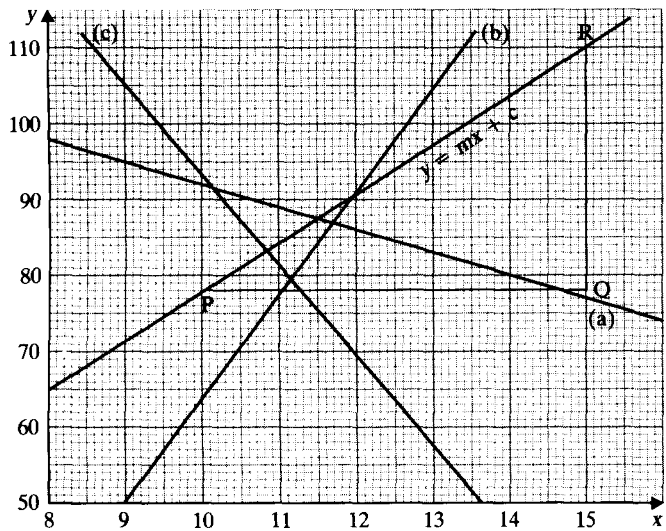

Figure 23.4

The gradient m is found from the triangle PQR (chosen so that the length of PQ is a whole number of units).

$$m = \frac{32}{5} = 6.4$$

Substituting in $y = mx + c$,

$$y = 6.4x + c$$

To find c, substitute the coordinates of a convenient point on the line, e.g. when $x = 10$, $y = 78$.

$\therefore 78 = 6.4 \times 10 + c$
$\therefore c = 14$

Therefore the required equation is $y = 6.4x + 14$.

Qu. 17 Find as accurately as possible the equations of the lines (a), (b), (c) in Fig. 23.4. (Note that two of these lines have negative gradients.)
Qu. 18 The upper end of a coiled spring was fixed and bodies were hung in turn from the lower end. The mass of the bodies (y g) and the corresponding lengths of the spring (x cm) were recorded as follows:

x	8.4	9.5	10.1	11.0	11.7	12.6	13.5	14.3
y	30	40	50	60	70	80	90	100

Find a law giving y in terms of x over this range, and estimate the unstretched length of the spring.

Linear check of non-linear laws

23.5 As we saw in §23.1, a non-linear law connecting two variables may often be considered in such a way that it involves a linear relationship. For example, if we suspect that two variables x and y are inversely proportional, we wish to show that $xy = k$, where k is a constant, i.e. $y = k \times 1/x$; this may be done by plotting y against $1/x$ and seeing if the points lie close to a straight line through the origin.

To take another example, let us suppose that the designer of a car windscreen wishes to find out if the air resistance (R N) is proportional to the square, or the cube, of the velocity (v km/h); he carries out an experiment which yields the following results:

v	20	30	40	50
R	4	14	33	60

The reader may check from a rough sketch that the graph of v against R does no more than indicate that R might vary as some power of v, which is of no real assistance. This problem is dealt with in the following question.

Qu. 19 With the data of the preceding paragraph, plot the following graphs, letting 1 cm represent 5 N:
(a) R against v^2 (on the v^2-axis let 1 cm represent 200),
(b) R against v^3 (on the v^3-axis let 1 cm represent 10 000).
Deduce an approximate relationship giving R in terms of v.
Qu. 20 A marble was allowed to run down a sloping sheet of glass and the time (t s) taken to roll s m from rest was measured by a stop watch. The results were

as follows:

s	1	2	3	4	5
t	1.4	2	2.5	2.8	3.2

Confirm that the law relating s and t is $s = kt^2$, and determine the value of the constant k to two significant figures.

Reduction of a law to linear form using logarithms*

23.6 The method of Qu. 19 is severely limited, since we assume a relationship $R = kv^n$, then we guess some integral value of n and test for it. It would be better to employ a method which tests for any rational value of n, and this is possible if we use logarithms. (At this point it may help some readers to refer back to Qu. 5 and Qu. 6 on p. 180)

Suppose that we wish to test the law

$$R = kv^n \tag{1}$$

where k and n are constants. If it is valid,

$$\log_{10} R = \log_{10} (kv^n)$$
$$\log_{10} R = \log_{10} v^n + \log_{10} k$$
$$\log_{10} R = n \log_{10} v + \log_{10} k \tag{2}$$

Writing $\log_{10} R$ as y, $\log_{10} v$ as x and $\log_{10} k$ as c, (2) becomes

$$y = nx + c$$

which represents a straight line of gradient n.

Thus if we plot $\log_{10} R$ against $\log_{10} v$ and we obtain a set of nearly collinear points, this means that we have established the linear relationship (2) and confirmed the law (1); we then draw the 'best' straight line. Its gradient determines the value of the constant n, and the constant k is found from the y-intercept c, or by the method of Example 6.

Qu. 21 From the data of Qu. 19 the following table has been prepared:

$x = \log_{10} v$	1.30	1.48	1.60	1.70
$y = \log_{10} R$	0.60	1.15	1.52	1.78

Using a scale of 0.1 to 1 cm, plot $\log_{10} R$ against $\log_{10} v$ and deduce that $R \approx 0.0005v^3$ (see Example 6, p. 454).

When a given mass of gas is compressed or allowed to expand slowly, so that there is time for the transfer of heat between the gas and its surroundings, its temperature remaining constant, the pressure (p) and the volume (V) are said to

*The reader should work some of Nos. 1 to 12 in Exercise 23c before proceeding with this section.

undergo an *isothermal* change and obey Boyle's law $pV = k$, a constant. If however the compression or expansion takes place suddenly, and there is no appreciable exchange of heat between the gas and its surroundings, then there is a change in the temperature of the gas, and the pressure and volume undergo an *adiabatic* change which does not conform to Boyle's law.

Boyle's law may be written $p = kV^{-1}$; the experimental data from an adiabatic change suggest that in this case we have the same form of relationship, $p = kV^n$, but that n has some value other than -1.

Example 7 *A given mass of air expands adiabatically and the following measurements are taken of the pressure* (p cm *of mercury*) *and volume* (V cm^3):

V	100	125	150	175	200
p	58.6	42.4	32.8	27.0	22.3

Confirm that $p = kV^n$ *and determine the values of the constants* k *and* n.

Assuming that $p = kV^n$, and taking logarithms to the base 10 of each side,

$$\log_{10} p = \log_{10} V^n + \log_{10} k$$
$$\log_{10} p = n \log_{10} V + \log_{10} k$$

Writing $\log_{10} p$ as y, $\log_{10} V$ as x, $\log_{10} k$ as c,

$$y = nx + c$$

Since this is a linear relationship between x and y, we hope to find that $\log_{10} V$ plotted against $\log_{10} p$ will yield points lying nearly on a straight line. From the following table the points have been plotted in Fig. 23.5, and the 'best' straight line has been drawn.*

$x = \log_{10} V$	2.000	2.097	2.176	2.243	2.301
$y = \log_{10} p$	1.768	1.627	1.516	1.431	1.348

The gradient n is found from triangle PQR

$$n = -\frac{0.28}{0.2} = -1.4$$

Therefore the equation of the straight line is

$$y = -1.4x + c$$

* Provided that the experimental errors are random, then a reliable aid to drawing the 'best' straight line is to make it pass through the point whose coordinates are the averages of the coordinates of the plotted points; this point is shown in Fig. 23.5. Sometimes there is also a point whose exact coordinates are known; such a point is (0, 0) in Fig. 23.2.

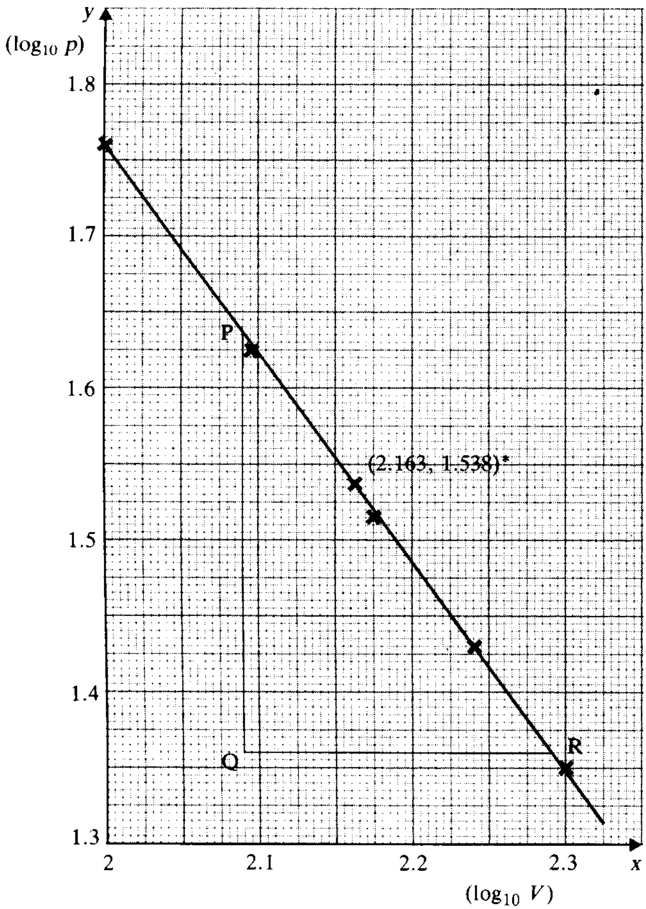

Figure 23.5
*See footnote, page 457

But from the graph, when $x=2$, $y=1.77$,

$\therefore\ 1.77=-1.4\times 2+c$
$\therefore\ \log_{10} k=c=4.57$
$\therefore\ k=37\,150=37\,000$ to two significant figures

Hence the experimental data confirms the relationship given between p and V, namely

$$p\approx 37\,000\ V^{-1.4}$$

There are other types of variation which may be confirmed by using logarithms to reduce them to a linear relationship; in laws of growth, for example, one of the variables is often in an index.

If $P = ka^x$, where k, a, are constants,

$$\log_{10} P = \log_{10} a^x + \log_{10} k$$
$$\therefore \log_{10} P = x \log_{10} a + \log_{10} k$$

and writing $\log_{10} P$ as y, $\log_{10} a$ as m, $\log_{10} k$ as c,

$$y = mx + c$$

This is a straight-line equation which reveals a linear relationship between x and $\log_{10} P$.

Example 8 *The frequency (f oscillations per second) and the interval (x semitones) of each note of a C major scale are given in the table below; show that f, x are related by a law in the form $f = ka^x$ and determine the constants k, a.*

Note	C	D	E	F	G	A	B	C
x	0	2	4	5	7	9	11	12
f	256	287	323	342	384	431	483	512

Assuming that $f = ka^x$, and taking logarithms to the base 10 of each side,

$$\log_{10} f = \log_{10} a^x + \log_{10} k$$
$$\therefore \log_{10} f = x \log_{10} a + \log_{10} k$$

Writing $\log_{10} f$ as y, $\log_{10} a$ as m, $\log_{10} k$ as c,

$$y = mx + c$$

This shows that we must, from the data, establish a linear relationship between x and $\log_{10} f$. From the following table the points have been plotted in Fig. 23.6 and the 'best' straight line has been drawn; we have confirmed that the law is of the form $f = ka^x$.

x	0	2	4	5	7	9	11	12
$y = \log_{10} f$	2.408	2.458	2.509	2.534	2.584	2.634	2.684	2.709

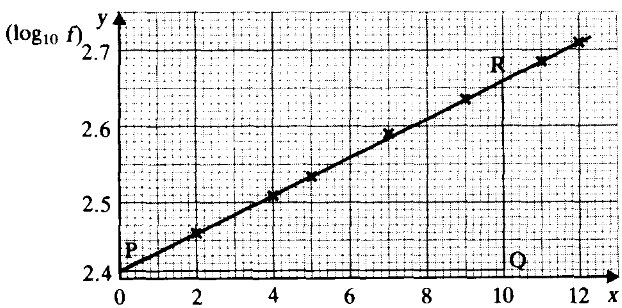

Figure 23.6

If we now consider the straight line in Fig. 23.6 to have the equation $y = mx + c$, we see from the triangle PQR that its gradient

$$m = \log_{10} a = \frac{0.25}{10} = 0.025$$

From antilogarithm tables or a calculator, $a = 1.059$.

We may now write $y = 0.025x + c$.

From the graph, when $x = 0$, $y = 2.408$.

$\therefore c = \log_{10} k = 2.408$

From antilogarithm tables or a calculator, $k = 256$.

Hence from the data we deduce the required law to be

$f = 256 \times 1.059^x$ *

Qu. 22 A remote and isolated tribe came under the influence of medical missionaries in 1935, when a very careful count of the population was made. Less reliable counts were made in later years, as shown in the following table:

Year	1935	1940	1950	1955
Lapse of years (t)	0	5	15	20
Population (P)	2070	2500	4200	5100

Show that the data points to the operation of a law of the form $P = ka^t$, and determine the constants k, a. Also estimate the population in 1948.

Two final points deserve mention, starting with a word of warning. A graphical method may confirm that a certain law is obeyed but *only within the given ranges of values of the variables*; guard against false deductions. For example, remember that an elastic band may be stretched beyond its elastic limit; or a gas undergoing changes of pressure and volume may also be approaching a change of state.

Secondly, the use of logarithmic graph paper has not been mentioned in this chapter. It can be a time-saver in repetitive work, and the reader who has mastered the idea of this last section will have no difficulty in using it should the need arise.

Exercise 23c

1 A round bolt with nominal diameter D mm has a countersunk head of diameter A mm. D and A are found to be as follows:

*In fact standard musical pitch has been set slightly higher than that used in this example, with $f = 440$ for A above middle C, giving $f = 261.6$ for middle C. Also, since the ratio of the frequency of a note to that of an octave below is 2:1, $a^{12} = 2$ and calculation gives a better value of a as 1.05946. Thus the corresponding law for a correctly tuned piano is $f = 261.6 \times 1.059^x$.

D	6.4	7.9	9.5	11.1	12.7	15.9	19.0	22.2	25.4
A	11.7	14.6	17.5	20.4	23.4	29.2	35.0	40.9	46.7

Find the linear equation giving A in terms of D. Does A vary as D?

2 The mass m kg of a 300 mm square of lead sheeting of thickness t mm is given as follows:

t	1.25	1.80	2.24	2.50	3.15	3.55
m	1.275	1.835	2.285	2.550	3.215	3.625

Obtain a linear relation giving m in terms of t. What is the connection between the gradient of the graph of m against t and the relative density of lead?

3 A marble was dropped from a height h_1 cm and observed to rise to a height h_2 cm. Four such observations are given in the table below:

h_1	4	9	16	22
h_2	$1\frac{1}{2}$	3	$5\frac{1}{2}$	$7\frac{1}{2}$

Does it appear that there is a law connecting h_1, h_2? If so, what is it?

4 A letter in a daily paper gave the following table relating the deaths in a certain group due to lung cancer with the number of cigarettes smoked per day.

No. of cigarettes per day n	0	1 to 14	15 to 24	over 25
Deaths per 100 000 per annum d	7	57	139	227

Investigate the justification for assuming from these figures that a linear relationship exists.

5 Some printers quoted the price of a small book as follows:

No. of copies	500	1000	2000	5000	6000
Cost in £	650	865	1300	2600	3035

Does this bear out the idea that one gets a reduction for ordering in quantity? Can you estimate the cost of (a) 3500 copies, (b) getting the type set up ready to print, without running off any copies?

6 A man bought a car when the distance travelled registered as 71 km, the fuel tank containing an unknown amount of petrol. According to his log book, he bought 20 litres of petrol at the following kilometre readings:

241, 432, 685, 907, 1123

Estimate the average number of km travelled per litre of petrol up to the last distance.

Given that the car ran out of petrol at 1378 km, estimate the quantity of petrol originally in the fuel tank.

7 While some water was cooling, the temperature was recorded at minute intervals as follows:

Time t minutes	0	1	2	3	4	5	6
Temperature θ°C	62	61.5	61	60.5	60	59.5	59

Find an equation giving θ in terms of t. Can you expect this equation to hold over a wider range of values? Give reasons for your answer.

8 The flow of water through a circular hole is thought to vary as the square root of the head of water. For a certain hole, the following results were obtained:

Head of water, h m	1.5	3	4.5	6
Flow of water, x litres/min	119	170	205	240

Do they confirm the conjecture? Estimate the flow of water through the hole when the head is 5 m.

9 A crane on a building site displayed the following figures:

Load in tonnes	2	1.5	1	0.75
Radius in metres	7.5	10	15	20

Do these figures confirm the expectation that the radius is inversely proportional to the load? Is there an equation giving the load l tonnes in terms of the radius r metres?

10 The mass m kg of 100 m lengths of a certain type of steel wire rope is given for nominal diameters d mm as follows:

d	8	9.5	11	13	16	19
m	21.6	30.5	40.9	57.2	86.6	122

Examine the suggestion that the mass varies as the square of the nominal diameter of the rope.

11 A hose squirts a stream of water horizontally and the height of the stream y m at distance x m along level ground is estimated to be as follows:

Distance x m	0	2	4	5	6	7	8
Height y m	3.50	3.40	3.10	2.88	2.60	2.28	1.90

Obtain an equation in the form $y = a + bx^2$ connecting these values approximately.

12 For purposes connected with a survey, the digits 0, 1, 2,..., 9 were required in a random order. However, when they were taken from a list of random numbers, it was noticed that the intervals between new digits tended to increase. Noting the intervals on a number of occasions the following averages were obtained:

Position of digit p	1	2	3	4	5	6	7	8	9	10
Average interval i	1.0	1.2	1.4	1.4	2.1	1.7	2.3	4.0	3.9	16.2

Find a law in the form $p = a - b/i$. [In finding a, use the fact that $i = 1$ when $p = 1$.] Hence express i in terms of p.

13 The periods and mean distances of some of the planets are given in the table below:

Period P days	87.97	224.7	365.3	687.0	4333	10 760
Mean distance s in millions of km	58	108	150	228	778	1426

Find a law in the form $P = ks^n$.

14 For a certain survey in which n people are to be interviewed, a market research organisation calculates that it has an even chance of obtaining correct within p% the percentage in favour of the product concerned in the survey. n and p are related as below:

n	500	1000	2000	5000	10 000
p	1.51	1.07	0.75	0.48	0.34

Find how p varies with n.

15 Some molecules are made out of two atoms. The moment of inertia and the distance between the nuclei of the atoms is given for four such molecules in the table below:

Moment of inertia I (10^{-40} g cm^2)	1.34	2.66	3.31	4.31
Distance between nuclei r (10^{-8} cm)	0.92	1.28	1.42	1.62

Find a law in the form $I = kr^n$. (Source of data: S. Glasstone, *Theoretical Chemistry*.)

16 The widths of successive whorls of a shell of *Turbo duplicatus* have been measured:

Position of whorl n	1	2	3	4	5	6	7	8
Width of whorl w cm	3.33	2.84	2.39	2.03	1.70	1.45	1.22	1.04

Find a law in the form $w = ab^n$. (Source of data: H. Moseley, *Phil. Trans.* 1838, 356.)

17 Two substances in a chemical reaction have the same initial concentration a moles per litre, and after t min the concentration of each is $(a - x)$ moles per litre. The following experimental results were obtained:

t	5	15	25	35	55	120
$a-x$	10.24	6.13	4.32	3.41	2.31	1.10

In order to establish that this is a second-order reaction (i.e. the rate of reaction $\frac{dx}{dt}$ is a quadratic function of x) show graphically that a linear relationship exists between t and the reciprocal of $(a-x)$; deduce that $\frac{dx}{dt} = k(a-x)^2$, and determine the value of k, the reaction velocity constant.

18 A given mass of ozone is subjected to an adiabatic change and the pressure $p\ 10^{-10}$ N/m^2 and volume V cm^3 are observed as follows:

Volume V cm^3	100	90	80	70	60	50
Pressure p (10^{-10} N/m^2)	1.18	1.35	1.57	1.82	2.27	2.87

Verify graphically that $pv^\gamma = k$, where γ, k are constants, and determine the value of γ.

19 Steinmetz's law, $E = \eta B^{1.6}$, gives an approximation for the energy lost per cycle of magnetisation in a transformer core, where the energy lost is E ergs/cm^3, the maximum magnetic flux density is B gauss, and η is the Steinmetz coefficient for the given material. Values of B and E are tabulated below:

B	1000	2000	3000	4000	5000	6000
$\frac{E}{10^3}$	0.316	0.956	1.83	2.90	4.14	5.55

Use a graphical method to show that these values agree with the given law, and determine the value of η for this material.

20 In the Ehrenfest game, n balls numbered from 1 to n are placed in a container A and another container B is left empty. Numbers in the range 1 to n are drawn at random. When a number is drawn, the corresponding ball is transferred from the container it is in to the other container. In such a game with $n = 100$, the total T balls left in container A after x numbers had been drawn was as follows:

x	0	10	20	30	40	50
T	100	92	84	78	72	68

Find a law in the form $T = ab^{-x}$, where a, b are constants, to fit these data as well as possible. As x becomes large, can T be expected to obey this law?

Chapter 24

Iterative methods for solving equations

Introduction

24.1 One of the most common tasks in mathematics is to solve an equation. In this book we have already solved a variety of different equations. We have solved quadratic equations by factorisation or by the formula, we have solved other polynomial equations by factorising them and we have solved some carefully selected trigonometrical equations.

Consider, however, the following problem. Fig. 24.1 represents a circle, whose centre is at O, and whose radius is one unit. Can we find the value of θ, in radians, so that the area of the shaded segment is exactly 0.5 square units?

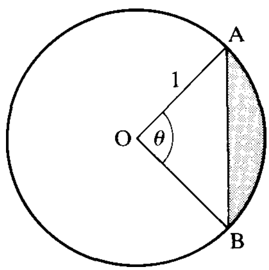

Figure 24.1

Since the angle θ is measured in radians, the area of the *sector* OAB is $\frac{1}{2}\theta r^2$, and since $r = 1$, this is just $\frac{1}{2}\theta$. The area of the *triangle* OAB can be obtained from the standard formula, $\frac{1}{2}ab \sin C$; in this case $a = 1$, $b = 1$, and $C = \theta$, so the area of the triangle OAB is $\frac{1}{2} \sin \theta$. The area of the shaded segment is the difference of these two areas, i.e.

$$\tfrac{1}{2}\theta - \tfrac{1}{2} \sin \theta$$

The problem is to find the value of θ so that this area is 0.5 square units. In other words we need to solve the equation

$$\tfrac{1}{2}\theta - \tfrac{1}{2}\sin\theta = \tfrac{1}{2}$$

or $$\theta - \sin\theta = 1$$

None of the methods for solving equations at our disposal (apart from drawing a graph) would enable us to solve this equation; indeed it is impossible to find an *exact* solution. However, there is no doubt that such an angle exists, and with a little experimentation using tables or a calculator, it is possible to see that an *approximate* solution is $\theta = 2$.

In this chapter we shall develop methods by which approximate solutions to equations can be obtained. An approximate solution should not be despised, for it can be very useful, and, as in the example above, it may be the only solution available. The value of such an answer is greatly enhanced if it is possible to give an estimate of its degree of accuracy.

Later in the chapter we shall return to the equation $\theta - \sin\theta = 1$, but first we shall tackle a simpler problem, namely, can we find the square root of a given number without using square root tables, or the square root function on a calculator?

An iterative method for finding square roots

24.2 What is the square root of 18? Or, to put it another way, solve the equation

$$x^2 = 18$$

Since we are not going to use tables or the square root function on a calculator, the most sensible first step is to check through the 'square numbers'

$$1,\quad 4,\quad 9,\quad 16,\quad 25,\quad 36,\quad 49,\quad \ldots$$

and note that $\sqrt{18}$ lies between 4 and 5, and that it is nearer 4 than 5. So we might say

'the square root of 18 is 4, correct to the nearest whole number'

This at least gives an approximate answer and it indicates the degree of accuracy of this approximate answer.

We shall now use this 'first approximation' to obtain a better 'second approximation', and this in turn will be used to form an even better 'third approximation'. Such a procedure is called **successive approximation**, or **iteration**.

The method we shall use to find the successive approximations will depend upon the fact that if x is exactly equal to $\sqrt{18}$, then $18/x$ is exactly equal to $\sqrt{18}$. If x does *not* equal $\sqrt{18}$, then

either x is less than $\sqrt{18}$, in which case $18/x$ is greater than $\sqrt{18}$,
or x is greater than $\sqrt{18}$, in which case $18/x$ is less than $\sqrt{18}$.

In both cases, we can say that $\sqrt{18}$ lies between x and $18/x$.

Consequently, using $\sqrt{18} \approx 4$ as a 'first approximation', we know that $\sqrt{18}$ lies between 4 and 18/4, i.e. between 4 and 4.5, so we take as our 'second approximation' the average of these two numbers, i.e.

$$\frac{1}{2}\left(4 + \frac{18}{4}\right) = 4.25$$

Now we repeat the process, using $\sqrt{18} \approx 4.25$. Once again we know that $\sqrt{18}$ must lie between 4.25 and 18/4.25, and so we take as our 'third approximation' the average of *these* two numbers. In other words the third approximation is

$$\frac{1}{2}\left(4.25 + \frac{18}{4.25}\right) = 4.24265, \quad \text{correct to six significant figures}$$

(The arithmetic at this stage is becoming rather heavy, and a calculator or tables may be used to lighten the load. However, square root tables and the square root function on the calculator are *not* allowed!)

We now have a very good approximate value of the square root of 18, and we know that the exact value lies between 4.24265 and 18/4.24265 (= 4.24263). So we are now able to say that

$$\sqrt{18} = 4.243, \quad \text{correct to four significant figures}$$

knowing that we are justified in claiming this degree of accuracy.

This procedure can be summed up as follows: writing x_r for the rth approximation, the $(r+1)$th approximation is given by

$$x_{r+1} = \frac{1}{2}\left(x_r + \frac{18}{x_r}\right)$$

This is called an **iterative formula** for finding $\sqrt{18}$. More generally, the iterative formula for finding the square root of any positive number, N, is

$$x_{r+1} = \frac{1}{2}\left(x_r + \frac{N}{x_r}\right)$$

Qu. 1 Use the iterative formula above, to find the square roots of
(a) 17, (b) 40, (c) 85, (d) 96, correct to four significant figures.

Historical note. This method for calculating square roots is a very old one. It was used by the Babylonians more than three thousand years ago. Today it is frequently called Newton's algorithm, but this is hardly fair to those great, but nameless, mathematicians from Mesopotamia.

If a programmable calculator or a microcomputer is available, the reader should try to write programs to solve some of the equations in this chapter by iteration. Iterative methods are ideally suited to such an approach, because the same basic sequence of steps is repeated over and over again; this can be done very rapidly and accurately on a programmable calculator or a microcomputer.

Further iterative formulae

24.3 If we were given the iterative formula

$$x_{r+1} = \frac{1}{2}\left(x_r + \frac{18}{x_r}\right)$$

but we did not know how it had been constructed, would it be possible to discover the equation which it is designed to solve? The answer is 'Yes', provided the sequence

$$x_1, x_2, x_3, x_4, \ldots$$

tends to a limit. Suppose that $x_n \to X$, as $n \to \infty$, then for a large value of n, the iterative formula could be written

$$X = \frac{1}{2}\left(X + \frac{18}{X}\right)$$

This equation could then be simplified, as follows:

$$2X = X + \frac{18}{X}$$

$$\therefore X = \frac{18}{X}$$

$$\therefore X^2 = 18$$

So, as expected, we see that the equation which is solved by the iterative formula above is

$$x^2 = 18$$

Example 1 *Starting with $x_1 = 4$, use the iterative formula*

$$x_{r+1} = 5 - \frac{2}{x_r}$$

to find x_2, x_3, and x_4, giving these values correct to three significant figures. Find the equation which is solved by this iterative formula.

$$x_2 = 5 - \frac{2}{4}$$

$$= 4.5, \quad \text{exactly}$$

$$x_3 = 5 - \frac{2}{4.5}$$

$$\approx 4.55556$$

$$= 4.56, \quad \text{correct to three significant figures}$$

$$x_4 = 5 - \frac{2}{4.55556}$$

$$\approx 4.56098$$

$$= 4.56, \quad \text{correct to three significant figures}$$

The successive values of x_r appear to be tending to a limit, namely 4.56.

(*Note.* If you are using a calculator for the arithmetic, the successive values x_2, x_3, x_4 etc. should be retained on the calculator. It is poor technique to use the *corrected* value of x_r to calculate x_{r+1}. However, if you are answering an examination question which requires a specific degree of accuracy in presenting answers, you should always follow this instruction; it is usually there to simplify the task of marking the answer and it is very unwise to upset the examiner!)

To find the equation which this iterative formula solves, we write this limit as X, then, for large values of r, the iterative formula becomes

$$X = 5 - \frac{2}{X}$$

When this is simplified we obtain

$$X^2 - 5X + 2 = 0$$

So $x = 4.56$ is a root, correct to three significant figures, of the equation

$$x^2 - 5x + 2 = 0$$

(This equation is of course a quadratic equation, and using an iterative method to solve it is using a sledge-hammer to crack a nut. However, at this stage it is more convenient to use fairly simple equations for the examples. If this equation is solved by the formula, the solution would be

$$x = \frac{5 \pm \sqrt{17}}{2} = 4.56 \text{ or } 0.44, \quad \text{correct to two decimal places}$$

The iterative formula has produced the first of these, but not the second. However, we could use the fact that the sum of the roots is 5 to calculate the second root, i.e. $5 - 4.56 = 0.44$.)

As we have seen above, if the sequence $x_1, x_2, x_3, x_4, \ldots$ converges, then we can deduce the equation from the iterative formula. This suggests that if we have a given equation and we wish to construct a suitable iterative formula, all we need to do is to rearrange the equation in the form

$$x = f(x)$$

and the corresponding iterative formula will be

$$x_{r+1} = f(x_r)$$

Example 2 *Form an iterative formula to solve the equation*

$$x^3 - 5x + 1 = 0$$

and use it to find the root which lies between 0 *and* 1, *correct to three significant figures.*

The given equation can be arranged in the form

$$5x = x^3 + 1$$

$$x = \frac{x^3 + 1}{5}$$

consequently we shall take as the iterative formula

$$x_{r+1} = \frac{x_r^{\,3} + 1}{5}$$

and, starting with $x_1 = 0$, we obtain

$$x_2 = \frac{1}{5} = 0.2$$

$$x_3 = \frac{1.008}{5} = 0.2016$$

$$x_4 = \frac{0.2016^3 + 1}{5}$$

$$= 0.201639, \quad \text{correct to six significant figures}$$

In view of the very small change from x_3 to x_4, it would be reasonable to conclude that we are now *very* near to the exact answer. Consequently we could claim, with some confidence, that the root of the equation is 0.202, correct to three significant figures.

However, the reader must not run away with the idea that *any* rearrangement of the original equation will yield a suitable iterative formula. Consider, for example, the following equation:

$$x^2 - 5x + 3 = 0$$

It is easy to verify that this has a root between 4 and 5.

The rearrangement

$$x = \frac{x^2 + 3}{5}$$

produces the iterative formula

$$x_{r+1} = \frac{x_r^{\,2} + 3}{5}$$

If we start at $x_1 = 5$, the succeeding values of x_r, correct to four significant figures, are

$$x_2 = \frac{25 + 3}{5} = 5.6$$

$$x_3 = \frac{5.6^2 + 3}{5} = 6.872$$

$$x_4 = \frac{6.872^2 + 3}{5} = 10.04$$

$$x_5 = \frac{10.04^2 + 3}{5} = 20.78$$

These values of x_r are getting further and further away from the root we were expecting; we say the sequence $x_1, x_2, x_3, \ldots$ is *diverging*. However, the rearrangement of the original equation was by no means the only possible one. Consider, for example,

$$x = 5 - \frac{3}{x}$$

This gives the iterative formula

$$x_{r+1} = 5 - \frac{3}{x_r}$$

and if we start, as before, with $x_1 = 5$, we obtain

$$x_2 = 5 - \frac{3}{5} = 4.4$$

$$x_3 = 5 - \frac{3}{4.4} = 4.318$$

$$x_4 = 5 - \frac{3}{4.318} = 4.305$$

$$x_5 = 5 - \frac{3}{4.305} = 4.303$$

$$x_6 = 5 - \frac{3}{4.303} = 4.303$$

(The root given by the quadratic formula is 4.303.)

So this second rearrangement has worked satisfactorily.

We can see from this that not all rearrangements of a given equation lead to a suitable iterative formula. We could decide to discard any iterative formula which produces a divergent sequence, but it would clearly be more satisfactory if we had some method for discriminating between a formula which produces a divergent sequence and one which produces a convergent sequence; we shall tackle this in the next section.

Exercise 24a

1 Use the iterative formula in §24.2 to find the square roots of
(a) 12, (b) 30, (c) 50, (d) 75,
giving your answers correct to three significant figures.

2 Use the iterative formula

$$x_{r+1} = \frac{2x_r}{3} + \frac{4}{x_r^2}$$

starting at $x_1 = 2$, to find x_2, x_3 and x_4, giving your answers correct to three significant figures. Find, in its simplest form, the equation which is solved by this iterative formula.

3 Adapt No. 2 so that it can be used to find $20^{1/3}$.

4 Show that the equation $x^2 - 5x + 1 = 0$ can be arranged as $x = (x^2 + 1)/5$, or, alternatively, as $x = 5 - 1/x$. Hence write down two possible iterative formulae which might be used for solving this quadratic, and, starting from $x_1 = 0.2$, find the values of x_2, x_3 and x_4 which are produced by each of these iterative formulae.

Only one of these sequences appears to converge; use this sequence to write down the (two) roots of the quadratic equation.

5 The cubic equation $x^3 - 10x + 1 = 0$ can be rearranged in the form $x = (x^3 + 1)/10$.

Use this rearrangement to form an iterative formula and use it to find, correct to four significant figures, the root which lies between 0 and 1. (Start with $x_1 = 0$.)

6 Solve the equation in §24.1, that is $\theta = \sin\theta + 1$, by an iterative method, starting from $\theta = 2$. (θ is measured in *radians*.)

7 Show that the equation $x^2 - 8x + 10 = 0$, has a root between 1 and 2.

Show that the iterative formula $x_{r+1} = 8 - 10/x_r$, can be formed from this equation, and, starting from $x_1 = 1$, calculate the values of x_2, x_3 and x_4. Comment on your results.

8 The iterative formulae

$$\text{(a)}\ x_{r+1} = \frac{2x_r^3 + 10}{3x_r^2} \qquad \text{and} \qquad \text{(b)}\ x_{r+1} = \frac{10}{x_r^2}$$

can both be obtained by rearranging the equation $x^3 - 10 = 0$.

Starting from $x_1 = 2$, find the values of x_2, x_3 and x_4, which are produced by these iterative formulae. Only one of these sequences converges; use this one to find $\sqrt[3]{10}$, correct to four significant figures.

9 The fifth root of a real number N can be calculated from the iterative formula

$$x_{r+1} = \left(4x_r + \frac{N}{x_r^4}\right) \Big/ 5$$

Use this formula to find the fifth root of 50, correct to three significant figures. [Hint: start with $x_1 = 2$.]

10 The product of the roots of the quadratic equation

$$x^2 - px + q = 0$$

is q, so if x_r is an approximate value of one of the roots, the other could be written q/x_r. Use the fact that the sum of the roots of this quadratic equation is p to find a new approximation to the first root. Hence deduce the iterative formula

$$x_{r+1} = p - \frac{q}{x_r}$$

Use this iterative formula to solve the quadratic equation

$$x^2 - 7x + 3 = 0$$

giving your answers correct to three significant figures.

Iteration — the test for convergence

24.4 In the preceding sections we have seen that an iterative formula

$$x_{r+1} = f(x_r)$$

can be used to produce a sequence of values of x_r

$$x_1, x_2, x_3, x_4, \ldots$$

the value of x_1 being selected by trial and error. We have also seen (but not formally proved) that, provided the sequence tends to a limit, which we shall call X, then $x = X$ is a root of the equation

$$x = f(x)$$

In this section we shall examine the conditions under which we can expect the sequence $x_1, x_2, x_3, x_4, \ldots$ to converge. (Example 1 will be used as an illustration, so the reader is advised to read through this example again before proceeding.)

Fig. 24.2 shows the graphs of $y = x$ and $y = f(x)$, where $f(x) = 5 - 2/x$. The graphs intersect at $P(X, Y)$.

The x-coordinate of the point P, that is X, is a solution of the equation

$$x = f(x)$$

This is the root of the equation which we expect to obtain from the iterative formula

$$x_{r+1} = f(x_r)$$

The diagram in Fig. 24.3 shows an enlargement of the region around the point P in the previous diagram. It also shows the points $P_1, P_2, P_3, \ldots$, whose coordinates are $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots$ respectively, where $x_1, x_2, x_3, \ldots$ are the successive approximations given by the iterative formula.

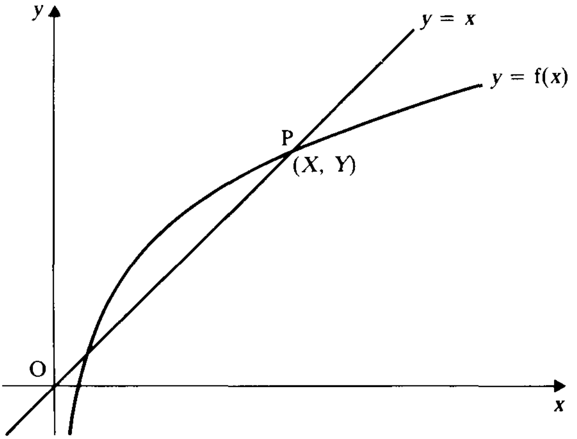

Figure 24.2

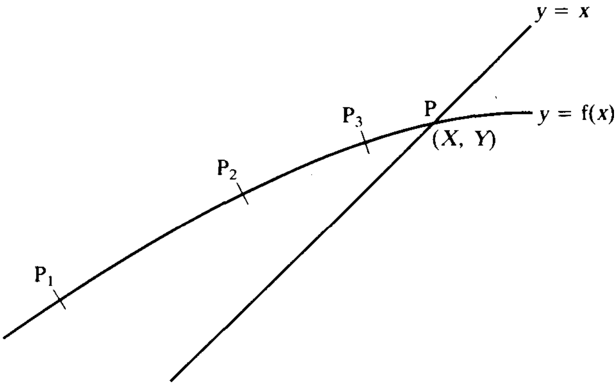

Figure 24.3

Since the point P_r, whose coordinates are (x_r, y_r), lies on the curve $y = f(x)$, the y-coordinate is given by

$$y_r = f(x_r)$$

and this in turn is equal to x_{r+1}, so the coordinates of P_r can be written (x_r, x_{r+1}). This lets us produce the following geometrical method for constructing the points $P_1, P_2, P_3, \ldots$ (see Fig. 24.4). First mark the point (x_1, x_2), remembering that x_1 is selected on a trial-and-error basis. From P_1 draw a line horizontally, i.e. parallel to the x-axis, and call the point where this meets the line, Q_1. The points P_1 and Q_1 have the same y-coordinate and Q_1 lies on the line $x = y$, so the coordinates of Q_1 are (y_1, y_1). But $y_1 = x_2$, so these coordinates could be written (x_2, x_2). From Q_1 we now draw a line vertically, i.e. parallel to the y-axis. The point where this meets the curve has the same x-coordinate as Q_1 and so its coordinates are (x_2, x_3). This is the point P_2. We now repeat the operation to construct the subsequent points P_3, P_4, P_5, ..., but because space is limited, only the first few points are printed.

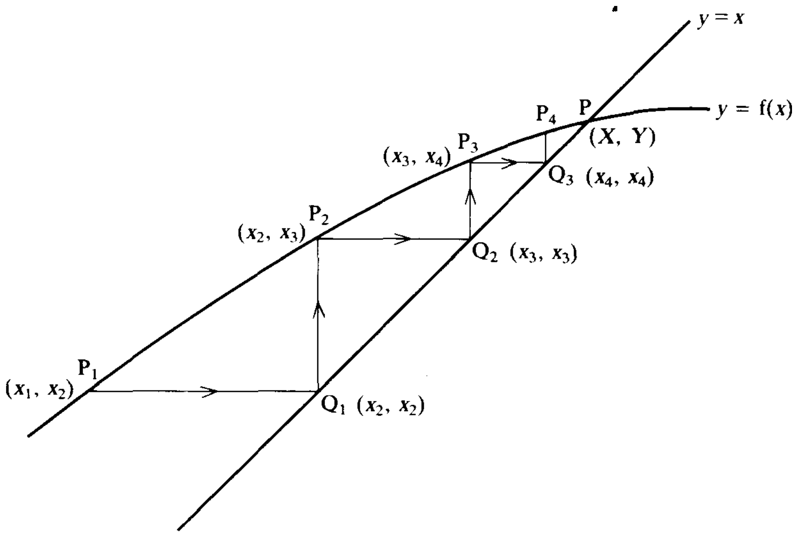

Figure 24.4

In this diagram we can see the points P_1, P_2, P_3, ... getting closer and closer to the point P itself, and so the x-coordinates $x_1, x_2, x_3, \ldots$ will be getting closer and closer to X, or, to put it more formally, $x_r \to X$, as $r \to \infty$.

Although the function $f(x) = 5 - 2/x$ has been used in this illustration, a diagram like that in Fig. 24.4 could be drawn for other functions *provided* $f'(x)$ *lies between* 0 *and* 1. If the gradient is greater than 1 the picture is quite different. Fig. 24.5 shows the same construction applied to the graph of a function whose gradient is greater than 1. In this case, each step moves P_r further and further away from P.

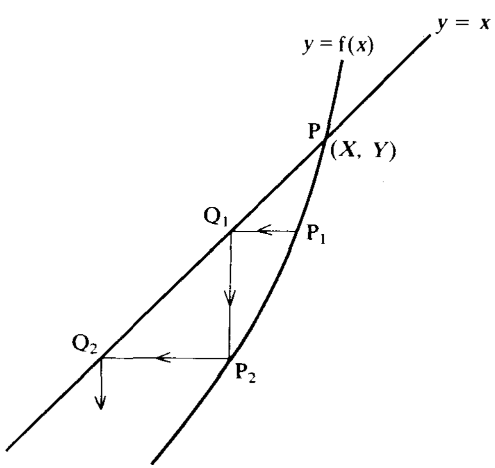

Figure 24.5

The diagrams in Fig. 24.6 show the corresponding constructions for graphs whose gradients are negative.

