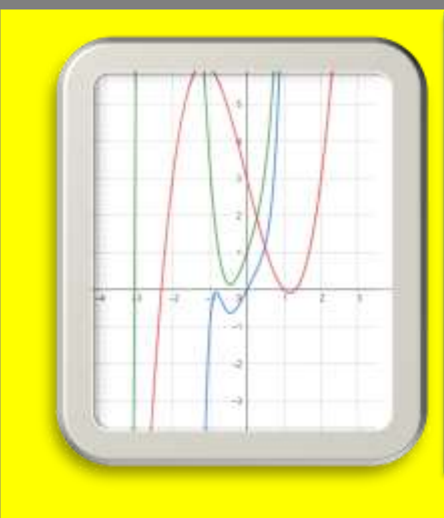
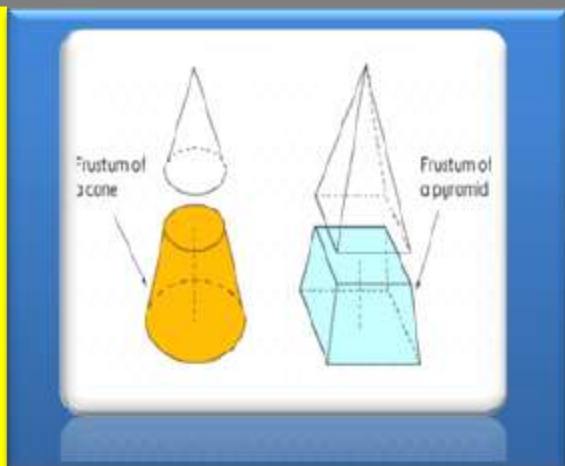
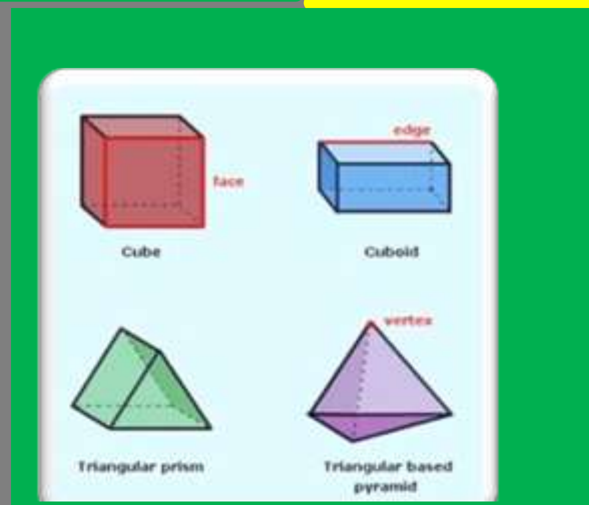
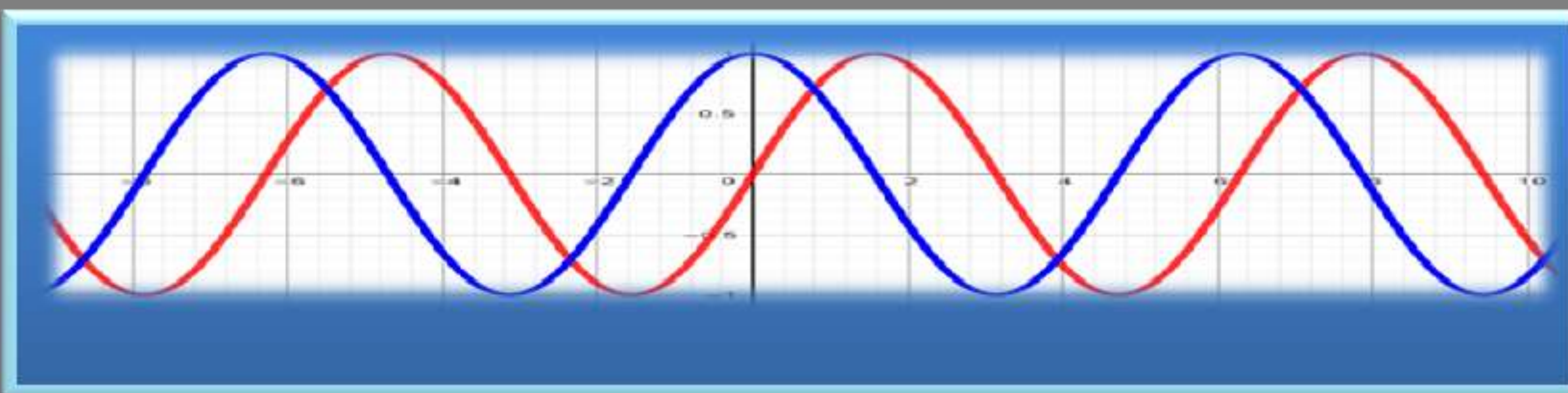
MATHEMATICS SHORT NOTES FOR GRADE 12, FROM GRADE 10

June, 2023









Oromia Development Association Boarding School

MATHEMATICS SHORT NOTES FOR GRADE 12, FROM GRADE 10

By: Merga Amara Angasu

June, 2023

Contents

Unit 1

POLYNOMIAL FUNCTIONS

Unit Outcomes:

After completing this unit, you should be able to:

- *define polynomial functions.*
- *apply theorems on polynomials to solve related problems.*
- *sketch and analyses the graphs of polynomial functions.*

Main Contents:

1.1. Introduction to polynomial functions

1.2. Theorems on polynomial

1.3. Zeros of polynomial functions

1.4. Graphs of polynomial functions

1.1. INTRODUCTION TO POLYNOMIAL FUNCTIONS

1.1.1. Definition of a Polynomial Function

Definition 1.1

Let n be a non – negative integer and let $a_n,\ a_{n-1}, ..., \ a_1,\ a_0$ be real numbers with $a_n \neq 0$.
The function P defined by $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ . . . + a_2x^2 + a_1x + a_0$ is called a **polynomial function** in variable x of degree n.
The expression $a_nx^n + a_{n-1}x^{n-1} + \ . . . a_1x^1 + a_0$ is called **polynomial expression** in variable x.

Note: In the definition of a polynomial functions

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \ . . . + a_2x^2 + a_1x + a_0\ ;$$

i) $a_n,\ a_{n-1},\ a_{n-2}, ..., \ a_1, a_0$ are called the **coefficients** of $P(x)$.

ii) The number a_n , where $a_n \neq 0$, is called the **leading coefficient** and the term a_nx^n is called the **leading term** of $P(x)$.

iii) The number a_0 is called the **constant term** of $P(x)$.

iv) If $a_n \neq 0$, then the number n (the highest exponent of power of x) is called the **degree** of $P(x)$.