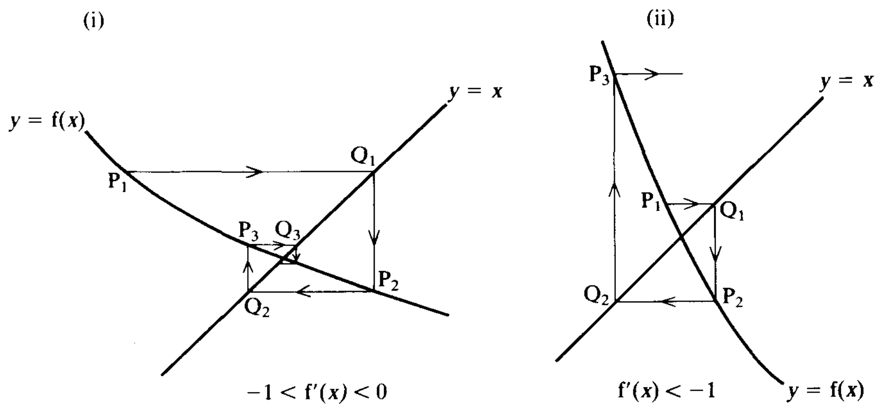

Figure 24.6

The first diagram (in which $-1 < f'(x) < 0$) shows the points $P_1, P_2, P_3, \ldots$ getting closer and closer to P; in other words, the sequence $x_1, x_2, x_3, \ldots$ converges when $|f'(x)| < 1$. In contrast the second diagram (in which $f'(x) < -1$) shows these points moving further and further away from P, and so the sequence $x_1, x_2, x_3, \ldots$ diverges when $|f'(x)| > 1$.

From these diagrams we can conclude that the sequence $x_1, x_2, x_3, \ldots$ will converge if $|f'(x)| < 1$. To ensure that this sequence converges rapidly, the initial approximation should be as close as possible to the exact root and the function $f(x)$ should be selected so that $|f'(x)|$ is as small as possible.

(A more rigorous proof is beyond the scope of this book; any reader who wishes to know more should consult a more specialised textbook. This topic usually comes under the heading 'Numerical methods'.)

Example 3 *Show that one of the iterative formulae*

(a) $x_{r+1} = (x_r^2 + 3)/5$, (b) $x_{r+1} = 5 - 3/x_r$,

produces a convergent sequence for $x \approx 5$, and the other does not.

In iterative formula (a),

$$f(x) = \frac{x^2 + 3}{5}$$

$$f'(x) = \frac{2x}{5}$$

hence,

$$f'(5) = 2$$

Since $|f'(5)| > 1$, formula (a) will *not* produce a convergent sequence when $x \approx 5$.

In formula (b),

$$f(x) = 5 - \frac{3}{x}$$

$$f'(x) = \frac{3}{x^2}$$

hence,

$$f'(5) = \frac{3}{25} = 0.12$$

In this case $|f'(5)| < 1$, so formula (b) *will* produce a convergent sequence when $x \approx 5$.

(*Note*. These formulae were used earlier in this chapter, see pp. 471–472.)

Exercise 24b

Which of the following iterative formulae should, according to the test in the preceding section, produce a convergent sequence, $x_1, x_2, x_3, \ldots$, in the region of the value of x indicated? (These iterative formulae were used in Exercise 24a.)

1 $x_{r+1} = \dfrac{1}{2}\left(x_r + \dfrac{12}{x_r}\right)$; $x \approx 3$.

2 $x_{r+1} = \dfrac{2x_r}{3} + \dfrac{4}{x_r^2}$; $x \approx 2$.

3 $x_{r+1} = \dfrac{x_r^2 + 1}{5}$; $x \approx 0.2$.

4 $x_{r+1} = 5 - \dfrac{1}{x_r}$; $x \approx 0.2$.

5 $x_{r+1} = \dfrac{x_r^3 + 1}{10}$; $x \approx 1$.

6 $\theta_{r+1} = \sin\theta_r + 1$; $\theta \approx 2$.

7 $x_{r+1} = 8 - \dfrac{10}{x_r}$; $x \approx 1$.

8 $x_{r+1} = \dfrac{2x_r^3 + 10}{3x_r^2}$; $x \approx 2$.

9 $x_{r+1} = \dfrac{10}{x_r^2}$; $x \approx 2$.

10 $x_{r+1} = \left(4x_r + \dfrac{50}{x_r^4}\right)\Big/5$; $x \approx 2$.

The Newton–Raphson method

24.5 We now come to a particular method of iteration known as the Newton–Raphson method (it is frequently called Newton's method). Throughout this section we shall be considering the task of solving an equation of the form $F(x) = 0$ and the exact root we are seeking will be denoted by X.

As with all iterative methods, the first step is to find an approximate root. This can be done quite conveniently by drawing the graph of $y = F(x)$. The *exact* root is the x-coordinate of the point where the graph crosses the x-axis. Fig. 24.7 shows the graph of $y = F(x)$ and the point $P(X, 0)$.

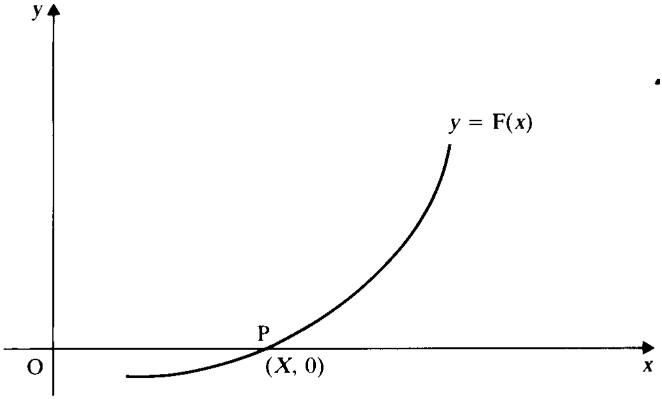

Figure 24.7

Now consider the enlargement of the region surrounding P, which is shown in Fig. 24.8.

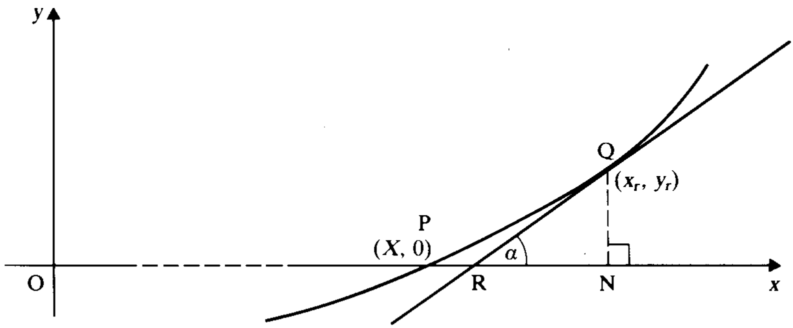

Figure 24.8

In this diagram, the point Q is near the point P and its x-coordinate x_r is an approximation to the exact root X, i.e. $x_r \approx X$. The coordinates of Q then are $(x_r, F(x_r))$. Newton's method consists of drawing a tangent to the curve at Q, and, if this line meets the x-axis at R, using the x-coordinate of R as the next approximation to X. In other words R is the point $(x_{r+1}, 0)$. It is clear from the diagram that x_{r+1} will be a better approximation than x_r.

(The reader is advised to draw the corresponding diagram for a graph whose gradient is negative, and also to consider the effect of $F(x_r)$ being negative. From these diagrams the reader should be able to see that Newton's method will yield the desired approximation, provided $F'(x)$ is not zero near the exact root.)

From the diagram in Fig. 24.8, we can produce a formula for x_{r+1}, in terms of the function $F(x)$ and x_r.

We know that

$$NQ = F(x_r)$$

and by elementary trigonometry

$$\frac{NQ}{RN} = \tan\alpha$$

i.e. $$RN = \frac{NQ}{\tan\alpha}$$

But, since the line RQ is the tangent to the curve at Q, $\tan\alpha$ is equal to the gradient at Q. In other words

$$\tan\alpha = F'(x_r)$$

So we can write

$$RN = \frac{F(x_r)}{F'(x_r)}$$

Now, from the diagram we can see that

$$OR = ON - RN$$

$$\therefore OR = x_r - \frac{F(x_r)}{F'(x_r)}$$

and since Newton's method is to use the x-coordinate of R as the new approximation, we have

$$\boldsymbol{x_{r+1} = x_r - \frac{F(x_r)}{F'(x_r)}}$$

Example 4 *Verify that the equation $x^3 - 5x - 40 = 0$ has a root between $x = 3$ and $x = 4$. Use the Newton–Raphson method to find this root correct to three significant figures.*

In this example,

$$F(x) = x^3 - 5x - 40$$

Putting $x = 3$ gives

$$F(3) = 27 - 15 - 40 = -28$$

and, putting $x = 4$,

$$F(4) = 64 - 20 - 40 = +4$$

Since $F(x)$ has changed sign between $x = 3$ and $x = 4$, the graph of the function must cross the x-axis in this interval, so there is a root between 3 and 4. (This assumes that $F(x)$ is continuous between these points; special care must be taken if $F(x)$ is known to have a discontinuity near the root being investigated.)

The Newton–Raphson iterative formula is

$$x_{r+1} = x_r - \frac{F(x_r)}{F'(x_r)}$$

and, in this case

$$F(x) = x^3 - 5x - 40$$

and, differentiating,

$$F'(x) = 3x^2 - 5$$

So, the iterative formula to solve this equation is

$$x_{r+1} = x_r - \frac{x_r^3 - 5x_r - 40}{3x_r^2 - 5}$$

As $|F(4)|$ is much smaller than $|F(3)|$, the root appears to be nearer 4 than 3, so we start with $x_1 = 4$, then

$$x_2 = 4 - \frac{64 - 20 - 40}{48 - 5}$$

$$= 4 - \frac{4}{43}$$

$$= 3.907$$

(*Note.* The value of x_2 is printed here, correct to four significant figures. If you are using a calculator, each intermediate value should be stored in the memory for use in the next iteration. It is important to understand that calculating x_2 as accurately as possible from a particular value of x_1 does *not* mean that the root has been found to the same degree of accuracy; at this stage it would be unwise to claim that more than the first one or two significant figures have been determined.)

This value of x_2 should now be substituted into the Newton–Raphson formula. This gives

$$x_3 = 3.904(45)$$

In view of the very small change between x_2 and x_3, we could now safely claim that, correct to three significant figures, the root is 3.90.

This example illustrates some of the virtues of the Newton–Raphson formula. Firstly, provided $F'(x)$ is not zero near the root, it is unnecessary to check whether the sequence converges. Secondly, the sequence converges very rapidly; in other words it is only necessary to calculate a few values of x_r in order to get a very accurate answer.

Example 5 *Use the Newton–Raphson formula to solve the equation*

$$\theta - \sin\theta = 1$$

giving your answer correct to three significant figures.

(This is the equation which arose from the problem in §24.1.) Firstly, the equation must be arranged in the form

$$\theta - \sin\theta - 1 = 0$$

and note that the function needed is

$$F(\theta) = \theta - \sin\theta - 1$$

and consequently

$$F'(\theta) = 1 - \cos\theta$$

The iterative formula we require is

$$\theta_{r+1} = \theta_r - \frac{\theta_r - \sin\theta_r - 1}{1 - \cos\theta_r}$$

Starting from $\theta_1 = 2$ (see §24.1, and remembering that θ must be measured in radians),

$$\theta_2 = 2 - \frac{2 - \sin 2 - 1}{1 - \cos 2} = 1.93595$$

and

$$\theta_3 = 1.93456$$

and hence

$$\theta_4 = 1.93456$$

(These values have, for convenience, been rounded off to six significant figures.) As the changes in $\theta_2, \theta_3, \theta_4$, have been so small, we can fairly confidently conclude that, correct to three significant figures, the root is 1.93.

Extreme care should be taken when rounding off numbers which have already been rounded. If, in the example above, θ_2, θ_3 and θ_4 had been rounded to *four* significant figures, they would have read

$$\theta_2 = 1.936$$
$$\theta_3 = 1.935$$
$$\theta_4 = 1.935$$

Rounding θ_4 to *three* significant figures would have given (wrongly) 1.94.

Exercise 24c

Use the Newton–Raphson method to find the root of each of these equations which is near the given value. Give your answers correct to three significant figures.

1 $x^3 - 4x^2 - x - 12 = 0$; $x_1 = 5$.
2 $x^4 - 3x^3 - 10 = 0$; $x_1 = 3$.
3 $2\sin\theta = \theta$; $\theta_1 = 2$.
4 $x^3 - 5x^2 = 4$; $x_1 = 5$.
5 $x^3 = 10x + 10$; $x_1 = 3.5$.
6 $3\tan\theta + 4\theta = 6$; $\theta_1 = 1$.
7 $x^4 - 4x^3 - x^2 + 4x - 10 = 0$; $x_1 = 4$.
8 $x^3 = 5x + 32$; $x_1 = 4$.

9 Verify that the equation $x^3 - 2x - 5 = 0$ has a root between $x = 2$ and $x = 3$, and find this root correct to three significant figures.

10 Find, correct to three significant figures, the smallest positive root of $5x^5 = 5x + 1$.

Exercise 24d (Miscellaneous)

1 Use the iterative formula

$$x_{r+1} = \frac{1}{2}\left(x_r + \frac{N}{x_r}\right), \quad \text{where } N \in \mathbb{R}^+$$

to find the square roots of (a) 200, (b) 450, (c) 700, (d) 1000.

2 Repeat No. 1, using the Newton–Raphson method to solve equations of the form $x^2 - N = 0$.

3 Prove that the iterative formula formed by applying the Newton–Raphson method to the equation $x^2 - N = 0$ can be written

$$x_{r+1} = \frac{1}{2}\left(x_r + \frac{N}{x_r}\right)$$

(In other words prove that the iterative formula explained in §24.1 can be deduced from the Newton–Raphson formula.)

4 Prove that if X is an *exact* root of an equation $f(x) = 0$, then substituting $x_r = X$ in the Newton–Raphson formula gives $x_{r+1} = X$.

5 Verify that the equation $10 \cos x - x = 0$ has a root between $x = 1$ and $x = 2$. Using $x = \pi/2$ as a first approximation, show that the next approximation, given by applying Newton's formula once, is $5\pi/11$.

6 Sketch the graphs of $y = x$ and $y = \frac{1}{2} \cos x$, and, from your sketch, estimate the value of x such that $x = \frac{1}{2} \cos x$.

Use the iterative formula $x_{r+1} = \frac{1}{2} \cos x_r$ to solve this equation.

7 Solve the equation $x = \frac{1}{2} \cos x$ (see No. 6), by the Newton–Raphson method.

8 The equation $5x = \cos x$ has a root near $x = 0.5$. Solve this equation using the iterative formula $x_{r+1} = 0.2 \cos x_r$.

Sketch, on a large scale, the graphs of $y = x$ and $y = 0.2 \cos x$ near this root and mark the points $P_1(x_1, x_2)$, $Q_1(x_2, x_2)$, $P_2(x_2, x_3)$, $Q_2(x_3, x_3)$, $P_3(x_3, x_4)$, $Q_3(x_4, x_4)$, etc. (see §24.4) to illustrate that the sequence $x_1, x_2, x_3, \ldots$ converges.

9 Repeat No. 8 for the iterative formula $x_{r+1} = 10 - 15/x_r$, starting at $x_1 = 8$.

Find the quadratic equation which is solved by this iterative formula and check your answer by applying the quadratic formula to this equation.

10 Show that the cubic equation $x^3 - 3x + 1 = 0$ can be arranged in the form

(a) $x = \frac{1}{3}(x^3 + 1)$, (b) $x = \dfrac{1 - 3x}{x^2}$.

By applying the test in §24.4, show that only one of these arrangements could be expected to produce a convergent iterative method, starting at $x_1 = 0.2$. Use this arrangement to solve the equation.

11 A cuboid has volume 100 cm^3, surface area 150 cm^2, and its length is twice its breadth. What are its dimensions?

12 When the height of water in a hemispherical bowl is h, the volume of water in the bowl is $\pi(rh^2 - \frac{1}{3}h^3)$, where r is the radius of the bowl. Find the height of the water when half the volume of the bowl is filled.

13 If I pay £100 on January 1st for fifteen consecutive years and draw £2100 on January 1st of the next year, what rate of compound interest do I receive?

14 A donkey is tied by a rope to a point on the circumference of a circular field of radius r. If the donkey is to be allowed to graze half the area of the field, how long should the rope be?

15 Show graphically, or otherwise, that the equation $x^3 - x - 1 = 0$ has only one root and find the integer n such that the root α satisfies $n < \alpha < n + 1$.

An iterative process for finding this root is defined by

$$x_1 = 1, \qquad x_{r+1} = (x_r + 1)^{1/3}$$

for all $r \in \mathbb{N}^+$. Obtain, to three places of decimals, the values of x_2 and x_3. Show, on a sketch graph, the line $y = x$ and the curve $y = (x + 1)^{1/3}$, indicating on this graph the relation between x_1, x_2, x_3 and the root α. (L)

16 Find, by the Newton–Raphson method, the solution of the equation*

$$x^2 + 20 \ln x = 400$$

giving your answer correct to three significant figures. [Hint: let $x_1 = 15$.]

(O & C)

17 Show graphically that the equation $x^2 = 7 \log_{10} x + 2.347$ has two real positive roots.

Taking $x = 2.2$ as an initial approximation to the larger of these roots, obtain a second approximation by writing the equation in the form $x = \sqrt{(7 \log_{10} x + 2.347)}$ and using an iterative method.

Work to three decimal places and give your answer to two decimal places.

(O & C: MEI)

18 Using the Newton–Raphson process, solve the equation

$$\sqrt{x} + \sqrt{(x + 1)} + \sqrt{(x + 2)} = 5$$

giving your answer correct to three significant figures and showing that you have achieved this degree of accuracy. (C)

19 A solution of the equation $x = f(x)$ is to be attempted using the iteration $x_{r+1} = f(x_r)$, starting with an initial estimate x_1. Draw sketch graphs showing $y = x$ and $y = f(x)$ to illustrate the following possibilities regarding the convergence towards, or divergence from, the root $x = a$.

(a) $x_1 > a$ and the successive iterates (approximations) steadily decrease, with the value a as a limit.

(b) $x_1 > a$ and the successive iterates are alternately less than a and greater than a, but approach a as a limit.

*The function $\ln x$ is the natural logarithm of x; in order to do this question the reader will need to know that its derivative is $1/x$. See Book 2, Chapter 2.

(c) $x_1 > a$ and the successive iterates get steadily larger.

Use an iterative method to find a non-zero root of the equation $x = \arctan(2x)$ correct to 2 significant figures. (C)

20 Show that the equation $x^3 - 6x + 1 = 0$ has a root between $x = 0$ and $x = 1$.

Three possible rearrangements of the given equation in the form $x = \mathrm{F}(x)$ are

$$x = \sqrt[3]{(6x - 1)}$$
$$x = \tfrac{1}{6}(x^3 + 1)$$
$$x = x^3 - 5x + 1$$

Only one of these rearrangements will provide an iterative method, of the form $x_{r+1} = \mathrm{F}(x_r)$, which converges to the root between 0 and 1. Use this rearrangement to find this root correct to 3 significant figures. (C)

Chapter 25

Groups

Introduction

25.1 Before starting to study this book, the reader was probably already familiar with the algebra of the real numbers, and in the course of the book, we have discussed the algebras of complex numbers, matrices and vectors. We have seen that, although many of the underlying principles of these topics are similar, there are important differences: in matrices, for example, **AB** is not always the same as **BA**. Since the early part of the nineteenth century, some of the most influential mathematicians have devoted much of their attention to the underlying structure of algebra, and, in the course of their research, they have produced new and unusual forms of algebra. Among the most important figures in these developments were Abel (1802–1829), Galois (1811–1832) and Klein (1849–1925). Readers who are A level candidates may be interested to note that Galois was only seventeen when he produced some of his most original work, and Abel was only nineteen when he solved one of the most famous problems in mathematics — he proved that it is impossible to find a general solution of the quintic equation.

In this chapter, we shall be looking at one of these algebraic structures, the **group**; Abel, Galois and Klein all made major contributions to group theory. The actual term 'group' was first used by Galois.

Latin squares

25.2 Look at the twelve tables in Fig. 25.1. They are all examples of Latin squares — their chief characteristic is that each of the elements employed appears once, and once only, in each row and each column.

These tables should be read like a ready reckoner; the 'product' XY is to be found in the space which is in the row labelled X and the column labelled Y (Fig. 25.2). For example, in table (vi), $qp = r$ and $rq = e$; in table (xi), $DB = C$. Notice that the *order* of the elements can make a difference. In table (x), for instance, $CA = D$, but $AC = B$.

Qu. 1 In table (vi), find $p(qr)$ and $(pq)r$.

(i)

	e	x
e	e	x
x	x	e

(ii)

	e	p	q
e	e	p	q
p	p	q	e
q	q	e	p

(iii)

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

(iv)

	P	Q	R
P	P	Q	R
Q	R	P	Q
R	Q	R	P

(v)

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

(vi)

	e	p	q	r
e	e	p	q	r
p	p	e	r	q
q	q	r	p	e
r	r	q	e	p

(vii)

	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

(viii)

	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

(ix)

	e	p	q	r	s
e	e	p	q	r	s
p	p	q	r	s	e
q	q	r	s	e	p
r	r	s	e	p	q
s	s	e	p	q	r

(x)

	I	A	B	C	D
I	I	A	B	C	D
A	A	I	D	B	C
B	B	C	I	D	A
C	C	D	A	I	B
D	D	B	C	A	I

(xi)

	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	B	I	E	C	D
B	B	I	A	D	E	C
C	C	D	E	I	A	B
D	D	E	C	B	I	A
E	E	C	D	A	B	I

(xii)

	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	B	C	D	E	I
B	B	C	D	E	I	A
C	C	D	E	I	A	B
D	D	E	I	A	B	C
E	E	I	A	B	C	D

Figure 25.1

Qu. 2 In table (vi), solve the equation $qx = r$.
Qu. 3 In table (xi), simplify $(AB)(CD)$.
Qu. 4 In table (xii), solve $x^2 = B$.
Qu. 5 In table (x), show that $C(BD) \neq (CB)D$.

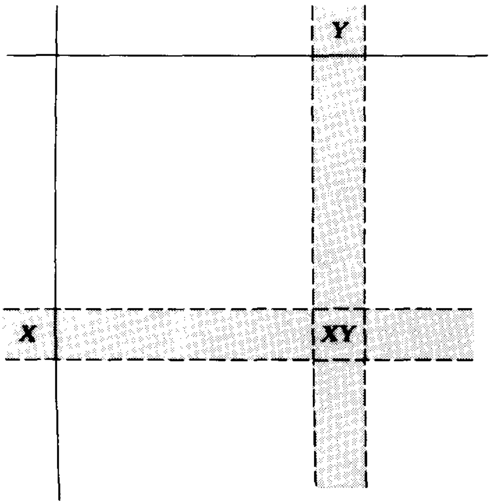

Figure 25.2

From Qu. 1–5, it can be seen that the Latin squares in Fig. 25.1 demonstrate many algebra-like properties and that questions containing instructions like 'solve', 'find', 'simplify', can be asked about them. However, they also contain some properties which look rather peculiar. In Qu. 5, for instance, we saw that $C(BD)$ was *not* the same as $(CB)D$.

In order to restrict the algebra to a structure which is fairly closely related to the algebra of real numbers, we shall impose on the Latin square two further important restrictions:

(a) there must be an **identity element**, that is, an element e, with the property $ex = xe = x$, where x is any of the other elements,

(b) if x, y and z are any of the elements used in the Latin square, then $x(yz) = (xy)z$. This is the **associative law**.

The first of these restrictions eliminates table (iv) and restriction (b) eliminates table (x).

A set of elements which can be arranged as a Latin square and which has the properties (a) and (b) above is called a **group**. (A more formal definition is given in §25.8.)

Isomorphisms

25.3 Look at tables (ii) and (iii) in Fig. 25.1. Are they really different? Certainly they employ different symbols, but if we change the 0, 1, 2 of table (ii) into e, p and q respectively, we see that the basic structure of the two tables is exactly the same; we say the two groups are **isomorphic**. The reader should now try to produce a group with three elements which has a structure which is different from the structure of tables (ii) and (iii). (It should not take long to discover that no other structure is possible.)

Now look at tables (v) and (vi): they appear to be different, but, as we have just seen in tables (ii) and (iii), this may be due to the use of different symbols. Let us change the e, p, q and r of table (vi) into 0, 2, 1 and 3 respectively. This is called a **one-to-one correspondence**, and we write it in the following way:

$$e \leftrightarrow 0, \quad p \leftrightarrow 2, \quad q \leftrightarrow 1, \quad r \leftrightarrow 3$$

Table (vi) now reads as shown in Fig. 25.3.

	0	2	1	3
0	0	2	1	3
2	2	0	3	1
1	1	3	2	0
3	3	1	0	2

Figure 25.3

At first sight this appears to be different from table (v), but if we re-write it with the numbers in the order 0, 1, 2, 3, we obtain the table shown in Fig. 25.4 and we can see that this *is* identical to table (v). So, tables (v) and (vi) are isomorphic.

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Figure 25.4

Qu. 6 (a) Use the one-to-one correspondence

$$e \leftrightarrow 0, \quad p \leftrightarrow 2, \quad q \leftrightarrow 3, \quad r \leftrightarrow 1$$

to show that tables (v) and (vi) are isomorphic.

(b) Set up a one-to-one correspondence between the elements of tables (v) and (vii), and hence show that they are isomorphic.

Qu. 7 Explain why tables (v) and (viii) are not isomorphic.

Qu. 8 Show that any group of four elements is isomorphic *either* to table (v) *or* to table (viii).

There are two, and only two, distinct groups with four elements. Their group tables are shown in Fig. 25.5.

(i)

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

(ii)

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Figure 25.5

The table in Fig. 25.5(ii) represents the group known as the **Klein group**; table (i) represents a **cyclic group** (see §25.5). Notice that in the Klein group the product of *any* element with itself (which can be seen in the diagonal of the table which goes from the top left-hand corner to the bottom right) is always equal to e, the identity element.

The table used to specify a group is often called a **Cayley table**, after Arthur Cayley (1821–1895), the Cambridge mathematician who made many important contributions to the development of modern algebra.

Exercise 25a

The questions in this exercise refer to the Cayley tables in Fig. 25.1.

1 Given that $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{J} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, write out the Cayley table for $\mathbf{I}$ and $\mathbf{J}$ under matrix multiplication.

Show that this is isomorphic to the group represented by table (i). What is the identity element?

2 In table (xi), solve the following equations, i.e. find x and y:
(a) $Cx = A$, (b) $Dx = B$, (c) $yC = D$, (d) $yD = A$.

3 In table (xi), simplify
(a) $B(CD)$, (b) $(BC)D$, (c) $C(DE)$, (d) $(CD)E$,
and verify that $B(CD) = (BC)D$ and that $C(DE) = (CD)E$.

4 If x is any element of a group and e is the identity element, then the element x^* such that $x.x^* = x^*.x = e$, is called the **inverse** of x.

Copy and complete the table below, showing each element of table (vi) and its inverse:

x	e	p	q	r
x^*	e		r	

5 Repeat No. 4 for table (xi).

6 Complete a Cayley table showing the products of the (complex) numbers, 1, i, −1, −i. Show that this table represents a group which is isomorphic to the group represented by table (vi).

7 Complete a Cayley table for the set of products of the complex numbers

$$e = 1, \qquad a = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}, \qquad b = -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$$

Show that this table represents a group which is isomorphic to the group represented by table (ii).

8 Complete a Cayley table for the products of the four matrices,

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Show that this table represents a group which is isomorphic to the group represented by table (vi).

9 Complete a Cayley table for the products of the four matrices,

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Show that this table represents a group which is isomorphic to the group represented by table (vii) (the Klein group).

10 Complete a Cayley table for the products of the eight matrices,

$$\mathbf{I}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \mathbf{A}=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad \mathbf{B}=\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\quad \mathbf{C}=\begin{pmatrix}0&1\\-1&0\end{pmatrix},$$

$$\mathbf{P}=\begin{pmatrix}-1&0\\0&1\end{pmatrix},\quad \mathbf{Q}=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \mathbf{R}=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \mathbf{S}=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$$

Further groups

25.4 The groups introduced so far in this chapter have arisen from the Latin squares in Fig. 25.1, from complex numbers (Exercise 25a, Nos. 6 and 7), and from matrices (Exercise 25a, Nos. 8, 9 and 10). In this section, and the next, we shall look at two further situations which give rise to groups.

One very fruitful source of examples of groups is finite arithmetic (mod n) where $n \in \mathbb{Z}^+$. [This concept may be new to some readers; however, it is not very complicated! In finite arithmetic (mod n), only the integers less than n are used. They are added or multiplied in the ordinary way, but any multiple of n is discarded, e.g. $3\times 2=6=1$ (mod 5); $6\times 7=42=2$ (mod 8); $5+6=11=4$ (mod 7). Table (vii) in Fig. 25.1 uses all the products of 1, 3, 7 and 9 (mod 10).]

Example 1 *Draw a Cayley table, showing all the products of* 1, 2, 3, 4 (mod 5). (See Fig. 25.6.)

	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Figure 25.6

Notice that this is isomorphic to the cyclic group. (Rearrange the elements in the order 1, 2, 4, 3.)

Example 2 *Draw a Cayley table, showing all the sums of the integers* 0, 1, 2, 3, 4, (mod 5). (See Fig. 25.7.)

	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Figure 25.7

Example 3 *Show that the integers* 1, 2, 3, 4, 5 *do not form a group when they are multiplied* (mod 6).

In the table representing a group, every element must appear once and once only in each row and column (the Latin square property). We can see that the table of products of 1, 2, 3, 4 and 5 (mod 6) (See Fig. 25.8) does not have this property.

Figure 25.8

	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

(An extra element, namely 0, has also appeared.)

Cyclic groups

25.5 It is convenient, at this stage, to introduce another technical term which is used in group theory; the number of distinct elements in a group is called the **order** of that group.

We have seen in §25.3 that there are just two groups of order four, the cyclic group and the Klein group. Their Cayley tables are shown in Fig. 25.9.

Figure 25.9

(i)

	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	A	I
C	C	B	I	A

(ii)

	I	P	Q	R
I	I	P	Q	R
P	P	I	R	Q
Q	Q	R	I	P
R	R	Q	P	I

(Notice that in this case I is being used for the identity element; this is the normal practice when capital letters are used to represent the elements.)

We must also introduce another technical term here, namely the **period** of an element. (Powers of an element are formed like powers of a number, i.e. $C^2 = C \times C$, etc.) The period of an element X is the smallest positive integer k, such that $X^k = I$. In Fig. 25.9, table (ii), each element has a period of 2. In table (i), A has a period of 2, but what are the periods of B and C? If we list the powers of these two elements, we obtain:

$$C^1 = C, \quad C^2 = A, \quad C^3 = CA = B, \quad C^4 = CB = I$$

and $B^1 = B, \quad B^2 = A, \quad B^3 = BA = C, \quad B^4 = BC = I$

So we can see that both B and C have a period of 4. Notice also that the successive powers of B (and C) produce all four elements of the group. We say that B (and C) is a **generator** of the group. A generator of a group will always have the same period as the order of the group. In the Klein group, shown in table (ii), there is *no* element which will generate the group.

Qu. 9 Find the period of each of the elements in table (xii), Fig. 25.1. Which elements generate the group?

Definition

A group which can be generated by the powers of a single element is called a **cyclic group**. *The standard symbol for a cyclic group of order n is* C_n.

For a given positive integer n, it is a simple matter to write down the Cayley table C_n generated by an element A. Suppose that

$$A^1 = A, \quad A^2 = B, \quad A^3 = C, \quad A^4 = D \quad \text{and} \quad A^5 = I$$

(Remember that the period of A is equal to the order of the group, which in this case is 5.) Then the Cayley table is as shown in Fig. 25.10.

	I	A	B	C	D
I	I	A	B	C	D
A	A	B	C	D	I
B	B	C	D	I	A
C	C	D	I	A	B
D	D	I	A	B	C

Figure 25.10

If, instead of using the distinct letters I, A, B, C and D, we use the powers of the generator A, then the table looks like Fig. 25.11,

	I	A^1	A^2	A^3	A^4
I	I	A^1	A^2	A^3	A^4
A^1	A^1	A^2	A^3	A^4	I
A^2	A^2	A^3	A^4	I	A^1
A^3	A^3	A^4	I	A^1	A^2
A^4	A^4	I	A^1	A^2	A^3

Figure 25.11

and if we write it out again, omitting the letters and recording the powers only, we obtain the table in Fig. 25.12. (In the case of I we write $I = A^0$ and we record the 0.)

	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Figure 25.12

Now we have seen this table before (see Fig. 25.7). It is the table for addition (mod 5). Consequently, C_5, the cyclic group of order 5, is isomorphic to the group of the sums of the integers 0, 1, 2, 3, 4, (mod 5).

The argument which has just been applied to C_5, could be applied to any cyclic group. Hence we can conclude that C_n is isomorphic to the group formed by adding the integers 0, 1, 2, 3, ..., $(n-1)$, (mod n).

Symmetry groups

25.6 In this section we shall be considering the symmetries of some plane figures (solid objects can also be symmetrical, but plane figures are easier to draw on a flat page!). Consider the rhombus in Fig. 25.13.

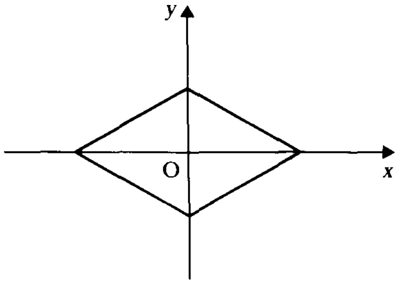

Figure 25.13

A rhombus illustrates the two kinds of symmetry which can be found in plane figures; it can be reflected in the two axes, and it can be rotated in its own plane, about O, through 180°. Most people have an intuitive feeling for symmetry, but if it is necessary to spell it out in words, we could say that *a symmetry is a transformation of the figure in which the image coincides with the original position of the figure.* In order to distinguish between the four vertices we shall mark each one in a distinctive manner. Fig. 25.14 shows the effect on the rhombus of the four symmetry transformations:

H – a reflection in the x-axis
V – a reflection in the y-axis
R – a rotation about O, through 180°
I – the 'no change' transformation, i.e. the identity element

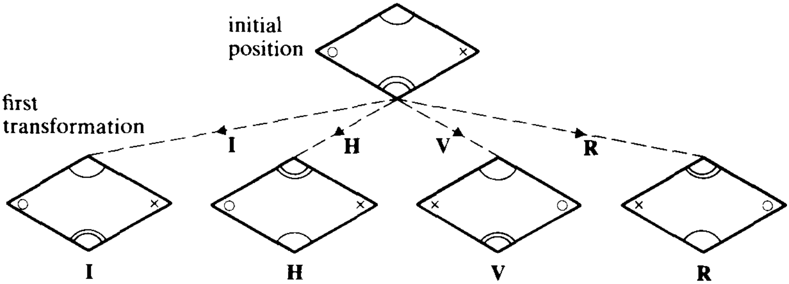

Figure 25.14

Now consider the effect of applying a second (but not necessarily different) transformation. (There is a convention in this subject, that 'apply transformation **X** and then apply transformation **Y**' is written **YX**, and any subsequent transformations are written *on the left* of any existing ones. This is the same convention as that used in composite functions; see §2.10.) There is not sufficient space to show all the sixteen possible pairs of transformations chosen from **I**, **H**, **V** and **R**, but Fig. 25.15 shows the effect of applying **H** and *then* one of the others.

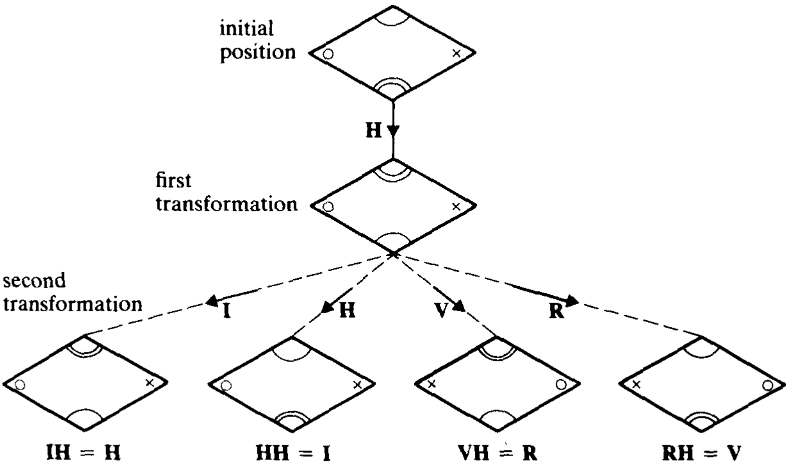

Figure 25.15

The remaining pairs are left as an exercise for the reader. (A copy of Fig. 25.13 cut out from a piece of cardboard is a useful aid.) The results of the combined transformations are shown in Fig. 25.16; the four pairs shown in Fig. 25.15 appear in the column labelled **H**.

		first transformation			
		I	**H**	**V**	**R**
second transformation	**I**	**I**	**H**	**V**	**R**
	H	**H**	**I**	**R**	**V**
	V	**V**	**R**	**I**	**H**
	R	**R**	**V**	**H**	**I**

Figure 25.16

The reader should, by now, be able to recognise this table; it is the Klein group.

Other groups can be produced from the symmetries of other symmetrical polygons. The regular polgons give rise to an especially important set of groups — the **dihedral** groups. The dihedral group produced by a regular n-sided polygon is always written D_n. The symmetry transformations will consist of n rotations, through angles which are multiples of $360°/n$, and n reflections, so there are $2n$ elements in the dihedral group D_n. We shall now examine D_3, the dihedral group of the equilateral triangle, in detail.

As in the previous example, we start with the figure in its standard initial position, with its vertices marked (Fig. 25.17), so that we can distinguish between them. The diagram also shows the three axes of symmetry marked c, d and e.

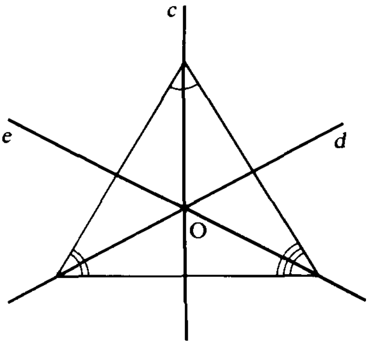

Figure 25.17

The six elements of D_3 will be

I – the identity
A – a rotation, in the plane of the triangle, about O, through 120°
B – a similar rotation, but through 240°
C – a reflection in axis *c*
D – a reflection in axis *d*
E – a reflection in axis *e*

The effects of these six transformations on the standard triangle are shown in Fig. 25.18.

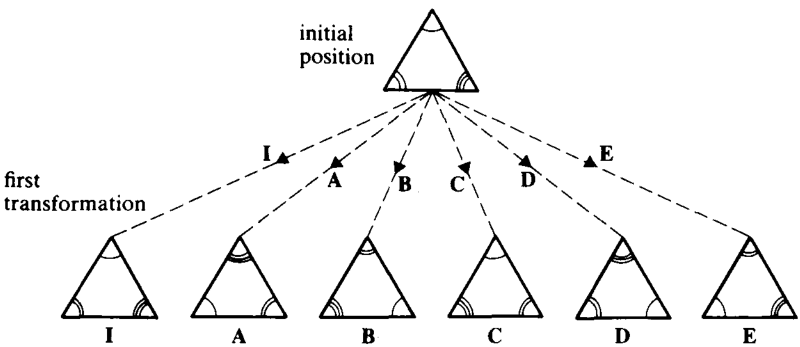

Figure 25.18

There is insufficient room to show the thirty-six combinations of two transformations, but to illustrate the procedure, the six transformations obtained by making transformation **C**, and *then* one of the others, are shown in Fig. 25.19.

The remaining results are left as an exercise for the reader. The full set of results appears in Fig. 25.20. (The results shown in Fig. 25.19 appear in the column headed **C**.)

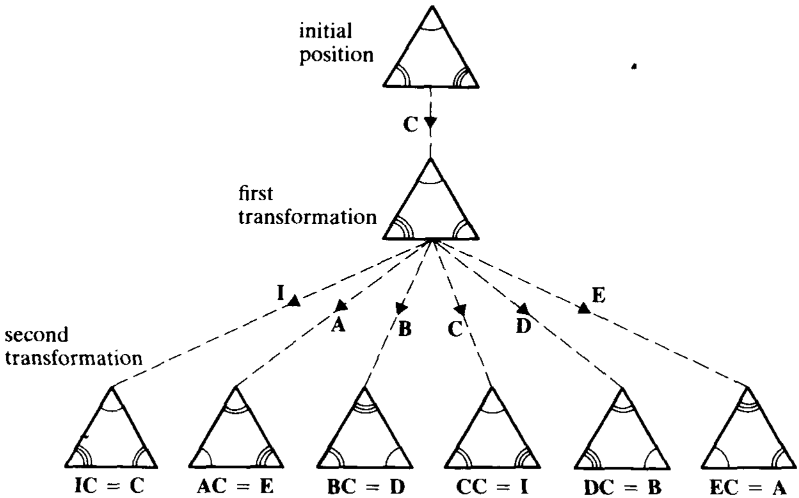

Figure 25.19

	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	B	I	E	C	D
B	B	I	A	D	E	C
C	C	D	E	I	A	B
D	D	E	C	B	I	A
E	E	C	D	A	B	I

Figure 25.20

This group has an important feature which does not appear in any group of order less than 6 — the *order* of the transformations matters, for example, **DA** = **E**, but **AD** = **C**. Groups in which $XY = YX$, for all pairs of elements X and Y are called **commutative** (or **Abelian**) groups. The dihedral group D_3 is *not* a commutative group.

Note that in Fig. 25.20, the elements are not symmetrical about the leading diagonal (the one that goes from the top left-hand corner to the bottom right). In an Abelian group, the Cayley table is always symmetrical about this diagonal.

Exercise 25b

Construct a Cayley table for each of the following groups:

1 The products of the integers 1, 2, 3, 4, 5, 6 (mod 7).
2 The sums of the integers 0, 1, 2, 3 (mod 4).
3 The cyclic group C_6, generated by an element x, such that $x^6 = e$, the identity element.
4 The dihedral group D_4, i.e. the symmetries of a square, using **I** for the identity element, **R**, $\mathbf{R}^2$, $\mathbf{R}^3$, where **R** is an anti-clockwise rotation through 90°, for the

rotations, and **A**, **B**, **C**, **D** for the reflections in the axes marked *a*, *b*, *c*, *d* in Fig. 25.21.

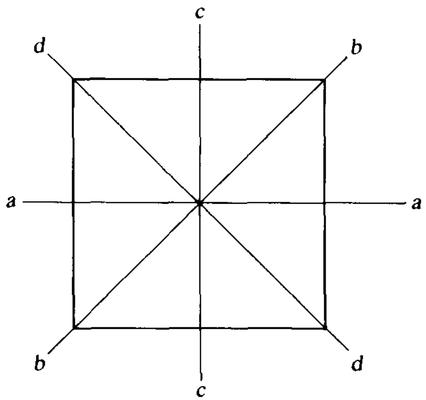

Figure 25.21

5 The dihedral group D_6, i.e. the symmetries of a regular hexagon, using **I** for the identity element, **R**, $\mathbf{R}^2$, $\mathbf{R}^3$, $\mathbf{R}^4$, $\mathbf{R}^5$, where **R** is an anti-clockwise rotation through 60°, for the rotations, and **A**, **B**, **C**, **D**, **E**, **F** for the reflections in the axes marked *a*, *b*, *c*, *d*, *e*, *f* in Fig. 25.22. (Since there are 144 results to find, it is suggested that this might be done as a class exercise.)

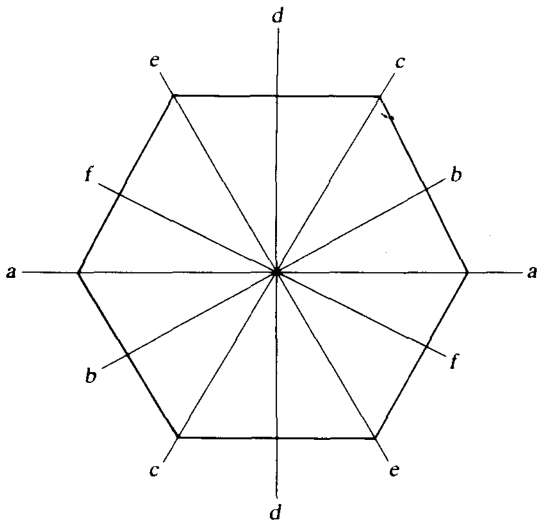

Figure 25.22

Subgroups

25.7 In the preceding sections the reader has met a fairly wide selection of groups and has probablv noticed that in several of them it is possible to spot a group within the group. In Fig. 25.1, table (vi), for instance, the elements e and p form a group of order 2, and in table (xi), the elements I, A, B form a group of order 3. Whenever a subset S of a group G is itself a group, we say S is a **subgroup** of G. (We may regard the whole group, and the identity element alone, as two rather special subgroups of G. If it is necessary to exclude these two special cases, we use the phrase **proper subgroup**.)

Qu. 10 List all the subgroups of the group in Fig. 25.1, table (vii).
Qu. 11 List all the subgroups of the group shown in Fig. 25.1, table (xi).
Qu. 12 List all the *proper* subgroups of C_6, the cyclic group generated by an element x.

Group theory

25.8 As with any other technical subject, the deeper one goes into it the more necessary it becomes to ensure that all the terms used are clearly defined. In this section we shall define and illustrate some of the more common terms used in group theory. (Some of them have already been mentioned.) In these definitions it will be convenient to use S to represent the set of elements under consideration.

(a) *A law of binary composition.*
This is any rule for combining two elements to produce a new element; for example, adding a pair of integers, finding the product of a pair of rational numbers, adding a pair of vectors, finding the product of a pair of 2×2 matrices. If we wish to have a symbol to represent a law of binary composition and we do not wish to use $+$, $-$, $\times$ or $\div$, it is usual to use a small circle $\circ$; however the symbol is frequently omitted altogether, as it is in ordinary multiplication.

(b) *Closure.*
We say a set S is closed, under a binary operation $\circ$, if, for any pair of elements a and b which belong to S, the 'product' $a \circ b$ also belongs to S. For example, if we add a pair of *even numbers*, the result is also an *even number*; the product of a *pair of* 2×2 *matrices* is another 2×2 *matrix*; the sum of two *vectors* is another *vector*: these are all examples of closed operations. On the other hand, the sum of two *odd numbers* is not an *odd number*; the scalar product of a pair of *vectors* is not another *vector*: these are examples of operations which are *not* closed.

(c) *An identity element.*
This is an element e of the set with the property that, given any element x of the set S, $e \circ x = x \circ e = x$. (When capital letters are used it is usual to use I for the identity element.) We have already seen many examples of identity elements, e.g. the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity element when multiplying 2×2 matrices, the number 1 is the identity element when *multiplying* real numbers, and the number 0 is the identity element when *adding* real numbers.

(d) *An inverse element.*
If x is any member of the set S, then the element x^*, such that $x \circ x^* = x^* \circ x = e$, is called the inverse of x. (It is frequently written x^{-1}.) When adding integers, the inverse of an integer a is the integer $-a$ (e.g. -5 is the additive inverse of $+5$); in multiplication, the inverse of the (non-zero) rational number a/b is b/a (e.g. 2/5 is the multiplicative inverse of 5/2). In the multiplication of 2×2 matrices $\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$ is the inverse of $\begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}$, because $\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(e) *Commutativity.*
A pair of elements a and b are said to commute if $a \circ b$ is equal to $b \circ a$, i.e. the order does not matter. If *every* pair of elements of a group commute, we say the group is commutative. (Commutative groups are frequently called Abelian groups, after the young Norwegian mathematician Henrik Abel, who did much original work in this branch of mathematics.) Note that not all groups are commutative, *vide* Fig. 25.1, table (xi).

(f) *Associativity.*
The law of binary composition is said to be associative in the set S if, for every triplet a, b, c of elements of set S, $a \circ (b \circ c) = (a \circ b) \circ c$ (i.e. the position of the brackets does not matter). Clearly both addition and multiplication of real numbers is associative, but division is not, e.g. $72 \div (6 \div 3) = 36$, but $(72 \div 6) \div 3 = 4$. Both addition and multiplication of complex numbers are associative and the same is true for addition and multiplication of (compatible) matrices.

Qu. 13 Which of the following sets are closed under the given law of binary composition?
(a) Even integers; multiplication.
(b) $\{1, 2, 3, 4, 5, 6\}$; multiplication (mod 7).
(c) Prime numbers; addition.
(d) Complex numbers; multiplication.

Qu. 14 In which of the following systems does *every* element have an inverse? In the cases where there are elements which do *not* have an inverse, give an example of such an element.
(a) Rational numbers; multiplication.
(b) Non-zero complex numbers; multiplication.
(c) 2×2 matrices; multiplication.
(d) $\{1, 2, 3, 4\}$; multiplication (mod 5).

With these technical terms at our disposal, we can now state the formal definition of a group.

Definition

A group is a set of elements $\{e, a, b, c, \ldots\}$ and a law of binary composition, with the following properties:
(1) *The set is* **closed** *under the law of binary composition.*

(2) *The law of binary composition is* **associative.**
(3) *There is an* **identity** *element.*
(4) *Every element has an* **inverse.**

Notice that a group does not have to be commutative and that the number of elements does not have to be finite.

Example 4 (a) *Show that the set* {0, 1, 2, 3} *under addition* (mod 4) *forms a group.* (b) *Show that the set* {1, 2, 3} *under multiplication* (mod 4) *does not form a group.*

(a) The table of addition (mod 4) is shown in Fig. 25.23.

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Figure 25.23

By inspection, we can see that it is closed; and since it is derived from ordinary arithmetic, it is associative. The identity element is 0.

The table below shows that every element x has a corresponding inverse x^*.

x	0	1	2	3
x^*	0	3	2	1

Hence it does satisfy all the group properties.

(b) The table for multiplication of the numbers 1, 2, 3 (mod 4) is shown in Fig. 25.24.

	1	2	3
1	1	2	3
2	2	0	2
3	3	2	1

Figure 25.24

Although this exhibits some of the group properties, it does not exhibit them all. It is not closed and the element 2 has no inverse. Hence it is *not* a group.

Example 5 *In a given set* S, *with a law of binary composition* $\circ$, *there is a left identity e, that is, an element e with the property that if x is any member of set* S *then* $e \circ x = x$, *and a right identity f, that is, an element f with the property that if x is any member of set* S, $x \circ f = x$. *Prove that* $e = f$.

Consider the 'product' $e \circ f$. Since e is a left identity, $e \circ f = f$, and, since f is a right identity, $e \circ f = e$. Hence $e = f$.

(Formal proofs, such as that in Example 5, are common-place in university level books on group theory; they will be kept to a bare minimum in this book.)

We shall now prove that a group, as defined in this section, will have a Cayley table which exhibits the 'Latin square' property which was used to introduce this topic in §25.1. The 'Latin square' property requires that in each row (and in each column) each element should appear once, and once only. We shall prove this for a row; it is left as an exercise for the reader to prove it for a column.

Consider the row corresponding to the element a. This row contains all the products of the form $a \circ x$, where x is any member of the set S. It is necessary to prove two things:

(1) that, given an element b, it must appear somewhere in a's row, i.e. it must be possible to find a solution of the equation $a \circ x = b$.

(2) that, if $a \circ x = a \circ y$, then $x = y$. (This ensures that each 'product' appears once only in a's row.)

The proofs are as follows:

(1) Consider the product $a^* \circ b$, where a^* is the inverse of a. We know that such an element exists (inverse property), and we know that $a^* \circ b$ is a member of set S (closure property). If we put $x = a^* \circ b$ into the left-hand side of the equation $a \circ x = b$, we obtain

$$\begin{aligned} a \circ x &= a \circ (a^* \circ b) \\ &= (a \circ a^*) \circ b && \text{(associative property)} \\ &= e \circ b && \text{(inverse property)} \\ &= b && \text{(identity property)} \end{aligned}$$

Hence $x = a^* \circ b$ is the solution we require.

(2) Given that $a \circ x = a \circ y$, multiply both sides by a^* (we know that such an element exists by the inverse property), hence

$$\begin{aligned} a^* \circ (a \circ x) &= a^* \circ (a \circ y) \\ (a^* \circ a) \circ x &= (a^* \circ a) \circ y && \text{(associative property)} \\ e \circ x &= e \circ y && \text{(inverse property)} \\ x &= y && \text{(identity property)} \end{aligned}$$

Hence, if $a \circ x = a \circ y$, then $x = y$.

Exercise 25c

In this exercise, it may be assumed that every set mentioned is associative under the given law of binary composition.

In Nos. 1–5, a set of elements S and a law of binary composition $\circ$ are given. In each question say whether the set S is, or is not, a group under $\circ$. If you decide it is not, give a clear reason for your decision.

1 The natural numbers; multiplication.
2 The odd integers (positive and negative), together with zero; addition.
3 The non-zero rational numbers; multiplication.
4 Numbers of the form 2^k, where $k \in \mathbb{Z}$; multiplication.
5 Non-singular 2×2 matrices; multiplication.

In Nos. 6–8, a finite group is given. In each case find a standard group e.g. a cyclic group, to which it is isomorphic. Draw up a table showing each element and its corresponding inverse.

6 $\{1, 4, 7, 13\}$; multiplication (mod 15).
7 The complex numbers, $e=1$, $a=\frac{1}{2}+(\sqrt{3}/2)\mathrm{i}$, $b=-\frac{1}{2}+(\sqrt{3}/2)\mathrm{i}$, $c=-1$, $d=-\frac{1}{2}-(\sqrt{3}/2)\mathrm{i}$, $f=\frac{1}{2}-(\sqrt{3}/2)\mathrm{i}$; multiplication.
8 The 2×2 matrices

$$\mathbf{I}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \mathbf{A}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \mathbf{B}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \mathbf{C}=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix};$$

multiplication.
9 Verify that the functions $I(x)=x$, $A(x)=1/(1-x)$, $B(x)=(x-1)/x$, $C(x)=1/x$, $D(x)=1-x$, $E(x)=x/(x-1)$, form a group when they are combined by forming composite functions. Tabulate each element and its inverse and, in the same table, show the period of each element. Is this a cyclic group?
10 Prove that, with a restriction which you should specify, the numbers of the form $a+b\sqrt{2}$, where $a, b \in \mathbb{Q}$, form a group under multiplication. What is the inverse of $a+b\sqrt{2}$?
11 Repeat No. 10 for the complex numbers $a+b\mathrm{i}$, where $a, b \in \mathbb{R}$, under multiplication.
12 Find a subset of the integers 1, 2, 3, ... 11, which, under multiplication (mod 12), forms a group.

Cosets

25.9 Most of the finite groups which we have met so far in this chapter have been groups of fairly small order. In order to develop the subject further we need to look more closely at a group of higher order than those we have examined up to this point. By way of example, we shall look at the group whose Cayley table is shown in Fig. 25.25. (It is actually the dihedral group D_6.) Notice that the elements of the group are not scattered in a random fashion, they appear in tidy 3×3 boxes; as we shall see, this is no accident. But first, we must meet another important technical term — **coset**.

(In the remainder of this chapter, the symbol $\circ$ for the rule of binary composition will be omitted. The product of a pair of elements x and y will be written xy, just as in elementary algebra.)

Definition

If $\mathsf{H}=\{e, a, b, c, \ldots\}$ *is a subgroup of a group* G, *and if* x *is any member of the group* G, *the set of products*

$$\{xe, xa, xb, xc, \ldots\}$$

is called the **left-coset** *of* H *with respect to x. Similarly the set*

$\{ex, ax, bx, cx, \ldots\}$

is called the **right-coset** *with respect to x.*

This rather bare definition may be difficult to assimilate at the first reading, so let us consider a particular group. Look at the Cayley table, in Fig. 25.25.

	I	*A*	*B*	*C*	*D*	*E*	*P*	*Q*	*R*	*S*	*T*	*U*
I	*I*	*A*	*B*	*C*	*D*	*E*	*P*	*Q*	*R*	*S*	*T*	*U*
A	*A*	*B*	*I*	*D*	*E*	*C*	*Q*	*R*	*P*	*T*	*U*	*S*
B	*B*	*I*	*A*	*E*	*C*	*D*	*R*	*P*	*Q*	*U*	*S*	*T*
C	*C*	*E*	*D*	*I*	*B*	*A*	*S*	*U*	*T*	*P*	*R*	*Q*
D	*D*	*C*	*E*	*A*	*I*	*B*	*T*	*S*	*U*	*Q*	*P*	*R*
E	*E*	*D*	*C*	*B*	*A*	*I*	*U*	*T*	*S*	*R*	*Q*	*P*
P	*P*	*Q*	*R*	*S*	*T*	*U*	*I*	*A*	*B*	*C*	*D*	*E*
Q	*Q*	*R*	*P*	*T*	*U*	*S*	*A*	*B*	*I*	*D*	*E*	*C*
R	*R*	*P*	*Q*	*U*	*S*	*T*	*B*	*I*	*A*	*E*	*C*	*D*
S	*S*	*U*	*T*	*P*	*R*	*Q*	*C*	*E*	*D*	*I*	*B*	*A*
T	*T*	*S*	*U*	*Q*	*P*	*R*	*D*	*C*	*E*	*A*	*I*	*B*
U	*U*	*T*	*S*	*R*	*Q*	*P*	*E*	*D*	*C*	*B*	*A*	*I*

Figure 25.25

This group has several subgroups. We shall consider just one of them, the set $\{I, A, B\}$ which can be clearly seen in the top left-hand corner of the Cayley table. We shall call this subgroup H.

(The reader's attention is drawn to the fact that the letters H, G etc., which represent groups will be printed in a special typeface, but the letters which represent individual elements are printed in *italics*.)

The cosets IH, AH, BH are $\{I, A, B\}$, $\{A, B, I\}$ and $\{B, I, A\}$ respectively. These can be clearly seen in Fig. 25.25; they are the first three elements in each of the first three rows of the table. Bearing in mind that sets which contain the same members are equal (see §2.6), we see that

$I\mathsf{H} = A\mathsf{H} = B\mathsf{H}$

In the next three rows of Fig. 25.25, we can see the three cosets CH, DH and EH. Once again notice that they are equal, i.e. they are equal to $\{C, D, E\}$. The next three rows contain the cosets PH, QH and RH and we can see that they equal the set $\{P, Q, R\}$ and, in the final three rows, we can see the cosets SH, TH and UH, which are equal to $\{S, T, U\}$.

The set of elements in G, then, can be broken down or *partitioned* into four distinct cosets, namely $\{I, A, B\}$, $\{C, D, E\}$, $\{P, Q, R\}$ and $\{S, T, U\}$. This is a particular example of Lagrange's theorem, which will be proved in the next section. Lagrange's theorem states that if H *is a subgroup of a group* G, *then the order of* H *is a factor of the order of* G. (In the case of the group in this section, the order of H was 3 and the order of G was 12.) The proof of Lagrange's theorem follows the method outlined above, that is, it sets out to partition G into distinct cosets, each containing h elements, where h is the order of H. The reader may find it helpful to refer back to this particular case while reading the general proof.

Lagrange's theorem

25.10 Lagrange's theorem states that *if* G *is a finite group of order g and* H *is a subgroup of* G, *of order h, then h is a factor of g.*

Notice that if H is an *improper* subgroup, that is, it is either $\{I\}$, where I is the identity element, or G itself, the theorem is trivial, so we need only prove it when H is a *proper* subgroup of G.

Before embarking on the proof, we shall prove three *lemmas*. (A lemma is a minor theorem which forms part of the proof of a more important one.)

Lemma (1) *The h members of the coset xH are distinct.*

Proof This follows immediately from the fact that they all come from the same row of G, and so, by the 'Latin square' property, they must be distinct.

Lemma (2) *If x is* not *a member of* H, *then the coset xH and* H *are disjoint sets.*

Proof (The method of proof in this lemma will be *reductio ad absurdum*, and so the first step is to assume there is an element of xH, say the element xa, which *does* belong to H.)

Suppose that $xa = b$, where $b \in$ H, then, multiplying both sides by a^{-1}, we obtain

$$(xa)a^{-1} = ba^{-1}$$

hence

$$x(aa^{-1}) = ba^{-1} \qquad \text{(associative law)}$$

$$xe = ba^{-1} \qquad \text{(because } a^{-1} \text{ is the inverse of } a\text{)}$$

$$x = ba^{-1} \qquad \text{(because } e \text{ is the identity element)}$$

Now b is an element of the subgroup H and so is a^{-1}, and since in any group 'multiplication' is closed, it follows that ba^{-1} is a member of H. Hence $x \in$ H. However this contradicts the fact that x does *not* belong to H, so the assumption that xa belongs to H is false. Hence no element of xH belongs to H, i.e. the sets xH and H are disjoint.

Lemma (3) *If y is not a member of the coset xH, then the cosets xH and yH are disjoint.*

Proof The proof of this lemma is very similar to that of Lemma (2) and so it is left as an exercise for the reader.

The stage is now set for the proof of Lagrange's theorem.

Let $\mathsf{H} = \{e, a, b, c, \ldots\}$ be a proper subgroup of G. Because it is a *proper* subgroup, there is at least one member of G which is not a member of H; let this element be x. Consider the coset $x\mathsf{H}$. By Lemma (1), we know that $x\mathsf{H}$ contains h distinct elements and, by Lemma (2) we know they are different from the h members of H. So the sets $x\mathsf{H}$ and H together account for $2h$ members of G. If this exhausts G then $g = 2h$ and the proof of the theorem is complete. But if $x\mathsf{H}$ and H do *not* exhaust G, there must be at least one more element of G unaccounted for. Let this element be y and consider the coset $y\mathsf{H}$. Once again, Lemma (1) tells us that it contains h distinct elements; Lemma (2) tells us they are different from the members of H, and Lemma (3) tells us that they are different from the members of $x\mathsf{H}$. So we have now accounted for $3h$ members of G; if this exhausts G, then $g = 3h$. Failing that, we repeat the argument. Since G is a finite group, we must eventually reach a stage when G *is* exhausted. When this happens, the elements of G will have been partitioned into a number of distinct cosets, each containing h elements. In other words g is a multiple of h, which completes the proof of the theorem.

It follows from Lagrange's theorem that if the order of G is a composite number (i.e. not a prime number) then any subgroup of it must have an order which is a factor of g. For example, if G is a group of order 12 then any proper subgroups it may have must have 2, 3, 4 or 6 elements (but Lagrange's theorem does *not* say that such subgroups must exist.)

Qu. 15 Find all the proper subgroups of the group D_6 in Fig. 25.25 and verify that their orders are factors of 12.

There is a very important corollary to Lagrange's theorem; if the order of a group is a *prime number*, then it can have no proper subgroups. Consequently no element can have a period which is less than the order of the group (if it did, this element and its powers would form a proper subgroup), and hence the group must be cyclic. So any group of order p, where p is a prime number, is isomorphic to the cyclic group C_p.

Generators

25.11 In §25.5 we saw that some groups can be generated by a single element; that is, every element; in the group can be expressed as a power of a single element, called the generator of the group. It will be remembered that such groups are called cyclic groups and that all cyclic groups of the same order are isomorphic to one another. Not all groups, however, are cyclic; the Klein group (see Fig. 25.9 (ii)), for instance, is not cyclic. We can say, however, that it is generated by two elements P and Q, because the only remaining element R can be expressed in terms of P and Q; indeed in this particular example R is equal to PQ. When it is possible to express every element of a group in terms of just two elements x and y, we say that the group is generated by x and y. In a large and complicated group it may be necessary to have a large number of generators. Example 6 illustrates a group of order 6, generated by two elements p and r.

Example 6 *A group of order 6 consisting of the elements e, p, q, r, s and t (e being the identity element) has the following properties:*

$$q = p^2, \quad s = pr, \quad t = p^2r \quad \text{and} \quad p^3 = r^2 = s^2 = e$$

Draw the Cayley table of this group.

Without any working, it is possible to complete part of the Cayley table, as shown in Fig. 25.26.

	e	p	q	r	s	t
e	e	p	q	r	s	t
p	p	q	e	s		
q	q	e	p	t		
r	r			e		
s	s				e	
t	t					

Figure 25.26

In order to complete the p row, consider the products ps and pt:

$$ps = p(pr) \;\; = p^2r = t$$
$$pt = p(p^2r) = p^3r = er = r$$

To complete the q row, we look at the products qs and qt:

$$qs = (p^2)(pr) \;\; = p^3r = er = r$$
$$qt = (p^2)(p^2r) = p^4r = pr = s$$

Before we tackle the next two rows, notice the following useful identities.

Since $s^2 = e$, and $s = pr$, we can write

$$(pr)(pr) = e$$

and, on removing the brackets,

$$prpr = e \tag{1}$$

[In the next few lines, we shall follow the usual rule of 'doing the same thing to both sides of the equation', but, because multiplication in this group is not commutative, we must make it clear whether the given element is to be placed on the *right* or on the *left* of the existing terms.]

Multiplying on the right by r,

$$prprr = er$$

hence, noting that $r^2 = e$,

$$prp = r$$

and multiplying again on the right by p^2, we obtain

$$pr = rp^2 = rq$$

If, on the other hand we had multiplied (1) on the *left* by p^2, we would have obtained

$$rpr = p^2 = q$$

and multiplying this on the right by r gives

$$rp = qr$$

These identities are very useful for completing the remaining rows of the Cayley table. For the r row, we require the products rp, rq, rs and rt:

$$rp = qr = t$$
$$rq = pr = s$$
$$rs = rpr = q$$
$$rt = rp^2r = (pr)r = pr^2 = pe = p$$

To complete the s row, we must find sp, sq, sr and st:

$$sp = prp = r$$
$$sq = prq = ps = t$$
$$sr = prr = pe = p$$
$$st = prp^2r = (prp)(pr) = rs = q$$

The final row is left as an exercise for the reader. The complete Cayley table is as shown in Fig. 25.27.

	e	p	q	r	s	t
e	e	p	q	r	s	t
p	p	q	e	s	t	r
q	q	e	p	t	r	s
r	r	t	s	e	q	p
s	s	r	t	p	e	q
t	t	s	r	q	p	e

Figure 25.27

Exercise 25d (Miscellaneous)

Whenever the symbols I or e are used in this exercise, you should assume they are intended to represent the identity element. When you are required to prove that a certain system is a group, the formal definition in §25.8 should be used.

1 Show that the set $\{1, 3, 5, 7\}$ forms a group under multiplication (mod 8).

2 Compile the Cayley table for the set $\{3, 6, 9, 12\}$ under multiplication (mod 15). Name the identity element and draw a table showing each element and its inverse.

3 Show that the set of matrices of the form $\begin{pmatrix} p & p-q \\ 0 & q \end{pmatrix}$, where $p, q \in \mathbb{R}$, forms a group under matrix multiplication. (The associative property may be assumed.)

4 Given that x and y are two elements of a group G, and given that x has a period of k, show that yxy^{-1} also has a period of k.

5 Consider the Latin square in Fig. 25.28.

	I	A	B	C	D	E	F	G
I	I	A	B	C	D	E	F	G
A	A	I	G	F	E	D	C	B
B	B	E	I	G	F	A	D	C
C	C	F	E	I	G	B	A	D
D	D	G	F	E	I	C	B	A
E	E	B	C	D	A	F	G	I
F	F	C	E	A	B	G	I	E
G	G	D	A	B	C	I	E	F

Figure 25.28

(a) Find the group which is generated by G.
(b) Find the group which is generated by A.
(c) Find the group which is generated by A and F.

6 Using the Cayley table in No. 5, find all the left-cosets of the subgroup $\{I, A\}$.

7 Given that $f(x) = (1+x)/(1-x)$, and combining functions by forming composite functions, show that f generates the cyclic group C_4. List the functions which make up this group.

8 Given that the elements x, y and (xy) of a group all have a period of 2, show that $xy = yx$.

9 Prove that the set of integers $\{1, 2, 4, 5, 7, 8\}$ forms a group under multiplication (mod 9). Find the period of each element and show that the group is isomorphic to the cyclic group C_6.

10 Given that **M** is the 2×2 matrix $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$, show that

$$\mathbf{M}^2 = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{pmatrix}$$

Prove by induction that $\mathbf{M}^n = \begin{pmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{pmatrix}$, where $n \in \mathbb{Z}^+$. Hence show that if $\alpha = 2\pi/N$, where $N \in \mathbb{Z}^+$, **M** generates the cyclic group C_N.

11† Prove that the group in Fig. 25.29 can be generated by R and S, provided $R^2 = S^2 = (RS)^3 = I$.

	I	P	Q	R	S	T
I	I	P	Q	R	S	T
P	P	Q	I	T	R	S
Q	Q	I	P	S	T	R
R	R	S	T	I	P	Q
S	S	T	R	Q	I	P
T	T	R	S	P	Q	I

Figure 25.29

†These questions could be undertaken by several students working together.

12† A group G is generated by two elements x and y, such that

$$x^4 = e, \quad x^2 = y^2 \quad \text{and} \quad xy = yx^3$$

Prove that $x^3y = yx$ and $x^2y = yx^2$. Hence show that the group contains exactly 8 members: e, x, x^2, x^3, y, yx, yx^2 and yx^3. Write out the Cayley table of G.

13† A group G is generated by three elements x, y and z, which satisfy

$$x^3 = y^2 = z^2 = (xy)^2 = (yz)^2 = xzx^2z = e$$

Prove that (a) $zx = xz$, (b) $yz = zy$, (c) $yx = x^2y$. Hence find all the members of G, expressing each one in the form $x^a y^b z^c$, where $a, b, c \in \mathbb{Z}^+$.

14 Consider the Latin square in Fig. 25.30.

	P	*Q*	*R*	*S*	*U*	*V*	*W*	*X*
P	*S*	*R*	*W*	*V*	*P*	*U*	*X*	*Q*
Q	*X*	*S*	*P*	*W*	*Q*	*R*	*U*	*V*
R	*Q*	*V*	*S*	*X*	*R*	*W*	*P*	*U*
S	*V*	*W*	*X*	*U*	*S*	*P*	*Q*	*R*
U	*P*	*Q*	*R*	*S*	*U*	*V*	*W*	*X*
V	*U*	*X*	*Q*	*P*	*V*	*S*	*R*	*Q*
W	*R*	*U*	*V*	*Q*	*W*	*X*	*S*	*P*
X	*W*	*P*	*U*	*R*	*X*	*Q*	*V*	*S*

Figure 25.30

(a) Simplify: $(PQ)R$, $P(QR)$, $R(SU)$, $(RS)U$, $(VW)X$, $V(WX)$.
(b) Name the identity element.
(c) Assuming that the associative law is satisfied, show that this Latin square is a group.

15 In No. 14, find two subgroups of order 4. Partition the group into the four left-cosets with respect to the subgroup generated by the element S.

16 The multiplication table for the set $\{e, a, b, c, d\}$ is given in Fig. 25.31.

	e	*a*	*b*	*c*	*d*
e	*e*	*a*	*b*	*c*	*d*
a	*a*	*e*	*c*	*d*	*b*
b	*b*	*d*	*e*	*a*	*c*
c	*c*	*b*	*d*	*e*	*a*
d	*d*	*c*	*a*	*b*	*e*

Figure 25.31

Using this table, determine $(ab)c$, $a(bc)$, $(bc)d$, $b(cd)$. Ascertain which of the group axioms are satisfied by the given set under the given multiplication. Find two subsets from the above set which form a group under the given multiplication. (L)

17 (a) Prove that the set of non-zero real numbers form a group under the operation defined by $x \circ y = 2xy$. State the identity of this system, and give the inverse of x.

†See previous page.

(b) A set of four 2×2 matrices forms a group under matrix multiplication. Two members of the set are $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Find the other members and write out the group table. (L)

18 Show that the four matrices

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -1 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} 0 & \sqrt{\frac{1}{2}} & 0 \\ \sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{1}{2}} \\ 0 & \sqrt{\frac{1}{2}} & 0 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 0 & -\sqrt{\frac{1}{2}} & 0 \\ -\sqrt{\frac{1}{2}} & 0 & -\sqrt{\frac{1}{2}} \\ 0 & -\sqrt{\frac{1}{2}} & 0 \end{pmatrix}$$

form a group under matrix multiplication (which may be assumed to be an associative operation). Give the operation table (i.e. the Cayley table) for this group.

Determine whether or not this group is isomorphic under multiplication to the group consisting of the four matrices

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \quad \text{(C)}$$

19 Prove that the integers $\{\ldots, -3, -2, -1, 0, 1, 2, \ldots\}$ form a group G under the operation of ordinary addition. Give an example of a proper subgroup. Prove that the powers of 2 $\{\ldots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, \ldots\}$ form a group H under ordinary multiplication.

Establish an isomorphism between G and H and give the subgroup of H corresponding under this isomorphism to your previous subgroup of G.

(C)

20 You are given that the set of real 2×2 matrices with non-zero determinants is a group G under multiplication. Show that the set S of all matrices of the form $\mathbf{A} = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}$ (with a_1 and a_2 taking real values, but not both zero) is a subgroup of G. Is it Abelian (i.e. does $\mathbf{AB} = \mathbf{BA}$ for all $\mathbf{A}, \mathbf{B} \in S$)? Given an arbitrary element $\mathbf{A} \in S$, find an element $\mathbf{C} \in S$ (if there is one) such that $\mathbf{C}^2 = \mathbf{A}$. If there is one is it unique?

Find an element $\mathbf{J} \in S$ (if there is one) such that $\mathbf{J}^2 = -\mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

Comment briefly on the relationship between the set S and the set of all complex numbers $(x + iy)$.

(Oxford Colleges Entrance Examination)

Appendix
Elementary algebra — revision

The object of this Appendix is to give the reader some extra practice, if it is needed, in the algebraic manipulation which he or she will meet in this book.

Readers who are confident about their command of algebra may omit the Appendix; others may find it useful to 'brush-up' particular topics. Some teachers may wish to use it as a preliminary course in basic algebra before embarking on the rest of the book.

Simplification

Example 1 *Simplify* $(x+h)^2+(x-h)^2$.

$$\begin{aligned}(x+h)^2+(x-h)^2 &= x^2+2xh+h^2+x^2-2xh+h^2\\ &= 2x^2+2h^2\\ &= 2(x^2+h^2)\end{aligned}$$

Exercise 1

Simplify:

1 $(x+h)^2-(x-h)^2$.

2 $(x+h)^3+(x-h)^3$.

3 $(x+h)^3-(x-h)^3$.

4 $x(1-2x^2)+2x(1-x^2)$.

5 $\dfrac{y-3}{x+5}=\dfrac{2}{5}$.

6 $\dfrac{2y-1}{3-x}=-\dfrac{3}{4}$.

7 $(2\sqrt{t}-3)(1+\sqrt{t})$.

8 $\dfrac{3t^2+2t^3}{(1+t)^2}\div\dfrac{t^2}{(1+t)^3}$.

9 $\dfrac{\sqrt{x}}{(\sqrt{x}-1)}\times\dfrac{\sqrt{x}}{(\sqrt{x}+1)}$.

10 $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}}{1-\dfrac{1}{xy}}$.

Factorisation

Example 2 *Factorise* $2x(2x+1)^2+(2x+1)(4x^2-3)$.

(When factorising an expression like this, it is important to spot any common factors; in this example $(2x+1)$ is a common factor.)

$$\begin{aligned} 2x(\mathbf{2x+1})^2+(\mathbf{2x+1})(4x^2-3) &= (\mathbf{2x+1})[2x(2x+1)+4x^2-3] \\ &= (2x+1)(4x^2+2x+4x^2-3) \\ &= (2x+1)(8x^2+2x-3) \\ &= (2x+1)(2x-1)(4x+3) \end{aligned}$$

(Note that if an expression can be factorised, this should be done.)

Exercise 2

Factorise:

1 $35x^2+x-6$.
2 $2x^2-98$.
3 $2x^2-xy-y^2$.
4 $xy+ay+xb+ab$.
5 $xy+3y-2x-6$.
6 $x(x+1)^2+(x+1)(x^2-3)$.
7 $(x+3)(x^2+3)+x(x+3)^2$.
8 $5(x+1)^2+7x(x+1)$.
9 $(x+3)^2-(x-7)^2$.
10 $(x-2)^3+5x(x-2)^2$.

Fractions

Example 3 *Express as a single fraction:*

(a) $\dfrac{1}{2+x}+\dfrac{2}{1-3x}$, (b) $\dfrac{1}{a^3b}+\dfrac{1}{ab^3}$.

(a) $$\begin{aligned} \frac{1}{2+x}+\frac{2}{1-3x} &= \frac{1}{2+x}\times\frac{1-3x}{1-3x}+\frac{2}{1-3x}\times\frac{2+x}{2+x} \\ &= \frac{(1-3x)+2(2+x)}{(2+x)(1-3x)} \\ &= \frac{1-3x+4+2x}{(2+x)(1-3x)} \\ &= \frac{5-x}{(2+x)(1-3x)} \end{aligned}$$

(b) $$\begin{aligned} \frac{1}{a^3b}+\frac{1}{ab^3} &= \frac{b^2}{a^3b^3}+\frac{a^2}{a^3b^3} \\ &= \frac{a^2+b^2}{a^3b^3} \end{aligned}$$

(In this part, notice that a^3b^3 is the lowest common multiple of the original denominators a^3b and ab^3.)

Exercise 3

Express as a single fraction:

1 $\dfrac{1}{x}-\dfrac{1}{y}$.

2 $\dfrac{x}{y}+\dfrac{y}{x}$.

3 $\dfrac{1}{a^2}+\dfrac{1}{a}$.

4 $\dfrac{1}{ab^2}+\dfrac{1}{a^2b}$.

5 $\dfrac{1}{x-h}+\dfrac{1}{x+h}$.

6 $\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}$.

7 $\dfrac{1}{1-x}-\dfrac{2}{2+x}$.

8 $\dfrac{x}{x^2+2}-\dfrac{2}{2+x}$.

9 $\dfrac{n}{n+1}+\dfrac{1}{(n+1)(n+2)}$.

10 $\dfrac{1}{(x+1)^2}+\dfrac{1}{(x+1)}+1$.

Further simplification

Exercise 4

Simplify:

1 $\dfrac{2T-2t}{T^2-t^2}$.

2 $y-2t=\dfrac{1}{t}(x-t^2)$.

3 $\dfrac{1-1/t}{1-t}$.

4 $\dfrac{T-t}{1/T-1/t}$.

5 $N(4N^2-1)+3(2N+1)^2$.

6 $\dfrac{a}{\sqrt{(a+b)}}+\dfrac{b}{\sqrt{(a+b)}}$.

7 $\dfrac{a/b+c/d}{1+ac/(bd)}$.

8 $\dfrac{(x+h)^3-x^3}{h}$.

9 $\dfrac{1}{\sqrt{(1+x^2)}}-\dfrac{x^2}{\sqrt{(1+x^2)}(1+x^2)}$.

10 $\sqrt{\left\{\dfrac{1-2t/(1+t^2)}{1+2t/(1+t^2)}\right\}}$.

Completing the square

(Completing the square is a very useful technique which appears in several different contexts. It depends on the identity

$$(A+B)^2=A^2+2AB+B^2$$

as the following example illustrates. In each of the questions below, there is a number missing wherever a box has been printed.)

Example 4 *Complete:* $(\square x + \square)^2 = 25x^2 + 70x + \square$.

Comparing this incomplete statement with the identity above and, in particular, comparing the term $25x^2$ with A^2 in the identity, we see that $A = 5x$. Comparing the middle terms, namely $70x$ and $2AB$, and bearing in mind that $A = 5x$, we can see that $B = 7$. Lastly, the final term on the right-hand side should be B^2 and so the missing number in the last box is 49. The complete statement is

$$(5x + 7)^2 = 25x^2 + 70x + 49$$

Exercise 5

Complete the following:

1 $(x + 3)^2 = x^2 + \square x + 9$.
2 $(x - 5)^2 = x^2 - \square x + \square$.
3 $(3x + 2)^2 = 9x^2 + \square x + 4$.
4 $(x + \square)^2 = x^2 + 10x + \square$.
5 $(x - \square)^2 = x^2 - 14x + \square$.
6 $(2x + \square)^2 = \square x^2 + 12x + \square$.
7 $(x + \frac{1}{2})^2 = x^2 + \square x + \square$.
8 $(\frac{1}{2}x - \square)^2 = \square x^2 - x + \square$.
9 $(\square x^2 + 3)^2 = 100x^4 + 60x^2 + \square$.
10 $(\frac{1}{3}x + \square y)^2 = \square x^2 + \frac{1}{3}xy + \square y^2$.

Changing the subject of a formula

Example 5 (a) *Make* y *the subject of* $\dfrac{y-k}{a} = \dfrac{x-h}{b}$.

(b) *Make* x *the subject of* $m = \dfrac{x+a}{b-x}$.

(a) $\dfrac{y-k}{a} = \dfrac{x-h}{b}$

Multiply both sides by a:

$$y - k = a \times \frac{(x-h)}{b}$$

$$= \frac{a}{b}(x - h)$$

Add k to both sides:

$$y = \frac{a}{b}(x - h) + k$$

(b) (This is slightly harder because x appears more than once; the purpose of the first few steps is to rearrange the equation so that x appears once only.)

$$m = \frac{x+a}{b-x}$$

Multiply both sides by $b - x$:

$$m(b - x) = x + a$$
$$\therefore\ mb - mx = x + a$$

Add mx to both sides and subtract a from both sides:

$$\begin{aligned} mb - a &= x + mx \\ &= x(1 + m) \end{aligned}$$

(This has achieved the first objective; x now appears once only.)

Now, divide both sides by $(1 + m)$:

$$\frac{mb - a}{1 + m} = x$$

i.e. $$x = \frac{mb - a}{1 + m}$$

Exercise 6

In each question, the letter which is to be made the subject is printed at the end of the line in brackets.

1 $y = mx + c$, (m).

2 $b = a(1 - e)$, (e).

3 $y^2 = (x + a)^2 - (x - a)^2$, (x).

4 $\dfrac{y - k}{K - k} = \dfrac{x - h}{H - h}$, (y).

5 $3mc = (4 + 3m)(c - 4)$, (c).

6 $ax - x + 1 - b = 0$, (x).

7 $T = 2\pi\sqrt{\dfrac{l}{g}}$, (l).

8 $T = 2\pi\sqrt{\dfrac{l}{g}}$, (g).

9 $2x + 2y + 2mx - 4my + 1 = 0$, (m).

10 $2x - 3y - 3mx + 2my - 2m + 4 = 0$, (m).

Linear and quadratic equations

Example 6 *Solve the equation* $\frac{1}{2}(2x - 3) - \frac{1}{3}(x - 2) = \frac{7}{6}$.

$$\tfrac{1}{2}(2x - 3) - \tfrac{1}{3}(x - 2) = \tfrac{7}{6}$$

Multiply both sides by 6:

$$3(2x-3)-2(x-2)=7$$
$$6x-9-2x+4=7$$

(Be very careful over the + sign in front of the 4: this is a very common source of error!)

Simplifying the left-hand side gives

$$4x-5=7$$
$$\therefore 4x=12$$

and hence

$$x=3$$

[It is a wise precaution to check the answer by substituting $x=3$ in the original equation. The L.H.S. gives

$$\begin{aligned}\tfrac{1}{2}(2x-3)-\tfrac{1}{3}(x-2)&=\tfrac{1}{2}(6-3)-\tfrac{1}{3}(3-2)\\&=\tfrac{1}{2}\times 3-\tfrac{1}{3}\times 1\\&=1\tfrac{1}{2}-\tfrac{1}{3}\\&=\tfrac{7}{6}\end{aligned}$$]

Example 7 *Solve the equation* $tx-t^2=Tx-T^2$, *expressing* x *in terms of* t *and* T.

$$tx-t^2=Tx-T^2$$
$$\therefore tx-Tx=t^2-T^2$$

Factorising this gives

$$x(t-T)=(t-T)(t+T)$$

Dividing both sides by $(t-T)$ we have

$$x=(t+T)$$

(However, it should be noted that the final step, namely dividing by $(t-T)$, is only permissible if $t \neq T$, because one must never divide by zero. If t *does* equal T, the original equation is true for all values of x.)

Exercise 7

Solve the following equations:

1 $2x+1=16-3x.$

2 $\dfrac{2x-1}{3}-\dfrac{x-7}{5}=2.$

3 $\dfrac{x}{x+1}-\dfrac{1}{x-2}=1.$

4 $\dfrac{x-5}{x+1}=\dfrac{x-7}{x-2}.$

5 $2x^2-17x+21=0.$

6 $x^2=5x+14.$

7 $\frac{1}{x} - \frac{1}{x+1} = \frac{1}{x+4}$.