Note: The domain of any polynomial function is the set of real number.

Example 1:

a. $(x) = \frac{5}{2}x - 2x^3$ is a polynomial function with degree 3 and constant term 0.
   - ✓ The leading term of f is $-2x^3$ and the leading coefficient is -2.
   - ✓ The coefficient of x^2 is 0 and the coefficient of x is $\frac{5}{2}$.
   - ✓ The domain of f is $\mathbb{R}$.

b. $f(x) = \sqrt{x^4 + 1}$ has domain $\mathbb{R}$, but $\sqrt{x^4 + 1}$ cannot be expressed in the standard form of polynomial $a_nx^n + \ . . . + a_1x + a_0$.
   Hence, $f(x) = \sqrt{x^4 + 1}$ is **NOT** polynomial.

c. $f(x) = \sqrt{(x^2 - 1)^2}$
   - f(x) = $\sqrt{(x^2 - 1)^2} = |x^2 - 1|$ which is **not** polynomial function, because $|x^2 - 1|$ has no the standard form $a_nx^n + a_{n-1}x^{n-1} + \ . . . a_1x^1 + a_0$

1.2. THEOREMS ON POLYNOMIALS

Theorem 1.1: Polynomial Division Theorem (Division Algorithm)

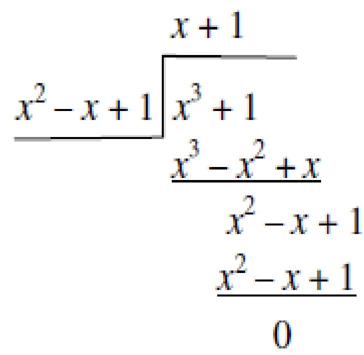
If $f(x)$ and $d(x)$ are two polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$.
If the remainder $r(x) = 0$, $f(x)$ divides exactly into $d(x)$ or we say that division of $f(x)$ by $d(x)$ is exact.

Example 2: Divide polynomial x^3+1 by x^2-x+1 using long division. Determine the quotient and the remainder. Is the division exact?

Solution: The long division to divide x^3+1 by x^2-x+1 is given below.



From the long division we can see that

- the **quotient** is $q(x)=x+1$, and
- the **remainder** is $r(x)=0$.

Hence, **the division is exact**.

As written in Division Algorithm,

$$f(x)=q(x)d(x)+r(x)$$
$$x^3+1=(x+1)(x^2-x+1)+0$$
$$x^3+1=(x+1)(x^2-x+1)$$

- Thus, the **quotient** $x+1$ is the **factor** of the dividend x^3+1.
- The **divisor** x^2-x+1 is also the **factor** of the dividend x^3+1.
 (That is $x+1$ and x^2-x+1 are the factors of x^3+1.)
- The product $(x+1)(x^2-x+1)$ a **factorized form** of x^3+1.

Remark: In division of a polynomial (dividend) with degree n by a polynomial (divisor) of degree m, where $n \geq m$,

i) The remainder is a zero polynomial or a polynomial of degree less than the degree of the divisor.
ii) The degree of the quotient = $n-m$ = degree of f − degree of g .

1.2.2 The Remainder Theorem

Theorem 1.2: (Remainder Theorem)

Let $f(x)$ be a polynomial of degree greater than or equal to 1 and let $c \in \mathbb{R}$. If $f(x)$ is divided by the linear polynomial$x-c$, then the remained is $f(c)$.

Example 3: Using Remainder Theorem, find the remainder if:

a) $f(x)=x^3+2x^2-6$ divided by $d(x)=x-3$
b) $g(x)=1+5x+x^2-2x^5$ divided by $d(x)=x+2$
c) $f(x)=2+7x^{25}$ divided by $d(x)=x-1$

Solution:

a) $f(x)=x^3+2x^2-6$. Here, $x-c=x-3$ implies $c=3$.
 Then by Division Algorithm, the remainder is
 $$R=f(3)=(3)^3+2(3)^2-6=27+18-6=39$$
b) $g(x)=1+5x+x^2-2x^5$. Here, $x-c=x+2$ implies $c=-2$.
 Then by Division Algorithm, the remainder is
 $$R=g(-2)=1+5(-2)+(-2)^2-2(-2)^5=1-10+4+64=59$$
c) $f(x)=2+7x^{25}$. Here, $x-c=x-1$ implies $c=1$.
 Then by Division Algorithm, the remainder is
 $$R=f(1)=2+7(-1)^{25}=2+7(-1)=2-7=-5$$

Example 4: When the polynomial $f(x) = x^7 - kx^6 + 5x^3 - x + 11$ is divided by $x + 1$, the remainder is 15, what is the values of k.

Solution: $x - c = x + 1$ implies $c = -1$

By Remainder Theorem, the remainder is

$$R = f(c) \quad \Rightarrow \quad f(-1) = 15$$

$$\Rightarrow \quad (-1)^7 - k(-1)^6 + 5(-1)^3 - (-1) + 11 = 15$$

$$\Rightarrow \quad -1 - k - 5 + 1 + 11 = 1 \Rightarrow \quad k = -11$$

1.2.3 The Factor Theorem

Theorem 1.3: Factor Theorem

Let $f(x)$ be a polynomial of degree greater than or equal to one, and let c be any real number. Then $x - c$ is a factor of $f(x)$, if and only if $f(c) = 0$.

Example 5: In each of the following, use the factor theorem to determine whether or not $g(x)$ is a factor of $f(x)$.

a. $f(x) = x^{15} + 1$; $g(x) = x + 1$,

b. $f(x) = 3x^4 + 7x^2 - x - 2$; $g(x) = x - 1$

Solution:

a. $x + 1 = x - c$ implies $c = -1$. Then $f(-1) = (-1)^{15} + 1 = -1 + 1 = 0$.

Thus, by Factor Theorem, $x + 1$ is a factor of $x^{15} + 1$.

b. $x - 1 = x - c$ implies $c = 1$. Then

$f(1) = 3(1)^4 + 7(1)^2 - (1) - 2 = 3 + 7 - 1 - 2 = 7 \neq 0$.

Therefore, $x - 1$ is **NOT** a factor of $f(x) = 3x^4 + 7x^2 - x - 2$

Example 6: In each of the following find a number k such that:

a) $x + 2$ is a factor of $3x^4 - 8x^2 - kx + 6$.

b) $3x - 2$ is a factor of $6x^3 - 4x^2 + 2kx - k - 3$.