In Nos. 8–10, express x in terms of the other letters.

8 $\frac{x-ct}{2} = \frac{cT-x}{3}$.

9 $5x^2 - 16tx + 3t^2 = 0$.

10 $tx^2 + (tT-1)x - T = 0$.

Simultaneous equations

(The reader will have solved simultaneous equations before, but the method of substitution may be new. In Example 8, substitution is merely an alternative to other possible methods, but in Example 9, it is the only way the equation can be solved.)

Example 8 *Solve the simultaneous equations*

$$7x + 2y = 11 \qquad (1)$$
$$4x + y = 7 \qquad (2)$$

Equation (2) can be rearranged to give

$$y = 7 - 4x$$

Substituting $(7 - 4x)$ for y in equation (1):

$$7x + 2(7 - 4x) = 11$$

Removing the brackets,

$$7x + 14 - 8x = 11$$
$$\therefore 14 - x = 11$$
$$\therefore x = 3$$

Putting $x = 3$ in equation (2) gives

$$12 + y = 7$$
$$\therefore y = -5$$

Hence the solution is $x = 3$, $y = -5$. (This should be checked by substituting these values into equation (1).)

Example 9 *Solve the simultaneous equations*

$$x^2 + y^2 = 25r^2$$
$$2y + x = 10r$$

giving the answers in terms of r.

From the second equation we have

$$x = 10r - 2y$$

Substituting this into the first equation gives

$$(10r - 2y)^2 + y^2 = 25r^2$$

Removing the brackets,

$$100r^2 - 40ry + 4y^2 + y^2 = 25r^2$$

i.e. $100r^2 - 40ry + 5y^2 = 25r^2$

$\therefore 5y^2 - 40ry + 75r^2 = 0$

After dividing both sides by 5, this becomes

$$y^2 - 8ry + 15r^2 = 0$$

$$\therefore (y - 3r)(y - 5r) = 0$$

Therefore either $y - 3r = 0$, or $y - 5r = 0$.

$$\therefore y = 3r \quad \text{or} \quad 5r$$

Substituting these values into the equation $2y + x = 10r$, gives

either $6r + x = 10r$, i.e., $x = 4r$

or $10r + x = 10r$, i.e., $x = 0$

Hence the solution is

$$x = 0 \quad \text{and} \quad y = 5r$$

or

$$x = 4r \quad \text{and} \quad y = 3r$$

(It should be carefully noted that each solution consists of a value of x and a value of y.)

Exercise 8

Solve the following equations:

1 $7x + 4y = 10$, $5x + 3y = 7$.

2 $6x + y = 9$, $4x - y = 11$.

3 $5x + 2y + 1 = 0$, $y = 7x + 3$.

4 $y^2 = 4x$, $y = x$.

5 $xy = 64$, $4x - y = 60$.

6 $y^2 = 4x + 1$, $y = x + 1$.

In Nos. 7–10, express x and y in terms of the other letters.

7 $2y = x + 4c$, $5y = x + 25c$.

8 $ty = x + t^2$, $Ty = x + T^2$, (where $t \neq T$).

9 $xy = 1$, $t^2x - y = t^3 - 1/t$.

10 $x^2 - y^2 = 16a^2$, $y = 3x - 12a$.

Equations of higher degree

(Equations of degree more than two can be very troublesome to solve unless they can be factorised. However when factors can be found, the same idea which is used in the solution of quadratic equations by factorisation may be used, namely that a product of real numbers (see §2.1) can only be zero if one of the factors is zero.)

Example 10 *Solve the equation* $x^3 - 5x^2 + 6x = 0$.

Since x is a factor of each term in the equation, we can re-write it as

$$x(x^2 - 5x + 6) = 0$$

and on factorising the quadratic, we have

$$x(x-2)(x-3) = 0$$

Hence

$$x = 0 \quad \text{or} \quad x - 2 = 0 \quad \text{or} \quad x - 3 = 0$$

Therefore $x = 0$, or 2, or 3.

(Notice that although it is tempting to 'divide through' by x, if we do so, we lose the solution $x = 0$.)

Example 11 *Solve the equation* $x^4 - 5x^2 - 36 = 0$.

(Although this is an equation of degree 4, we may treat it as a quadratic in x^2.)

$$X^2 - 5X - 36 = 0, \quad \text{where } X = x^2$$

Factorising:

$$(X-9)(X+4) = 0$$
i.e. $$(x^2 - 9)(x^2 + 4) = 0$$

The factor $(x^2 + 4)$ cannot be zero (not, that is, unless we use complex numbers, see §10.6), so if x is a real number we must deduce that

$$x^2 - 9 = 0$$
$$\therefore x^2 = 9$$

Therefore $x = +3$, or -3.

Exercise 9

Solve the following equations. In Nos. 6–10 express x in terms of the other letters.

1 $x^3 - 4x = 0$. **2** $x^3 = 7x^2$.
3 $x^3 - x^2 - 20x = 0$. **4** $x^4 - 17x^2 + 16 = 0$.
5 $9x^4 + 5x^2 - 4 = 0$. **6** $x^3 + kx^2 = 0$.

7 $(x-a)^3 - b^2(x-a) = 0.$ **8** $x^3 + a^2x = 0.$
9 $x^4 - a^4 = 0.$ **10** $(x-p)^3 = q^3.$

Exercise 10 (Miscellaneous)

1 Simplify:
(a) $\sqrt{\{(a-b)^2 + 4ab\}}$, (b) $(a+b)^3 - 3ab(a+b)$.

2 Factorise:
(a) $K^2(K+1)^2 + 4(K+1)^3$, (b) $N(N+1)(2N+7) + 6(N+1)(N+3)$.

3 Express as a single fraction:

(a) $\dfrac{1}{(x-h)^2} - \dfrac{1}{(x+h)^2}$, (b) $\dfrac{2}{(N+1)(N+3)} - \dfrac{2N+3}{(N+1)(N+2)}$.

4 Simplify:

(a) $\dfrac{2n+1}{n^2(n+1)^2} - \dfrac{1}{n^2}$, (b) $N\{(N+1)(2N+1) + 9(N+1) + 12\}$.

5 Complete the following:
(a) $(7x + \square)^2 = 49x^2 + 42x + \square$, (b) $(x + \square)^2 + \square = x^2 + 7x + 13$.

6 Make x the subject:
(a) $u = \sqrt{(3x+3)}$, (b) $u = 5x^2 + 1$.

7 Solve the following equations, expressing x in terms of a:

(a) $\dfrac{1}{2}(x-a) + \dfrac{1}{5}(2x+a) = \dfrac{3a}{10}$, (b) $7x^2 + 4ax - 3a^2 = 0$.

8 Solve the following equations, expressing x in terms of c and t:
(a) $c/t = (x - ct)/t^2$, (b) $x - c/t = -t^2(x + ct)$.

9 Solve the following simultaneous equations, giving the answers in terms of a:
(a) $7x - 4y + 5a = 0$, $9x - 5y - a = 0$. (b) $y^2 = 4ax$, $4y = 3(x - a)$.

10 Solve the following equations, giving the answers in terms of k:
(a) $x^5 + k^2x = 0$, (b) $x^6 - 7k^3x^3 - 8k^6 = 0$.

Answers

Chapter 1

Qu. 1 $(-3, 2), (2, -3), (0, 0)$.
Qu. 3 (a) 13, (b) $\sqrt{41}$, (c) $\sqrt{\{(r-p)^2+(s-q)^2\}}$.
Qu. 4 (a) $(5, 6)$, (b) $(-1, 4)$, (c) $(-\frac{3}{2}, -\frac{5}{2})$, (d) $\left(\frac{p+r}{2}, \frac{q+s}{2}\right)$.
Qu. 5 (a) $\frac{9}{4}$, (b) $\frac{3}{2}$, (c) $-\frac{4}{5}$, (d) $-\frac{10}{11}$, (e) 0, (f) $(s-q)/(r-p)$, (g) -1, (h) b/a.
Qu. 6 $\frac{4}{3}, -\frac{3}{4}, -1$.
Qu. 7 $\frac{12}{5}, -\frac{5}{12}, -1$.
Qu. 8 (a) $-\frac{1}{3}$, (b) -4, (c) $\frac{1}{6}$, (d) $\frac{3}{2}$, (e) $-1/(2m)$, (f) a/b, (g) $2/m$.
Qu. 9 (a) parallel, (b) perpendicular, (c) neither.
Qu. 10 $5, 20, -1$.
Qu. 11 $2, 0, -\frac{5}{2}$.
Qu. 12 (a) $(-\frac{1}{2}, 0)$, $(1, 0)$, (b) $(0, -1)$.
Qu. 13 (a) yes, (b) no, (c) no, (d) no, (e) yes, (f) yes.
Qu. 14 (a) 0, (b) 2, (c) 3, (d) $\frac{1}{2}$, (e) -1.
Qu. 15 (a) $y=\frac{1}{3}x$, (b) $y=-2x$, (c) $y=mx$.
Qu. 16 (a) $\frac{1}{4}$, (b) $-\frac{5}{4}$, (c) $\frac{3}{2}$, (d) $\frac{7}{4}$, (e) q/p.
Qu. 17 (a) $y=3x+2$, (b) $y=3x+4$, (c) $y=3x-1$, (d) $y=\frac{1}{5}x+2$, (e) $y=\frac{1}{5}x+4$.
Qu. 18 (a) $\frac{2}{3}, 2$, (b) $\frac{1}{4}, \frac{1}{2}$, (c) $-3, -6$, (d) $\frac{7}{3}, -\frac{5}{3}$, (e) $0, -4$, (f) $-l/m, -n/m$.
Qu. 19 (a) $y=0$, (b) $x=0$, (c) $x=4$, (d) $y=-7$.
Qu. 20 (a) $5x-2y-26=0$, (b) $5x+2y-1=0$.
Qu. 21 (a) $3x-2y-19=0$, (b) $12x+5y-1=0$.
Qu. 22 $y-y_1=m(x-x_1)$.
Qu. 23 (a) $(4\frac{1}{2}, 1)$, (b) $(1, 5)$, (c) $(0, c)$, (d) $(-a, c-a)$.
Qu. 24 No. They are parallel.
Qu. 25 $(-\frac{3}{4}, 0)$, $(\frac{2}{3}, 0)$.

Exercise 1a, page 4

1 (a) 4, (b) 5, (c) 6, (d) 13, (e) $\sqrt{74}$, (f) 10.
2 (a) $(3, 2)$, (b) $(5, \frac{5}{2})$, (c) $(1, 3)$, (d) $(0, \frac{7}{2})$, (e) $(-\frac{1}{2}, -\frac{9}{2})$, (f) $(-6, -7)$.

Page 4

3 17. **4** $(-\frac{5}{2}, \frac{9}{2})$. **5** $(-\frac{3}{2}, -\frac{3}{2})$. **6** P, R, S.
7 A, B, D; $\sqrt{50}$. **8** 13, $6\frac{1}{2}$.

Exercise 1b, page 14

1 (a) $\frac{9}{5}$, (b) $-\frac{7}{3}$, (c) $-\frac{1}{14}$, (d) $\frac{1}{5}$.
3 (a) 1, (b) -1, (c) $\sqrt{3}$, (d) $-1/\sqrt{3}$.
4 (a) perpendicular, (b) parallel, (c) perpendicular, (d) parallel, (e) perpendicular, (f) neither.
6 $\sqrt{50}$; $(3\frac{1}{2}, 4\frac{1}{2})$. **7** $\frac{1}{2}\sqrt{34}$; $(\frac{3}{4}, -1\frac{3}{4})$. **8** 10, 1, 2, 26; ± 2, ± 4.
9 -27, -1, 1, 27; -2, 0, 2. **10** (a) yes, (b) no, (c) no, (d) yes.
11 $-\frac{10}{3}$, $+5$; $\frac{25}{3}$.
12 (a) (4, 0), $(-3, 0)$, $(0, -12)$, (b) $(\frac{2}{3}, 0)$, $(\frac{1}{2}, 0)$, (0, 2), (c) (0, 9), and touches x-axis at (3, 0), (d) (9, 0), and cuts y-axis, touches x-axis at (0, 0), (e) $(-1, 0)$, (0, 25) and touches x-axis at (5, 0), (f) (1, 0), $(-1, 0)$, (3, 0), $(-3, 0)$, (0, 9).
13 (a) $y = x$, (b) $y = -x$, (c) $y = \frac{1}{2}x$, (d) $y = \frac{1}{2}x - 4$, (e) $x = -5$, (f) $y = -\frac{2}{3}x + 5$.
14 (a) $y = 11$, (b) $x = 4$, (c) $y = 6x - 10$, (d) $y = -8x + 2$, (e) $y = -\frac{2}{3}x - 1$.
15 $y = \frac{1}{8}x$. **16** $M(0, -\frac{3}{2})$; $S(5, -1)$. **17** (a) $(b - q)/(a - p)$, 7, (b) $-\frac{3}{2}$.

Exercise 1c, page 18

1 (a) $4x - y - 1 = 0$, (b) $3x - y + 11 = 0$, (c) $x - 3y - 17 = 0$, (d) $3x + 4y - 41 = 0$, (e) $3x - 6y - 4 = 0$, (f) $20x + 12y + 31 = 0$.
2 (a) $3x - 4y + 21 = 0$, (b) $5x + 4y - 23 = 0$, (c) $3x + 11y - 35 = 0$, (d) $x - 5y - 19 = 0$, (e) $2x + 3y - 7 = 0$, (f) $2x - y + 1 = 0$.
3 (a) $(7, -7)$, (b) $(-\frac{3}{2}, -\frac{11}{2})$, (c) $(\frac{11}{7}, -\frac{13}{7})$, (d) $(4, -7)$.
4 (a) $3x - 4y + 1 = 0$, (b) $5x - 2y + 16 = 0$, (c) $7x - y - 28 = 0$, (d) $3x - 4y - 6 = 0$.
5 $2x - 5y + 19 = 0$. **6** $26x + 4y - 21 = 0$.
7 $7x - 10y - 70 = 0$; $7x + 10y = 0$. **8** $2x - 7y - 3 = 0$. **9** $\frac{8}{21}$.
10 $(2, -5)$. **11** $4x - 3y - 13 = 0$; 5. **12** $x + 4y - 15 = 0$.
13 $7x - 4y - 43 = 0$. **14** (0, 0), (16, 64). **15** $\sqrt{512} = 16\sqrt{2}$.

Exercise 1d, page 19

1 $5x + y - 33 = 0$.
2 $2x + 7y - 14 = 0$; $2x - 7y - 14 = 0$.
3 (a) $3x - 5y + 14 = 0$, (b) $3x + 5y - 14 = 0$, (c) $2x + 5y + 14 = 0$.
4 $m_1 m_2 = -1$, (a) $5x + 2y - 11 = 0$, (b) $2x - 5y - 16 = 0$.
5 (0, 0), $(2, \frac{3}{2})$, $(5, -\frac{3}{2})$; $y = 0$, $9x + 2y - 21 = 0$, $(2\frac{1}{3}, 0)$.
6 (a) $3x + 2y + 5 = 0$, (b) $2x + 7y - 19 = 0$, (c) $2x + 5y + 11 = 0$.
7 (a) $2x - 3y - 14 = 0$, (b) $3x + 2y - 8 = 0$, $(\frac{32}{13}, \frac{4}{13})$.
8 $\sqrt{85}$, $6x + 7y - 85 = 0$. **9** (1, 8), 52. **11** (a) AB = AC = 13, (b) 12, 78.
12 $2x + y - 17 = 0$, $72\frac{1}{4}$. **13** $x - y = 0$, $2x + 2y - 9 = 0$, $(\frac{3}{2}, 3)$, $(3, \frac{3}{2})$, $\frac{3}{2}\sqrt{5}$.

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14 $x+3y+2=0, x-3y-4=0.$ **15** $x+4y-9=0.$
16 $3x+4y+1=0, 4x-3y-7=0.$
17 $13x-8y=0, 4x+y-30=0, x-2y+12=0, (\frac{16}{3}, \frac{26}{3}).$
18 $x+2y+4=0, 8x-y+15=0, 10x+3y-45=0.$ **19** $(\frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}).$
20 $(-11, 3)$, 174. **21** $3x+4y-15=0, 4x-3y-1=0.$
22 $3x+2y-2=0, 4x+y+1=0, (-\frac{4}{5}, \frac{11}{5}).$ **23** $13x+y-22=0.$
24 $9, \frac{2}{13}\sqrt{13}, 2\frac{1}{12}.$

Chapter 2

Qu. 1 (a) F, (b) T, (c) F, (d) F.
Qu. 2 (a) -2.5, (b) $-3, -5$, (c) $+5, -5$, (d) $+\sqrt{3}, -\sqrt{3}$.
Qu. 3 (a) $\{x: x \in \mathbb{R}, x \neq 3\}$, $\mathbb{R}$, (b) $\{x: x \in \mathbb{R}, x \leqslant 10\}$, $\{y: y \in \mathbb{R}, 0 \leqslant y\}$, (c) $\{x: x \in \mathbb{R}, |x| \leqslant 5\}$, $\{y: y \in \mathbb{R}, |y| \leqslant 5\}$, (d) $\{x: x \in \mathbb{R}, x \neq \pm 5\}$, $\{y: y \in \mathbb{R}, y < 0 \text{ or } y > \frac{1}{25}\}$, (e) $\mathbb{R}$, $\{y: y \in \mathbb{R}, 0 < y \leqslant \frac{1}{25}\}$.
Qu. 4 (a) many-to-one, (b) one-to-one, (c) not a function, (d) many-to-one.
Qu. 7 Odd. **Qu. 8** 2. **Qu. 9** 9. **Qu. 10** 7.389. **Qu. 11** 1.
Qu. 12 0.5.

Exercise 2a, page 27

1 (a) $\{1, 4, 9, 16, 25\}$, (b) $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$, (c) $\{2, 4, 6, 8, 10\}$, (d) $\{5, 9, 13, 17, 21\}$.
2 (a) $\{0, 1, 4, 9\}$, (b) $\{-24, -6, 0, +6, +24\}$, (c) $\{0, 1, 16, 81\}$, (d) $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}\}$.
3 (a) $\{1, 4, 9, 16, 25, 36, 49, 64, 81\}$, (b) $\{9, 16, 21, 24, 25\}$, (c) $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, (d) $\{\frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, 3\frac{1}{2}, 4, 4\frac{1}{2}\}$.
4 (a) F, (b) T, (c) T, (d) T.
5 $\{1, 2, 3\}$.
6 (a) $\{5, 10, 15, \ldots 95\}$, $\{7, 14, 21, \ldots 98\}$, $\{35, 70\}$, $\{5, 7, 10, 14, \ldots 95, 98\}$, (b) multiples of 35, (c) 19, 14, 2, 31.
7 (a) $\{3, 6, 9, \ldots 18\}$, (b) $\{4, 8, 12, 16, 20\}$, (c) $\{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20\}$, (d) $\{1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 18, 19\}$, (e) $\{1, 2, 5, 7, 10, 11, 13, 14, 17, 19\}$, (f) the same as (e).
8 (a) $0.\dot{3}$, (b) $0.\dot{2}8571\dot{4}$, (c) $0.\dot{2}\dot{7}$.
9 $\frac{7}{9}$.
10 (a) $\frac{4}{33}$, (b) $\frac{73}{111}$, (c) $\frac{3}{7}$.

Exercise 2b, page 32

1 (a) 1, (b) 126, (c) $1\frac{27}{64}$, (d) -7. **2** $\{1, 6, 11, 16, 21, 26\}$.
3 $\{y: y \in \mathbb{R}, y \geqslant 1\}$. **4** $\{y: y \in \mathbb{R}, 0 < y \leqslant 1\}$. **5** $\{x: x \in \mathbb{R}, x < 25\}$.
6 fg: $x \mapsto 5x^2 + 1$, gf: $x \mapsto (5x+1)^2$.
7 (a) $(5+h)^2 = 25 + 10h + h^2$, (b) $10 + h$. **8** $2a + h$.
9 (a) 8, (b) -1000, (c) $\frac{1}{8}$, (d) $125a^3$, (e) $a^3/27$, (f) $a^3 + 3a^2h + 3ah^2 + h^3$, (g) $6a^2h + 2h^3$, (h) $3a^2 + h^2$.
10 (a) 56, (b) 91, (c) 2, F: $x \mapsto (x-21)/7$.

Exercise 2c, page 37

18 (a) translation a to the right, (b) translation a vertically upwards, (c) 'stretch', $\times k$, parallel to the y-axis, (d) reflection in the x-axis, (e) reflection in the y-axis.

Exercise 2d, page 46

1 (a) $x = 7$, (b) $x = -3$, (c) $x = -0.2$, (d) $x = (a-1)/5$.
2 (a) $t = 7$, (b) $t = 5.5$, (c) $t = 4$, (d) $t = 1/a + 5$.
3 (a) $f^{-1}(x) = 24 - 2x$, (b) $f^{-1}(x) = 3 + 2x$, (c) $f^{-1}(x) = \frac{1}{2}(5x - 1)$, (d) $f^{-1}(x) = (7 - 10x)/3$.
4 (a) $f^{-1}(x) = 9x/5 + 32$, (b) $f^{-1}(x) = x/180 + 2$, (c) $f^{-1}(x) = x/2\pi$, (d) $f^{-1}(x) = \frac{3}{5}(x + 9) - 7$.
5 (a) $F^{-1}: t \mapsto \sqrt{(t-5)}, (t \geqslant 5)$, (b) $F^{-1}: t \mapsto t^2/25$, (c) $F^{-1}: t \mapsto \sqrt[3]{t} + 5$, (d) $F^{-1}: t \mapsto t^3 - 1$.
6 (a) $g^{-1}: x \mapsto 1/x + 3, (x \neq 0)$, (b) $g^{-1}: x \mapsto (1/x - 1)/2, (x \neq 0)$, (c) $g^{-1}: x \mapsto 4 - 3/x, (x \neq 0)$, (d) $g^{-1}: x \mapsto x/(2-x), (x \neq 2)$.

Exercise 2e, page 52

1 (a) 2.5, (b) 0, (c) ∞, (d) 0.
2 (a) 6, (b) 10, (c) 75, (d) ∞.
3 (a) $f(0) = 1$, (b) not possible, (c) not possible, (d) 3.
4 (a) continuous, (b) discontinuous (c) discontinuous, (d) continuous.
5 ± 3, $f(3) = 4$, $f(-3) = -2$.

Exercise 2f, page 53

1 (a) $\{8, 23, 48, 83, 128\}$, (b) $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}\}$.
2 (a) 6, (b) 12, (c) 30, (d) -60.
3 (a) 2, (b) -10, (c) -12, (d) $2(a-5)$.
4 $\{2t: t \in \mathbb{Z}, t \geqslant 0\}$, non-negative integers.
5 (a) $5(x+1)^2$, (b) $25x^2 + 1$.
6 (a) $5x$, (b) $7 - x$, (c) $(7-x)/5$, (d) $7 - 5x$.
7 $\{y: 0 < y \leqslant 1\}$.
8 $gf(x) = x/(x^2 + 2x - 15)$, $x \neq 0, 3, -5$.
9 (a) one-to-one, (b) one-to-one, (c) many-to-one, $fgh(x) = 10/(x^2 + 5) + 2$, $\{y: 2 < y \leqslant 4\}$.
10 $a = 5, b = 2$.
11 (a) 7, (b) 1.
12 (a) $\{y: 0 \leqslant y \leqslant 2\}$, (b) $\{x: x \geqslant 0\}$, $\{y: 0 < y \leqslant 4\}$, $\sqrt{(4/x - 1)}$, $\{x: 0 < x \leqslant 4\}$, $\{y: 0 \leqslant y < \infty\}$.
13 $f(x) = x^p$, $p \in \mathbb{Q}$.

Chapter 3

Qu. 1 The circle.
Qu. 2 90°.

Qu. 3 PQ → 0, QO → 0, PQ/QO = $\frac{1}{3}$.
Qu. 4 3, $2\frac{1}{2}$, 2.1, 2.01; 2.
Qu. 5 Q → P; gradient of PQ → gradient of tangent at P; 2.
Qu. 6 4.
Qu. 7 $-3, -2, 1, 4$.
Qu. 8 (a) $6x$, (b) $10x$, (c) x, (d) $2cx$, (e) $2x$, (f) $2x$.
Qu. 9 $4x^3$.
Qu. 10 $6x^2$.
Qu. 11 (a) $12x^2$, (b) $20x^3$, (c) $2ax$, (d) $4nx^{n-1}$, (e) $k(n+1)x^n$.
Qu. 12 (a) $3x^2+4x+3$, (b) $16x^3-6x$, (c) $2ax+b$.
Qu. 13 (a) $12x^2-4x$, (b) $2x-1$, (c) 5.

Exercise 3a, page 68

1 $12x^{11}$. **2** $21x^6$. **3** 5. **4** 5. **5** 0. **6** $10x-3$.
7 $12x^3-6x^2+2x-1$. **8** $8x^3+x^2-\frac{1}{2}x$. **9** $3ax^2+2bx+c$.
10 $18x^2-8$. **11** $15x^2+3x$. **12** -1. **13** 0. **14** $12x^2-3$.
15 $ax-2b$. **16** $4x+2$. **17** $6x-3$. **18** x^2-1. **19** $2x-1$.
20 $6x$. **21** $x+\frac{7}{4}$. **22** $\frac{4}{3}x-\frac{1}{3}$. **23** x. **24** 1; 2. **25** 1; 1.
26 3; -4. **27** -5; 4. **28** 28; -36. **29** 9; -24. **30** (4, 16).
31 $(-2, -8)$, (2, 8). **32** (0, 0). **33** $(\frac{3}{2}, -\frac{5}{4})$. **34** $(-1, 8)$, (1, 6).
35 $(2, -12)$. **36** (0, 1), $(\frac{3}{2}, -\frac{11}{16})$. **37** $(-\frac{1}{3}, \frac{4}{27})$, (1, 0).
38 (1, 4), (3, 0).

Exercise 3b, page 71

1 (a) $4x-y-4=0$, (b) $24x-y-46=0$, (c) $x+y-1=0$, (d) $8x+y-5=0$, (e) $18x+y+54=0$.
2 (a) $x+4y-18=0$, (b) $x+24y-1204=0$, (c) $x-y+1=0$, (d) $x-8y-25=0$, (e) $x-18y+3=0$.
3 $9x-y-27=0$; $x+9y-3=0$. **4** $16x-y=0$; $x+16y=0$.
5 $2x-y-10=0$. **6** $4x+y-3=0$. **7** $y+4=0$; $y-23=0$.
8 $y-10=0$; $y+17=0$.

Exercise 3c, page 72

1 6, $6x-y+2=0$, $(-\frac{2}{5}, -\frac{2}{5})$. **2** $10x-y-16=0$, $(-4, -56)$.
3 $5x-y-1=0$, (2, 4), $(4, -8)$. **4** $4x-2y+5=0$, $(\frac{2}{3}, \frac{41}{27})$.
5 (0, 0), (1, 0), (2, 0), $y=2x$, $x+y-1=0$, $y=2x-4$.
6 $x-y+3=0$, $x-2y+12=0$, (6, 9).
7 $y=-5x+4$, $(2, -6)$, $y=-4x+2$.
8 $(-2, 0)$, $x-2y+2=0$, $x+2y-2=0$, (0, 1).
9 $y=3x+2$, $y=3x+6$. **10** $(-\sqrt{\frac{2}{3}}, -\frac{1}{3}\sqrt{\frac{2}{3}})$, (0, 0), $(\sqrt{\frac{2}{3}}, \frac{1}{3}\sqrt{\frac{2}{3}})$.
11 $3x-y-9=0$, $x+3y-33=0$, 20. **12** $(0, -3)$.
13 $6x-y-9=0$, (1, 1). **14** $2x+y+16=0$, $(-16, 16)$.
15 $2x-y-1=0$, $6x-3y-8=0$, $\frac{1}{3}\sqrt{5}$.
16 $4h$, $y-k=4h(x-h)$, $y=\pm 12x$. **17** (1, 4), (3, 12).
18 (a) $f': x \mapsto 3$, $g': x \mapsto 2x$, (b) 3, 20, (c) $(3x+4)^2$, $6(3x+4)$.

Chapter 4

Qu. 1 6.1 m, 12.2 m/s.

Qu. 2 (a) 1.0 m, 10 m/s, (b) $4.9\,(2h + h^2)$ m, $4.9(2 + h)$ m/s.

Qu. 3 9.8 m/s.

Qu. 4 (a) 24.5 m, 24.5 m/s, (b) 11 m, 22 m/s, (c) 2.0 m, 20 m/s, (d) $4.9\,(4h + h^2)$ m, $4.9\,(4 + h)$ m/s; 19.6 m/s.

Qu. 5 (a) 6.9, 23.6, 50.1, 86.4 m below top, (b) 11.8, 21.6, 31.4, 41.2 m/s, (c) 26.5 m/s.

Qu. 6 (a) 19.8, 29.6, 39.4, $10 + 9.8t$, m/s, (b) straight line through (0, 10) of gradient 9.8.

Exercise 4a, page 78

1 (a) 10.5 m, 10.5 m/s, (b) 13, 15, $(15.4 - 4.9h)$ m/s, (c) 15.4 m/s.
2 $v = 24.5 - 9.8t$, (a) t = 0, 5 seconds, (b) 19.6, 29.4, 29.4, -29.4 m; 14.7, 4.9, -4.9, -34.3 m/s, (c) below ledge; falling, (d) $t = 2.5$; 30.6 m, (e) 2.4 m.
3 $v = 3 + 2t$; (a) At O, 3 m/s, (b) $t = 0$, or -3, (c) $t = -\frac{3}{2}$, $\frac{9}{4}$ m from O on the negative side, (d) -3 m/s.
4 (a) 0, 8, 9, 8, 0, -7 m; on AO produced.
(b) 6, 2, -2, -6 m/s; moving in direction $\overrightarrow{AO}$.
(c) $t = 3$; 9 m from O, on OA.
5 (a) 11.59 a.m., 12.03 p.m., (b) $\frac{5}{27}$, 1 km, (c) $\frac{8}{27}$ km/min $= 17\frac{7}{9}$ km/h, (d) $\frac{1}{3}$ km/min = 20 km/h.
6 (a) 11.57 a.m., 12.02 p.m., (b) $\frac{9}{8}$, $\frac{11}{18}$ km, (c) $\frac{25}{72}$ km/min $= 20\frac{5}{6}$ km/h, (d) $\frac{1}{2}$ km/min = 30 km/h.
7 29.4 m/s.

Exercise 4b, page 80

1 2.5 m/s^2. **2** 3 m/s^2. **3** (a) 18 km/h per s, (b) 64 800 km/h^2.
4 (a) 3.6 km/h per s, (b) 1 m/s^2, (c) 12 960 km/h^2. **5** 6.25 s.
6 -1.5 m/s^2; -5. **7** 130 km/h.

Exercise 4c, page 83

1 (a) $+5.6$ m, $+0.7$ m/s (up), -9.8 m/s^2 (decreasing speed),
(b) $+1.4$ m, -9.1 m/s (down), -9.8 m/s^2 (increasing speed),
(c) -12.6 m, -18.9 m/s (down), -9.8 m/s^2 (increasing speed).
2 24.9 m, 29.8 m/s, 9.8 m/s^2.
3 (a) 31.5 m, -4.2 m/s, (b) $t = 2\frac{4}{7}$, (c) 32.4 m, (d) 2.5 m, (e) -9.8 m/s^2 (constant).
4 (a) 18, 54, 114 m/s^2, (b) 58 m/s^2.
5 (a) $t = 2$, (b) $t = \frac{2}{3}$, $\frac{32}{27}$ m from O on OA; $t = 2$, at O, (c) $\frac{32}{27}$, $\frac{64}{27}$ m, (d) 3 m/s, (e) 1 m from O, on OA; towards O (-1 m/s); increasing ($a = -2$ m/s^2).
6 9 m from O on AO produced ($s = -9$); towards O ($+15$ m/s); decreasing ($a = -14$ m/s^2).
7 (a) After 0, 1, 2 s, (b) 2, -1, 2 m/s; -6, 0, $+6$ m/s^2, (c) 0 m/s, (d) 0 m/s^2.

Exercise 4d, page 84

1 0.7 m/s. **2** 16 m/s, 14 m/s^2. **3** 84 m/s, 4 m/s^2.
4 0 cm/s, -4 cm/s^2, $1\frac{1}{2}$ s, 18 cm.
5 (a) 1, 3 s, 4, 0 cm, (b) -6, $+6$ cm/s^2, (c) -3 cm/s.
6 0 cm/s, 16 cm, (-24 cm/s).
7 $(24n-5)$ cm, $(24n+7)$ cm/s, 24 cm/s^2.

Chapter 5

Qu. 1 (a) $2x-4$, (b) $6x$, (c) $6x^2-10x$, (d) $2x-2$, (e) $3x^2-4x-3$.
Qu. 2 (a) (1, 2), (b) $(-\frac{1}{3}, -5\frac{1}{3})$, (c) $(\frac{3}{4}, -\frac{1}{4})$.
Qu. 3 (a) $\frac{5}{3}$, highest, (b) $\frac{7}{6}$, lowest, (c) $-\frac{3}{2}$, lowest, (d) $-\frac{4}{5}$, highest.
Qu. 4 (a) $-4x^{-5}$, (b) $-6x^{-3}$, (c) $-6x^{-4}$, (d) $-\frac{3}{2}x^{-4}$, (e) $-mx^{-m-1}$, (f) $4x-3-5x^{-2}$, (g) $1-3x^{-2}+8x^{-3}$.
Qu. 5 A, E min., D, F max., B, C infl.; G max., I min., H infl.; K max., J, L, infl.
Qu. 6 (a) neg., pos., decreasing, (b) pos., pos., increasing, (c) neg., zero, neither.

Exercise 5a, page 91

1 (a) $6x-2$, (b) $10x+4$, (c) $2-4x$, (d) $6x+1$, (e) $48x+6$.
2 (a) $(-2\frac{1}{2}, -8\frac{1}{4})$, (b) $(\frac{9}{14}, 7\frac{25}{28})$, (c) $(\frac{1}{3}, -\frac{1}{3})$, (d) $(-\frac{5}{8}, 7\frac{9}{16})$.
3 (a) $2\frac{1}{2}$, lowest, (b) -6, lowest, (c) $\frac{4}{3}$, highest, (d) -25, highest.
4 (a) $-2\frac{1}{4}$, least, (b) 4, greatest, (c) 16, greatest, (d) $-6\frac{1}{8}$, least.
6 12.1 m, $1\frac{4}{7}$ s. **7** 50 m by 50 m. **8** 10 cm.
9 250 m, 500 m, 125 000 m^2. **10** 50 m, 5 s. **11** 2 cm, 3 cm.

Exercise 5b, page 97

1 (a) 0, infl., (b) 0, y max., (c) 2, y max.; 3, y min., (d) -3, y max.; 5, y min., (e) -6, y max.; -1, y min., (f) 1, y min.; 3, y max., (g) -3, y min.; 4, y max., (h) -6, y min.; 1, y max., (i) -5, y min.; 3, y max., (j) $-\sqrt{\frac{27}{5}}$, y max.; 0, infl.; $\sqrt{\frac{27}{5}}$, y min., (k) -2, y max.; 2, y min.
2 (a) $-\frac{16}{9}$, min.; $\frac{16}{9}$, max., (b) 0, max.; -27, min., (c) 0, max.; $-\frac{256}{27}$, min., (d) -2, max.; $+2$, min., (e) $\frac{100}{27}$, max., -9 min.
3 (a) $(-2, 16)$ max.; $(2, -16)$ min., (b) $(\frac{1}{3}, 2\frac{13}{27})$ max.; $(3, -7)$ min., (c) (0, 0) min.; (2, 4) max., (d) $(\frac{1}{2}, 3)$ min., (e) $(3\frac{1}{3}, 181\frac{13}{27})$ max.; $(12, -144)$ min.
4 (a) 0, min., (b) 3, infl., (c) 0, infl.; $\frac{27}{16}$, max., (d) 19, infl.; 3, min.
5 18 cm^3; $x=1$. **6** $7\frac{11}{27}$ cm^3; $x=\frac{2}{3}$. **7** 2 cubic feet.
8 $8/\pi$ cubic feet. **9** $\sqrt[3]{(5/\pi)}$ cm; $2\sqrt[3]{(5/\pi)}$ cm. **11** 4 cm.
12 6, 6, 3 cm.

Exercise 5c, page 100

1 (0, 0), (3, 0); (0, 0) min., (2, 4) max.

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2 (0, 0), (6, 0); (0, 0) max., (4, -32) min.
3 (0, 0), (1, 0); ($\frac{1}{3}$, $\frac{4}{27}$) max., (1, 0) min.
4 (-1, 0), (2, 0), (0, 2); (-1, 0) min., (1, 4) max.
5 (0, 0), (2, 0); (0, 0) min., (1, 1) max., (2, 0) min.
6 (0, 0), (8, 0); (0, 0) infl., (6, -432) min.
7 (± 1, 0), (± 3, 0), (0, 9); (0, 9) max., ($\pm\sqrt{5}$, -16) min.
8 (0, 0), ($-\sqrt[3]{32}$, 0); (-2, -48) min.
9 (0, 0), ($1\frac{1}{4}$, 0); (0, 0) max., (1, -1) min.
10 (0, 0), ($\pm\sqrt{\frac{5}{3}}$, 0); (-1, 2) max., (0, 0) infl., (1, -2) min.
11 (0, 0), ($-\sqrt[3]{\frac{5}{2}}$, 0); (-1, 3) max., (0, 0) min.

Exercise 5d, page 102

2 $v = 6t^2 - 22t + 12$, $a = 12t - 22$; (a) 4 m from O on BO produced ($s = -4$), (b) away, (c) 8 m/s ($v = -8$), (d) decreasing, (e) 2 m/s^2 ($a = +2$).
3 (a) 3 m from O on OB ($s = +3$), (b) away, (c) 4 m/s ($v = -4$), (d) increasing, (e) 10 m/s^2 ($a = -10$).
4 After $\frac{11}{6}$ s; $s = -\frac{143}{54}$.
5 (a) 100 m from O on OA ($s = +100$); approaching A at 40 m/s ($v = +40$); retarding at 14 m/s^2 ($a = -14$), (b) $t = 3\frac{1}{3}$ to $t = 12$, (c) $t = 7\frac{2}{3}$.

Exercise 5e, page 103

1 (a) (1, 6) max., (3, 2) min.; (b) (-1, 15) max.; (2, -12) min.; (c) (-1, 2) max., (1, -2) min.; (d) (1, -1) min., ($-\frac{1}{2}$, $5\frac{3}{4}$) max.; (e) (0, 0) max., (-2, -16), (2, -16) min.; (f) ($-\frac{1}{3}$, $\frac{91}{27}$) max., (1, 1) min.
2 (-2, 27) max., (1, 0) min., ($-3\frac{1}{2}$, 0).
3 ± 1. **4** (a) max., (b) infl. **5** $y = 1$, $y = 1$, $y = 0$.
6 $2\frac{4}{5}$, $x + 3y - 7 = 0$. **7** $a = 2$, $b = -4$, $c = -1$. **8** $x + t^2y - 4t = 0$.
9 $4x - y \pm 15 = 0$. **10** 256 cm^3. **11** 32 m.
12 (a) $4\frac{1}{2}$ cm^2, (b) 4 cm^2. **13** $\frac{1}{144}(17x^2 - 16xl + 8l^2)$, min. **16** 48 m^2.
17 $2\pi r(r + h)$; (a) $(12/r) - r$, $\pi r(12 - r^2)$, (b) $r = 2$. **18** (a) 5 cm, (b) 6 cm.
19 $V = \pi r^2(5 - 2\pi r)$, $125/(27\pi) \approx 1.47$ m^3. **20** 10 m by 10 m by 5 m.
21 6 cm by 3 cm by 4 cm. **24** 20.
25 $AP^2 = x^4 - x^2 + 1$, $(\frac{1}{2}\sqrt{2}, \frac{1}{2})$, $(-\frac{1}{2}\sqrt{2}, \frac{1}{2})$.

Chapter 6

Qu. 1 (a) $2x + c$, (b) $mx + c$, (c) $x^3 + c$, (d) $\frac{3}{2}x^2 + c$, (e) $\frac{3}{5}x^5 + c$, (f) $3x + x^2 + c$, (g) $\frac{1}{2}x^2 - \frac{1}{3}x^3 + c$, (h) $\frac{1}{2}ax^2 + bx + c$.

Qu. 2 (a) $-\frac{1}{2}x^{-2} + c = \frac{-1}{2x^2} + c$, (b) $-\frac{1}{3}x^{-3} + c = \frac{-1}{3x^3} + c$, (c) $-2x^{-1} + c$, (d) $\frac{-x^{-(n-1)}}{n-1} + c$.

Qu. 3 $\frac{x^0}{0} + c$ is meaningless.

Qu. 4 $y = 4x + 18$. A straight line of gradient 4 through $(-2, 10)$.
Qu. 5 $v = 15 + 9.81t$, $s = 15t + 4.905t^2$.
Qu. 6 $+9.8$ m/s (rising), -9.8, -29.4 m/s (falling); 14.7, 14.7 m (above start), -24.5 m (below).
Qu. 7 (a) 9, (b) 42, (c) -6, (d) 35.
Qu. 8 $12\frac{2}{3}$ m.
Qu. 9 (a) 13 m past O, (b) 5 m past O, (c) 7 m past O, (d) 100 m short of O.
Qu. 10 (a) 72, (b) 9, (c) 36, (d) 21.
Qu. 11 (a) $3\frac{1}{4}$, (b) 9, (c) 2, (d) -8, (e) -38, (f) $9\frac{1}{4}$.
Qu. 12 25.
Qu. 13 (a) 9, (b) 81.
Qu. 14 $2\frac{2}{3}$.
Qu. 15 $\frac{1}{2}$.

Exercise 6a, page 108

1 (a) $\frac{1}{2}x + c$, $\frac{1}{6}x^3 + c$, $\frac{1}{3}x^3 + \frac{3}{2}x^2 + c$, $\frac{4}{3}x^3 + 6x^2 + 9x + c$, $-\frac{1}{4}x^{-4} + c$, $\frac{2}{3}x^{-3} + c$;
(b) $\frac{1}{2}at^2 + c$, $\frac{1}{12}t^4 + c$, $\frac{1}{3}t^3 - \frac{1}{2}t^2 - 2t + c$, $-\frac{1}{n}t^{-n} + c$, $-t^{-1} + 3t + t^2 + c$;
(c) $ay^{-1} + c$, $-ky^{-1} + c$, $\frac{1}{3}y^3 - y + 6y^{-1} + c$.
2 (a) $y = ax^3 + c$, (b) $s = \frac{3}{4}t^4 + c$, (c) $s = ut + \frac{1}{2}at^2 + c$, (d) $x = t + t^{-1} + c$, (e) $y = t + 3t^{-1} - 2t^{-2} + c$, (f) $A = -x^{-1} - x - \frac{2}{3}x^3 + c$.
3 $x - 6y + 34 = 0$. 4 $y = x^2 + 5x - 25$. 5 $y = x^3 + 1/x - 8\frac{1}{2}$.
6 (1, 0), (3, 0). 7 (4, 0); $y = 9\frac{13}{27} = \frac{256}{27}$. 8 $s = \frac{3}{2}t^2 + 8/t - 8$.
9 $A = c - 3x^{-1} - \frac{1}{2}x^{-2} + x^{-3} + \frac{1}{4}x^{-4}$; $\frac{49}{64}$.

Exercise 6b, page 112

1 $v = 20 + 9.81t$; $s = 20t + 4.90t^2$.
2 $v = -12 + 9.8t$; $s = -12t + 4.9t^2$; -2.2 m/s (rising), $+7.6$, $+17.4$ m/s (falling), -7.1, -4.4 m (above ground level), $+8.1$ m (below).
3 (a) $s = 3t + 3$, (b) $s = 2t^2 - t - 6$, (c) $s = t^3 + \frac{5}{2}t^2 - 2t - 13$, (d) $s = \frac{1}{3}t^3 + 5t + 2t^{-1} - 7$.
4 (a) 32, (b) 328, (c) -21, (d) 16.
5 (a) $s = 2t^2 + 3t + c$, 14 m, (b) $s = \frac{1}{3}t^3 - 3t + c$, $3\frac{1}{3}$ m; (c) $s = \frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t + c$, $3\frac{5}{6}$ m; (d) $s = \frac{1}{2}t^2 + 3t + 1/t + c$, $179\frac{19}{20}$ m.
6 $v = \frac{1}{2}At^2 + B$; $s = \frac{1}{6}At^3 + Bt + c$.
7 (a) $v = \frac{3}{2}t^2 + 3$, $s = \frac{1}{2}t^3 + 3t$; (b) $v = 2t + \frac{1}{2}t^2$, $s = -3 + t^2 + \frac{1}{6}t^3$; (c) $v = -7\frac{1}{2} + 10t - \frac{1}{2}t^2$, $s = -7\frac{1}{2}t + 5t^2 - \frac{1}{6}t^3$; (d) $v = \frac{1}{4}t^2 + 5$, $s = \frac{1}{12}t^3 + 5t + c$; (e) $v = \frac{1}{3}t^3 + c$, $s = \frac{1}{12}t^4 + ct + 9\frac{11}{12} - c$.
8 (a) $13\frac{1}{2}$ m past O, (b) $2\frac{1}{2}$ m past O, (c) 8 m past O, (d) $7\frac{1}{2}$ m.
9 (a) $s = -5 + 6t - t^2$, 5 m; (b) 13 m.
10 (a) $1\frac{2}{7}$ s; (b) 8.1 m; (c) 7.7, 2.9 m.
11 (a) 40 km, (b) 20 km/h, (c) 30 km/h.
12 (a) $13\frac{1}{3}$ km/h, (b) 20 km/h.
13 35 m/s, 28 m/s^2.

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14 $k = 6$; 4800 m.
15 (a) After 4 s, -64 m/s; (b) 27 m; (c) 16 m/s, 64 m/s.
16 (a) 11.59$\frac{1}{2}$ a.m., 12.01$\frac{1}{2}$ p.m.; (b) $s = \frac{1}{48}(5 + 18t + 12t^2 - 8t^3)$; (c) 20 km/h; (d) 30 km/h.

Exercise 6c, page 121

1 (a) $3\frac{63}{64}$, (b) -2, (c) $10\frac{2}{3}$, (d) $36\frac{137}{144}$. **2** 50.
3 (a) 26, (b) $58\frac{1}{3}$, (c) $22\frac{7}{60}$, (d) $2\frac{1}{2}$. **4** $5\frac{1}{3}$. **5** $-\frac{1}{6}$. **6** $-\frac{5}{12}, 2\frac{2}{3}$.
7 (a) $-2\frac{1}{3}$, (b) 4, (c) $2\frac{9}{20}$, (d) $-\frac{1}{2}$. **8** $1\frac{1}{3}$. **9** $4\frac{1}{2}$.
10 (a) (0, 0), (4, 8), $5\frac{1}{3}$; (b) $(-2, 12)$, (1, 3), $13\frac{1}{2}$; (c) $(-1, 0)$, (3, 4), $10\frac{2}{3}$.
11 (a) 96, (b) 60, (c) $1\frac{1}{2}$.
12 $833\frac{1}{3}$.

Exercise 6d, page 122

1 $y = 3 - 3/x + 1/x^2$. **2** $f(x) = x^2 - 1 + 1/x$. **3** $20\frac{5}{6}$. **4** $-4\frac{1}{2}$.
5 $6\frac{3}{4}$. **6** 36. **7** $\frac{1}{6}(4t^3 - 27t^2 + 60t)$, $4t - 9$, $7\frac{1}{3}$, $7\frac{7}{24}$.
8 $4\frac{5}{6}$ m, 1 m/s^2. **9** 14 m/s^2, 44 m. **10** $y = x^3 - 3x^2 + 4x + 8$, $10\frac{3}{4}$.
11 $10\frac{2}{3}$. **14** $(t^2 - 4t + 3)$ m/s, 1, $\frac{4}{3}$ m. **15** 20 m/s, 467 m.
16 3, -1 m/s^2, $11\frac{1}{3}$ m, 1, 6. **17** 16 m/s, $42\frac{2}{3}$ m, $85\frac{1}{3}$ m. **18** 9 m/s, 3 m/s^2.
19 9 m/s, 6 m/s^2, $2\frac{2}{3}$ m.

Chapter 7

Qu. 1 (a) $-4x^{-5}$, (b) $-6x^{-4}$, (c) $-\dfrac{2}{x^3}$, (d) $-\dfrac{4}{x^2}$, (e) $\dfrac{4}{x^3}$, (f) $-\dfrac{1}{x^4}$, (g) $\dfrac{4}{x^5}$, (h) $-\dfrac{3}{x^6}$, (i) $-\frac{1}{2}x^{-2} + c$, (j) $-2x^{-1} + c$, (k) $-\dfrac{1}{x} + c$, (l) $-\dfrac{1}{x^2} + c$, (m) $-\dfrac{1}{6x^2} + c$, (n) $-\dfrac{2}{15x^3} + c$.

Qu. 2 (a) $\frac{1}{2}x^{-1/2}$, (b) $-\frac{2}{3}x^{-4/3}$, (c) $\dfrac{1}{2\sqrt{x}}$, (d) $\dfrac{1}{3\sqrt[3]{x^2}}$, (e) $-\dfrac{1}{3\sqrt[3]{x^4}}$, (f) $\dfrac{2}{3\sqrt[3]{x^4}}$, (g) $3\sqrt{x}$, (h) $-\dfrac{1}{3\sqrt{x^3}}$, (i) $\frac{4}{3}x^{3/4} + c$, (j) $\frac{4}{5}x^{5/2} + c$, (k) $\frac{2}{3}\sqrt{x^3} + c$, (l) $\frac{3}{4}\sqrt[3]{x^4} + c$, (m) $2\sqrt{x} + c$, (n) $-\dfrac{2}{\sqrt{x}} + c$.

Qu. 3 (a) $2(x + 4)$, (b) $3(x + 2)^2$, (c) $6(3x + 1)$, (d) $-4(5 - 2x)$, (e) $3(x + 4)^2$, (f) $6x^2(x^3 + 1)$, (g) $6x(5 + x^2)^2$, (h) $-(2/x^2)(2 + 1/x)$, (i) $-6x^2(1 - x^3)$, (j) $\frac{3}{2}(\frac{1}{2}x - 7)^2$.

Qu. 4 (a) $2x + 3$, (b) $2x(2x^2 + 1)$, (c) $4(x - 2)(x^2 - x - 1)$, (d) $2(x + 1)(x + 2)(2x + 3)$.

Qu. 6 (a) and (b) $\dfrac{2(3x - 1)}{(x + 3)^3}$.

Qu. 7 (a) 1, (b) $\dfrac{dy}{dx}$, (c) $2x$, (d) $2y\dfrac{dy}{dx}$, (e) $y + x\dfrac{dy}{dx}$, (f) $2xy + x^2\dfrac{dy}{dx}$, (g) $y^2 + 2xy\dfrac{dy}{dx}$.

Qu. 8 $\dfrac{2x - 6y + 3}{6x - 2y + 2}$.

Qu. 9 $\dfrac{2x + y}{3y^2 - x}$.

Qu. 10 (a) $2x + \dfrac{2}{x^3}, 2 - \dfrac{6}{x^4}$; (c) $-\dfrac{1}{(x-1)^2}, \dfrac{2}{(x-1)^3}$.

Qu. 11 $\dfrac{1}{t}, -\dfrac{1}{2at^3}$.

Exercise 7a, page 129

1 (a) $4(2x+3)$, (b) $24(3x+4)^3$, (c) $-2(2x+5)^{-2}$, (d) $2(3x-1)^{-1/3}$, (e) $(3-2x)^{-3/2}$, (f) $12(3-4x)^{-4}$.

2 (a) $\frac{1}{12}(3x+2)^4 + c$, (b) $\frac{1}{6}(2x+3)^3 + c$, (c) $-\frac{1}{3}(3x-4)^{-1} + c$, (d) $\frac{1}{3}(2x+3)^{3/2} + c$.

3 (a) $\dfrac{-3}{(3x+2)^2}$, (b) $\dfrac{-4}{(2x+3)^3}$, (c) $\dfrac{-3}{2\sqrt{(3x+1)^3}}$, (d) $\dfrac{-4}{3(2x-1)^{5/3}}$.

4 (a) $-\frac{1}{2}(2x-3)^{-1} + c$, (b) $\frac{2}{3}\sqrt{(3x+2)} + c$, (c) $2(2x-1)^{1/4} + c$.

5 (a) $18x(3x^2+5)^2$, (b) $(18x^2+10)(3x^3+5x)$, (c) $\dfrac{14x}{3}(7x^2-4)^{-2/3}$, (d) $-(36x^2-8)(6x^3-4x)^{-3}$, (e) $-\frac{2}{3}(6x-5)(3x^2-5x)^{-5/3}$.

6 (a) $\dfrac{-6x}{(3x^2+2)^2}$, (b) $\dfrac{-3x}{\sqrt{(2+x^2)^3}}$, (c) $\dfrac{1}{\sqrt{x}(1+\sqrt{x})^3}$, (d) $\dfrac{3}{x^2}\left(1-\dfrac{1}{x}\right)^2$, (e) $\dfrac{-2x}{3(x^2-1)^{4/3}}$.

7 (a) $3(3\sqrt{x}-2x)^2\left(\dfrac{3}{2\sqrt{x}}-2\right)$, (b) $\dfrac{1}{\sqrt{x}(2-\sqrt{x})^2}$, (c) $\dfrac{1}{3}\left(2x^2-\dfrac{3}{x^2}\right)^{-2/3}\left(4x+\dfrac{6}{x^3}\right)$, (d) $\dfrac{1}{2}\left(x-\dfrac{1}{x}\right)^{-1/2}\left(1+\dfrac{1}{x^2}\right)$.

8 (a) $\dfrac{-\frac{3}{2}\sqrt{x}}{(x^{3/2}-1)^2}$, (b) $\dfrac{1}{2x^{3/2}\sqrt{(x-1)}}$, (c) $\dfrac{-1}{6\sqrt{x}\sqrt[3]{x}(1-\sqrt{x})^2}$, (d) $\dfrac{x^2-1}{x^2}$.

9 (a) $\dfrac{-3(2x-7)}{(x^2-7x)^4}$, (b) $\dfrac{1-4x^{3/2}}{\sqrt{x(x^2-\sqrt{x})^3}}$, (c) $\dfrac{x}{(1-x^2)^{3/2}}$, (d) $\dfrac{1}{\sqrt{x}(1-\sqrt{x})^3}$.

10 (a) $\dfrac{x^4+1}{x^2\sqrt{(x^4-1)}}$, (b) $\dfrac{-2(\sqrt{x}+1)}{\sqrt{x}(x+2\sqrt{x})^2}$, (c) $\dfrac{1}{3x^{3/2}(1-2/\sqrt{x})^{2/3}}$, (d) $\dfrac{1}{4x^{3/2}\sqrt{(1-1/\sqrt{x})}}$.

Exercise 7b, page 132

1 $1458\ \text{cm}^3/\text{s}$. **2** $16\pi\ \text{cm}^2/\text{s}$. **3** $\frac{2}{27}$ cm/s. **4** $\frac{3}{2}$ cm/s.
5 Decreasing, $8\pi\ \text{cm}^2/\text{s}$. **6** 24. **7** $1/(8\pi)$ cm/s. **8** $1/(2\pi)$ cm/s.
9 $\frac{4}{45}$ cm/s. **10** (a) 6 cm, (b) $\frac{1}{6}$ cm/min. **11** 30. **12** $\frac{4}{15}$.
13 0.27 cm/s. **14** 4.8 litres/min.

Exercise 7c, page 136

1 $x(5x+2)(x+1)^2$. **2** $(9x^2+1)(x^2+1)^3$. **3** $2(2x-1)(x+1)^2$.
4 $\dfrac{1}{(x+1)^2}$. **5** $\dfrac{-4x}{(1+x^2)^2}$. **6** $\dfrac{2(x-1)}{(x+1)^3}$. **7** $2x(1+x^2)(1-3x^2)$.
8 $2x-\frac{3}{2}\sqrt{x}$. **9** $-x(x+1)(x-1)^2(4+7x+7x^2)$. **10** $\dfrac{2x^2-x+1}{\sqrt{(x^2+1)}}$.
11 $\dfrac{x(2+3x^2)}{\sqrt{(1+x^2)}}$. **12** $\dfrac{x(2+x^2)}{\sqrt{(1+x^2)^3}}$. **13** $\dfrac{(x-1)(3x+1)}{2\sqrt{x^3}}$.
14 $\dfrac{x(-2x^3+2x^2+3x-4)}{\sqrt{(x^2-1)^3}}$. **15** $\dfrac{2x+5}{2\sqrt{(x+3)}\sqrt{(x+2)}}$.
16 $\dfrac{1}{2\sqrt{\{x(x+1)^3\}}}$. **17** $\dfrac{-1}{\sqrt{x}(1+\sqrt{x})^2}$. **18** $\dfrac{1}{2\sqrt{\{(1+x)(2+x)^3\}}}$.
19 $\dfrac{(4x+5)\sqrt{(x+2)}}{2\sqrt{(x+1)}}$. **20** $\dfrac{(2x+5)\sqrt{(x+1)}}{2\sqrt{(x+2)^3}}$.

Exercise 7d, page 139

1 $\pm\frac{1}{3}$. **2** $-1, \frac{11}{3}$. **3** $\frac{3}{7}$. **4** $-\frac{3}{2}$. **5** (a) $\dfrac{3t}{2}$, (b) $\frac{3}{2}\sqrt{x}$.
6 $(9, 3), (-1, 3)$. **7** (a) $\dfrac{-2y}{3x}$, (b) $\dfrac{y(2x-y)}{x(2y-x)}$. **8** (a) $\dfrac{1}{t}$, (b) $\dfrac{t}{t+1}$.
9 $2t-t^2$. **10** $\dfrac{2(x-y-1)}{2x-2y-3}$. **11** $\dfrac{9(t+2)^2}{4(t+3)^2}$. **12** $\dfrac{4y-3x}{3y-4x}$.

Exercise 7e, page 141

1 $8\pi\ \text{cm}^2$. **2** 9%. **3** (a) 2.000 83, (b) 5.01. **4** $\frac{1}{2}x$.
5 $\delta p/p = -\delta v/v$. **6** (a) $1\frac{1}{4}$, (b) $1\frac{1}{4}$. **7** $1.6\pi\ \text{cm}^3$. **8** 4%.
9 2%. **10** (a) 25.04, (b) 10.0166. **11** $1\frac{1}{3}$%.

Exercise 7f, page 142

1 (a) $-\dfrac{2n}{x^{2n+1}}$, (b) $(n+1)x^n$, (c) $\frac{1}{2}(2a-1)x^{2a-2}$, (d) $2mx^{2m-1}$, (e) $\frac{1}{2}nx^{(1/2)n-1}$.
2 (a) $\dfrac{1}{2k-1}x^{2k-1}+c$, (b) $\dfrac{1}{(1-2n)}x^{1-2n}+c$, (c) $-n^2x^{-1/n}+c$, (d) $\dfrac{1}{k}x^k+c$.
3 (a) $\dfrac{1}{n}x^{(1/n)-1}$, (b) $\frac{1}{2}nx^{(1/2)n-1}$, (c) $2(1-n)/x^n$, (d) $-\frac{1}{2}nx^{-(1/2)n-1}$,
(e) $-\frac{2}{3}nx^{-(2/3)n-1}$.

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4 (a) $-\frac{6}{n}x^{-(3/n)-1}$, (b) $-\frac{1}{3}x^{-4/3}$, (c) $-\frac{3n}{2}x^{-(3n/2)-1}$, (d) $\frac{3\sqrt{2}}{2}\sqrt{x}$,

(e) $\frac{n}{n+1}x^{-1/(n+1)}$.

5 (a) $8x(x^2+3)^3$, (b) $\frac{3x^2}{\sqrt{(2x^3-3)}}$, (c) $\frac{3}{2\sqrt{x}}(\sqrt{x}+1)^2$,

(d) $n\left(\frac{1}{2}+\frac{2}{x^2}\right)\left(\frac{x}{2}-\frac{2}{x}\right)^{n-1}$.

6 (a) $-\frac{(2\sqrt{x}+1)}{2\sqrt{x}(x+\sqrt{x})^2}$, (b) $-\frac{2x}{(x^2-1)^2}$, (c) $-\frac{1}{\sqrt{x}(\sqrt{x}-1)^3}$,

(d) $\frac{2x-2}{3(2x-x^2)^{4/3}}$.

7 (a) $x(5x-2)(x-1)^2$, (b) $(4x+1)(x+1)^{1/2}(x-1)^{3/2}$, (c) $\frac{(5x-3)(x-1)}{2\sqrt{(x-2)}}$.

8 (a) $\frac{4x+1}{2}\sqrt{\left(\frac{x-2}{x+1}\right)}$, (b) $\frac{14x^2-6x-2}{3\sqrt[3]{(1-2x)^2}}$, (c) $\frac{2x^2-1}{\sqrt{(x^2-1)}}$.

9 (a) $-\frac{1+x^2}{(x^2-1)^2}$, (b) $\frac{x-2}{2(x-1)^{3/2}}$, (c) $-\frac{1+x}{2\sqrt{x}(x-1)^2}$,

(d) $-\frac{1}{2\sqrt{x}\sqrt{(x-1)}(\sqrt{x}-1)}$.

10 (a) $\frac{4(x-1)}{(x+2)^3}$, (b) $\frac{3(x-1)^3(x+1)}{(x^3-1)^2}$.

11 (a) $\frac{1}{2\sqrt{(x+1)}\sqrt{(x+2)^3}}$, (b) $\frac{(2x-5)\sqrt{(x+2)}}{2(x-1)^{3/2}}$.

12 (a) $-\frac{2x}{\sqrt{\{(x^2-1)^3(x^2+1)\}}}$, (b) $\frac{(1-\sqrt{x})(1-x\sqrt{x})}{\sqrt{x}(x^2-1)^{3/2}}$.

13 -1. **14** $\frac{3y-2x-4}{2y-3x-2}$. **15** $\frac{3x-2y}{2x}$.

16 $-\frac{2t}{1-t^2}$. **17** $-\frac{1}{t}$. **18** -1. **19** 1%. **20** 3%. **21** 2%.

22 (a) 4.021, (b) 6.083. **23** $12x^2-12x-9$, -24, 24. **24** $\frac{1}{t}$, $-\frac{1}{2at^3}$.

25 $\frac{5}{3}$, $-\frac{3}{4}$. **26** $5x-4y=9$. **28** $\frac{3x^2+4x}{2(x+1)^{3/2}}$, $\frac{3x^2+8x+8}{4(x+1)^{5/2}}$.

29 $1\frac{1}{2}$, $1\frac{7}{10}$, 2, 2.

30 (a) $2\sec 2x\tan 2x$, (b) $2\sin x\cos x$, (c) $\cos x - x\sin x$, (d) $3\tan^2 x\sec^2 x$,

(e) $\frac{1}{2\sqrt{x}}\cos\sqrt{x}$.

31 $\frac{3t}{4(t^2-1)}$, $-\frac{3(1+t^2)}{16t(t^2-1)^3}$. **32** 1%. **33** $-\frac{6}{5}\sqrt{5}$.

35 $y=3x+\frac{1}{3}$, $(\frac{1}{9},\frac{2}{3})$, $3y+x=2\frac{1}{9}$. **36** $\frac{bt}{s}$ m/s, $\frac{b}{s^3}(s^2-bt^2)$ m/s^2.

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37 $x \sin t + y \cos t = a \sin t \cos t,\ x \cos t - y \sin t = a \cos 2t.$

39 $0, 0$, min.; $\dfrac{64\sqrt{2}}{25\sqrt{5}}$, max.

40 $x \sin \theta - y \cos \theta = 2a\theta \sin \theta,\ x \cos \theta + y \sin \theta = 2a\theta \cos \theta + 2a \sin \theta.$

41 $0.08\pi\ \text{cm}^3$.

43 (a) $6x \sin (6x^2 + 8)$, (b) $\dfrac{1}{2\sqrt{(x+1)}} \sec^2 \sqrt{(x+1)}$, (c) $\dfrac{1}{1 + \cos x}$,
(d) $\dfrac{\sec^4 x}{2\sqrt{\tan x}\,(1 - \tan^2 x)^{3/2}}$.

44 $-5, 78$. **45** $\dfrac{x}{\sqrt{(x^2+1)}}$.

47 min. $(0, 0)$, max. $(-2, -4)$. **48** $-1, 2, k = -13, 19, -8$. **49** $2, -3$.

50 min. $(-5, \frac{8}{9})$, max. $(-1, 0)$.

Chapter 8

Qu. 1 $60 + 50/n$; 60.

Qu. 2 (a) $\frac{3}{2}x^2 - 4x + c$, (b) $\frac{8}{3}x^3 + 3x^{-1} + c$, (c) $\frac{7}{8}x^{8/7} + c$,
(d) $-t^2 + \frac{10}{3}t^{3/2} - 3t + c$.

Qu. 3 (a) $17\frac{5}{6}$, (b) $\frac{7}{48}$, (c) $-5\frac{3}{5}$.

Qu. 4 (a) (i) A cone, vertex C,
(ii) two cones with common base, vertices A and C, (b) sphere,
(c) hemisphere, (d) ring internal dia. 4, external dia. 8, (e) cylinder.

Qu. 5 (a) $31\pi/5$, (b) $56\pi/15$.

Qu. 6 (a) $(\frac{6}{5}, 0)$, (b) $(\frac{6}{5}, 3\sqrt{2}/8)$.

Exercise 8a, page 149

4 (a) $\frac{3}{4}x^{4/3} + c$, (b) $\frac{4}{5}x^{5/4} + c$, (c) $\frac{5}{3}x^{6/5} + c$, (d) $\frac{3}{4}kx^{4/3} + c$, (e) $2x^{1/2} + c$,
(f) $\frac{3}{2}x^{2/3} + c$, (g) $\frac{6}{5}x^{5/6} + c$, (h) $\frac{5}{2}x^{4/5} + c$, (i) $\frac{3}{5}x^{5/3} + c$, (j) $\frac{3}{10}x^{10/3} + c$,
(k) $\frac{2}{5}x^{5/2} + c$, (l) $-3x^{-1/3} + c$, (m) $\dfrac{a}{a+1}x^{(a+1)/a} + c$, (n) $\dfrac{n}{n-1}x^{(n-1)/n} + c$,
(o) $\frac{2}{7}x^{7/2} + \frac{4}{5}x^{5/2} - 2x^{3/2} + c$, (p) $\frac{2}{3}x^{3/2} + 4x^{1/2} + c$, (q) $\frac{1}{2}x^2 - \frac{2}{3}x^{3/2} - 6x + c$,
(r) $\frac{2}{3}(x+2)^{3/2} + c$, (s) $\frac{1}{3}(x^2 - 3)^{3/2} + c$.

5 (a) $-\frac{1}{2}$, (b) 21, (c) $12\frac{2}{3}$.

Exercise 8b, page 158

1 (a) $\frac{1}{3}x^3 - \frac{3}{2}x^2 + c$, (b) $-2x^{-1} + x^{-2} + c$, (c) $\frac{1}{3}at^3 + bt - ct^{-1} + k$,
(d) $\frac{1}{5}x^5 - \frac{3}{5}x^{5/3} + 2x + x^{-1} + c$, (e) $\frac{1}{3}y^3 + \frac{2}{3}y^{3/2} + y - 2y^{-1/2} + c$,
(f) $\frac{3}{8}s^{8/3} + \frac{6}{5}s^{5/3} + \frac{3}{2}s^{2/3} + c$.

2 (a) $26\frac{2}{3}$, (b) $25\frac{2}{3}$, (c) $1\frac{11}{15}$, (d) $3\frac{19}{24}$, (e) $21\frac{1}{3}$, (f) $24\frac{2}{3}$.

3 (a) 12, (b) $-31\frac{1}{4}$, (c) $1\frac{7}{24}$.

4 (a) 9, (b) $11\frac{1}{4}$, (c) 12, (d) $2(\sqrt{3} - \sqrt{2})$.

5 (a) $4\frac{1}{2}$, on the negative side of the y-axis, (b) $4\frac{1}{2}$, (c) $1\frac{1}{3}$.

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6 $18\frac{2}{3}$. **7** -36. **8** $28\frac{4}{9}$. **9** (a) 4, (b) $\frac{3}{4}$.

10 (a) $10\frac{2}{3}$, (b) $1\frac{1}{3}$, (c) $4\frac{1}{2}$, (d) $4\frac{1}{2}$, (e) $20\frac{5}{6}$, (f) $20\frac{5}{6}$.

11 (a) $\frac{9}{8}$, (b) $\frac{1}{3}$, (c) $\frac{1}{24}$, (d) $\frac{16}{3}$, (e) $41\frac{2}{3}$, (f) $13\frac{1}{2}$.

12 Other points of intersection $(2, \frac{1}{4})$ and $(\frac{1}{2}, 4)$; $1\frac{11}{16}$.

Exercise 8c, page 164

1 (a) 144π, (b) $28\pi/15$, (c) 2π, (d) $16\pi/15$, (e) $\pi/105$, (f) $3\pi/4$.

2 (a) 18π, (b) $9\pi/2$, (c) $96\frac{3}{5}\pi$, (d) $34\frac{2}{15}\pi$, (e) $3\pi/5$, (f) $3\pi/10$.

3 (a) $8\pi/3$, (b) $256\pi/15$, (c) $8\pi/3$, (d) $16\pi/15$, (e) $128\pi/105$, (f) $7\pi/3$.

4 (a) 5π, (b) $64\pi/15$, (c) $32\pi/3$, (d) $\frac{1}{2}\pi$.

5 $\frac{1}{3}\pi r^2 h$. **6** $\frac{4}{3}\pi r^3$. **7** $661\frac{1}{3}\pi \text{ cm}^3$. **8** $1296\pi \text{ cm}^3$. **9** $57\frac{6}{7}\pi \text{ cm}^3$.

10 $27\frac{1}{2}\pi \text{ cm}^3$. **11** $16\pi/15$. **12** $37\pi/10$. **13** $37\frac{1}{3}\pi$. **14** 8π.

15 $45\pi/2$.

Exercise 8d, page 170

1 (a) $(\frac{12}{5}, 0)$, (b) $(0, \frac{3}{5})$, (c) $(\frac{8}{5}, 0)$, (d) $(0, \frac{7}{3})$.

2 (a) $(\frac{2}{5}, 0)$, (b) $(0, \frac{83}{70})$.

3 (a) $(8, \frac{4}{3})$, (b) $(\frac{3}{4}, \frac{3}{5})$, (c) $(\frac{9}{4}, \frac{27}{10})$, (d) $(\frac{8}{5}, \frac{16}{7})$.

4 (a) $(1\frac{1}{2}, 0)$, (b) $(\frac{8}{3}, 0)$, (c) $(0, \frac{10}{3})$, (d) $(\frac{5}{4}, 0)$, (e) $(\frac{9}{7}, 0)$, (f) $(0, \frac{1275}{248})$.

5 $\frac{1}{4}h$ above the base.

6 $\frac{3}{8}r$.

7 $9\frac{1}{7}$ cm.

8 $(4r/(3\pi), 4r/(3\pi))$.

Exercise 8e, page 170

1 0, (a) $\frac{1}{4}$, (b) $-\frac{1}{4}$. **2** $\frac{2}{15}$. **3** $A(1, 1\frac{1}{2})$, $B(4, 3)$, $\frac{3}{2}\sqrt{5}$. **4** 134π.

5 $\frac{128}{5}\pi$, 16π. **6** $12\frac{2}{3}\pi$. **7** $\frac{1}{105}\pi$. **9** $y = 2x - 2$. **10** $10\frac{2}{3}$.

11 255π. **13** $6\frac{3}{4}$, $1\frac{4}{5}$. **14** $5\frac{3}{5}$, $2\frac{4}{21}$. **15** $3\frac{1}{4}$, $\frac{73}{195}$.

16 (a) $\frac{7}{3}$, (b) $(\frac{45}{28}, \frac{93}{70})$. **17** $(\frac{9}{5}, \frac{54}{35})$. **18** $6\frac{3}{4}$, $\frac{1}{5}$. **19** $(1\frac{17}{28}, 6\frac{9}{14})$.

20 $\frac{32}{5}\pi$, $\frac{31}{24}$. **21** 8π, $\frac{9}{8}$. **22** 4π, $\frac{5}{4}$. **23** $\frac{4}{3}\pi$, $\frac{4}{5}$. **24** $\frac{4}{3}\pi$, $\frac{6}{5}$.

Chapter 9

Qu. 1 (a) 2, (b) 6, (c) a, (d) ab, (e) 18, (f) 80, (g) $4a$, (h) 6, (i) 35, (j) 16, (k) 36, (l) ab.

Qu. 2 (a) 3, (b) 3, (c) 9, (d) 2, (e) 8, (f) 243, (g) 16, (h) 8.

Qu. 3 '$0^0 = 1$' would have to be derived from '$0^n - 0^n = 0^0$', but division by 0, or by 0^n, is meaningless.

Qu. 4 Bases: (a) 10, (b) 10, (c) 3, (d) 4, (e) 2, (f) $\frac{1}{2}$, (g) a.
Logarithms: (a) 2, (b) 1.6021, (c) 2, (d) 3, (e) 0, (f) -3, (g) b.

Qu. 5 $x = \log_c a$, $y = \log_c b$, $x + y = \log_c (ab)$, $x - y = \log_c (a/b)$.

Qu. 6 $x = \log_c a$, $nx = \log_c a^n$.

Qu. 7 (a) 10, (b) 100, (c) 0.1, (d) 1, (e) 0, (f) $\frac{1}{2}$.

Qu. 8 (a) a, (b) a^2, (c) $1/a$, (d) 1, (e) 0, (f) $\frac{1}{2}$.
Qu. 9 (a) $\frac{2}{3}, -\frac{7}{3}$; (b) $-\frac{11}{5}, \frac{3}{5}$; (c) $-\frac{5}{2}, -\frac{1}{2}$; (d) $-\frac{1}{2}, -\frac{7}{2}$.
Qu. 10 (a) $x^2 - 7x + 12 = 0$, (b) $x^2 - 3x - 2 = 0$, (c) $8x^2 + 4x - 3 = 0$, (d) $3x^2 - 2x = 0$.
Qu. 11 $-3, \frac{7}{3}$.
Qu. 12 Polynomials.

Exercise 9a, page 174

1 (a) 5, (b) $\frac{1}{2}$, (c) 48, (d) $\frac{1}{2}$, (e) a/b, (f) 15, (g) 21, (h) p/q, (i) $1/(4p)$, (j) $9a/(2b)$.
2 (a) $2\sqrt{2}$, (b) $2\sqrt{3}$, (c) $3\sqrt{3}$, (d) $5\sqrt{2}$, (e) $3\sqrt{5}$, (f) $11\sqrt{10}$, (g) $5\sqrt{3}$, (h) $4\sqrt{2}$, (i) $6\sqrt{2}$, (j) $7\sqrt{2}$, (k) $2\sqrt{15}$, (l) $16\sqrt{2}$.
3 (a) $\sqrt{18}$, (b) $\sqrt{12}$, (c) $\sqrt{80}$, (d) $\sqrt{24}$, (e) $\sqrt{72}$, (f) $\sqrt{216}$, (g) $\sqrt{128}$, (h) $\sqrt{1000}$, (i) $\sqrt{\frac{1}{2}}$, (j) $\sqrt{\frac{1}{3}}$, (k) $\sqrt{\frac{1}{6}}$, (l) $\sqrt{\frac{2}{3}}$.
4 (a) $\sqrt{5}/5$, (b) $\sqrt{7}/7$, (c) $-\sqrt{2}/2$, (d) $2\sqrt{3}/3$, (e) $\sqrt{6}/2$, (f) $\sqrt{2}/4$, (g) $-\sqrt{3}/2$, (h) $3\sqrt{6}/8$, (i) $\sqrt{2} - 1$, (j) $2 + \sqrt{3}$, (k) $(4 + \sqrt{10})/6$, (l) $\sqrt{6} - 2$, (m) $(\sqrt{5} + \sqrt{3})/2$, (n) $3\sqrt{6} + 3\sqrt{5}$, (o) $3 + 2\sqrt{2}$, (p) $(3\sqrt{2} + 2\sqrt{3})/6$.

Exercise 9b, page 175

1 (a) $3\sqrt{2}$, (b) $6\sqrt{3}$, (c) $4\sqrt{7}$, (d) $5\sqrt{10}$, (e) $28\sqrt{2}$, (f) 0.
2 (a) 25.5, (b) 2.26, (c) 3.15, (d) 19.5, (e) 0.354, (f) 0.260.
3 (a) $\frac{6}{7} + \frac{2}{7}\sqrt{2}$, (b) $9 + 4\sqrt{5}$, (c) $-1 + \sqrt{2}$, (d) $4 - 2\sqrt{3}$, (e) $-1 - \sqrt{2}$, (f) $\frac{1}{2} + \frac{3}{4}\sqrt{2}$, (g) $\frac{2}{9}\sqrt{3}$, (h) $\frac{6}{25}\sqrt{5}$, (i) $\frac{8}{11} + \frac{5}{11}\sqrt{3}$, (j) $\frac{3}{2} + \frac{1}{2}\sqrt{5}$, (k) $\frac{3}{7} + \frac{5}{14}\sqrt{2}$, (l) 0.
4 (a) $5 + 2\sqrt{6}$, (b) $\frac{1}{2}(5 + \sqrt{3} + \sqrt{5} + \sqrt{15})$, (c) $-7 + 3\sqrt{6}$, (d) $4 + \sqrt{10}$, (e) $3 + 2\sqrt{2}$, (f) $\sqrt{2}$.
5 (a) $2 - \sqrt{2}$, (b) $4(2 + \sqrt{3})$, (c) $-(2 + \sqrt{3})$, (d) $2 + \sqrt{3}$, (e) $3 + 2\sqrt{2}$, (f) $6 + 4\sqrt{2}$.

Exercise 9c, page 178

1 (a) 5, (b) 3, (c) 2, (d) 7, (e) $\frac{1}{2}$, (f) 1, (g) -2, (h) -1, (i) 16, (j) 9, (k) 125, (l) 343, (m) $\frac{1}{8}$, (n) $\frac{2}{3}$, (o) $1\frac{1}{2}$, (p) $\frac{2}{3}$.
2 (a) 1, (b) $\frac{1}{3}$, (c) 1, (d) $\frac{1}{4}$, (e) $\frac{1}{8}$, (f) 2, (g) 9, (h) 1, (i) $\frac{1}{27}$, (j) $-\frac{1}{6}$, (k) 1, (l) $\frac{9}{4}$, (m) 4, (n) 3, (o) $4\frac{1}{2}$, (p) $\frac{5}{9}$.
3 (a) $\frac{1}{2}$, (b) $\frac{1}{4}$, (c) $\frac{1}{2}$, (d) $\frac{1}{8}$, (e) $\frac{1}{9}$, (f) 2, (g) 2, (h) 9, (i) $1\frac{1}{2}$, (j) $1\frac{1}{2}$, (k) $1\frac{1}{2}$, (l) $\frac{16}{81}$.

Exercise 9d, page 178

1 (a) 16, (b) 36, (c) 4, (d) 6, (e) $1\frac{1}{2}$, (f) $1\frac{1}{3}$, (g) $\frac{1}{2}$, (h) $\frac{1}{8}$, (i) $\frac{1}{16}$, (j) $\frac{1}{27}$, (k) $2\frac{3}{4}$, (l) 64, (m) $\frac{4}{9}$, (n) 1.1, (o) 125, (p) $\frac{1}{2}$.
2 (a) $\frac{1}{2}$, (b) 1, (c) $\frac{1}{8}$, (d) 2, (e) 2, (f) 1.
3 (a) 2^{-n}, (b) 3^{n+1}, (c) 4, (d) 3, (e) 12, (f) $10^{n/2}$.
4 (a) $x^{-7/12}$, (b) 2, (c) $x^{n/2+3/2}$, (d) 1, (e) y^{-q}, (f) 1.
5 (a) $-\dfrac{1}{x^2(x^2+1)^{1/2}}$, (b) $\dfrac{x-2}{2x^2(1-x)^{1/2}}$, (c) $-\dfrac{1}{2x^{3/2}(1+x)^{1/2}}$, (d) $\dfrac{3+2x}{3(1+x)^{4/3}}$, (e) $\dfrac{1}{(1-x)\sqrt{(1-x^2)}}$.

Exercise 9e, page 180

1 (a) $\log_2 16 = 4$, (b) $\log_3 27 = 3$, (c) $\log_5 125 = 3$, (d) $\log_{10} 1\,000\,000 = 6$, (e) $\log_{12} 1728 = 3$, (f) $\log_{16} 64 = \frac{3}{2}$, (g) $\log_{10} 10\,000 = 4$, (h) $\log_4 1 = 0$, (i) $\log_{10} 0.01 = -2$, (j) $\log_2 \frac{1}{2} = -1$, (k) $\log_9 27 = \frac{3}{2}$, (l) $\log_8 \frac{1}{4} = -\frac{2}{3}$, (m) $\log_{1/3} 81 = -4$, (n) $\log_e 1 = 0$, (o) $\log_{16} \frac{1}{2} = -\frac{1}{4}$, (p) $\log_{1/8} 1 = 0$, (q) $\log_{81} 27 = \frac{3}{4}$, (r) $\log_{1/16} 4 = -\frac{1}{2}$, (s) $\log_{-2/3} \frac{4}{9} = 2$, (t) $\log_{-3} (-\frac{1}{3}) = -1$, (u) $\log_a c = 5$, (v) $\log_a b = 3$, (w) $\log_p r = q$, (x) $\log_b a = c$.

2 (a) $2^5 = 32$, (b) $3^2 = 9$, (c) $5^2 = 25$, (d) $10^5 = 100\,000$, (e) $2^7 = 128$, (f) $9^0 = 1$, (g) $3^{-2} = \frac{1}{9}$, (h) $4^{1/2} = 2$, (i) $e^0 = 1$, (j) $27^{1/3} = 3$, (k) $a^2 = x$, (l) $3^b = a$, (m) $a^c = 8$, (n) $x^y = z$, (o) $q^p = r$.

3 (a) 6, (b) 2, (c) 7, (d) 2, (e) $\frac{1}{3}$, (f) 0, (g) $\frac{1}{3}$, (h) 2, (i) 3, (j) -1, (k) 3, (l) -1.

Exercise 9f, page 182

1 (a) $\log a + \log b$, (b) $\log a - \log c$, (c) $-\log b$, (d) $2 \log a + \frac{3}{2} \log b$, (e) $-4 \log b$, (f) $\frac{1}{3} \log a + 4 \log b - 3 \log c$, (g) $\frac{1}{2} \log a$, (h) $\frac{1}{3} \log b$, (i) $\frac{1}{2} \log a + \frac{1}{2} \log b$, (j) $1 + \lg a$, (k) $-2 - 2 \lg b$, (l) $\frac{1}{2} \log a - \frac{1}{2} \log b$, (m) $\frac{1}{2} \log a + \frac{3}{2} \log b - \frac{1}{2} \log c$, (n) $\log b + \frac{1}{2} \log a - \frac{1}{3} \log c$, (o) $\frac{1}{2} + \frac{1}{2} \lg a - \frac{5}{2} \lg b - \frac{1}{2} \lg c$.

2 (a) log 6, (b) log 2, (c) log 6, (d) log 2, (e) $\log (ac)$, (f) $\log (xy/z)$, (g) $\log (a^2/b)$, (h) $\log (a^2b^3/c)$, (i) $\log \sqrt{(x/y)}$, (j) $\log (p/\sqrt[3]{q})$, (k) $\lg (100\, a^3)$, (l) $\lg (10a/\sqrt{b})$, (m) $\lg (a^2/2000c)$, (n) $\lg (10x^3/\sqrt{y})$.

3 (a) 3, (b) 2, (c) 2, (d) 1, (e) log 2, (f) log 7, (g) $\log \frac{1}{2}$, (h) 0, (i) 0, (j) 3, (k) 2, (l) $\frac{2}{3}$.

4 (a) 2.322, (b) 0.6309, (c) 0.3155, (d) 1.161, (e) -2.585, (f) 6.838.

5 (a) 3.170, (b) 0.7211, (c) 1.042, (d) 2.303, (e) 1.145, (f) -0.6309.

7 (a) 3.119, (c) 1.297, (c) 23.14, (d) 0.7936, (e) 0.3674, (f) 0.000 759 7.