Solution:

a) Let $f(x) = 3x^4 - 8x^2 - kx + 6$ then $f(2) = 0$.

$$f(-2) = 3(-2)^4 - 8(-2)^2 - k(-2) + 6 = 3(16) - 8(4) + 2k + 6 = 0$$

$\Rightarrow$ 48 – 32 + 2k + 6 = 22 + 2k = 0

$\Rightarrow$ k = -11

b) $3x - 2 = ax + b \Rightarrow -\frac{b}{a} = \frac{2}{3}$.

Let $f(x) = 6x^3 - 4x^2 + 2kx - k - 3$. Then

$$f\left(\frac{2}{3}\right) = 0 \Rightarrow 6\left(\frac{2}{3}\right)^3 - 4\left(\frac{2}{3}\right)^2 + 2k\left(\frac{2}{3}\right) - k - 3 = 0$$

$$\Rightarrow \frac{16}{9} - \frac{16}{9} + \frac{4}{3}k - k - 3 = 0 \Rightarrow \frac{1}{3}k = 3.$$

Therefore, $k = 9$

1.3. ZEROS OF A POLYNOMIAL FUNCTION

Definition 1.2: For a polynomial function P and a real number c, if $P(c) = 0$, then c is a **zero** of P.

For instance, for a polynomial function $f(x) = 2x^3 - x^2 + x - 2$,

$$f(1) = 2(1)^3 - 1^2 + 1 - 2 = 2 - 1 + 1 - 2 = 0$$

Therefore, $x = 1$ is a zero of f.

1.3.1 Zeros and Their Multiplicities

Definition 1.3: If $f(x-c)^k$ is a factor of f(x), but $(x-c)^{k+1}$ is not, then c is said to be a zero of multiplicity k of f .

Example 7: Given that -1 and 2 are zeros of $f(x) = x^4 + x^3 - 3x^2 - 5x - 2$, determine their multiplicity.

Solution: By the factor theorem (x+1) and (x-2) are factors of f(x) hence f(x) can be divided by $(x+1)(x-2) = x^2 - x - 2$ gives;

$$\begin{aligned} f(x) &= (x^2 - x - 2)(x^2 + 2x + 1) \\ &= (x+1)(x-2)(x+1)^2 \\ &= (x+1)^3(x-2) \end{aligned}$$

$\therefore -1$ is a zero of multiplicity 3 and 2 is a zero of multiplicity 1

1.4. GRAPHS OF POLYNOMIAL FUNCTIONS

Properties of $f(x) = ax + b,\ a \neq 0$ graph

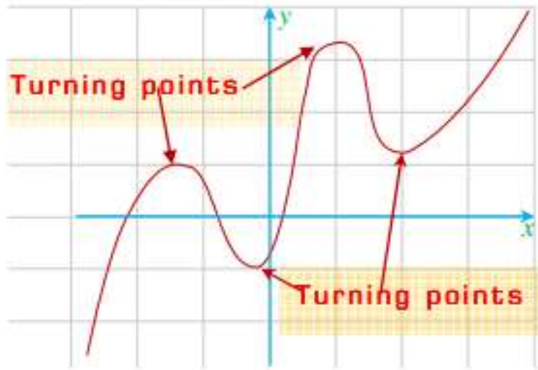
- its graph is a straight line
- the domain of f is real number
- the range of f is real number
- $x -$ intercept of f is $\left(\frac{-b}{a}, 0\right)$
- $y -$intercept of f is (0,b)
- slope of the graph is a.
- if $a > 0$, then the function is increasing.
- if $a < 0$, then the function is decreasing.

Graph of quadratic function $f(x) = ax^2 + bx + c$, $a, b,$ and c $\in R$, $a \neq 0$

- ✓ Graph of quadratic function is a curve known as parabola
- ✓ If $a > 0$ the parabola opens up ward
- ✓ If $a < 0$ the parabola opens down ward
- ✓ Vertex of the parabola (turning point) $v = \left(\frac{-b}{2a}, \frac{4ac-b^2}{4a}\right)$
- ✓ The domain of the function is real number
- ✓ The range of the function is $y \geq \frac{4ac-b^2}{4a}$, $a > 0$, $y \leq \frac{4ac-b^2}{4a}$, $a < 0$

Note: The graph of polynomial function of degree n meets the x – axis at most n times.

- Every polynomial function of degree n has at most n zeros.
- The graph of polynomial function f(x) = $a_n x^n + a_{n-1}x^{n-1} + ...a_1 x + a_0$ eventually rises or falls



Note:

1.

The graph of a polynomail function with leading term $a_n x^n$ has the following right and left beahviour:		
	$n-$even	$n-$odd
$a_n > 0$	Up to left and up to right	Down to left and up to right
$a_n < 0$	Downto left and down to right	Up to left and down to right

2. If the multiplicity of the root c is an **odd** number, then the graph of the function **crosses** the $x-$ axis at $x = c$.
3. If the multiplicity of the root c is an **even** number, then the graph of the function **touches**(**but does not cross**) the $x-$ axis at $x = c$.

PRACTICE QUESTIONS ON UNIT 1

CHOOSE THE BEST ANSWER FROM THE GIVEN ALTERNATIVES

1. Which of the following is **NOT** a polynomial?

A. $P(x) = |x-2| + 4$

B. $P(x) = (-3x)\left(1 - \frac{2x}{5}\right)$

C. $P(x) = \frac{x^3 - 5x^2 + x - 5}{x^2 + 1}$

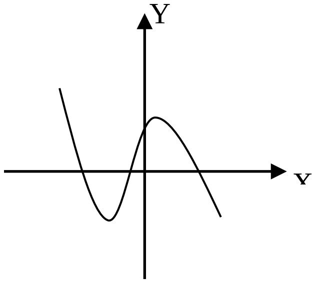
D. $P(x) = \frac{x^{50}}{3} + \sqrt{17}x^5 + \pi$

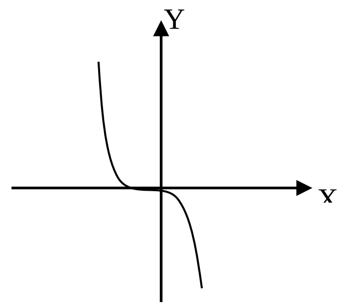
2. When divide the polynomial $f(x) = 3x^3 + 2x^2 - 19x + 6$ by $x - 1$, the remainder that you will get is__________