Exercise 9g, page 187

1 (a) $\frac{11}{2}, \frac{3}{2}$; (b) $-\frac{1}{2}, -\frac{1}{2}$; (c) $\frac{7}{3}, -2$; (d) $-1, -1$; (e) $1, -3$; (f) $1, -5$; (g) 4, 1; (h) $3, -2$.

2 (a) $x^2 - 3x + 4 = 0$, (b) $x^2 + 5x + 6 = 0$, (c) $2x^2 - 3x - 5 = 0$, (d) $3x^2 + 7x = 0$, (e) $x^2 - 7 = 0$, (f) $5x^2 - 6x + 4 = 0$, (g) $36x^2 + 12x + 1 = 0$, (h) $10x^2 + 25x - 16 = 0$.

3 (a) $\frac{25}{4}$, (b) $\frac{3}{4}$, (c) $-\frac{5}{2}$, (d) $-\frac{25}{8}$.

4 (a) $\frac{5}{9}$, (b) $\frac{16}{9}$, (c) $\frac{80}{27}$, (d) $\frac{80}{9}$.

5 (a) 72, (b) 5, (c) $-\frac{9}{8}$, (d) -32.

6 (a) $x^2 - 39x + 49 = 0$, (b) $x^2 - 7x - 1 = 0$, (c) $x^2 + 35x - 343 = 0$.

7 (a) $2x^2 + 4x + 1 = 0$, (b) $x^2 - 4x + 2 = 0$, (c) $x^2 - 6x + 1 = 0$.

8 $4x^2 - 49x + 36 = 0$.

9 $\frac{35}{4}$.

10 ± 6.

12 (a) $-bc/a^2$, (b) $(b^2 - 2ac)/a^2$, (c) $b(3ac - b^2)/a^3$, (d) $-b/c$, (e) $(b^2 - 2ac)/(ac)$, (f) $(b^4 - 4ab^2c + 2a^2c^2)/a^4$.

13 (a) $ax^2 - bx + c = 0$, (b) $ax^2 + (b - 2a)x + a - b + c = 0$,

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(c) $a^2x^2 + (2ac - b^2)x + c^2 = 0$, (d) $cx^2 - bx + a = 0$, (e) $a^2x^2 - (b^2 - 4ac) = 0$, (f) $a^2x^2 + 3abx + (2b^2 + ac) = 0$.

17 2, -9, 9; 3, $\frac{3}{2}$.

18 (a) $ay^2 + y(b - 2a) + a - b + c = 0$, $\alpha + 1$, $\beta + 1$; (b) $ay^4 + by^2 + c = 0$, $\pm\sqrt{\alpha}$, $\pm\sqrt{\beta}$; (c) $a^2y^2 + (2ac - b^2)y + c^2 = 0$, α^2, β^2.

19 (a) $ay^2 + (b - 4a)y + 4a - 2b + c = 0$, (b) $cy^2 + by + a = 0$, (c) $ay^4 - 4ay^3 + (6a + b)y^2 - 2(2a + b)y + a + b + c = 0$.

Exercise 9h, page 190

1 (a) -12, -12, -6, 0, 0; $(x - 2)$, or $(x + 2)$; (b) -1, 0, -2, 19, -21, $(x - 1)$, (c) 0, 6, -2, 88, -24, x, (d) 3, 0, 0, 3, 3, $(x - 1)$, or $(x + 1)$.

2 (a) 2, (b) 18, (c) -11, (d) -1, (e) 2, (f) $-2\frac{1}{2}$.

3 (a) -3, (b) -10, (c) 2, (d) 4, (e) 4, (f) 2.

4 $(x + 3)(2x - 1)$. **5** $(2x - 1)(2x + 3)(3x + 1)$.

6 (a) $(x - 1)(x + 2)(x - 3)$, (b) $(x + 1)(x - 2)(x - 3)$, (c) $(2x + 1)(x - 2)(x + 2)$, (d) $(x + 1)(x + 2)(2x - 1)$, (e) $(x + 2)(x + 3)(2x + 1)$, (f) $(x^2 + 1)(2x - 1)$.

7 $a = 3$, $b = 2$. **8** $p = 1$, $q = -3$. **9** $a = 3$, $b = -1$, $c = -2$.

10 $a = 2$, $b = -1$, $c = -2$.

Exercise 9i, page 191

1 (a) $5\sqrt{5}$, (b) $\sqrt{2}$, (c) $18\sqrt{3}$. **2** (a) 18.9, (b) 6.29, (c) 0.642.

3 (a) $(11 + 6\sqrt{2})/7$, (b) $13 + 2\sqrt{2}$. **4** (a) $\frac{1}{16}$, $\frac{8}{27}$, $\frac{1}{4}$, (b) 8, 27.

5 $x - 3x^{1/3} - 2$, 0. **6** (a) 8, (b) 2.

7 (a) 1.079 18, (b) 0.653 21, (c) 0.592 72.

8 (a) $2 + 2 \lg a - 3 \lg b - \frac{1}{2} \lg c$, (b) 1.602 060.

9 (a) 0.698 970, (b) 1.255 273, (c) 0.176 091.

10 (a) $1\frac{1}{8}$, (b) 3.17. **11** (a) 7.525 cm, (b) 4.402 cm.

12 $-\frac{5}{3}$, $-\frac{1}{3}$; $9x^2 - 31x + 1 = 0$. **13** 1, 9; -3.

14 (a) $7\frac{1}{4}$, (b) $x^2 + 5x - 2 = 0$. **15** $a = 3$, $b = -63$.

16 $(x - 1)(x + 2)(3x - 2)$. **17** 3. **18** $(x + 1)(x - 5)(3x + 1)$.

19 $a = 1$, $b = -1$. **20** $p = 12$, $q = 4$.

Chapter 10

Qu. 1 (a) 37, (i); (b) 0, (ii); (c) -8, (iii); (d) 17, (i).

Qu. 2 $f(t) \leqslant 30$.

Qu. 4 (a) $\pm 8i$, (b) $\pm\sqrt{7}i$, (c) $\pm\frac{3}{2}i$, (d) $-3 \pm 5i$.

Qu. 5 $3 \pm 5i$.

Qu. 6 (a) $2 \pm 3i$, (b) $\pm 5i/3$, (c) $(1 \pm 5i)/2$, (d) $(3 \pm 5i)/34$.

Qu. 7 $\frac{5}{2} + \frac{1}{2}i$.

Qu. 9 (a) $[a + c, 0]$, (b) $[ac, 0]$, (c) $[a - c, 0]$, (d) $[a/c, 0]$.

Qu. 11 $-y + ix$, $-x - iy$, $y - ix$.

Qu. 12 (a) 5, (b) 1, (c) 1, (d) 1, (e) 3, (f) $\sqrt{2}$.

Qu. 13 (a) 45°, (b) 0°, (c) $-90°$, (d) $-45°$, (e) 60°, (f) 120°, (g) $-20°$, (h) 70°.

Exercise 10a, page 198

1 (a) $1\frac{1}{2}$, 1, (b) 3, -7, (c) ± 2.5, (d) 0, $-\frac{5}{7}$.

2 (a) $\dfrac{3 \pm \sqrt{11}}{2}$, (b) $\dfrac{-6 \pm \sqrt{6}}{5}$, (c) $\dfrac{-7 \pm \sqrt{61}}{2}$, (d) $\dfrac{3 \pm \sqrt{89}}{4}$.

3 (a) $\dfrac{7 \pm \sqrt{61}}{6}$, (b) $\dfrac{-3 \pm \sqrt{149}}{10}$, (c) $\dfrac{-13 \pm \sqrt{153}}{2}$, (d) $\dfrac{7 \pm \sqrt{73}}{6}$.

4 (a) $\dfrac{15 \pm \sqrt{165}}{4}$, (b) 0, $\frac{48}{11}$, (c) no real solution, (d) 5, $\frac{3}{7}$.

5 (a) $2\left(x - \dfrac{3 + \sqrt{11}}{2}\right)\left(x - \dfrac{3 - \sqrt{11}}{2}\right)$,

(b) $5(x + 1.2 + \sqrt{0.24})(x + 1.2 - \sqrt{0.24})$,

(c) $\left(x + \dfrac{7 + \sqrt{61}}{2}\right)\left(x + \dfrac{7 - \sqrt{61}}{2}\right)$,

(d) $-2\left(x - \dfrac{3 + \sqrt{89}}{4}\right)\left(x - \dfrac{3 - \sqrt{89}}{4}\right)$.

8 $|k| < 12$.

10 (a) 3, (b) 5, (c) 10, (d) -17.

Exercise 10b, page 203

1 (a) $-\mathrm{i}$, (b) 1, (c) i, (d) -1, (e) -1, (f) $-\mathrm{i}$, (g) i.

2 (a) $4 + 3\mathrm{i}$, (b) 9, (c) $1 - 5\mathrm{i}$, (d) $2\mathrm{i}$.

3 (a) $-7 + 22\mathrm{i}$, (b) $8 + \mathrm{i}$, (c) 2, (d) 25, (e) $u^2 + v^2$, (f) $2x^2 - 2y^2 + 5\mathrm{i}xy$, (g) $-3q + 2\mathrm{i}p$, (h) $p^2 + 4q^2$.

4 (a) $-\mathrm{i}$, (b) $\dfrac{2 + 3\mathrm{i}}{13}$, (c) $\dfrac{4 + 7\mathrm{i}}{5}$, (d) $\dfrac{9 + 40\mathrm{i}}{41}$, (e) $\dfrac{x - \mathrm{i}y}{x^2 + y^2}$, (f) $\dfrac{x + \mathrm{i}y}{x^2 + y^2}$, (g) 4/13.

5 (a) $-5 + 12\mathrm{i}$, (b) $-9 - 40\mathrm{i}$, (c) $x^2 - y^2 + 2\mathrm{i}xy$.

6 (a) $-2 + 2\mathrm{i}$, (b) $-2 - 2\mathrm{i}$, (c) $-\frac{1}{4}(1 + \mathrm{i})$.

7 (a) $2 \pm 5\mathrm{i}$, (b) $\pm\frac{1}{2}\sqrt{7}\mathrm{i}$, (c) $\dfrac{-3 \pm \sqrt{31}\mathrm{i}}{4}$, (d) $\frac{1}{2}(-1 \pm 2\mathrm{i})$.

10 $\frac{1}{2}$, $-1 \pm 2\mathrm{i}$.

Exercise 10c, page 209

1 (a) $\sqrt{2}$, 45°, (b) $\sqrt{13}$, 146.3°, (c) $\sqrt{13}$, $-146.3°$, (d) 5, $-53.1°$, (e) 5, 143.1°, (f) 1, 60°, (g) 1, 120°, (h) 1, 180°.

2 (a) 1, (b) i, (c) -1, (d) $-\mathrm{i}$, (e) 1, (f) $\frac{1}{2}\sqrt{3} + \frac{1}{2}\mathrm{i}$, (g) $\frac{1}{2}\sqrt{3} - \frac{1}{2}\mathrm{i}$, (h) $-\frac{1}{2} + \frac{1}{2}\sqrt{3}\mathrm{i}$, (i) $-\frac{1}{2} - \frac{1}{2}\sqrt{3}\mathrm{i}$, (j) $-\frac{1}{2}\sqrt{3} + \frac{1}{2}\mathrm{i}$.

3 (a) 5, 53.1°, (b) 13, 22.6°, (c) $\frac{1}{5}$, $-53.1°$, (d) $\frac{1}{13}$, $-22.6°$, (e) 65, 75.7°, (f) 5, $-53.1°$, (g) 13, $-22.6°$, (h) 65, $-75.7°$, (i) 25, 106.3°, (j) 169, 45°.

4 $2\mathrm{i}$, $-2 + 2\mathrm{i}$, -4; 45°, 90°, 135°, 180°.

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5 (a) $\frac{1}{2}+\frac{1}{2}\sqrt{3}i$, i, $-\frac{1}{2}+\frac{1}{2}\sqrt{3}i$; 30°, 60°, 90°, 120°, (b) $2+2\sqrt{3}i$, 8i, $-8+8\sqrt{3}i$; 30°, 60°, 90°, 120°.

6 $(a^2-b^2)+2abi, \dfrac{a-ib}{a^2+b^2}$.

8 $(ac-bd)+i(ad+bc)$.

Exercise 10d, page 210

1 (a) $\varnothing$, (b) $\frac{2}{3}$, (c) $\frac{2}{3}$, (d) $\frac{2}{3}$, $+i$, $-i$.

2 (i) (a) -4, (b) -4, 7/5, (c) -4, 7/5, (d) -4, 7/5.
(ii) (a) $\varnothing$, (b) $\varnothing$, (c) $\pm\sqrt{(7/5)}$, (d) $\pm\sqrt{(7/5)}$, $\pm 2i$.

3 (a) 10i, 10, 90°, (b) $(3-4i)/5$, 1, $-53.1°$, (c) 16, 16, 0.

4 (a) $(3+2i)/13$, $\frac{1}{2}i$, (b) $\sqrt{5}$, 63.4°, $\sqrt{5}$, $-26.6°$, $3+i$, $4+3i$.

6 Either $a=1/\sqrt{2}$, $b=1/\sqrt{2}$, or $a=-1/\sqrt{2}$, $b=-1/\sqrt{2}$;
$z=-1+1/\sqrt{2}+i/\sqrt{2}$, or $-1-1/\sqrt{2}-i/\sqrt{2}$.

7 (a) 3, -1, $1+i$, (b) $(-1\pm\sqrt{3}i)/2$.

8 $\pm(3+2i)$.

9 $\sqrt{2}/2$, 45°; $\frac{1}{2}$, 90°; $\sqrt{2}/4$, 135°; $\frac{1}{4}$, 180°.

10 $1+j$, $2-j$; $x^2-3(1+j)x+(2+3j)=0$.

Chapter 11

Qu. 1 $\begin{pmatrix} 17 & 23 & 29 \\ 14 & 20 & 34 \end{pmatrix}$.

Qu. 2. (a) $\begin{pmatrix} 5 & 1 \\ 2 & 0 \\ 6 & 1 \end{pmatrix}$, (b) $\begin{pmatrix} 29 \\ 20 \end{pmatrix}$, (c) $(20 \quad 7)$, (d) $\begin{pmatrix} 6 & 5 \\ 8 & 5 \end{pmatrix}$.

Qu. 5 (a) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, (b) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, (c) $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$.

Qu. 6 $\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$.

Exercise 11a, page 219

1 (a) $\begin{pmatrix} 9 & 3 & 6 \\ 15 & 3 & 21 \end{pmatrix}$, (b) $\begin{pmatrix} 8 & -2 & 4 \\ 6 & 2 & 6 \end{pmatrix}$, (c) $\begin{pmatrix} 17 & 1 & 10 \\ 21 & 5 & 27 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & 5 & 2 \\ 9 & 1 & 15 \end{pmatrix}$.

2 **PS** = (3240 7500 10 500 6600 9540 9900).

3 (a) $(19 \quad 31)$, (b) $\begin{pmatrix} 14 & 19 & 24 \\ 17 & 22 & 27 \end{pmatrix}$, (c) not possible, (d) $\begin{pmatrix} 38 \\ 68 \end{pmatrix}$.

4 $\begin{pmatrix} \frac{1}{2} & -2 \\ 7 & 10 \end{pmatrix}$, $\begin{pmatrix} 1\frac{1}{2} & -\frac{1}{2} \\ 11 & 9 \end{pmatrix}$. **5** $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. **6** $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. **7** $\begin{pmatrix} 6 \\ -8\frac{1}{2} \end{pmatrix}$.

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8 $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$. **9** $\begin{pmatrix} 3 \\ -7 \end{pmatrix}$. **10** (a) $\begin{pmatrix} 11 & -1 & 29 \\ 29 & -1 & 62 \\ 1 & 1 & -7 \end{pmatrix}$, (b) $\begin{pmatrix} 19 & -2 \\ 9 & -4 \\ 24 & 0 \end{pmatrix}$.

14 $\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$.

Exercise 11b, page 224

1 (a) 1, (b) 14, (c) 30, (d) 1.
2 (a) 0, (b) $\frac{2}{15}$, (c) a^2+b^2, (d) $ad-bc$.
3 (b), (c), (d).
4 (a) 28, (b) ± 4, (c) 1, 4, (d) none.
5 (a) $\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}$, (b) $\begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$, (c) $\begin{pmatrix} 7/20 & -11/20 \\ -2/20 & 6/20 \end{pmatrix}$,
(d) $\frac{1}{x^2+1}\begin{pmatrix} x & 1 \\ -1 & x \end{pmatrix}$.

Exercise 11c, page 225

1 (a) $\begin{pmatrix} 3 & -4 \\ -5 & 7 \end{pmatrix}$, (b) $\frac{1}{2}\begin{pmatrix} 3 & -2 \\ -11 & 8 \end{pmatrix}$, (c) $\frac{1}{2}\begin{pmatrix} 6 & -2 \\ -3 & 3 \end{pmatrix}$, (d) not possible.
2 (a) $\frac{1}{2}\begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}$, (b) $\frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$, (c) $\frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$, (d) $\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$.
3 $\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$. **4** $\frac{1}{20}\begin{pmatrix} 4 & -2 \\ -8 & 9 \end{pmatrix}$, $\frac{1}{20}\begin{pmatrix} -2 \\ 19 \end{pmatrix}$. **5** $\begin{pmatrix} -5 \\ 4 \end{pmatrix}$.
6 $\begin{pmatrix} -2 & -8 \\ 3 & 11 \end{pmatrix}$. **7** $\frac{1}{25}\begin{pmatrix} 304 & 372 \\ 372 & 521 \end{pmatrix}$. **9** $\begin{pmatrix} 17 \\ -7 \\ 19 \end{pmatrix}$. **10** $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.
11 $\frac{1}{14}\begin{pmatrix} 7 & -3 & -1 \\ 0 & 6 & 2 \\ 0 & 8 & -2 \end{pmatrix}$. **12** $\frac{1}{14}\begin{pmatrix} 7 & -3 & -1 \\ 0 & 6 & 2 \\ 0 & 8 & -2 \end{pmatrix}$.

Exercise 11d, page 238

1 (a) Reflection in x-axis, (b) reflection in y-axis, (c) rotation through 90°, (d) reflection in $x+y=0$, (e) shear parallel to x-axis.
2 (6, 17), (22, 29), (9, 38); 6, 150.
3 $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; rotation, through 90° clockwise.
4 πa^2, πab.
5 (a) $\begin{pmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{pmatrix}$, (b) $\begin{pmatrix} 0 & 5 \\ 5 & 0 \end{pmatrix}$.
6 Rotation and enlargement, $a^2+b^2=1$.

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7 Enlargement, with scale-factor $\sqrt{2}$ and reflection in the line $y=(\tan 22\frac{1}{2}°)x$; $\lambda=\sqrt{2}$, $m=\tan 22\frac{1}{2}°$.

8 Reflection in $y=(\tan\alpha)x$, where $\cos 2\alpha=3/5$.

9 $\beta-\alpha$.

10 $\cos 2\theta=\cos^2\theta-\sin^2\theta$, $\sin 2\theta=2\sin\theta\cos\theta$.

Exercise 11e, page 239

1 (a) $\begin{pmatrix}-13\\-31\end{pmatrix}$, (b) (22 15), (c) not possible, (d) $\begin{pmatrix}0&3&1&2\\0&7&3&4\end{pmatrix}$.

2 (a) $\begin{pmatrix}5&-7\\-2&3\end{pmatrix}$, (b) $\frac{1}{2}\begin{pmatrix}4&-3\\-6&5\end{pmatrix}$, (c) $\frac{1}{2}\begin{pmatrix}-5&3\\4&-2\end{pmatrix}$, (d) not possible.

3 (a) $\frac{1}{a^2+b^2}\begin{pmatrix}a&-b\\b&a\end{pmatrix}$, (b) $\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}$, (c) $\begin{pmatrix}0&a\\1/a&0\end{pmatrix}$, (d) not possible.

4 (a) $\begin{pmatrix}-11\\13\end{pmatrix}$, (b) $\begin{pmatrix}1&3\\-1&-4\end{pmatrix}$.

5 $2x=7y$.

6 (19, 11), (39, 15), (31, 35); 216.

7 $1, -\frac{1}{2}$.

8 $(-19, -35), (21, -11)$.

9 $\begin{pmatrix}-2&0&0\\0&-2&0\\0&0&-2\end{pmatrix}$, $\frac{1}{2}\begin{pmatrix}-1&2&1\\3&-2&-1\\-1&0&1\end{pmatrix}$.

10 (a) $\begin{pmatrix}76.05\\23.95\end{pmatrix}$, (b) $\begin{pmatrix}45.8\\54.2\end{pmatrix}$, (c) $\begin{pmatrix}78.3\\21.7\end{pmatrix}$.

12 $\mathbf{M}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\mathbf{R}=\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}$; rotation about the origin, through an angle $-\theta$; $\cos 2\theta=\cos^2\theta-\sin^2\theta$, $\sin 2\theta=2\sin\theta\cos\theta$.

13 $\mathbf{A}^{-1}=\begin{pmatrix}\frac{1}{3}&0&0\\0&\frac{1}{2}&\frac{1}{2}\\0&\frac{3}{4}&\frac{1}{4}\end{pmatrix}$.

14 $\begin{pmatrix}a^2+bc&ab+bd\\ac+cd&bc+d^2\end{pmatrix}$.

15 $\mathbf{M}^2=\mathbf{I}$; -1. $2x=3y$, $2v=-3u$; reflection in $2x=3y$.

16 $k=2/\sqrt{5}$; rotation, 26.6°, clockwise.

Chapter 12

Qu. 1 (a) 56; (b) 210.

Qu. 2 $\frac{n!}{(n-r)!r!}$.

Exercise 12a, page 244

1 720. **2** 360. **3** 24, 120. **4** 243. **5** 72. **6** 24.
7 27 000. **8** 120. **9** 900. **10** 120. **11** 719. **12** 48.
13 5040. **14** 168. **15** 336, 144. **16** 3 628 800, 3 628 800. **17** 78.
18 80. **19** 10 368 000. **20** 40 320, 384.

Exercise 12b, page 247

1 (a) 6, (b) 24, (c) 120, (d) 90, (e) 210, (f) 1320, (g) 330, (h) $\frac{1}{28}$, (i) 4, (j) 20, (k) 120, (l) 2520.
2 (a) $\frac{6!}{3!}$, (b) $\frac{10!}{8!}$, (c) $\frac{12!}{8!}$, (d) $\frac{n!}{(n-3)!}$, (e) $\frac{(n+2)!}{(n-1)!}$, (f) $\frac{10!}{8!2!}$, (g) $\frac{7!}{4!3!}$, (h) $\frac{52!}{49!3!}$, (i) $\frac{n!}{(n-2)!2!}$, (j) $\frac{(n+1)!}{(n-2)!3!}$, (k) $\frac{(2n)!}{(2n-2)!2!}$, (l) $\frac{n!}{(n-r)!}$.
3 (a) $20! \times 22$, (b) $25! \times 25$, (c) $13! \times 12$, (d) $14! \times 19$, (e) $n!(n+2)$, (f) $(n-2)!(n-2)$, (g) $(n-1)!(n+2)$, (h) $n!(n+2)^2$.
4 (a) $\frac{16!}{12!4!}$, (b) $\frac{22!}{14!8!}$, (c) $\frac{18!}{7!11!}$, (d) $\frac{37!}{19!18!}$, (e) $\frac{(n+1)!}{r!(n-r+1)!}$, (f) $\frac{(n+2)!}{r!(n-r+2)!}$.

Exercise 12c, page 250

1 282 240. **2** 362 880, 40 320. **3** 6720, 1680.
4 $24 \times 17!$, $48 \times 16!$. **5** $\frac{1}{60} \times 13!$. **6** 181 440. **7** $20 \times 10!$.
8 768. **9** 16. **10** 144. **11** 30 240. **12** 60 480. **13** 528.
14 1 404 000. **15** 2400. **16** 11 520, 276 480. **17** 23 520.
18 100. **19** 138 600. **20** 34 560, 31 680.

Exercise 12d, page 253

1 (a) 45, (b) 15, (c) 35, (d) 126, (e) 70, (f) $\frac{1}{2}n(n-1)$, (g) $\frac{1}{6}n(n-1)(n-2)$, (h) $\frac{1}{2}n(n-1)$, (i) $\frac{1}{2}n(n+1)$, (j) $\frac{1}{2}n(n+1)$.
2 78. **3** 70. **4** 252. **5** 126. **6** 30. **7** 252. **8** 286.
9 792. **10** 200. **11** 495. **12** 840. **13** 182. **14** 420.
15 11 550. **16** 34 650. **17** 25 200. **18** 2142. **19** 31 733.

Exercise 12e, page 254

1 2160. **2** 1960. **4** 15 120. **5** 5040, 240. **6** 360, 240.
7 728. **8** $\frac{1}{2}n(n-3)$. **9** 48. **10** 120 960. **11** 2520.
12 240, 15 552. **13** 277 200. **14** 4200. **15** 5120. **16** 504.
17 876. **18** 1013. **19** 1 693 440. **20** 300. **21** 319.
22 646. **23** 28 732. **24** 6006. **25** 240. **26** (a) 917, (b) 296.

Chapter 13

Qu. 1 (a) 9, 11; (b) 14, 17; (c) 16, 32; (d) $\frac{1}{48}$, $\frac{1}{96}$; (e) 5^3, 6^3; (f) $\frac{5}{6}$, $\frac{6}{7}$; (g) 25, 36; (h) 720, 5040; (i) $\frac{5}{81}$, $\frac{6}{243}$; (j) -4, -6; (k) 1, -1; (l) $\frac{1}{16}$, $-\frac{1}{32}$.

Qu. 2 (i) (a) 6, 8; (b) 8, 16. (ii) (a) 0, -6; (b) 3, $1\frac{1}{2}$.
Qu. 3 (a) 34, (b) 16.
Qu. 4 8, $12\frac{1}{2}$, 10.
Qu. 5 $2ac/(a+c)$.
Qu. 6 (a) $n(2n+1)$, (b) $\frac{1}{6}(n+1)(n+2)(2n+3)$, (c) $\frac{1}{4}(n-1)^2n^2$, (d) $n(2n-1)$, (e) $\frac{1}{3}n(2n+1)(4n+1)$, (f) $n^2(2n-1)^2$.

Exercise 13a, page 258

1 (a) $1\frac{1}{2}$, (b) -3, (c) 0.1, (e) $\frac{1}{3}$, (g) n, (i) $1\frac{1}{8}$, (j) -7, (l) -0.2.
2 (a) 75, 147; (b) -34, -82; (c) $7\frac{1}{8}$, $\frac{1}{8}(5n-3)$. (d) -148, $52-2n$; (e) $-13\frac{1}{2}$, $\frac{1}{2}(15-n)$; (f) 799, $3+4n$.
3 (a) 23, (b) 13, (c) 31, (d) 21, (e) 91, (f) 13, (g) $2n$, (h) n, (i) n, (j) $(l-a)/d+1$.
4 (a) 2601, (b) 632, (c) 420, (d) 288, (e) 250.5, (f) $60\frac{1}{2}$, (g) $121x$, (h) $\frac{1}{2}n(2a+n-1)$, (i) $\frac{1}{2}n\{2a+(n-1)d\}$.
5 (a) 444, (b) -80, (c) 20 100, (d) -520, (e) $n(2n+4)$, (f) $\frac{1}{8}n(11-n)$.
6 2, 13, 220. **7** 33, -72. **8** 5. **9** 14, 4. **10** 7500.
11 7650. **12** $3\frac{1}{2}$, $\frac{1}{10}$, $148\frac{1}{2}$. **14** 60.

Exercise 13b, page 260

1 (a) 3, (b) $\frac{1}{4}$, (c) -2, (d) -1, (f) a, (g) 1.1, (j) 6.
2 (a) 5×2^{10}, 5×2^{19}, (b) $10(\frac{5}{2})^6$, $10(\frac{5}{2})^{18}$, (c) $\frac{2}{3}(\frac{9}{8})^{11}$, $\frac{2}{3}(\frac{9}{8})^{n-1}$, (d) $3(-\frac{2}{3})^7$, $3(-\frac{2}{3})^{n-1}$, (e) $\frac{2}{7}(-\frac{3}{2})^8$, $\frac{2}{7}(-\frac{3}{2})^{n-1}$, (f) $3(\frac{1}{2})^{18}$, $3(\frac{1}{2})^{2n-1}$.
3 (a) 9, (b) 8, (c) 7, (d) 8, (e) $n+1$, (f) n.
4 (a) $2^{10}-2$, (b) $\frac{1}{2}(3^5-\frac{1}{27})$, (c) $0.03(2^7-1)$, (d) $-\frac{16}{405}\{(\frac{3}{2})^8-1\}$, (e) $5(2^{n+1}-1)$, (f) $a\left(\dfrac{1-r^n}{1-r}\right)$.
5 (a) $2(3^{12}-1)$, (b) $\frac{45}{2}\{1-(\frac{1}{3})^{20}\}$, (c) $-\frac{1}{3}(2^{50}-1)$, (d) $16\{1+(\frac{1}{2})^{17}\}$, (e) $11(1.1^{23}-1)$, (f) $1-(\frac{1}{2})^{13}$, (g) $3(2^n-1)$, (h) $\frac{3}{4}\{1-(-\frac{1}{3})^n\}$.
6 2, $2\frac{1}{2}$, $157\frac{1}{2}$. **7** ± 3, $\pm\frac{2}{3}$. **8** 6, $13\frac{1}{2}$. **9** £10 700 000.
10 $6\frac{3}{4}$. **12** $\frac{5}{2}$, $-\frac{1}{3}$. **13** $\sqrt{2}-1$, $5\sqrt{2}-7$. **14** 1023.

Exercise 13c, page 265

1 2550. **2** 8. **3** 98. **4** $\frac{3}{4}$, $-\frac{3}{2}$, 3. **5** 16 400. **6** 432.
7 $\frac{7}{2}$, 2. **8** 17, -2, 10th. **9** $1\frac{1}{2}$, 2, 24. **10** 3, 4; 3, 7, 11, 15, 19.
11 -2, 1, 4, 7, 10. **12** -3, -2. **13** 18. **14** 18th, 655 360.
15 14. **16** -9, 5. **17** 2, 4, 6, 8, 10. **18** 5808. **19** 6, 8, 10.
20 $2\frac{1}{2}$, 5, $7\frac{1}{2}$, 10. **21** £2270. **22** £19 100.

Exercise 13e, page 270

1 (a) $1^3+2^3+3^3+4^3$, (b) $2^2+3^2+\ldots+n^2$, (c) $2+6+\ldots+(n^2+n)$, (d) $\dfrac{1}{1\times 2}+\dfrac{1}{2\times 3}+\dfrac{1}{3\times 4}$, (e) $2^2+2^3+2^4+2^5$, (f) $-1+4-9+16$, (g) $1+2^2+\ldots+n^n$, (h) $-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}$, *Page 270*

(i) $n(n-1)+(n+1)n+(n+2)(n+1)$, (j) $\dfrac{n-2}{n-1}+\dfrac{n-1}{n}+\dfrac{n}{n+1}$.

2 (a) $\sum_{1}^{n} m$, (b) $\sum_{1}^{n+1} m^4$, (c) $\sum_{1}^{5} \dfrac{1}{m}$, (d) $\sum_{2}^{5} 3^m$, (e) $\sum_{2}^{6} m(m+5)$, (f) $\sum_{1}^{5} \dfrac{m}{3^{m-1}}$,
(g) $\sum_{1}^{5} \dfrac{m(2m+1)}{2(m+1)}$, (h) $\sum_{1}^{6} (-1)^m m$, (i) $\sum_{0}^{5} (-2)^m$, (j) $\sum (-1)^{m+1} m(2m+1)$.

3 (a) $(n+1)(2n+1)$, (b) $\frac{1}{6}n(n-1)(2n-1)$, (c) $n^2(2n+1)^2$, (d) $n(n+2)$,
(e) $\frac{1}{2}n(3n+1)$, (f) $n(2n+3)$, (g) $\frac{1}{6}n(2n^2+3n+7)$, (h) $\frac{1}{3}n(n+1)(n+2)$,
(i) $\frac{1}{6}n(n+1)(2n+7)$, (j) $\frac{2}{3}n(n+1)(2n+1)$, (k) $\frac{1}{3}n(2n-1)(2n+1)$,
(l) $\frac{1}{4}n(n+1)(n^2+n+2)$, (m) $\frac{1}{12}n(n+1)(n+2)(3n+1)$.

Exercise 13f, page 272

1 (a) $1\frac{1}{2}$, (b) 24, (c) $\frac{1}{3}$, (d) $\frac{13}{99}$, (e) $\frac{5}{9}$, (f) $\frac{6}{11}$, (g) $\frac{2}{3}$, (h) $40\frac{1}{2}$.
2 (a) $\frac{8}{9}$, (b) $\frac{4}{33}$, (c) $3\frac{2}{9}$, (d) $2\frac{23}{33}$, (e) $1\frac{1}{225}$ (f) $2\frac{317}{330}$.
3 $\frac{2}{3}$.
4 $2, \frac{1}{2}, \frac{1}{4}$.
5 $\frac{2}{5}$, 60; $\frac{3}{5}$, 40.

Exercise 13g, page 273

1 1683. **2** 20. **5** 17. **6** 2. **7** $6n+7$. **8** $\frac{1}{3}n(n-1)(n+1)$.
9 $27+29+\ldots+113$. **10** 4234. **11** $\frac{5}{2}(3^n-1)$, 16. **12** $3, 2, \frac{4}{3}, \frac{8}{9}$.
14 $\frac{9}{4}, 3, \frac{15}{4}$. **15** 35. **16** $4, -12, 15\frac{7}{8}, 57\frac{7}{8}$. **18** $\frac{1}{6}n(2n^2+3n+13)$.
19 $(ar+b)/(r+1)$, $(br+a)/(r+1)$. **20** $3, 12, 48, 3\times 4^{n-1}$. **21** 13, 9.
23 $1, \frac{1}{2}, 2$. **26** $\dfrac{1}{2}n(n+1), \dfrac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$.

Chapter 14

Exercise 14a, page 278

1 (a) $a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$, (b) $x^3+3x^2y+3xy^2+y^3$,
(c) $x^4+8x^3y+24x^2y^2+32xy^3+16y^4$, (d) $1-4z+6z^2-4z^3+z^4$,
(e) $16x^4+96x^3y+216x^2y^2+216xy^3+81y^4$, (f) $64z^3+48z^2+12z+1$,
(g) $a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$,
(h) $a^3-6a^2b+12ab^2-8b^3$, (i) $81x^4-108x^3y+54x^2y^2-12xy^3+y^4$,
(j) $8x^3+4x^2+\frac{2}{3}x+\frac{1}{27}$, (k) $x^5-5x^3+10x-10x^{-1}+5x^{-3}-x^{-5}$,
(l) $\frac{1}{16}x^4+x^2+6+16x^{-2}+16x^{-4}$,
(m) $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$,
(n) $a^{10}-5a^8b^2+10a^6b^4-10a^4b^6+5a^2b^8-b^{10}$,
(o) $a^6-3a^4b^2+3a^2b^4-b^6$.
2 (a) 14, (b) 194, (c) $10\sqrt{2}$, (d) $160\sqrt{6}$, (e) 98, (f) $40\sqrt{2}$.
3 $32+80x+80x^2+40x^3+10x^4+x^5$, 32.080 08.
4 $1+x+\frac{3}{8}x^2+\frac{1}{16}x^3+\frac{1}{256}x^4$, 1.104.
5 $64-192x+240x^2-160x^3+60x^4-12x^5+x^6$, 63.616 96, 5.

Exercise 14b, page 281

1 (a) $448x^5$, (b) $1080u^3$, (c) $-3168t^7$, (d) $1320x^3y^8$.
2 (a) $84x^3$, (b) $-14\,080x^3$, (c) $945x^4$, (d) $190x^2$.
3 (a) $\frac{105}{512}$, (b) 540, (c) 6048, (d) 1386.
4 (a) 120, (b) -9120, (c) 4320, (d) 5670.
5 (a) $15x^2$, (b) 20.
6 (a) 70, (b) $3\frac{3}{4}$.
7 (a) 6, (b) 14, (c) -16.
8 $3/(5x)$.
9 $8/(45x)$.
10 $b(r+1)/\{a(n-r)\}$.
11 (a) $1+10x+45x^2+120x^3$, (b) $1+\frac{9}{2}x+9x^2+\frac{21}{2}x^3$,
(c) $1-11x+55x^2-165x^3$, (d) $1+12x+66x^2+220x^3$,
(e) $256+512x+448x^2+224x^3$, (f) $128-224x+168x^2-70x^3$.
12 (a) 1.105, (b) 1029.13, (c) 0.965, (d) 253.96.
13 (a) $1+3x+6x^2+7x^3$, (b) $1+12x+54x^2+100x^3$, (c) $1-4x+2x^2+8x^3$,
(d) $32+80x+160x^2+200x^3$, (e) $1-8x+36x^2-112x^3$,
(f) $128+448x-224x^2-2128x^3$, (g) $81-216x+324x^2-312x^3$,
(h) $81+108x+54x^2+120x^3$.

Exercise 14c, page 285

1 (a) 10, (b) 5, (c) $-1/8$, (d) $-15/128$.
2 (a) $1-2x+3x^2-4x^3$, $-1<x<1$; (b) $1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3$, $-1<x<1$;
(c) $1+\frac{3}{2}x+\frac{3}{8}x^2-\frac{1}{16}x^3$, $-1<x<1$; (d) $1-x-\frac{1}{2}x^2-\frac{1}{2}x^3$, $-\frac{1}{2}<x<\frac{1}{2}$;
(e) $1-\frac{3}{2}x+\frac{3}{2}x^2-\frac{5}{4}x^3$, $-2<x<2$; (f) $1+\frac{3}{2}x+\frac{27}{8}x^2+\frac{135}{16}x^3$, $-\frac{1}{3}<x<\frac{1}{3}$;
(g) $1-3x+9x^2-27x^3$, $-\frac{1}{3}<x<\frac{1}{3}$; (h) $1-\frac{1}{2}x^2$, $-1<x<1$;
(i) $1-\frac{1}{3}x-\frac{1}{9}x^2-\frac{5}{81}x^3$, $-1<x<1$; (j) $1-x+\frac{3}{2}x^2-\frac{5}{2}x^3$, $-\frac{1}{2}<x<\frac{1}{2}$;
(k) $1-x+\frac{3}{4}x^2-\frac{1}{2}x^3$, $-2<x<2$; (l) $1-3x+\frac{3}{2}x^2+\frac{1}{2}x^3$, $-\frac{1}{2}<x<\frac{1}{2}$;
(m) $\frac{1}{2}-\frac{1}{4}x+\frac{1}{8}x^2-\frac{1}{16}x^3$, $-2<x<2$;
(n) $\sqrt{2}(1-\frac{1}{4}x-\frac{1}{32}x^2-\frac{1}{128}x^3)$, $-2<x<2$;
(o) $\sqrt[3]{3}(1+\frac{1}{9}x-\frac{1}{81}x^2+\frac{5}{2187}x^3)$, $-3<x<3$;
(p) $\frac{1}{2}\sqrt{2}(1-\frac{1}{4}x^2)$, $-\sqrt{2}<x<\sqrt{2}$; (q) $\frac{1}{9}+\frac{2}{27}x+\frac{1}{27}x^2+\frac{4}{243}x^3$, $-3<x<3$;
(r) $\sqrt[3]{9}(1+\frac{1}{9}x^3)$, $-\sqrt[3]{3}<x<\sqrt[3]{3}$.
3 (a) 1.000 500, (b) 0.9612, (c) 0.998 999, (d) 1.0099, (e) 1.0102.
4 (a) $1+2x+2x^2+2x^3$, (b) $2-3x+4x^2-5x^3$, (c) $1-\frac{3}{2}x+\frac{7}{8}x^2-\frac{11}{16}x^3$,
(d) $1+x+\frac{1}{2}x^2+\frac{1}{2}x^3$, (e) $-\frac{3}{2}+\frac{7}{4}x-\frac{7}{8}x^2+\frac{7}{16}x^3$, (f) $1-2x+\frac{3}{2}x^2-x^3$,
(g) $3+4x+7x^2+16x^3$.
5 $1-4x-8x^2-32x^3$, 4.7958.
6 $1-\frac{1}{3}x-\frac{1}{9}x^2-\frac{5}{81}x^3$, 3.332 22.
7 $1-4x-24x^2-224x^3$, 2.499 00, six.

Exercise 14d, page 286

1 $252(3x)^5(2y)^5$, 252.
2 (a) $32x^5+40x^3+20x+\dfrac{5}{x}+\dfrac{5}{8x^3}+\dfrac{1}{32x^5}$, (b) $40\sqrt{6}$.

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3 $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$, 77 400.

4 (a) $a^{11} + 11a^{10}b + 55a^9b^2 + 165a^8b^3$, (b) $8064x^5y^5$, (c) 5376.

5 $x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$, $x^4 - 8x^3 + 24x^2 - 32x + 16$, 96.

6 $4 - 28x + 85x^2 - 146x^3 + 155x^4$.

7 (a) $16 + 96x + 216x^2 + 216x^3 + 81x^4$, (b) $1 + 12x + 78x^2 + 340x^3$.

8 (a) $1 - 5x + 20x^2 - 50x^3$, (b) $1 - 4x + 10x^2 - 20x^3$.

9 (a) $1 - 3x + 6x^2 - 10x^3 + 15x^4$, (b) $1024 + 1280x + 720x^2 + 240x^3$, 1159.

10 (a) $70\,(2x)^4 3^4$, $\frac{35}{8}$, (b) $1 + 4x + 12x^2 + 32x^3$.

11 (a) 2520, (b) $1 + \frac{1}{3}x - \frac{1}{9}x^2$, 2.080.

12 $1 + 4x - 8x^2 + 32x^3$, 1.732 05.

13 $a = 2$, $b = \frac{3}{16}$.

14 $3 - 5x + 7x^2 - 9x^3$.

Chapter 15

Qu. 1 (a) $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, (b) $\begin{pmatrix} 3 \\ -3 \end{pmatrix}$, (c) $\begin{pmatrix} -7 \\ -1 \end{pmatrix}$, (d) $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$, (e) $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

Qu. 2 (a) $\sqrt{20}$, 63.4°, (b) $\sqrt{18}$, $-45°$, (c) $\sqrt{50}$, $-171.9°$, (d) 2, 90°, (e) 3, 0°.

Qu. 3 (2, 4).

Qu. 4 (2, 3, 4).

Qu. 5 $\overrightarrow{AB} = \overrightarrow{DC} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}$.