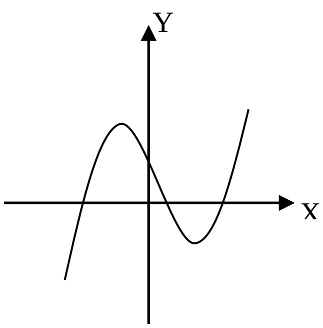
A. 0 B. -8 C. 24 D. 8

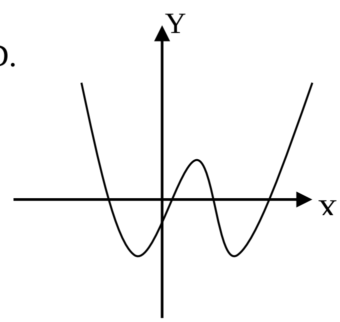
3. If $f(x)$ a polynomial of degree is n, where $n \geq 1$ and if c any real number, then which one of the following statements is true?
   A. If $f(c)=0$ and $n=1$, then $f(x)=k(x-c)$, for some non-zero real number k.
   B. If $f(c)=0$ and $n=2$, then $f(x)=(x-c)q(x)+r(x)$, where both $q(x)$ and $r(x)$ are polynomials of degree 1.
   C. If $f(c)=0$ and $n>1$, then $f(x)=(x-c)q(x)+r(x)$, where $q(x)$ and $r(x)$ are polynomials and the degree of $r(x)$ is 1.
   D. If $f(c)=0$ and $n>2$, then $f(x)=(x-c)q(x)$, where $q(x)$ is a polynomial of degree 1.
4. What number must be added to x^3+5x^2+6x+4 so that $x+1$ is a factor?
   A. 2 B. -19 C. 19 D. -2
5. Which of the following function is neither even nor odd?
   A. $k(x)=x^5$ C. $h(x)=2x^2-3|x|$
   B. $g(x)=x-1$ D. $f(x)=x^4$

6.Which of the following can be the graph of a polynomial function of degree three with positive leading coefficient?

A.


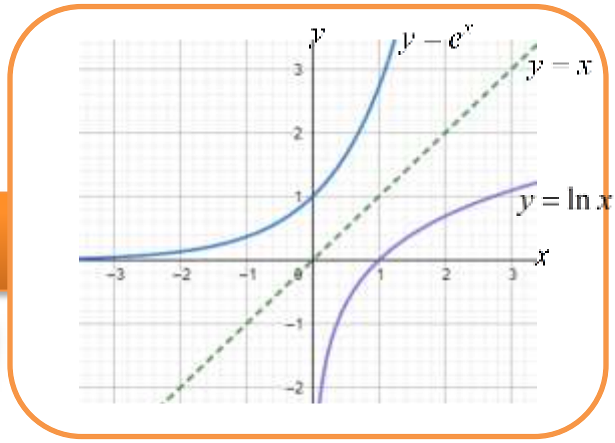
C.


B.


D.


7. Let $f(x)=ax^{10}+bx^5-1$. If $x+1$ is a factor of $f(x)$ and when $f(x)$ is divided by $x-1$ the remainder is 4, then a and b respectively equal to:
   A. -1, 1 B. 3, 2 C. 1, 5 D. 3, 1

Unit 2



EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Unit Outcomes:

After completing this unit, you should be able to:

- *understand the laws of exponents for real exponents.*
- *know specific facts about logarithms.*
- *know basic concepts about exponential and logarithmic functions.*

Main Contents:

2.1. Exponents.
2.2. Exponential Functions and Their Graphs.
2.3. Logarithms.
2.4. Logarithmic Functions and Their Graphs.
2.5. Equations Involving Exponents and Logarithms.

2.1. EXPONENTS

Laws of Exponents

NOTE: If a and b are non-zero real numbers and the exponents m and n are integers, then:

1. $a^m \cdot a^n = a^{m+n}$
2. $(ab)^n = a^n b^n$
3. $a^{\frac{m}{n}} = \left(a^m\right)^{\frac{1}{n}} = \left(a^{\frac{1}{n}}\right)^m$
4. $\left(a^m\right)^n = a^{mn}$
5. $a^{\frac{1}{n}} = \sqrt[n]{a}$, if n is odd, $a \in \mathbb{R}$ and if n is even, $a \geq 0$
6. $a^m = b^n$ if and only if $m = n,\ a \neq \pm 1$
7. $a^0 = 1$
8. $a^{-n} = \frac{1}{a^n},\ n > 0$
9. $\left(\frac{a}{b}\right)^{-n} = \frac{b^n}{a^n}$, for $n > 0$
10. $\frac{a^m}{a^n} = a^{m-n}$
11. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Example 1: Simplify each of the following:

a. $\dfrac{\left[\left(x^2\right)^3\right]^4}{x^5 \times x^{10} \times x^{-2}}$

b. $\left(\dfrac{a^{-\frac{1}{3}}b^{\frac{1}{2}}}{a^{-\frac{1}{4}}b^{\frac{1}{3}}}\right)^6$

Solution:

a. $\dfrac{\left[\left(x^2\right)^3\right]^4}{x^5 \times x^{10} \times x^{-2}} = \dfrac{\left[x^{2\times3}\right]^4}{x^{5+10+(-2)}} = \dfrac{\left[x^6\right]^4}{x^{13}} = \dfrac{x^{6\times4}}{x^{13}} = x^{24-13} = x^{11}$

b. $\left(\dfrac{a^{-\frac{1}{3}}b^{\frac{1}{2}}}{a^{-\frac{1}{4}}b^{\frac{1}{3}}}\right)^6 = \left(a^{-\frac{1}{3}+\frac{1}{4}}b^{\frac{1}{2}-\frac{1}{3}}\right)^6 = \left(a^{-\frac{1}{3}+\frac{1}{4}}b^{\frac{1}{2}-\frac{1}{3}}\right)^6 = \dfrac{b}{\sqrt{a}}$, for $a \geq 0$ and $b > 0$.

2.2. THE EXPONENTIAL FUNCTIONS AND THEIR GRAPHS

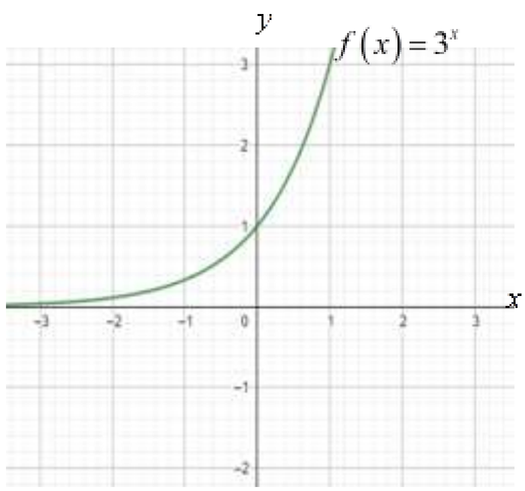
Definition: The exponential function f with the base b is defined by $f(x) = b^x$ where $b > 0,\ b \neq 1$ and x is any real number.

In this section you will draw graphs and investigate the major properties of functions ofthe form $f(x) = 2^x, g(x) = \left(\frac{1}{2}\right)^x,\ h(x) = \left(\frac{2}{3}\right)^x,\ k(x) = 3^{5x}$, etc.

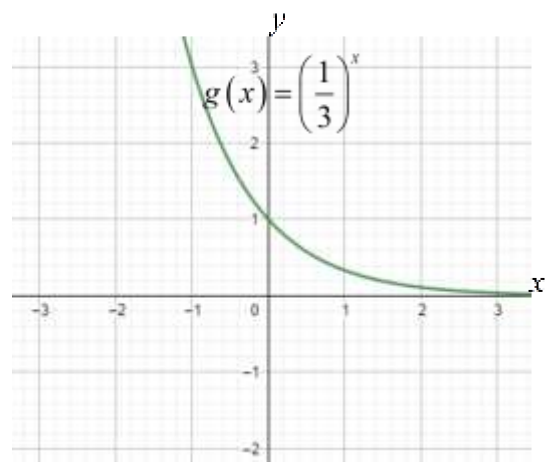
Example 2: Draw the graphs of $f(x)=3^x$ and $g(x)=\left(\frac{1}{3}\right)^x$.

Solution: We begin by calculating values of $f(x)=3^x$ and $g(x)=\left(\frac{1}{3}\right)^x$ for integer values of x as shown in the following table.

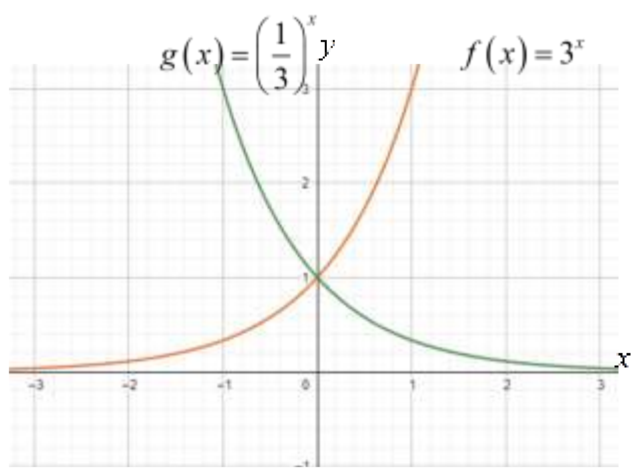
x	-3	-2	-1	0	1	2	3
$f(x)=3^x$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	8	27
$g(x)=\left(\frac{1}{3}\right)^x$	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$



The graph of $f(x)=3^x$



The graph of $g(x)=\left(\frac{1}{3}\right)^x$

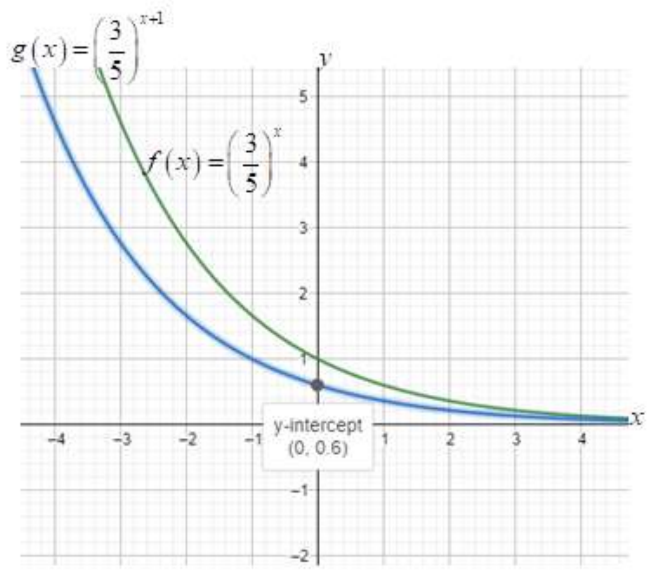


Graphs of $f(x)=3^x$ and $g(x)=\left(\frac{1}{3}\right)^x$ drawn using the same coordinate axes

Example 3: Sketch the graphs of $f(x)=\left(\frac{3}{5}\right)^x$ and $f(x)=\left(\frac{3}{5}\right)^{x+1}$.

Solution:

x	-3	-2	-1	0	1	2	3
$f(x)=\left(\frac{3}{5}\right)^x$	$\frac{125}{27}$	$\frac{25}{9}$	$\frac{5}{3}$	1	$\frac{3}{5}$	$\frac{9}{25}$	$\frac{27}{125}$
$g(x)=\left(\frac{3}{5}\right)^{x+1}$	$\frac{25}{9}$	$\frac{5}{3}$	1	$\frac{3}{5}$	$\frac{9}{25}$	$\frac{27}{125}$	$\frac{81}{625}$



Graphs of $g(x)=\left(\frac{3}{5}\right)^x$ and $g(x)=\left(\frac{3}{5}\right)^{x+1}$ drawn using the same coordinate axes

Note: Basic properties

The graph of $f(x)=b^x$, $b>1$ has the following basic properties:

1. The domain is the set of all real numbers.
2. The range is the set of all positive real numbers.
3. The graph includes the point $(0, 1)$, i.e. the $y-$intercept is 1.
4. The function is increasing.
5. The values of the function are greater than 1 for $x>0$ and between 0 and 1 for $x<0$.
6. The graph approaches the $x-$as an asymptote on the left and increases without bound on the right.

The graph of $f(x) = b^x$, $0 < b < 1$ has the following basic properties:

1. The domain is the set of all real numbers.
2. The range is the set of all positive real numbers.
3. The graph includes the point $(0, 1)$, i.e. the $y-$intercept is 1.
4. The function is decreasing.
5. The values of the function are greater than 1 for $x < 0$ and between 0 and 1 for $x > 0$.
6. The graph approaches the $x-$as an asymptote on the right and increases without bound on the left.

Theorem 2.1 Let $a > 0$ and $x, y \in R$.

a. If $a > 1$, then :
   i. $a^x > a^y$ if and only if $x > y$.
   ii. $a^x = a^y$ if and only if $x = y$.

b. If $0 < a < 1$, then :
   iii. $a^x < a^y$ if and only if $x > y$.
   iv. $a^x = a^y$ if and only if $x = y$.