Qu. 6 $\dfrac{1}{13}\begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix}$.

Qu. 8 $2x + 3y + z = 5$.

Qu. 9 $x + 2y + z = 8$.

Qu. 10 -75.

Qu. 11 $x_1x_2 + y_1y_2 + z_1z_2 = 0$.

Qu. 12 101°.

Qu. 14 76.7°, 72.1°, 22.6°.

Exercise 15a, page 298

1 (a) $\begin{pmatrix} 6 \\ 10 \end{pmatrix}$, (b) $\begin{pmatrix} 12 \\ -18 \end{pmatrix}$, (c) $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$, (d) $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$, (e) $\begin{pmatrix} 7 \\ -1 \end{pmatrix}$, (f) $\begin{pmatrix} 18 \\ -8 \end{pmatrix}$, (g) $\begin{pmatrix} -1 \\ 11 \end{pmatrix}$, (h) $\begin{pmatrix} 1 \\ 27 \end{pmatrix}$.

2 (a) 5, 53.1°, (b) 13, 112.6°, (c) 10, $-90°$, (d) $\sqrt{2}$, $-45°$.

3 $\begin{pmatrix} 8.66 \\ 5 \end{pmatrix}$.

4 $-4.33\mathbf{i} + 2.5\mathbf{j}$.

5 (a) $(12, 11\frac{1}{2})$, (b) (21, 16), (c) $(-21, -5)$.

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6 (a) $\frac{1}{2}(\mathbf{c}-\mathbf{a})$, (b) $\mathbf{c}-\mathbf{a}$, (c) $\frac{1}{2}(\mathbf{c}-\mathbf{a})$.

7 $\frac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c})$.

Exercise 15b, page 302

1 (a) $\begin{pmatrix}6\\2\end{pmatrix}$, (b) $\begin{pmatrix}4\frac{2}{3}\\3\end{pmatrix}$, (c) $\begin{pmatrix}34\\-19\end{pmatrix}$, (d) $\begin{pmatrix}5.2\\2.6\end{pmatrix}$, (e) $\begin{pmatrix}8.4\\0.2\end{pmatrix}$,
(f) $\dfrac{1}{m+n}\begin{pmatrix}2n+10m\\5n-m\end{pmatrix}$.

2 (a) $\begin{pmatrix}-4\\-6\end{pmatrix}$, (b) $\begin{pmatrix}-5\\-3\end{pmatrix}$, (c) $\begin{pmatrix}17\\-69\end{pmatrix}$, (d) $\begin{pmatrix}-4.6\\-4.2\end{pmatrix}$, (e) $\begin{pmatrix}-2.2\\-11.4\end{pmatrix}$,
(f) $\dfrac{1}{m+n}\begin{pmatrix}-7n-m\\3n-15m\end{pmatrix}$.

3 $-2, 3; 1.5, -0.5$.

4 $3, -2; \frac{2}{3}, \frac{1}{3}$.

5 $\begin{pmatrix}5\\-5\sqrt{3}\end{pmatrix}$, $\begin{pmatrix}20+5t\\15-5\sqrt{3}t\end{pmatrix}$, $\sqrt{3}$, $20+5\sqrt{3}$.

6 $2, -1$.

7 2:3.

8 $-\frac{5}{12}\mathbf{b}+\frac{2}{3}\mathbf{c}$; $(\frac{3}{4}-\frac{5}{12}t)\mathbf{b}+\frac{2}{3}t\mathbf{c}$; $\overrightarrow{OM}=\frac{6}{5}\overrightarrow{OC}$; $-\frac{1}{6}$.

9 $\frac{2}{3}\mathbf{b}$, $\frac{3}{5}\mathbf{a}+\frac{2}{5}\mathbf{b}$, $\frac{1}{3}$.

Exercise 15c, page 315

1 (a) $\begin{pmatrix}2\\6\\2\end{pmatrix}$, (b) $\begin{pmatrix}-2\\0\\0\end{pmatrix}$, (c) $\begin{pmatrix}4\\3\\0\end{pmatrix}$, (d) $\begin{pmatrix}-3\\4\\-6\end{pmatrix}$, (e) $\begin{pmatrix}2k\\0\\-2k\end{pmatrix}$.

2 (a) $\begin{pmatrix}2\\3\\3\end{pmatrix}$, (b) $\begin{pmatrix}4\\0\\4\end{pmatrix}$, (c) $\begin{pmatrix}4\\2.5\\3\end{pmatrix}$, (d) $\begin{pmatrix}3.5\\6\\4\end{pmatrix}$, (e) $\begin{pmatrix}2k\\2k\\2k\end{pmatrix}$.

3 (a) $\begin{pmatrix}11\\30\\12\end{pmatrix}$, (b) $\begin{pmatrix}-5\\0\\4\end{pmatrix}$, (c) $\begin{pmatrix}22\\16\\3\end{pmatrix}$, (d) $\begin{pmatrix}-10\\24\\-23\end{pmatrix}$, (e) $\begin{pmatrix}11k\\2k\\-7k\end{pmatrix}$.

4 $x+y+z=3$.

5 $3x-3y+z=1$.

6 (7, 4, 9).

8 (4, 5, 10).

10 3:1.

11 $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\2\\4\end{pmatrix}+t\begin{pmatrix}2\\-3\\1\end{pmatrix}$.

Exercise 15d, page 321

1 (a) $\mathbf{a}+\mathbf{c}$, (b) $\mathbf{c}-\mathbf{a}$, (c) $\frac{2}{3}\mathbf{a}+\frac{1}{2}\mathbf{c}$, (d) $\frac{1}{2}\mathbf{c}-\frac{2}{3}\mathbf{a}$, (e) $\frac{1}{2}\mathbf{c}$, (f) $\frac{1}{2}\mathbf{c}-\frac{1}{3}\mathbf{a}$, (g) $\frac{1}{2}\mathbf{c}-\frac{2}{3}\mathbf{a}$, (h) $\frac{1}{3}\mathbf{a}+\mathbf{c}$, (i) $\mathbf{c}-\frac{2}{3}\mathbf{a}$, (j) $-\frac{1}{3}\mathbf{a}-\frac{1}{2}\mathbf{c}$.
2 30, -21.
3 (a) $\frac{1}{2}(\mathbf{a}+\mathbf{b})$, (b) $2\mathbf{b}-\mathbf{a}$, (c) $\frac{7}{10}\mathbf{a}+\frac{3}{10}\mathbf{b}$, (d) $\frac{5}{8}\mathbf{a}+\frac{3}{8}\mathbf{b}$, (e) $3\mathbf{a}-2\mathbf{b}$.
4 (a) 0, (b) 12, 15, (c) 90°.
5 44, 64.4°.
7 $\frac{3}{5}$, $\frac{6}{5}$, 3:2.
8 (5, 7, 18).
9 $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$, $x-2y-3z=0$.
10 $2x+2y+z=5$.
12 $(-3, -4, 0)$.
13 3, 500 s, $1000\sqrt{10}$ m.
14 $\mathbf{t}=\frac{1}{3}\mathbf{a}+\frac{2}{3}\mathbf{b}$, $\mathbf{m}=\frac{1}{2}\mathbf{a}+\mathbf{b}$, OB:BK = 1:1.
15 (a) $(4, -1, -3)$, $(\frac{22}{9}, \frac{8}{9}, -\frac{17}{9})$, (c) $\frac{4}{9}$.
17 $\frac{3}{2}\mathbf{b}-\frac{1}{2}\mathbf{c}$.
18 1:4.
19 (a) $\frac{1}{\sqrt{61}}(3\mathbf{i}-4\mathbf{j}+6\mathbf{k})$, (b) $8\mathbf{i}+(4+\lambda)\mathbf{j}+8\mathbf{k}$, (c) -9, (d) $4\mathbf{i}+\frac{1}{3}\mathbf{j}$.

Chapter 16

Qu. 1 (a) sin 10°, (b) − tan 60°, (c) − cos 20°, (d) − sin 50°, (e) cos 20°, (f) − sin 35°, (g) tan 40°, (h) − cos 16°, (i) − cosec 50°, (j) − tan 37°, (k) − cos 50°, (l) − sin 70°, (m) − tan 50°, (n) cot 20°, (o) cos 67°, (p) sin 50°, (q) − sec 38°, (r) − cot 24°, (s) − cosec 53°, (t) − sec 8°.
Qu. 3 360°, 180°.
Qu. 4 (a) $\frac{1}{2}$, (b) $\sqrt{3}/2$, (c) $1/\sqrt{2}$, (d) $1/\sqrt{3}$, (e) 2, (f) $2/\sqrt{3}$, (g) 1, (h) $\sqrt{2}$.
Qu. 5 (a) cot θ, (b) cosec θ, (c) − cosec θ, (d) − tan θ, (e) sec θ, (f) − cosec θ, (g) − sin θ, (h) sin θ, (i) − tan θ, (j) − cos θ, (k) − cos θ, (l) cosec θ.

Exercise 16a, page 331

1 (a) 0, (b) 0, (c) -1, (d) -1, (e) $\frac{1}{2}$, (f) $-\sqrt{3}/2$, (g) $-\sqrt{3}$, (h) $\sqrt{3}/2$, (i) $-\sqrt{3}/2$, (j) $1/\sqrt{2}$, (k) $-1/\sqrt{2}$, (l) $-1/\sqrt{2}$, (m) $-\sqrt{3}$, (n) 1, (o) $1/\sqrt{3}$.
3 360°.
4 180°.
5 (a) 180°, (b) 720°, (c) 240°, (d) 360°, (e) 360°.
6 (a) 240°, (b) 225°, (c) none, (d) 230°, 310°, (e) 306.9°, (f) 300°, (g) 240°, 360°, (h) 270°, 330°.
7 (a) 30°, 150°, 210°, 330°; (b) 30°, 150°, 210°, 330°; (c) 15°, 75°, 195°, 255°; (d) $67\frac{1}{2}$°, $157\frac{1}{2}$°, $247\frac{1}{2}$°, $337\frac{1}{2}$°; (e) 10°, 110°, 130°, 230°, 250°, 350°; (f) 90°, 210°, 330°; (g) 45°, 135°, 225°, 315°; (h) 35.3°, 144.7°, 215.3°, 324.7°;

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(i) 15°, 45°, 75°, ... 345°; (j) 37.8°, 142.2°, 217.8°, 322.2°;
(k) 11.6°, 48.4°, 191.6°, 228.4°; (l) 23.9°, 83.9°, 143.9°, 203.9°, 263.9°, 323.9°.

8 (a) $-180°$, $-45°$, 0°, 135°, 180°; (b) $\pm 60°$, $\pm 90°$;
(c) 0°, $\pm 180°$, $-19.5°$, $-160.5°$; (d) $-150°$, $-30°$, 90°; (e) $\pm 120°$, $\pm 180°$;
(f) $\pm 60°$, $\pm 90°$, $\pm 120°$; (g) 0°, $\pm 180°$; (h) $\pm 45°$, $\pm 135°$;
(i) $\pm 90°$, 11.5°, 168.5°; (j) $\pm 40.9°$, $\pm 139.1°$; (k) $\pm 90°$, 41.8°, 138.2°;
(l) $-104°$, $-45°$, 76.0°, 135°; (m) 23.6°, 156.4°; (n) $\pm 109.5°$.

9 (Maxima first), (a) 1, 90°; -1, 270°. (b) 3, 0°; -3, 180°. (c) 2, 0°; -2, 360°.
(d) $\frac{1}{2}$, 135°; $-\frac{1}{2}$, 45°. (e) 3, 270°; -1, 90°. (f) 5, 0°; 1, 60°. (g) 1, 270°; $\frac{1}{3}$, 90°.
(h) 1, 0°; $\frac{1}{7}$, 180°. (i) -1, 120°; 1, 0°. (j) no max.; 0, 0°. (k) $\frac{1}{2}$, 90°; no min.
(l) none. (m) none.

10 (a), (c), (d), (e), (g).

12 (a) 180°, (b) 1080°, (c) 60°, (d) 360°, (e) 540°.

Exercise 16b, page 337

1 (a) $\cos\theta$, (b) $\tan\theta$, (c) $\cos\theta\cot\theta$.
2 (a) $\sin\theta$, (b) $\tan\theta$, (c) $\operatorname{cosec}\theta\cot\theta$.
3 (a) $\sec\theta$, (b) $\sec^2\theta\tan\theta$, (c) $\sin\theta$.
4 (a) $\cot\theta$, (b) $\cos\theta$, (c) $\operatorname{cosec}\theta\tan^2\theta$.
5 (a) $a^2\cos^2\theta$, (b) $\frac{1}{a}\sec\theta$, (c) $a\cos\theta\cot\theta$.
6 (a) $b^2\operatorname{cosec}^2\theta$, (b) $b^2\cot\theta\operatorname{cosec}\theta$, (c) $\frac{1}{b}\sin\theta\cos\theta$.
7 (a) $a^2\tan^2\theta$, (b) $\frac{1}{a}\cot\theta$, (c) $\sin\theta$.
8 0°, 60°, 300°, 360°. **9** 270°. **10** 45°, 63.4°, 225°, 243.4°.
11 26.6°, 135°, 206.6°, 315°. **12** 60°, 300°. **13** 30°, 41.8°, 138.2°, 150°.
14 (a) $\pm\frac{4}{5}$, (b) $\pm\frac{3}{4}$. **15** $\frac{15}{17}$, $-\frac{8}{15}$. **16** (a) $-\frac{25}{24}$, $-\frac{7}{25}$.

32 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. **33** $\dfrac{y^2}{b^2}-\dfrac{x^2}{a^2}=1$. **34** $\dfrac{b^2}{y^2}-\dfrac{x^2}{a^2}=1$.

35 $(x-1)^2+(y-1)^2=1$. **36** $x(y-b)=ac$. **37** $\dfrac{a^2}{x^2}+\dfrac{b^2}{y^2}=1$.

38 $y^2(x-1)^2+y^2=1$. **39** $x^2+y^2=2$. **40** $xy=1$.

47 $\dfrac{4}{(x+y)^2}-\dfrac{4}{(x-y)^2}=1$.

Exercise 16c, page 338

1 (a) $-\cos 25°$, (b) $-\tan 27°$, (c) $\sec 51°$, (d) $\sin 35°$, (e) $\cot 46°$,
(f) $-\operatorname{cosec} 36°$.

2 (a) -1, (b) $-\frac{1}{2}\sqrt{3}$, (c) $\sqrt{3}$, (d) $\frac{1}{2}\sqrt{2}$, (e) $-\frac{2}{3}\sqrt{3}$, (f) $-\sqrt{2}$, (g) -1, (h) $\frac{1}{2}$, (i) $\frac{1}{2}$.

3 (a) 30°, 150°; (b) 135°, 315°; (c) 36.9°, 323.1°; (d) $22\frac{1}{2}°$, $112\frac{1}{2}°$, $202\frac{1}{2}°$, $292\frac{1}{2}°$;
(e) 37.8°, 142.2°, 217.8°, 322.2°; (f) 60°, 300°; (g) 80.5°, 299.5°;
(h) 14.4°, 105.6°; (i) 96.0°.

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4 (a) 0°, $\pm 180°$; $-30°$, $-150°$. (b) $\pm 90°$; $-123.7°$, 56.3°. (c) 30°, 150°; 90°. (d) $\pm 131.8°$. (e) 30°, 150°. (f) $\pm 66.4°$, $\pm 120°$. (g) 45°, $-135°$; 63.4°, $-116.6°$. (h) $\pm 60°$; $-23.6°$, $-156.4°$.

5 (a) max. 5, 90°; min. 1, 270°. (b) max. 4, 180°; min. -2, 0°. (c) max. 4, 60°; min. -4, 180°. (d) max. 3, 180°; min. 0, 0°. (e) max. -1, 180°; min. $\frac{1}{5}$, 0°. (f) max. 1, 45°; min. $\frac{1}{5}$, 135°.

6 (a) $\tan\theta$, (b) $\cos\theta$, (c) $\sin\theta$, (d) $-\cot\theta$. (e) $-\operatorname{cosec}\theta$, (f) $-\sec\theta$, (g) $-\sin\theta$, (h) $-\tan\theta$, (i) $\sin\theta$.

7 (a) $\cot^2\theta$, (b) $\sin\theta$, (c) $-\operatorname{cosec}\theta$, (d) 1, (e) 1, (f) $\sec\theta\operatorname{cosec}\theta$.

8 (a) 90°; 210°, 330°. (b) 41.4°, 318.6°. (c) 0°, 360°; 131.6°, 228.2°. (d) 23.6°, 156.4°; 16.6°, 163.4°. (e) 60°, 300°. (f) 56.3°, 236.3°. (g) 53.1°, 135°, 315°, 233.1°.

9 (a) $\frac{3}{5}$, $\frac{3}{4}$; (b) $-\frac{13}{12}$, $\frac{5}{13}$; (c) $\frac{8}{17}$, $\frac{8}{15}$; (d) $-\frac{21}{29}$, $-\frac{29}{20}$.

18 $b^2x^2 - a^2y^2 = a^2b^2$.

19 $(x-1)^2 + (y-1)^2 = 1$.

20 $a^2b^2 - x^2y^2 = a^2y^2$.

21 $x^2y^2 - a^2b^2 = a^2y^2$.

22 $b^2x^2 - a^2y^2 = x^2y^2$.

23 $xy = 1$.

24 $(y+1)^2 = x^2(1+y^2)$.

25 $y^2(1+x) = 1-x$.

26 max. $\sqrt{2}$, 45°; min. $-\sqrt{2}$, $-135°$.

27 $-36.9°$, 90°.

28 18°.

29 $x = 60°$, $y = 75°$, 345°; $x = 120°$, $y = 15°$, 285°; $x = 240°$, $y = 165°$, 255°; $x = 300°$, $y = 105°$, 195°.

30 (a) neither, $0 \leqslant y \leqslant 2$, (b) even, $-1 \leqslant y \leqslant 5$, (c) neither, $5 \leqslant y \leqslant 15$, (d) even, $0 \leqslant y \leqslant 2$.

Chapter 17

Qu. 3 (a) 0°, 112.6°, 360°; (b) 53.1°, 323.1°; (c) 48.4°, 205.3°; (d) 119.6°, 346.7°.

Exercise 17a, page 345

1 (a) $\frac{1}{4}\sqrt{2}(\sqrt{3}+1)$, (b) $\frac{1}{4}\sqrt{2}(\sqrt{3}+1)$, (c) $\frac{1}{4}\sqrt{2}(\sqrt{3}+1)$, (d) $\frac{1}{4}\sqrt{2}(1-\sqrt{3})$, (e) $-\frac{1}{4}\sqrt{2}(\sqrt{3}+1)$, (f) $\frac{1}{4}\sqrt{2}(\sqrt{3}-1)$, (g) $\frac{1}{4}\sqrt{2}(\sqrt{3}-1)$, (h) $\frac{1}{4}\sqrt{2}(\sqrt{3}-1)$.

2 (a) $\frac{56}{65}$, (b) $\frac{33}{65}$, (c) $\frac{33}{56}$.

3 (a) $\frac{63}{65}$, (b) $-\frac{63}{16}$, (c) $-\frac{33}{56}$.

4 (a) $\frac{56}{65}$, (b) $\frac{56}{33}$, (c) $-\frac{63}{65}$.

5 $\frac{1}{3}$.

6 -2.

7 45°.

8 135°.

9 (a) $\cos(x+60°) = \sin(30°-x)$, (b) $\cos(45°-x) = \sin(45°+x)$. (c) $\tan(x+60°)$, (d) $\sin 26°$, (e) $\sec 39°$, (f) $\cos 15° = \sin 105° = \sin 75°$.

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10 (a) $\frac{1}{2}$, (b) $\frac{1}{2}$, (c) $\frac{1}{3}\sqrt{3}$, (d) 0, (e) $\frac{1}{2}$, (f) $\frac{1}{2}\sqrt{2}$, (g) $\frac{1}{3}\sqrt{3}$, (h) $\frac{1}{2}\sqrt{6}$.
11 2.
12 $\frac{12}{31}$.
13 (a) $\frac{1}{3}$, (b) 1, (c) $-\frac{7}{4}$, (d) $2-\sqrt{3}$.
16 (a) 9.9°, 189.9°; (b) $157\frac{1}{2}$°, $337\frac{1}{2}$°; (c) 49.1°, 229.1°; (d) 56.5°, 236.5°.

Exercise 17b, page 349

1 sin 34°. **2** tan 60°. **3** cos 84°. **4** sin θ. **5** cos 45°.
6 tan θ. **7** cos 30°. **8** sin $4A$. **9** cos θ. **10** cos 6θ.
11 $\frac{1}{2}\tan 4\theta$. **12** $\frac{1}{2}\sin 2x$. **13** 2 cot 40°. **14** 2 cosec 2θ.
15 cos θ.

Exercise 17c, page 349

1 (a) $\frac{1}{2}$, (b) 1, (c) $-\frac{1}{2}\sqrt{3}$, (d) $-\frac{1}{2}\sqrt{2}$, (e) $\frac{1}{2}\sqrt{2}$, (f) $2\sqrt{3}$, (g) 1, (h) $2\sqrt{2}$.
2 (a) $\pm\frac{24}{25}, \frac{7}{25}$; (b) $\pm\frac{120}{169}, \frac{119}{169}$; (c) $\pm\frac{1}{2}\sqrt{3}, -\frac{1}{2}$.
3 (a) $-\frac{24}{7}$, (b) $\frac{240}{161}$, (c) $\pm\frac{120}{119}$.
4 (a) $\pm\frac{3}{4}, \pm\frac{1}{4}\sqrt{7}$; (b) $\pm\frac{4}{5}, \pm\frac{3}{5}$; (c) $\pm\frac{12}{13}, \pm\frac{5}{13}$.
5 (a) $\frac{1}{3}, -3$; (b) $\frac{1}{2}, -2$; (c) $-\frac{2}{3}, \frac{3}{2}$.
6 $\sqrt{2}-1$.
7 90°, 120°, 240°, 270°.
8 0°, 180°, 360°; 60°, 300°.
9 30°, 150°; 270°.
10 56.4°, 123.6°; 270°.
11 30°, 150°; 90°, 270°.
12 0°, 180°, 360°; 85.2°, 274.8°.
13 0°, 180°, 360°; 120°, 240°; 36.9°, 323.1°.
14 0°, 180°, 360°; 30°, 150°, 210°, 330°.
15 45°, 225°; 121.0°, 301.0°.
16 18.4°, 161.6°, 198.4°, 341.6°.
17 (a) $y=2x^2-1$, (b) $2y=3(2-x^2)$, (c) $y(1-x^2)=2x$, (d) $x^2y=8-x^2$.

Exercise 17d, page 353

1 90°, 330°.
2 94.9°, 219.9°.
3 114.3°, 335.7°.
4 204.6°, 351.7°.
5 72.6°, 319.3°.
6 76.7°, 209.6°.
7 28.1°, 208.1°; 159.5°, 339.5°.
8 0°, 180°, 360°; 45°, 225°.
10 max. 2, 330°; min. −2, 150°.
11 max. $\sqrt{13}$, 33.7°; min. $-\sqrt{13}$, −146.3°.
12 5, 53.1°.

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13 max. $\sqrt{5}$, 63.4°; min. $-\sqrt{5}$, $-116.6°$.
14 $\sqrt{2}$, 45°; $-\sqrt{2}$, 225°.
15 5, 126.9°; -5, 306.9°.
16 2, 60°; -2, 240°.
17 17, 298.1°; -17, 118.1°.
18 $\sqrt{37}$, 170.5°; $-\sqrt{37}$, 350.5°.
19 1, 240°; -1, 60°.
20 5, 53.1°; -5, 233.1°.

Exercise 17e, page 356

1 $\cos(x+y)-\cos(x-y)$. **2** $\cos(x+y)+\cos(x-y)$.
3 $\cos 4\theta+\cos 2\theta$. **4** $\cos 2S-\cos 2T$. **5** $\cos 2x-\cos 8x$.
6 $\cos 2x+\cos 2y$. **7** $\cos A+\cos B$. **8** $\cos B-\cos C$.
9 $\cos 2x$. **10** $\cos 4x+\cos 60°$.

Exercise 17f, page 356

1 $\sin(x+y)+\sin(x-y)$. **2** $\sin(x+y)-\sin(x-y)$.
3 $\sin 4\theta+\sin 2\theta$. **4** $\sin 2S+\sin 2T$. **5** $\sin 8x-\sin 2x$.
6 $\sin 2x-\sin 2y$. **7** $\sin 2x-\sin 6x$. **8** $\sin A+\sin B$.
9 $\sin A-\sin B$. **10** $\sin R-\sin S$.

Exercise 17g, page 359

1 $2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$. **2** $2\sin 4x\cos x$.
3 $2\cos(y+z)\sin(y-z)$. **4** $2\cos 6x\cos x$. **5** $-2\sin\frac{3}{2}A\sin\frac{1}{2}A$.
6 $2\cos 3x\sin x$. **7** $2\sin 4A\sin A$. **8** $2\sin 6\theta\cos\theta$. **9** $\sqrt{3}\sin x$.
10 $\sqrt{2}\cos(y-35°)$. **11** $-2\cos 4\theta\sin\theta$. **12** $-\sin x$.
13 $-2\sin x\sin\frac{1}{2}x$. **14** $2\sin 2x\cos 80°$.
15 $2\cos(45°-\frac{1}{2}x+\frac{1}{2}y)\cos(45°-\frac{1}{2}x-\frac{1}{2}y)$.
16 $2\cos(45°-\frac{1}{2}A+\frac{1}{2}B)\cos(45°-\frac{1}{2}A-\frac{1}{2}B)$.
17 $2\sin(\frac{3}{2}x+45°)\cos(\frac{3}{2}x-45°)$.
18 $2\sin(x+45°)\cos(x-45°)$.
19 $2\cos(45°-\frac{1}{2}A+\frac{1}{2}B)\sin(45°-\frac{1}{2}A-\frac{1}{2}B)$.
20 $2\cos(30°+\theta)\cos(30°-\theta)$.

Exercise 17h, page 360

14 30°, 90°, 150°, 210°, 270°, 330°; 45°, 135°, 225°, 315°.
15 0°, 120°, 240°, 360°; 72°, 144°, 216°, 288°.
16 0°, 180°, 360°; 45°, 135°, 225°, 315°.
17 0°, 72°, 144°, 180°, 216°, 288°, 360°.
18 175°, 355°.
19 45°, 135°, 225°, 315°.
20 25°, 205

Exercise 17i, page 361

1 (a) $\frac{140}{221}$, (b) $-\frac{21}{221}$, (c) $\frac{171}{140}$.
2 (a) $\frac{468}{493}$, (b) $-\frac{475}{493}$, (c) $\frac{475}{132}$.
3 (a) $\frac{1}{2}$, (b) 1, (c) 1.
4 (a) $\pm\frac{2}{3}$, $\pm\frac{1}{3}\sqrt{5}$; (b) $\pm\frac{4}{9}$, $\pm\frac{1}{9}\sqrt{65}$.
5 (a) $\frac{5}{2}$, $-\frac{2}{5}$; (b) $\frac{2}{9}$, $-\frac{9}{2}$.
6 (a) $\frac{840}{1369}$, (b) $-\frac{1081}{1369}$.
7 60°, 300°.
8 0°, 180°, 360°; 41.4°, 318.6°.
9 0°, 180°, 360°; 60°, 120°, 240°, 300°.
10 41.6°, 244.7°.
11 79.8°, 347.6°.
12 $9x = 4y^2 - 18$.
13 $y(4 - x^2) = 4x$.
14 $2(t + 2)^2/(1 + t^2)$.
15 $(1 + t)/(1 - t)$.
16 13, 292.6°; −13, 112.6°.
17 37, 71.1°; −37, 251.1°.
18 73, 311.1; −73, 113.1°.
21 60°, 120°; 30°, 90°, 150°.
22 0°, 45°, 90°, 135°, 180°; 60°, 120°.
23 0°, 90°, 180°.
24 60°, 180°.
25 45°, 135°; 30°, 150°.

Chapter 18

Qu. 1 54.1.
Qu. 2 6.95.
Qu. 3 (a) 6.49(5), (b) 72.4, (c) 32.2, (d) 43.8, (e) 76.3, (f) 123, (g) 32 600.
Qu. 4 (a) 6 deg/s, (b) 1 rev/min.
Qu. 5 (a) 3000 deg/s, (b) $2\frac{1}{7}$ deg/h.
Qu. 6 $120° + 360n°$, or $240° + 360n°$.
Qu. 7 $\frac{1}{4}\pi + n\pi$.

Exercise 18a, page 370

1 (a) $A = 48°$, $b = 13.8$, $c = 15.4$. (b) $B = 56.1°$, $a = 6.53$, $c = 5.04$.
(c) $C = 45.1°$, $a = 231$, $b = 213$.
2 (a) $B = 95°$, $a = 1.40$, $c = 1.80$. (b) $B = 19.7°$, $b = 4.63$, $c = 8.29$.
(c) $A = 32.7°$, $b = 244$, $c = 172$.
3 (a) $B = 59.1°$, $A = 72.6°$, $a = 19.6$; or $B = 120.9°$, $A = 10.9°$, $a = 3.87$.
(b) $C = 26.7°$, $A = 24.3°$, $a = 4.18$.
(c) $B = 55.5°$, $C = 96.25°$, $c = 17.9$; or $B = 124.5°$, $C = 27.25°$, $c = 8.22$.
4 (a) $A = 38.2°$, $B = 81.8°$, $C = 60°$. (b) $A = 54.6°$, $B = 78.1°$, $C = 47.2°$.
(c) $A = 64.2°$, $B = 43.5°$, $C = 72.4°$.

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5 (a) $a = 13$, $B = 32.2°$, $C = 87.8°$. (b) $b = 11.7$, $A = 72.3°$, $C = 54.7°$.
(c) $c = 7.60$, $A = 82.6°$, $B = 54.2°$.
6 (a) $A = 29.5°$, $B = 38.0°$, $C = 112.4°$. (b) $A = 17.9°$, $B = 120°$, $C = 42.1°$.
(c) $A = 35.8°$, $B = 49.3°$, $C = 94.9°$.
7 (a) $A = 11.6°$, $B = 73°$, $C = 48.4°$. (b) $a = 17.4$, $B = 33.8°$, $C = 41.9°$.
(c) $A = 31.2°$, $B = 44.6°$, $c = 58.0$.
8 1.43 km.
9 25.8 m.
10 1.0°.
11 347.3°, 3.64 n.mi.
12 200 m.

Exercise 18b, page 373

1 (a) 90°, (b) 45°, (c) 60°, (d) 120°, (e) 30°, (f) 270°, (g) 450°, (h) 720°, (i) 900°, (j) 240°, (k) 630°, (l) 135°.
2 (a) 2π, (b) $\frac{1}{2}\pi$, (c) $\frac{1}{4}\pi$, (d) $\frac{1}{12}\pi$, (e) $\frac{1}{3}\pi$, (f) $\frac{2}{3}\pi$, (g) $\frac{5}{3}\pi$, (h) $\frac{3}{2}\pi$, (i) 3π, (j) $\frac{1}{6}\pi$, (k) $\frac{5}{6}\pi$, (l) $\frac{5}{2}\pi$.
3 8 cm. **4** 9.6 cm. **5** 6 cm. **6** $\frac{4}{5}$ rad. **7** 3 cm^2.
8 4 rad. **9** 12 cm. **10** 4 cm^2.

Exercise 18c, page 374

1 (a) $\frac{1}{8}\pi$, (b) 6π, (c) $\pi/900$, (d) $5\pi/24$. **2** (a) 72°. (b) 5°, (c) 105°, (d) 630°.
3 2.705 cm. **4** $3/\pi$. **5** 1.2 rad, 68.8°. **6** 6.43 cm. **7** 4.03 cm^2.
8 (a) 151 cm^2, (b) 62.4 cm^2, (c) 88.4 cm^2. **9** 24.1 cm^2. **10** 22.4 cm^2.
11 $\frac{1}{2}r^2(2\pi - \theta + \sin\theta)$.

Exercise 18d, page 376

1 (a) $\frac{1}{60}$ rev/min, (b) $\frac{1}{10}$ deg/s, (c) $\pi/1800$ rad/s.
2 (a) 1200 deg/s, (b) $20\pi/3$ rad/s.
3 (a) 1536 rev/min, (b) 161 rad/s.
4 0.262 rad/h.
5 (a) 100π rad/s, (b) 65π cm/s.
6 (a) 3.89 rad/s, (b) 15.6 cm/s.
7 (a) 40π rad/s, (b) 1.57 m/s.
8 (a) 35.2 rad/s, (b) 336 rev/min.
9 1600 rev/min.
10 128 rad/s.
11 1.99×10^{-7} rad/s, 30 km/s.
12 4.8 km/h.

Exercise 18e, page 378

1 $45° + n360°$, or $135° + n360°$.
2 $n360°$.

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3 $60° + n180°$.
4 $270° + n360°$.
5 $120° + n360°$, or $240° + n360°$.
6 $150° + n180°$.
7 $\pi/3 + 2n\pi$, or $5\pi/3 + 2n\pi$.
8 $3\pi/4 + n\pi$.
9 $\pi/12 + n\pi$, or $5\pi/12 + n\pi$.
10 $\pi/6 + 2n\pi$, or $11\pi/6 + 2n\pi$; $5\pi/6 + 2n\pi$, or $7\pi/6 + 2n\pi$.
11 $\pi/3$. **12** $\pi/4$. **13** $\pi/4$. **14** $-\pi/6$. **15** $5\pi/6$.
16 $-\pi/4$. **17** $-\pi/2$. **18** π. **19** 0. **20** $\pi/2$.

Exercise 18f, page 380

1 (a) $a = 13$, $B = 32.2°$, $C = 87.8°$; (b) $b = 11.7$, $A = 72.3°$, $C = 54.7°$; (c) $c = 7.59$, $A = 82.6°$, $B = 54.2°$.
2 (a) $b = 73$, $A = 11.6°$, $C = 48.4°$; (b) $a = 17.4$, $B = 33.8°$, $C = 41.9°$; (c) $c = 57.9$, $A = 31.3°$, $B = 44.7°$.
3 (a) $C = 99.4°$, $a = 9.54$, $b = 5.23$; (b) *either*, $B = 38.9°$, $C = 109.9°$, $c = 9.00$, *or*, $B = 141.1°$, $C = 7.7°$, $c = 1.28$; (c) $B = 146.8°$, $C = 13.2°$, $b = 24.0$.
4 (a) 11.5, (b) 9.92, (c) not possible.
5 (a) 72°, (b) 150°, (c) $67\frac{1}{2}°$, (d) 105°.
6 (a) $\frac{11\pi}{6}$, (b) $\frac{5\pi}{18}$, (c) $\frac{5\pi}{12}$, (d) $\frac{2\pi}{15}$.
7 10.5 cm.
8 π rad/s.
9 $\frac{1}{720}$ rev/min, $\frac{\pi}{21\,600}$ rad/s.
10 -75 rad/s.
11 4π.
12 No solution.
13 (a) No solution, (b) 2 solutions, (c) 1 solution.
14 16.
15 (a) 210, (b) 21.
16 $(\cos\alpha, \sin\alpha)$, $(\cos\beta, \sin\beta)$.

Chapter 19

Qu. 1 (a) $1\frac{1}{2}$, (b) 2, (c) $\frac{1}{2}$, (d) $\frac{1}{2}$, (e) $\sin\alpha$, (f) $\cos\alpha$, (g) $\frac{2}{9}$, (h) 2, (i) $\sec^2\alpha$.
Qu. 3 $2\cos\frac{1}{2}(A + B)\sin\frac{1}{2}(A - B)$.
Qu. 6 (a) $-3\sin 3x$, (b) $2\sin x\cos x = \sin 2x$, (c) $4\cos 2x$, (d) $-3\cos^2 x\sin x$.

Exercise 19a, page 387

1 (a) $-2\sin 2x$, (b) $6\cos 6x$, (c) $-3\sin(3x - 1)$, (d) $2\cos(2x - 3)$, (e) $15\sin 5x$, (f) $8\cos 4x$, (g) $-6\cos\frac{3}{2}x$, (h) $\cos\frac{1}{2}(x + 1)$, (i) $2x\cos x^2$.

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2 (a) $-\frac{1}{3}\cos 3x + c$, (b) $\frac{1}{3}\sin 3x + c$, (c) $-\frac{1}{2}\cos 4x + c$, (d) $\sin 2x + c$, (e) $\frac{1}{12}\cos 6x + c$, (f) $\frac{3}{2}\sin 4x + c$, (g) $-\frac{1}{2}\cos(2x+1) + c$, (h) $\frac{3}{2}\sin(2x-1) + c$, (i) $-\frac{4}{3}\cos\frac{1}{2}x + c$.

3 (a) $2\sin x\cos x = \sin 2x$, (b) $-8\cos x\sin x = -4\sin 2x$, (c) $-3\cos^2 x\sin x$, (d) $6\sin^2 x\cos x$, (e) $-12\cos^3 x\sin x$, (f) $\dfrac{\cos x}{2\sqrt{(\sin x)}}$, (g) $\dfrac{-\sin x}{2\sqrt{(\cos x)}}$, (h) $-6\cos 3x\sin 3x = -3\sin 6x$, (i) $4\sin 2x\cos 2x = 2\sin 4x$, (j) $-18\sin^2 3x\cos 3x$, (k) $24\sin^3 2x\cos 2x$, (l) $\dfrac{\cos 2x}{\sqrt{(\sin 2x)}}$.

4 (a) $\cos x - x\sin x$, (b) $\sin 2x + 2x\cos 2x$, (c) $x(2\sin x + x\cos x)$, (d) $\cos^2 x - \sin^2 x = \cos 2x$, (e) $(x\cos x - \sin x)/x^2$, (f) $-(2x\sin 2x + \cos 2x)/x^2$, (g) $(\sin x - x\cos x)/\sin^2 x$, (h) $x(2\cos x + x\sin x)/\cos^2 x$, (i) $\sec^2 x$, (j) $-\operatorname{cosec}^2 x$, (k) $\sec x\tan x$, (l) $-\operatorname{cosec} x\cot x$.

5 (a) 1 m, (b) 2 m/s^2, (c) 0.983 s.

6 (a) $\frac{1}{3}\pi$ s, (b) $-\frac{3}{2}\sqrt{3}$ cm/s, (c) $-5, 3\frac{3}{4}, -3$ cm/s^2.

7 (a) 5, (b) -20.

8 (a) 0.841, (b) $\frac{5}{3}\sqrt{5}$, (c) $-\frac{17}{3}\sqrt{5}$.

9 $\frac{2}{3}$.

10 2π.

11 $\frac{1}{3}\pi + \frac{1}{2}\sqrt{3}, \frac{1}{8}\pi^2 + 1$.

12 $\sqrt{3} - \frac{1}{3}\pi, 2 - \frac{1}{8}\pi^2$.

14 $\frac{1}{2}(1 + \cos 2x)$.

17 $\frac{1}{2}$.

Exercise 19b, page 390

1 (a) $2\sec^2 2x$, (b) $-3\operatorname{cosec}^2 3x$, (c) $6\sec 2x\tan 2x$, (d) $-\operatorname{cosec}\frac{1}{2}x\cot\frac{1}{2}x$, (e) $-2\sec^2(2x+1)$, (f) $\sec(3x-2)\tan(3x-2)$, (g) $6\operatorname{cosec}^2(3x+2)$, (h) $-2x\operatorname{cosec}^2 x^2$, (i) $(\sec^2\sqrt{x})/(2\sqrt{x})$.

2 (a) $2\tan x\sec^2 x$, (b) $2\sec^2 x\tan x$, (c) $-6\cot^2 x\operatorname{cosec}^2 x$, (d) $-6\operatorname{cosec}^2 x\cot x$, (e) $-4\sec^2 2x\tan 2x$, (f) $-3\operatorname{cosec}^2 3x\cot 3x$, (g) $\sec^3 2x\tan 2x$, (h) $8\operatorname{cosec}^4 x\cot x$, (i) $(\sec^2 x)/(2\sqrt{\tan x})$.

3 (a) $\tan x + x\sec^2 x$, (b) $\sec x(\sec^2 x + \tan^2 x)$, (c) $x(2\cot x - x\operatorname{cosec}^2 x)$, (d) $3\operatorname{cosec} x(1 - x\cot x)$, (e) $-\operatorname{cosec} x(\operatorname{cosec}^2 x + \cot^2 x)$, (f) $(x\sec^2 x - \tan x)/x^2$, (g) $\sec x(x\tan x - 2)/x^3$, (h) $x\sin x$, (i) $2x\sec^2 x\tan x$.

4 (a) $\frac{1}{2}\tan 2x + c$, (b) $3\sec x + c$, (c) $2\cot\frac{1}{2}x + c$, (d) $-\frac{1}{9}\operatorname{cosec} 3x + c$, (e) $\sec^2 x + c$, *or* $\tan^2 x + c$, (f) $\tan x + c$, (g) $\sec x + c$, (h) $-\frac{1}{2}\cot 2x + c$, (i) $-\frac{1}{2}\operatorname{cosec} 2x + c$.

5 $1 - \frac{1}{4}\pi$.

6 2π.

7 (a) $2\sqrt{3}$, (b) $5\sqrt{5}$, (c) $5\sqrt{3}$.

9 $\cot^2 x = \operatorname{cosec}^2 x - 1$, $-\cot x - x + c$.