Theorem 2.2 Let $a > 0$, $b > 0$ and $x \in R$.

i. If $x > 0$, then $a^x > b^x$ if and only if $a > b$.
ii. If $x < 0$, then $a^x > b^x$ if and only if $a < b$.
iii. If $x = 0$, then $a^x = b^x$.

2.3. LOGARITHMS

NOTE:

1. For a fixed positive number $b \neq 1$ and for each $a > 0$, $b^c = a$ if and only if $c = \log_b a$. The value of $\log_b a$ is the answer of the question "to what power must b be raised to produce a"?
2. The equations $y = \log_a x$ and $a^y = x$ are equivalent.

Example 4: Write an equivalent logarithmic statement for:

a. $8^{\frac{1}{3}} = 2$

b. $2^{-5} = \frac{1}{32}$

Solution:

a. From $8^{\frac{1}{3}} = 2$, we have $\log_8 2 = \frac{1}{3}$.

b. Since $2^{-5} = \frac{1}{32}$, $\log_2 \frac{1}{32} = -5$.

Laws of Logarithms

If a, b, x and y are positive numbers and $a, b \neq 1$ then:

1. $\log_b xy = \log_b x + \log_b y$ .... Product formula.
2. $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$ ....quotient formula.
3. $\log_b x^k = k \log_b x$, k is any real number............... Power formula.
4. $\log_b x = \frac{\log_a x}{\log_a b}$ .... Change of base formula.
5. $b^{\log_b c} = c$
6. $\log_a a^x = x$
7. $\log_{\frac{1}{b}} a = -\log_b a$
8. $\log_a x = \frac{1}{\log_x a}$, for $x > 0, x \neq 1$, $a > 0$ and $a \neq 1$ Thus $(\log_a x)\left(\frac{1}{\log_x a}\right) = 1$
9. $\log_{a^k} x = \frac{1}{k}\log_a x$, for $a > 0$ and $a \neq 1$
10. $\log_{a^{\frac{1}{k}}} x = k \log_a x$, for $a > 0$ and $a \neq 1$.

Useful identities for logarithms

1. $\log_a a = 1$, for all $a > 0$
2. $\log_a 1 = 0$, for all $a > 0$

2.4. THE LOGARITHMIC FUNCTIONS AND THEIR GRAPHS

2.4.1. Graphs of logarithmic functions

Example 5: Draw the graph of each of the following using:

i. Different coordinate system
ii. The same coordinate system

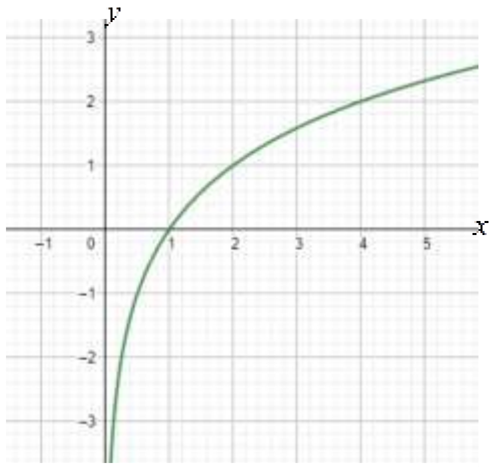
a. $f(x) = \log_2 x$ b. $g(x) = \log_{\frac{1}{2}} x$

Solution:

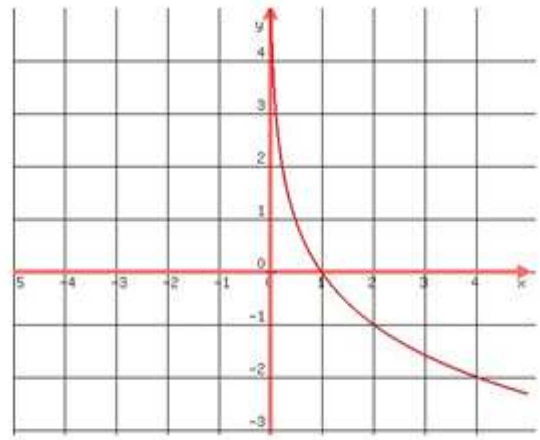
We begin by calculating the values of $f(x) = \log_2 x$ and $g(x) = \log_{\frac{1}{2}} x$ for positive values of x.

x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
$f(x) = \log_2 x$	–4	–3	–2	–1	0	1	2	3	4
$g(x) = \log_{\frac{1}{2}} x$	4	3	2	1	0	–1	–2	–3	–4

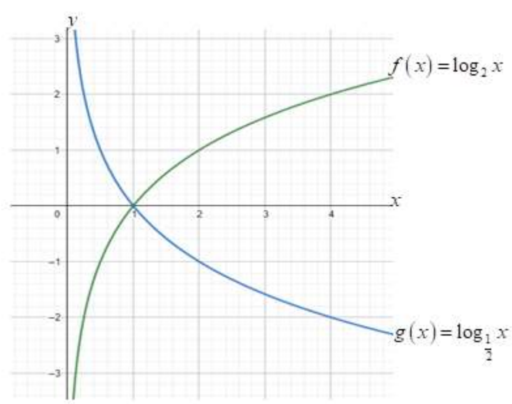
Then we plot the points corresponding to the pairs we have found and connect the points with smooth curves to obtain the graphs as shown below.



The graph of $f(x) = \log_2 x$



The graph of $f(x) = \log_{\frac{1}{2}} x$



Example 6: Draw the graph of each of the following using:

a. Different coordinate system
ii. The same coordinate system

a. $h(x) = \log_3 x$

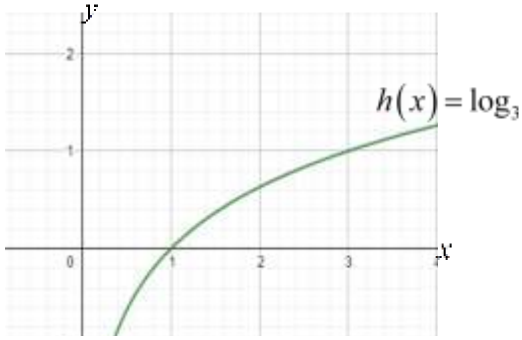
b. $k(x) = \log_{\frac{3}{2}} x$

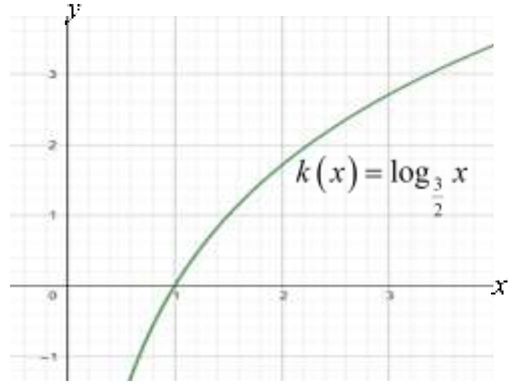
Solution:

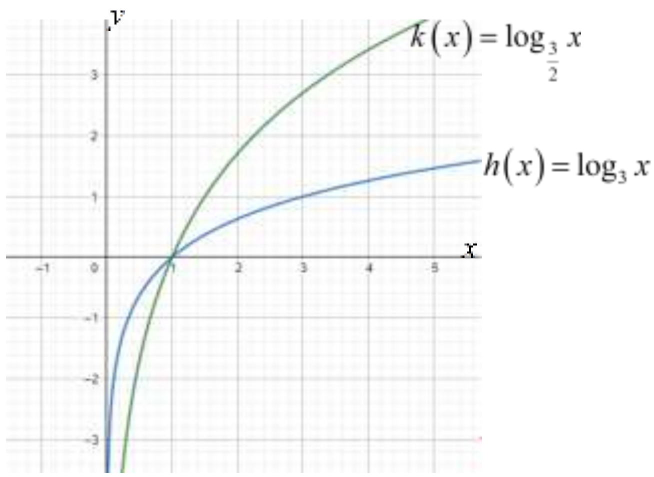
We begin by calculating the values of $f(x) = \log_2 x$ and $g(x) = \log_{\frac{1}{2}} x$ for positive values of x.

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9		$\frac{4}{9}$	$\frac{2}{3}$	1	$\frac{3}{2}$	$\frac{9}{4}$
$h(x) = \log_3 x$	-2	-1	0	1	2	$k(x) = \log_{\frac{3}{2}} x$	-2	-1	0	1	2

Then we plot the points corresponding to the pairs we have found and connect the points with smooth curves to obtain the graphs as shown below.







In general, the graph of $f(x) = \log_b x$, for any $b > 1$ has the following shape

Basic properties of the graph of $\mathbf{y = \log_b x, (b > 1)}$

1. The domain is the set of positive real numbers
2. The range is the set of all real numbers.
3. The graph includes the points $(1, 0)$ i.e. the $x-$intercept of the graph is 1.
4. The value of the function increases as x increases.
5. The $y-$axis is a vertical asymptote of the graph.
6. The values of the function are negative for $0 < x < 1$ and they are positive for $x > 1$.

Example 7: Draw the graph of each of the following using:

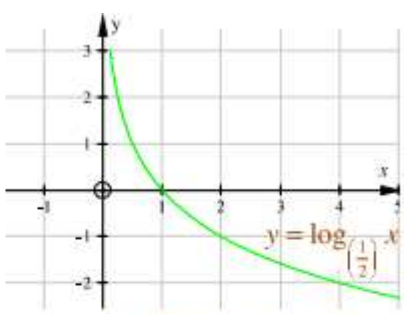
i. Different coordinate system
ii. The same coordinate system

a. $h(x) = \log_{\frac{1}{2}} x$ b. $k(x) = \log_{\frac{2}{3}} x$

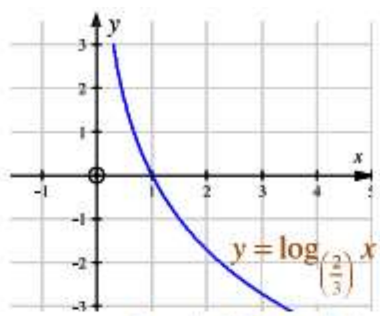
Solution: calculate the values of the given functions for some values of x as shown in the table below. The plot the corresponding points on the co-ordinate system.

x	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
$h(x) = \log_{\frac{1}{2}} x$	−3	−2	−1	0	1	2	3

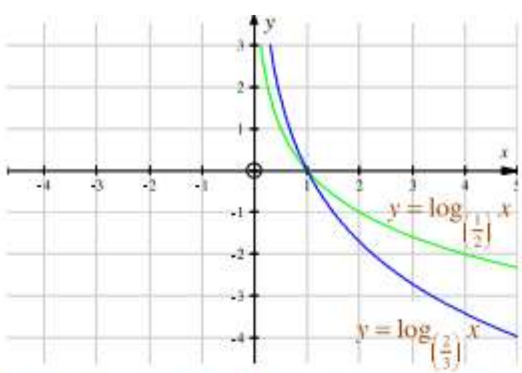
x	$\frac{27}{8}$	$\frac{9}{4}$	$\frac{3}{2}$	1	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{8}{27}$
$k(x) = \log_{\frac{2}{3}} x$	−3	−2	−1	0	1	2	3



Graph of $h(x) = \log_{\left(\frac{1}{2}\right)} x$

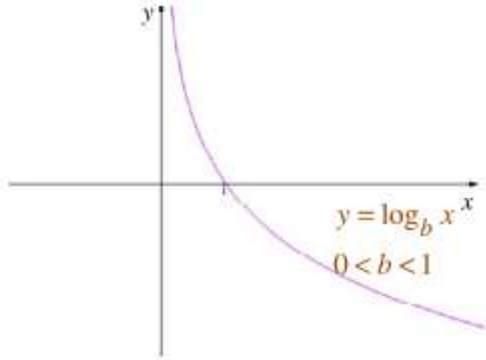


Graph of $k(x) = \log_{\left(\frac{2}{3}\right)} x$



Graphs of $y = \log_{\left(\frac{1}{2}\right)} x$ and $y = \log_{\left(\frac{2}{3}\right)} x$ drawn using the same coordinate axes

In general, the graph of $f(x) = \log_b x$ for $0 < b < 1$ looks like the one given below .



Basic properties of the graph of $y = \log_b x, (0 < b < 1)$

1. The domain is the set of positive real numbers
2. The range is the set of all real numbers.
3. The graph includes the points $(1, 0)$ i.e. the $x-$intercept of the graph is 1.
4. The value of the function decreases as x increases.
5. The $y-$axis is a vertical asymptote of the graph.
6. The values of the function are positive for $0 < x < 1$ and they are negative for $x > 1$.

2.5. EQUATIONS INVOLVING EXPONENTS AND LOGARITHMS

2.5.1.Solving Exponential Equations

NOTE: property of equality of exponential equations

For $b > 0,\ b \neq 1,\ x$ and y real numbers,

1. $b^x = b^y$, if and only if $x = y$.
2. $a^x = b^x, (x \neq 0)$, if and only if $a = b$.