Exercise 19c, page 390

1 (a) $72°$, (b) $150°$, (c) $67\frac{1}{2}°$, (d) $105°$.
2 (a) $11\pi/6$, (b) $5\pi/18$, (c) $5\pi/12$, (d) $2\pi/15$.
3 (a) 0.909, (b) 1.14, (c) 3.90, (d) -0.987.
4 10.5 cm.
5 $1\frac{3}{5}$ cm., $29.8°$.
7 171 cm^2.
8 1.93 rad.
9 0.515 rad.
10 $60°$, $72°$, $144°$, $\frac{1}{3}\pi$, $\frac{2}{5}\pi$, $\frac{4}{5}\pi$ rad.
11 π rad/s.
12 (a) $\frac{1}{720}$ rev/min, (b) $\pi/21\,600$ rad/s.
13 44 rad/s, 238 km/h.
14 6.9 rad/s.
15 42.4 rev/min.
16 (a) 1, (b) $\frac{2}{3}$, (c) $-\sin\alpha$.
18 (a) $3\cos 3x$, (b) $\frac{1}{2}\sec^2\frac{1}{2}x$, (c) $-2x\sin x^2$, (d) $-\sin x/(2\sqrt{\cos x})$, (e) $-6\operatorname{cosec}^3 x\cot x$, (f) $2\sin x$, (g) $-18\sec^3 2x\tan 2x$, (h) $(\cos 2x)/\sqrt{(\sin 2x)}$, (i) $12\tan 2x\sec^2 2x$.
19 (a) $\frac{1}{2}\sin 2x + c$, (b) $-\frac{1}{2}\cos(2x-1)+c$, (c) $6\sin\frac{1}{2}x + c$, (d) $2\tan\frac{1}{2}x + c$, (e) $-\operatorname{cosec} x + c$, (f) $\frac{1}{2}\sec 2x + c$, (g) $-\operatorname{cosec} x + c$, (h) $\frac{1}{2}\tan 2x + c$, (i) $-\frac{1}{2}\cos x^2 + c$.
20 (a) $\sin x + x\cos x$, (b) $\cos x\cos 2x - 2\sin x\sin 2x$, (c) $2x\tan x(\tan x + x\sec^2 x)$, (d) $\sec x(x\tan x - 1)/x^2$, (e) $-(2\sin 3x\sin 2x + 3\cos 3x\cos 2x)/\sin^2 3x$, (f) $\cos x\tan 2x + 2\sin x\sec^2 2x$, (g) $(x\cos x - 2\sin x)/x^3$, (h) $x^2\sin x$.
23 (a) 5, (b) -20, (c) 10.
24 (a) $\frac{15}{8}\sqrt{15}$ cm/s, (b) 14 cm.
25 (a) 1, (b) $\frac{4}{3}\sqrt{3}$, (c) $\frac{1}{2}\pi$, (d) $\frac{1}{4}$.

Chapter 20

Qu. 2 $4x - 6y - 13 = 0$.
Qu. 4 (a) $y = x \pm \sqrt{7}$, (b) $y = \sqrt{3}x \pm \sqrt{13}$.
Qu. 5 (a) (0, 0); (b) (0, 0), (3, 6).

Exercise 20a, page 396

1 $x^2 + y^2 = 25$. **2** $x^2 + y^2 - 6x - 2y + 6 = 0$. **3** $4x - 10y + 29 = 0$.
4 $5x - 3y - 4 = 0$. **5** $x + 1, y^2 = 2x + 1$. **6** $x^2 = 4y$.
7 $2x^2 + 2y^2 - x - 1 = 0$. **8** $3x^2 + 3y^2 + 36x - 38y + 159 = 0$.
9 $3x^2 - y^2 = 48$. **10** $3x^2 + 4y^2 = 48$. **11** $x^2 + y^2 = 9$.
12 $y^2 = 4ax$. **13** $3x^2 + 4y^2 = 12$. **14** $y = 0$. **15** $2x + 3y - 13 = 0$.

Exercise 20b, page 399

1 (a) $(y-2)(y+5)+(x+3)(x-4)=0$; (b) $(y-1)(y-4)+(x-\frac{1}{2})(x+\frac{3}{2})=0$; (c) $y(y-a)+x(x-a)=0$; (d) $(y-y_1)(y-y_2)+(x-x_1)(x-x_2)=0$.

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2 (a) $x^2 + y^2 = 36$, (b) $4x^2 + y^2 = 64$.
3 $x^2 + y^2 = 16$.
4 $xy = 3y + 4x$.
5 $xy = 3$.
6 $y = 6x^2 + 1$.
7 $y^2 = 8x + 4$.
9 $x^2 + y^2 = 4$.
10 $4x^2 + 4y^2 - 8x + 3 = 0$.
11 $2y = x^2 + x + 2$.
12 $2x + 3y - 13 = 0$.
13 $xy = 2x + 3y$.
14 $x^2 - 4xy + 5y^2 = 4$.
15 $y^2 - xy - y + 2x = 0$.

Exercise 20c, page 402

1 (a) $4x - y - 4 = 0$, $x + 4y - 18 = 0$; (b) $4x - y - 2 = 0$, $x + 4y - 9 = 0$; (c) $y + 2 = 0$, $x + 1 = 0$; (d) $x + y + 1 = 0$, $x - y - 3 = 0$; (e) $6x + y + 4 = 0$, $x - 6y + 50 = 0$; (f) $x + y - 4 = 0$, $x - y = 0$; (g) $2x - 3y + 1 = 0$, $3x + 2y - 5 = 0$.
2 (a) $(\frac{1}{2}, 2)$; (b) $(2, -2)$; (c) $(6, \frac{2}{3})$; (d) $(-\frac{5}{2}, -\frac{9}{4})$.
3 $(1, 0), (3, 0)$; $2x + y - 2 = 0$, $x - 2y - 1 = 0$; $2x - y - 6 = 0$, $x + 2y - 3 = 0$.
4 $5x - y - 11 = 0$, $3x + y + 3 = 0$.
5 $x + 2y - 1 = 0$, $x - 2y + 1 = 0$, $(0, \frac{1}{2})$.
6 $(0, 0), (1, 1)$; $x = 0$, $2y - x - 1 = 0$; $y = 0$, $y - 2x + 1 = 0$.
7 $4y - x + 48 = 0$, $(48, 0)$.
8 $9x - y - 27 = 0$, $9x - y + 5 = 0$.
9 $x + y \pm 4 = 0$.
10 $0, 2$; $y = 0$, $y - 4x + 4 = 0$.
11 $x + 4y - 4c = 0$.
13 $3x - 8y \pm 10 = 0$.
14 $x - y \pm 4 = 0$.
15 $n^2 = a^2l^2 + b^2m^2$.

Exercise 20d, page 403

1 $2x - 16y + 41 = 0$.
2 $y^2 = 8(x - 2)$.
3 $3x^2 + 4y^2 - 24x + 36 = 0$.
4 $8x^2 - y^2 - 18ax + 9a^2 = 0$.
5 $x^2 + y^2 + 4x = 0$.
6 $3x^2 + 3y^2 + 8x = 0$.
7 $(x - a)(x - c) + (y - b)(y - d) = 0$.
8 (a) $x^2 + y^2 = 144$, (b) $x^2 + 9y^2 = 324$.
9 $x^2 + y^2 = 36$.
10 $xy = 4$.

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11 $bx + ay = xy$.
12 $4x^2 + y^2 - 4x - 8 = 0$.
13 $7, 7x - y + 1 = 0, (-\frac{1}{6}, -\frac{1}{6})$.
14 $x + 2y = 12, (-6, 9)$.
15 $x + y - 4 = 0, x + 9y - 12 = 0$.
16 $\pm 1, x - y + 1 = 0, 4x + 4y - 11 = 0$.
17 $9y - 27x = 19, (-\frac{2}{3}, \frac{1}{9})$.
18 $8x - 28y + 49 = 0$.
19 $y = x + 1 + 2/x, (-2, -2); x - 2y + 6 = 0, x - 2y - 2 = 0$.
20 $6x + 12y \pm 5 = 0$.

Chapter 21

Qu. 1 $g = -a, f = -b, c = (a^2 + b^2 - r^2)$.
Qu. 2 (a) 0, (b) no real length.
Qu. 3 $X^2 + Y^2 - 1, X^2 + Y^2 - 6X - 8Y + 21; 3x + 4y - 11 = 0$.
Qu. 4 $x + y = 0$.

Exercise 21a, page 408

1 (a) $x^2 + y^2 - 4x - 6y + 12 = 0$; (b) $x^2 + y^2 + 6x - 8y = 0$; (c) $9x^2 + 9y^2 - 12x + 6y + 1 = 0$; (d) $x^2 + y^2 + 10y = 0$; (e) $x^2 + y^2 - 6x + 7 = 0$; (f) $144x^2 + 144y^2 + 72x - 96y - 47 = 0$.
2 (a) $1, (-2, 3)$; (b) $2, (1, 2)$; (c) $\frac{3}{2}, (\frac{3}{2}, 0)$; (d) $\frac{7}{2}, (-\frac{3}{2}, 2)$; (e) $\frac{1}{4}\sqrt{2}, (-\frac{1}{4}, -\frac{1}{4})$; (f) $1, (\frac{1}{3}, \frac{1}{2})$; (g) $\sqrt{(a^2 + b^2)}, (a, b)$; (h) $\sqrt{(g^2 + f^2 - c)}, (-g, -f)$.
3 (a), (d) if $a > 0$, (f) if $b = 0$, (g) if $c < 0$.
4 $x^2 + y^2 - 4x - 2y - 15 = 0$.
5 $(5, 3), \sqrt{10}; x^2 + y^2 - 10x - 6y + 24 = 0$.
6 $x^2 + y^2 - 4x + 6y + 4 = 0$.
7 4, 6.
8 The y-axis is a tangent.
9 $x^2 + y^2 \pm 8x - 10y + 16 = 0$.
10 $x^2 + y^2 - 4x - 4y + 4 = 0$.
11 $(2, 1), x^2 + y^2 - 4x - 2y - 45 = 0$.
12 $x^2 + y^2 - 16x + 8y - 5 = 0$.
13 (a) $x^2 + y^2 + 4x - 2y = 0$, (b) $x^2 + y^2 - 10x - 8y + 28 = 0$, (c) $x^2 + y^2 - 2x - 49 = 0$.
14 (4, 0), 2.
15 (4, 1), 3.

Exercise 21b, page 410

1 (a) $3x - y = 0$; (b) $x - 4y + 17 = 0$; (c) $4x + y - 11 = 0$; (d) $3x + y - 8 = 0$; (e) $4x + 9y + 5 = 0$.
2 (a) $\sqrt{10}$, (b) $\sqrt{15}$, (c) $\sqrt{29}$, (d) $2\sqrt{7}$, (e) $\sqrt{(x_1{}^2 + y_1{}^2 - a^2)}$, (f) $\sqrt{c}$.
3 5.

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4 $x-y-1=0, x+y-5=0$.
5 $(23, 0), (0, 7\frac{2}{3}), 88\frac{1}{6}$.
6 $\sqrt{13}$.
7 $\sqrt{(X^2+Y^2-4)}, 2x-5=0$.
8 $2x+3y-6=0$.
10 $(7, 4)$.

Exercise 21c, page 413

2 $(0, 0)$.
4 $(1, 2)$.
5 $(-2, 5)$.
6 $y=2x$.
7 $x^2+y^2-10x-8y+33=0$.
8 $4x-3y-18=0, 13\frac{1}{2}$.
9 $2\sqrt{3}$.
10 $x^2+y^2=a^2$.
11 $(0, 0), 3x+y=0$.
12 $x+y-1=0$.
13 $2x-5=0$.
14 5.
15 $x^2+y^2-10x-4y+4=0$.
16 $(3, 1), (7\frac{1}{2}, 2\frac{1}{2}); (5, 0)$.
17 $x^2+y^2-5x-y=0$.
18 $(2, -3), (-11, -3); x^2+y^2-4x+6y=0, x^2+y^2+22x+6y+117=0$.

Chapter 22

Qu. 1 (a) 1, (b) $\frac{2}{13}$, (c) $\frac{1}{2}\sqrt{26}$, (d) $3\sqrt{2}$, (e) $\frac{16}{17}\sqrt{34}$, (f) $\frac{2}{13}\sqrt{13}$, (g) $\frac{3}{5}a$, (h) $\frac{4}{5}q$, (i) $\frac{1}{13}(12X-5Y+7)$, (j) $\frac{1}{17}(8x_1-15y_1)$.

Exercise 22a, page 416

1 (a) $y-x=-1$, (b) $y+2x=-1$, (c) $2y-x=-12$, (d) $3y+x=13$, (e) $5y+7x=-9$, (f) $4y-3x=7$, (g) $6y+5x=-39$, (h) $3y-4x=23$, (i) $yt-x=at^2$, (j) $y+tx=at^3+2at$, (k) $y\sin\theta+x\cos\theta=a$, (l) $t^2y+x=2ct$.
2 (a) $2x-3y=-2$, (b) $3x+4y=0$, (c) $6x-5y=-43$, (d) $2x+3y=7$, (e) $y+tx=k+th$, (f) $bx-ay=bx_1-ay_1$, (g) $y-t^2x=c/t-ct^3$.
3 (a) $x/3+y/2=1$, (b) $y/2-x=1$, (c) $2x+5y=1$, (d) $4y-3x=1$.
4 (a) $x/3+y/2=1$, (b) $y/5-x=1$, (c) $3y/2-2x=1$.
5 $p\sec\alpha, p\operatorname{cosec}\alpha, x\cos\alpha+y\sin\alpha=p$.
6 (a) $y-3x+9=0$; (b) $2y+x=0$; (c) $5y-2x-3=0$; (d) $4y+3x+12=0$; (e) $y-6x+16=0$; (f) $4y-9x-3=0$; (g) $y=2$; (h) $2y+x-4=0$; (i) $3y-4x-13=0$; (j) $6y+x-19=0$; (k) $y+x-1=0$; (l) $y-t^2x=k-t^2h$.

Exercise 22b, page 421

2 (a) $r = a$, (b) $\theta = \alpha$, (c) $r = a \sec \theta$, (d) $r = a \operatorname{cosec} \theta$, (e) $r = a \cos \theta$, (f) $r = 2a \sin \theta$, (g) $a^2 = r^2 + c^2 - 2cr \cos \theta$, (h) $r = 2a/(1 + \cos \theta)$.
5 (a) $r = a$, (b) $r^2 = a^2 \sec 2\theta$, (c) $\theta = 0$, (d) $r = 2a/(1 + \cos \theta)$, (e) $r = 2 \sin \theta$, (f) $r^2 = 2c^2 \operatorname{cosec} 2\theta$.
6 (a) $x^2 + y^2 = 4$, (b) $(x^2 + y^2 - ax)^2 = a^2(x^2 + y^2)$, (c) $x^2 + y^2 - ax = 0$, (d) $x^4 + x^2y^2 = a^2y^2$, (e) $(x^2 + y^2)^3 = 4a^2(x + y)^4$, (f) $4xy = c^2$, (g) $x^2 + y^2 = (l - ex)^2$, (h) $y^2 = 4ax$.
7 (a) 1, 60°; (b) $2\sqrt{2}$, $-45°$; (c) 2, $\tan^{-1} \frac{4}{3}$; (d) 2, $\tan^{-1}(-\frac{12}{5})$; (e) $\frac{1}{5}\sqrt{10}$, $\tan^{-1} 3$; (f) $c/\sqrt{(a^2 + b^2)}$, $\tan^{-1}(b/a)$.

Exercise 22c, page 425

1 (a) $4\frac{1}{5}$, (b) $2\frac{1}{13}$, (c) $\frac{10}{17}\sqrt{17}$, (d) 0, (e) $\frac{38}{29}\sqrt{29}$, (f) $1\frac{1}{5}$, (g) $\frac{6}{41}\sqrt{41}$, (h) $\frac{5}{13}\sqrt{13}$, (i) p, (j) $\frac{1}{13}(5X - 12Y + 1)$, (k) $\frac{22}{17}c$, (l) $\frac{1}{5}(4y_1 - 3x_1 + 2)$.
2 (a) $3x - y - 2 = 0$, $x + 3y - 4 = 0$; (b) $7x - 7y + 4 = 0$, $x + y - 2 = 0$; (c) $17x + 17y - 4 = 0$, $7x - 7y - 4 = 0$; (d) $x + (1 \pm \sqrt{2})y - 1 = 0$.
3 (a) $8x - 4y + 17 = 0$, (b) $8y + 1 = 0$, (c) $4x + 12y + 5 = 0$.
4 $4x^2 - 4xy + y^2 - 20x + 30y + 65 = 0$.
5 $4x + 3y - 24 = 0$.
6 $7x^2 - 2xy + 7y^2 - 40x - 40y + 48 = 0$.
7 $4y - 3x - 15 = 0$, $4y - 3x + 35 = 0$.
8 $2x - 3y \pm 13 = 0$.
9 $x^2 + y^2 - 4x - 14y + 49 = 0$.
10 $n^2 = a^2(l^2 + m^2)$.

Exercise 22d, page 429

2 (a) $\frac{4}{3}, \frac{4}{3}$; (b) $\pm\frac{3}{2}$, $\pm 3a$, (c) -2, $-\frac{1}{3}$; (d) 60°, $(\sqrt{3}/2)b$.
3 (a) $(y - 2)^2 = x - 1$, (b) $x^3 = y^2$, (c) $xy = 1$, (d) $2x + y - 5 = 0$, (e) $y^2 = 4ax$, (f) $xy^2 = 1$, (g) $5x + y - 13 = 0$, (h) $4x^2 - 9y^2 = 144$, (i) $4x^2 + 9y^2 = 36$, (j) $9x^2 - 16y^2 = 144$.
4 (a) $x = t^4$, $y = t^5$; (b) $x = t - 2$, $y = t^2 - 2t$; (c) $x = \dfrac{2}{t^2 - 1}$, $y = \dfrac{2t}{t^2 - 1}$;
(d) $x = \dfrac{1}{1 - t^3}$, $y = \dfrac{t}{1 - t^3}$; (e) $x = \dfrac{3t}{1 + t^3}$, $y = \dfrac{3t^2}{1 + t^3}$.
5 $3x - 2y + 1 = 0$.
6 $-\frac{4}{13}$, $-\frac{4}{3}$.
7 $(\frac{1}{2}t^2, \frac{3}{2}t)$, $2y^2 = 9x$.
8 $y^2 = 8ax$.
9 $x = y(2x - 1)^2$.
10 $(1, 1)$, $(-1, -1)$, $\sqrt{2}$.
11 $(a, 2a)$, inflexion; $(4a, -4a)$, minimum.
12 (a) $(p + q)y - 2x = 2pq$, (b) $pqy - x + (p + q) = 0$, (c) $(p^2 + pq + q^2)y - x = pq(p + q)$, (d) $(pq - 1)y - 2pqx + 2(p + q) = 0$.
13 $(a, 2a)$, $\frac{3}{2}\sqrt{2}a$.

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14 $-1, 2; (1, -2), (4, 4)$.

15 $(1, \pm 2), (4, \pm 4)$.

Exercise 22e, page 431

1 (a) $2y + 3x - 1 = 0, 2x - 3y - 5 = 0$; (b) $y + 4x = 3, 16y - 4x = 31$;
(c) $x + y + a = 0, x - y - 3a = 0$; (d) $y + x + 2c = 0, y - x = 0$;
(e) $2y + x + 9 = 0, 2x - y + 3 = 0$;
(f) $3\sqrt{3}y + 2x - 12 = 0, 6\sqrt{3}x - 4y - 5\sqrt{3} = 0$.

2 (a) $ty - 2x - t^3 = 0, 2y + tx = 6t^2 + t^4$;
(b) $ty - x - at^2 = 0, y + tx = 2at + at^3$;
(c) $y - tx + t^4 = 0, ty + x = 3t^5 + 4t^3$;
(d) $t^2y + x - 2ct = 0, y - t^2x = c/t - ct^3$;
(e) $bx \cos t + ay \sin t = ab, ax \sin t - by \cos t = \frac{1}{2}(a^2 - b^2) \sin 2t$;
(f) $bx \sec t - ay \tan t = ab, ax \sin t + by = (a^2 + b^2) \tan t$.

3 (a) $(p + q)y - 2x = 2pq, py - x = p^2$;
(b) $y + pq(p + q)x = p^2 + pq + q^2, y + 2p^3x = 3p^2$;
(c) $pqy + x = c(p + q), p^2y + x = 2cp$;
(d) $bx \cos \frac{1}{2}(p + q) + ay \sin \frac{1}{2}(p + q) = ab \cos \frac{1}{2}(p - q)$,
$bx \cos p + ay \sin p = ab$.

4 $2x + y - 12a = 0, (9a, -6a)$.

5 $(-\frac{1}{8}c, -8c)$.

6 $(-\frac{1}{2}, 4)$.

7 $yt - x = at^2$; 2, 4; $2y - x = 4a, 4y - x = 16a$.

8 $y + x = 2c, 9y + x = 6c$.

9 $y + 2x = 12a, y - 4x + 72a = 0$.

10 $(-c/t^3, -ct^3)$.

Exercise 22f, page 432

4 $yt - x = at^2, y + tx = 2at + at^3$.

17 $2, -1; y + 2x = 12a, y - x + 3a = 0$.

18 $y - x - a = 0, 4y - x - 16a = 0$.

Exercise 22g, page 434

4 $r = 2a/(1 - \cos \theta)$.

5 $6\frac{1}{2}$.

6 $r = a(1 - \cos \theta), x^2 + 2ay = a^2$.

7 $\frac{7}{25}x - \frac{24}{25}y = \frac{2}{5}, \frac{2}{5}$.

8 (a) $2x - 8y + 7 = 0, 4x + y + 4 = 0$; (b) $12x + 4y - 13 = 0, 2x - 6y + 7 = 0$.

9 $x^2 - 2xy + y^2 + 8x + 8y = 0$.

10 $x - 3y + 5 = 0, x - 3y + 25 = 0$.

11 $2xy = x + 1$.

13 $x - 2y + 2 = 0, 2x + y - 11 = 0$.

14 $2t^3y + x = 3t^2, 2xy^2 = 1$.

16 $x - y + a = 0, x - 5y + 25a = 0$.

17 $(\frac{121}{9}a, -\frac{22}{3}a)$.

Chapter 23

Note: Approximate answers have generally been rounded to 2 or 3 significant figures. The reader should not assume from the form of an answer that the result is exact.

Qu. 1 (a) s varies as the square of t, (b) V varies as the cube of r, (c) y varies inversely as the square of x, (d) T varies as the square root of l, (e) p varies inversely as v, (f) the square of T varies as the cube of d.

Qu. 2 W is increased by a factor of (a) 8, (b) 27.

Qu. 3 (i) (a) $p = kq$, (b) $p = \frac{k}{v}$, (c) $v = kx^3$, (d) $U = k\sqrt{l}$, (e) $F = kc^2$, (f) $H = \frac{k}{d^2}$, (g) $T = \frac{k}{\sqrt{g}}$, (h) $A = ks^n$, (i) $A^3 = kv^2$.

(ii) (a) $\frac{p_1}{p_2} = \frac{q_1}{q_2}$, (b) $\frac{p_1}{p_2} = \frac{v_2}{v_1}$, (c) $\frac{v_1}{v_2} = \frac{x_1^3}{x_2^3}$, (d) $\frac{U_1}{U_2} = \frac{\sqrt{l_1}}{\sqrt{l_2}}$, (e) $\frac{F_1}{F_2} = \frac{c_1^2}{c_2^2}$, (f) $\frac{H_1}{H_2} = \frac{d_2^2}{d_1^2}$, (g) $\frac{T_1}{T_2} = \frac{\sqrt{g_2}}{\sqrt{g_1}}$, (h) $\frac{A_1}{A_2} = \frac{s_1^n}{s_2^n}$, (i) $\frac{A_1^3}{A_2^3} = \frac{v_1^2}{v_2^2}$.

Qu. 4 l is increased by a factor of (a) 4, (b) 9. T is increased by a factor of $\sqrt{2}$.

Qu. 5 1.1 s.

Qu. 6 $w = \frac{3.21 \times 10^{10}}{d^2}$.

Qu. 7 w is multiplied by (a) $\frac{1}{4}$, (b) $\frac{1}{9}$.

Qu. 8 (a) c varies as p, (b) C varies as a^2 over a limited range, (c) w varies as r^3, (d) l varies inversely as b, (e) S varies as l^2, (f) A varies as a^2, (g) a varies as $\sqrt{A}$, (h) V varies as a^3, (i) a varies as $\sqrt[3]{V}$.

Qu. 9 $y = kxz^3$.

Qu. 10 $W \propto \frac{hr^2}{t}$.

Qu. 11 (a) $T = kmr^2$, (b) $T \propto mr^2$.

Qu. 12 $F = k\frac{mv^2}{r}$.

Qu. 13 (a) $\frac{z_1}{z_2} = \frac{x_1 y_1^2}{x_2 y_2^2}$, (b) $\frac{z_1}{z_2} = \frac{y_1 x_2^2}{y_2 x_1^2}$, (c) $\frac{z_1}{z_2} = \frac{x_1^3 y_1^2}{x_2^3 y_2^2}$, (d) $\frac{z_1}{z_2} = \frac{x_1 y_1}{x_2 y_2}$, (e) $\frac{z_1}{z_2} = \frac{x_1^2 y_1^2}{x_2^2 y_2^2}$, (f) $\frac{z_1}{z_2} = \frac{y_2\sqrt{x_1}}{y_1\sqrt{x_2}}$.

Qu. 14 (a) $C = K + kx^3$, k, K constants; (b) £11.95.

Qu. 15 $v = 4.2t$.

Qu. 16 The cost in labour and materials before any copies are run off is £10.50.

Qu. 17 (a) $y = -3x + 122$, (b) $y = 13.5x - 71.5$, (c) $y = -12x + 213$.

Qu. 18 $y = 12x - 72$; 6 cm.

Qu. 19 $R = 0.0005v^3$.

Qu. 20 $k = 0.49$.
Qu. 22 $k = 2070$; $a = 1.05$. 3700.

Exercise 23a, page 443

1 5.70 cm^2. **2** 29.9 km, $d \approx 3.57\sqrt{h}$. **3** 155 cm, $l \approx 24.8(5)\ T^2$.
4 25 m. **5** 0.242 kg, 11.0 cm, $m = \frac{1}{810}d^2$.
6 (a) $C = 200\pi r$, yes, (b) $C = 5.08\pi r$, yes. **7** 2.8×10^5 N/m^2. **8** 3584.
9 15.6 cm^3, 11.2 cm. **10** 10.4 k, $v = \frac{6}{5}\sqrt{(5l)}$. **11** 3168, 3.3 mm.
12 (a) y varies as t^6, (b) p varies inversely as r^2.
13 $F = \frac{175}{54}v^2$. (a) 2360 N, (b) 43.6 km/h.
14 $H \approx 0.000\ 182v^3$, 4.92 kW.
15 Increases approx. 7% in speed and 15% in acceleration.
16 1.59.
17 Increase $\approx 0.05\%$.
19 1 h 37 min, $T \approx 2.87 \times 10^{-6}d^{3/2}$.
20 275:1472.

Exercise 23b, page 449

1 15.7 cm^2. **2** 454.5 rev/min. **3** 48 litres/s. **4** 675 kJ.
5 185 cm^3, $V \approx 51.6\dfrac{T}{p}$. **6** 0.70 kW, $0.001\dfrac{V^2}{R}$.
7 570 s^{-1}, $f \approx 14.2\dfrac{\sqrt{F}}{l}$. **8** 76 s^{-1}. **9** 4.91 s.
10 The former; ratio 16:15. **11** The latter; ratio 27:32.
12 The former; ratio 16:15. **13** £1150. **14** £9.75, $C = 6.75 + 0.015n$.
15 88 m, $s = 10t + 3t^2$. **16** 54.4 m, $s = 0.2v + 0.006v^2$, 50 km/h.
17 $C = 0.25 + 0.2m$. **18** £3.50, 125. **19** $28\frac{1}{2}$ cm^2, $S = 2x^2 + \dfrac{40}{x}$.
20 $V = \pi rh^2 - \frac{1}{3}\pi h^3$.

Exercise 23c, page 460

1 $A = 1.84D$. Yes.
2 $m = 1.02t$; relative density = gradient $\times \frac{100}{9}$.
3 $h_2 = 0.34h_1$.
4 $d = 7 + 6.8n$, taking n to be 0, $7\frac{1}{2}$, $19\frac{1}{2}$.
5 (a) £1950, (b) £435.
6 11 km/litre, about 17 litres.
7 $\theta = 62 - \frac{1}{2}t$. No: cooler bodies lose heat more slowly.
8 Yes; about 219 litres.
9 Yes. $l = 15/r$.
10 $m = 0.338d^2$.
11 $y = 3.50 - 0.025x^2$.
12 $p = 10.6 - \dfrac{9.6}{i}$; $i = \dfrac{9.6}{10.6 - p}$. $\left(\text{Theoretically, } i = \dfrac{10}{11 - p}.\right)$

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13 $P = 0.199s^{1.50}$.

14 $p \propto \frac{1}{\sqrt{n}}$.

15 $I = 1.6r^2$.

16 $w = 3.94 \times 0.846^n$.

17 0.0071.

18 1.28.

19 0.005.

20 $T = 100 \times 1.008^{-x}$; no.

Chapter 24

Qu. 1 (a) 4.123, (b) 6.325, (c) 9.220, (d) 9.798.

Exercise 24a, page 473

1 (a) 3.46, (b) 5.48, (c) 7.07, (d) 8.66.
2 2.33, 2.29, 2.29; $x^3 - 12 = 0$.
3 2.71.
4 $x_{r+1} = (x_r^2 + 1)/5$; 0.208, 0.209, 0.209; $x_{r+1} = 5 - 1/x_r$; 0, $-\infty$, 5.
5 0.1001.
6 1.93.
7 $x_{r+1} = 8 - 10/x_r$; -2, 13, 7.23; the sequence does not appear to converge, however it eventually converges to 6.45, the other root.
8 (a) 2.17, 2.15, 2.15; (b) 2.5, 1.6, 3.91; 2.15.
9 2.19.
10 6.54, 0.459.

Exercise 24b, page 478

Nos. 1, 2, 3, 5, 6, 8, 10 (i.e. *not* 4, 7, 9).

Exercise 24c, page 482

1 4.74. **2** 3.28. **3** 1.90. **4** 5.15. **5** 3.58.
6 0.771. **7** 4.15. **8** 3.70. **9** 2.09. **10** 1.04.

Exercise 24d, page 483

1 (a) 14.1, (b) 21.2, (c) 26.5, (d) 31.6. **2** (a) 14.1, (b) 21.2, (c) 26.5, (d) 31.6.
6 0.450. **7** 0.450. **8** 0.196. **9** 8.16. **10** 0.347.
11 2.34, 4.68, 9.13 cm. **12** $0.653r$. **13** 4.1%. **14** $1.16r$.
15 $n=1$, $x_2 = 1.260$, $x_3 = 1.312$. **16** 18.5. **17** 2.17. **18** 1.84.
19 1.2. **20** 0.167.

Chapter 25

Qu. 1 p, p.
Qu. 2 p.
Qu. 3 A.
Qu. 4 A, D.
Qu. 6 $0 \leftrightarrow 1, 1 \leftrightarrow 3, 2 \leftrightarrow 9, 3 \leftrightarrow 7$, *or* $0 \leftrightarrow 1, 1 \leftrightarrow 7, 2 \leftrightarrow 9, 3 \leftrightarrow 3$.
Qu. 7 In (h) every element is 'self-inverse'.

Qu. 9

A	B	C	D	E
6	3	2	3	6

A and E.

Qu. 10 $\{1\}, \{1, 9\}, \{1, 3, 7, 9\}$.
Qu. 11 $\{I\}, \{I, C\}, \{I, D\}, \{I, E\}, \{I, A, B\}, \{I, A, B, C, D, E\}$.
Qu. 12 $\{e, x^3\}, \{e, x^2, x^4\}$.
Qu. 13 (a), (b), (d).
Qu. 14 (b), (d); (a) not zero, (c) not singular matrices.
Qu. 15 $\{I, C\}, \{I, D\}, \{I, E\}, \{I, P\}, \{I, S\}, \{I, T\}, \{I, U\}; \{I, A, B\}$;
$\{I, D, P, T\}, \{I, E, P, U\}, \{I, C, P, S\}$;
$\{I, A, B, P, Q, R\}, \{I, A, B, S, T, U\}, \{I, A, B, C, D, E\}$.

Exercise 25a, page 490

1 $\mathbf{I} \leftrightarrow e$, $\mathbf{J} \leftrightarrow x$, $\mathbf{I}$.
2 (a) D, (b) C, (c) B, (d) C.
3 (a) I, (b) I, (c) D, (d) D.
4 $p^* = p$, $r^* = q$.

5

x	I	A	B	C	D	E
x^*	I	B	A	C	D	E

6

	1	i	−1	−i
1	1	i	−1	−i
i	i	−1	−i	1
−1	−1	−i	1	i
−i	−i	1	i	−1

$e \leftrightarrow 1$, $p \leftrightarrow -1$, $q \leftrightarrow \mathrm{i}$, $r \leftrightarrow -\mathrm{i}$ (or $q \leftrightarrow -\mathrm{i}$, $r \leftrightarrow \mathrm{i}$).

7

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

8

	I	**A**	**B**	**C**
I	**I**	**A**	**B**	**C**
A	**A**	**B**	**C**	**I**
B	**B**	**C**	**I**	**A**
C	**C**	**I**	**A**	**B**

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9

	I	P	Q	R
I	I	P	Q	R
P	P	I	R	Q
Q	Q	R	I	P
R	R	Q	P	I

10

	I	A	B	C	P	Q	R	S
I	I	A	B	C	P	Q	R	S
A	A	B	C	I	S	R	P	Q
B	B	C	I	A	Q	P	S	R
C	C	I	A	B	R	S	Q	P
P	P	R	Q	S	I	B	A	C
Q	Q	S	P	R	B	I	C	A
R	R	Q	S	P	C	A	I	B
S	S	P	R	Q	A	C	B	I

Exercise 25b, page 497

1

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

2

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

3

	e	x	x^2	x^3	x^4	x^5
e	e	x	x^2	x^3	x^4	x^5
x	x	x^2	x^3	x^4	x^5	e
x^2	x^2	x^3	x^4	x^5	e	x
x^3	x^3	x^4	x^5	e	x	x^2
x^4	x^4	x^5	e	x	x^2	x^3
x^5	x^5	e	x	x^2	x^3	x^4

4

	I	R	R^2	R^3	A	B	C	D
I	I	R	R^2	R^3	A	B	C	D
R	R	R^2	R^3	I	B	C	D	A
R^2	R^2	R^3	I	R	C	D	A	B
R^3	R^3	I	R	R^2	D	A	B	C
A	A	D	C	B	I	R^3	R^2	R
B	B	A	D	C	R	I	R^3	R^2
C	C	B	A	D	R^2	R	I	R^3
D	D	C	B	A	R^3	R^2	R	I

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5

	I	R	R^2	R^3	R^4	R^5	A	B	C	D	E	F
I	I	R	R^2	R^3	R^4	R^5	A	B	C	D	E	F
R	R	R^2	R^3	R^4	R^5	I	B	C	D	E	F	A
R^2	R^2	R^3	R^4	R^5	I	R	C	D	E	F	A	B
R^3	R^3	R^4	R^5	I	R	R^2	D	E	F	A	B	C
R^4	R^4	R^5	I	R	R^2	R^3	E	F	A	B	C	D
R^5	R^5	I	R	R^2	R^3	R^4	F	A	B	C	D	E
A	A	F	E	D	C	B	I	R^5	R^4	R^3	R^2	R
B	B	A	F	E	D	C	R	I	R^5	R^4	R^3	R^2
C	C	B	A	F	E	D	R^2	R	I	R^5	R^4	R^3
D	D	C	B	A	F	E	R^3	R^2	R	I	R^5	R^4
E	E	D	C	B	A	F	R^4	R^3	R^2	R	I	R^5
F	F	E	D	C	B	A	R^5	R^4	R^3	R^2	R	I

Exercise 25c, page 502

1 No inverses.

2 Not closed.

6

1	4	7	13
1	4	13	7

, C_4.

7

e	a	b	c	d	f
e	f	d	c	b	a

, C_6.

8

I	A	B	C
I	A	B	C

, Klein.

9

I	A	B	C	D	E
I	B	A	C	D	E
1	3	3	2	2	2

, no.

10 $(a - b\sqrt{2})/(a^2 - 2b^2)$.

11 $(a - \mathrm{i}b)/(a^2 + b^2)$.

12 $\{1, 5, 7, 11\}, \{1, 5\}, \{1, 7\}, \{1, 11\}, \{4, 8\}, \{3, 9\}, \{1\}$.

Exercise 25d, page 508

2 $e = 6$.

x	3	6	9	12
x^{-1}	12	6	9	3

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5 (a) $\{I, G, E, F\}$, (b) $\{I, A\}$, (c) $\{I, A, C, F\}$.
6 $\{I, A\}$, $\{B, E\}$, $\{C, F\}$, $\{D, G\}$.
7 $(1+x)/(1-x)$, $-1/x$, $(x-1)/(x+1)$, x.

9

element	1	2	4	5	7	8
period	1	6	3	6	3	2

12

	e	x	x^2	x^3	y	yx	yx^2	yx^3
e	e	x	x^2	x^3	y	yx	yx^2	yx^3
x	x	x^2	x^3	e	yx^3	y	yx	yx^2
x^2	x^2	x^3	e	x	yx^2	yx^3	y	yx
x^3	x^3	e	x	x^2	yx	yx^2	yx^3	y
y	y	yx	yx^2	yx^3	x^2	x^3	e	x
yx	yx	yx^2	yx^3	y	x	x^2	x^3	e
yx^2	yx^2	yx^3	y	yx	e	x	x^2	x^3
yx^3	yx^3	y	yx	yx^2	x^3	e	x	x^2

13 $e, x, x^2, y, z, xy, xz, x^2y, x^2z, yz, xyz, x^2yz$.
14 (a) S, S, X, X, U, U. (b) U.
15 $\{U, P, S, V\}$, $\{U, X, R, S\}$, $\{U, Q, S, W\}$; $\{U, S\}$, $\{P, V\}$, $\{Q, W\}$, $\{R, X\}$.
16 e, e, b, d; not associative; $\{e, a\}$, $\{e, b\}$, $\{e, c\}$, $\{e, d\}$.
17 (a) $\frac{1}{2}, \frac{1}{2x}$; (b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.
18 Yes.
20 Yes; **C** exists but is not unique; $\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Appendix

Exercise 1, page 512

1 $4xh$. **2** $2x(x^2+3h^2)$. **3** $2h(3x^2+h^2)$. **4** $3x-4x^3$.
5 $5y = 2x + 25$. **6** $8y = 3x - 5$. **7** $2t - \sqrt{t} - 3$.
8 $(3+2t)(1+t)$. **9** $\frac{x}{x-1}$. **10** $\frac{x+y}{xy-1}$.

Exercise 2, page 513

1 $(5x-2)(7x+3)$. **2** $2(x+7)(x-7)$. **3** $(2x+y)(x-y)$.
4 $(x+a)(y+b)$. **5** $(x+3)(y-2)$. **6** $(x+1)(x-1)(2x+3)$.
7 $(x+3)(2x^2+3x+3)$. **8** $(x+1)(12x+5)$. **9** $20(x-2)$.
10 $2(x-2)^2(3x-1)$.

Exercise 3, page 514

1 $\frac{y-x}{xy}$. **2** $\frac{x^2+y^2}{xy}$. **3** $\frac{1+a}{a^2}$. **4** $\frac{a+b}{a^2b^2}$.

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5 $\dfrac{2x}{(x-h)(x+h)}$. **6** $\dfrac{-h(2x+h)}{x^2(x+h)^2}$. **7** $\dfrac{3x}{(1-x)(2+x)}$.

8 $\dfrac{-(x^2-2x+4)}{(x^2+2)(2+x)}$. **9** $\dfrac{n+1}{n+2}$. **10** $\dfrac{x^2+3x+3}{(x+1)^2}$.

Exercise 4, page 514

1 $2/(T+t)$.
2 $ty = x + t^2$.
3 $-1/t$.
4 $-Tt$.
5 $(N+1)(2N+1)(2N+3)$.
6 $\sqrt{(a+b)}$.
7 $(ad+bc)/(bd+ac)$.
8 $3x^2+3xh+h^2$.
9 $\dfrac{1}{\sqrt{(1+x^2)} \times (1+x^2)} = \dfrac{1}{(1+x^2)^{3/2}}$.
10 $(1-t)/(1+t)$ [or $(t-1)/(t+1)$, whichever is positive.]

Exercise 5, page 515

1 6. **2** -10, 25. **3** 12. **4** 5, 25. **5** 7, 49. **6** 3, 4, 9.
7 $1, \frac{1}{4}$. **8** $1, \frac{1}{4}, 1$. **9** 10, 9. **10** $\frac{1}{2}, \frac{1}{9}, \frac{1}{4}$.

Exercise 6, page 516

1 $m = (y-c)/x$. **2** $e = (a-b)/a$. **3** $x = y^2/(4a)$.
4 $y = \dfrac{(K-k)(x-h)}{H-h} + k$. **5** $c = 4 + 3m$. **6** $x = (b-1)/(a-1)$.
7 $l = T^2 g/(4\pi^2)$. **8** $g = 4\pi^2 l/T^2$. **9** $m = \dfrac{2x+2y+1}{2(2y-x)}$.
10 $m = \dfrac{2x-3y+4}{3x-2y+2}$.

Exercise 7, page 517

1 3. **2** 2. **3** $\frac{1}{2}$. **4** 17. **5** 7, $1\frac{1}{2}$. **6** 7, -2.
7 $+2, -2$. **8** $c(2T+3t)/5$. **9** $t/5, 3t$. **10** $1/t, -T$.

Exercise 8, page 519

1 2, -1. **2** 2, -3. **3** $-7/19, 8/19$. **4** 0, 0; 4, 4.
5 16, 4; $-1, -64$. **6** 0, 1; 2, 3. **7** $10c, 7c$. **8** $tT, (t+T)$.
9 $t, 1/t; -1/t^3, -t^3$. **10** $5a, 3a; 4a, 0$.

Exercise 9, page 520

1 0, $+2, -2$. **2** 0, 0, 7. **3** 0, -4, 5. **4** $+4, -4, +1, -1$.

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5 $+2/3, -2/3$. **6** $0, 0, -k$. **7** $a, (a+b), (a-b)$. **8** 0.
9 $+a, -a$. **10** $p+q$.

Exercise 10, page 521

1 (a) $(a+b)$, (or $-(a+b)$ if $a+b<0$), (b) a^3+b^3.
2 (a) $(K+1)^2(K+2)^2$, (b) $(N+1)(N+2)(2N+9)$.
3 (a) $\dfrac{4xh}{(x-h)^2(x+h)^2}$, (b) $\dfrac{-(2N+5)}{(N+2)(N+3)}$.
4 (a) $-\dfrac{1}{(n+1)^2}$, (b) $2N(N^2+6N+11)$.
5 (a) 3, 9, (b) $3\frac{1}{2}, \frac{3}{4}$.
6 (a) $(u^2-3)/3$, (b) $\pm\sqrt{\{(u-1)/5\}}$.
7 (a) $2a/3$, (b) $3a/7, -a$.
8 (a) $2ct$, (b) $c(1-t^2)/t$.
9 (a) $29a, 52a$, (b) $9a, 6a; a/9, -2a/3$.
10 (a) 0, (b) $-k, 2k$.

Notes

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