Examples 8: Solve the following exponential equations

a. $\left(\frac{2}{3}\right)^{2x+1} = \left(\frac{9}{4}\right)^x$

b. $4^x = \left(\frac{1}{2}\right)^{x-3}$

c. $7^{x^2+x} = 49$

d. $\frac{10^{x+2}}{2^{x-3}} = 5^{x+1}$

Solution:

a. $\left(\frac{2}{3}\right)^{2x+1} = \left(\frac{9}{4}\right)^x \Rightarrow \left(\frac{2}{3}\right)^{2x+1} = \left(\frac{3}{2}\right)^{2x} = \left(\frac{2}{3}\right)^{-2x}$

$$\Leftrightarrow 2x + 1 = -2x$$

$$\Rightarrow 2x+2x=-1$$
$$\Rightarrow x=-\frac{1}{4}$$

b. $4^x=\left(\frac{1}{2}\right)^{x-3} \Rightarrow 4^x=\left(2^{-1}\right)^{x-3}=2^{(-x+3)}$

$$\Rightarrow \left(2^2\right)^x=2^{(-x+3)}$$
$$\Rightarrow 2^{2x}=2^{(-x+3)}$$
$$\Leftrightarrow 2x=-x+3$$
$$\Rightarrow 2x=-x+3$$
$$\Rightarrow x=1$$

c. $7^{x^2+x}=49 \Rightarrow 7^{x^2+x}=7^2$

$$\Leftrightarrow x^2+x=2$$
$$\Rightarrow x^2+x-2=0$$
$$\Rightarrow (x-1)(x+2)=0$$
$$\Rightarrow x=1 \text{ or } x=-2$$

d. $\frac{10^{x+2}}{2^{x-3}}=5^{x+1} \Rightarrow \frac{10^x \times 10^2}{2^x \times 2^{-3}}=5^x \times 5$

$$\Rightarrow \frac{100\left(10^x\right)}{\frac{2^x}{8}}=5^x \times 5$$
$$\Rightarrow \frac{160\left(10^x\right)}{2^x}=5^x$$
$$\Rightarrow 160\left(10^x\right)=10^x$$
$$\Rightarrow 160=1^x \Rightarrow \text{No solution. Why?}$$

2.5.2. Solving Logarithmic Equations

Logarithmic equations can be solved by changing them to equivalent exponential form. However, it is necessary first to state the **universe** and to use the basic properties of logarithms to simplify one side of an equation.

A **universe** is the largest set in $\mathbb{R}$ for which the given expression is defined.

Examples 9: Sate the universe and solve the following equations:

a. $\log_2(x-3)=5$

b. $\log(x+3)+\log x=1$

c. $\log_3(x+1)-\log_3(x+3)=1$

d. $\log\left(x^2-121\right)-\log(x+11)=1$

Solution:

a. First find domain of $\log_2(x-3)=5$ i.e. $x-3>0 \Rightarrow x>3$

So the domain is $x \in (3, \infty)$.

Now by changing $\log_2(x-3)=5$ to exponential equation we get $2^5 = x-3$

$$\Rightarrow 32 = x-3$$
$$\Rightarrow x = 35$$

Check! For $x=35$, $\log_2(x-3)=\log_2(35-3)=\log_2 32 = 5$ is true.

b. $\log(x+3)+\log x$ is valid for $x+3>0$ and $x>0$ i.e. $x>-3$ and $x>0$

Therefore the universe $\text{U}=(0, \infty)$

$\log(x+3)+\log x = 1 \Rightarrow \log(x+3)(x)=1$ by the law $\log x + \log y = \log xy$

$$\Rightarrow \log(x^2+3x)=1$$
$$\Rightarrow 10^1 = x^2+3x$$
$$\Rightarrow x^2+3x-10=0$$
$$\Rightarrow (x-2)(x+5)=0$$
$$\Rightarrow x-2=0 \text{ or } x+5=0$$
$$\Rightarrow x=2 \text{ or } x=-5 \text{ but } x=-5 \notin \text{Domain}$$

Therefore the only solution set is $\{2\}$.

c. $\log_3(x+1)-\log_3(x+3)$ is valid for $x+1>0$ and $x+3>0$ i.e. $x>-1$ and $x>-3$

Therefore the universe $\text{U}=(-1, \infty)$

$\log_3(x+1)-\log_3(x+3)=1 \Rightarrow \log_3\left(\frac{x+1}{x+3}\right)=1$ since $\log x - \log y = \log\left(\frac{x}{y}\right)$

$$\Rightarrow 3^1 = \frac{x+1}{x+3}$$
$$\Rightarrow x+1 = 3(x+3) = 3x+9$$

Therefore $-2x=8$ and $x=-4$ but -4 is **not** in the universe.

Hence there no x that satisfies the given equation and the solution set is empty set.

d. $\log(x^2-121)-\log(x+11)$ is valid for $x^2-121>0$ and $x+11>0$ i.e. $|x|>11$ and $x>-11$

Therefore the universe $\text{U}=(11, \infty)$

$\log(x^2-121)-\log(x+11)=1 \Rightarrow \log\left(\frac{x^2-121}{x+11}\right)=1$ since $\log x - \log y = \log\left(\frac{x}{y}\right)$

$$\Rightarrow 10^1 = \frac{x^2-121}{x+11}$$
$$\Rightarrow x^2-121 = 10(x+11) = 10x+110$$
$$\Rightarrow x^2-10x-231=0$$

$\Rightarrow (x-21)(x+11)=0$

$\Rightarrow x-21=0$ or $x+11=0$

$\Rightarrow x=21$ or $x=-11$ but -11 is **not** in the universe.

Therefore the solution set is only $\{21\}$.

PRACTICE QUESTIONS ON UNIT 2

CHOOSE THE BEST ANSWER FROM THE GIVEN ALTERNATIVES

1. Which of the following is **NOT** true?

   A. $\log_{\frac{1}{a}} x > \log_{\frac{1}{a}} y$, for $a>1, x>0, y>0$ and $x>y$

   B. $\log_a x > \log_a y$, for $a>1, x>0, y>0$ and $x>y$

   C. $\log_{\frac{1}{3}} \sqrt[3]{9} > \log_{\frac{1}{3}} 27$

   D. $\log_{3.5} 7.6 = 0.3357$

2. Which of the following is **NOT** true?

   A. If $\log_2 4x + \log_4 2x = 2x - \frac{1}{2}$, then $x=1$

   B. $(\log_b 4)(\log_8 b^3 = 2)$, for each $b>1$

   C. $2^{\log_2 3} = 3$

   D. $\log_2(\log_4(\log_8 64)) = -1$

3. If $0<x<y<1$, then which of the following is **NOT** true?

   A. $x^{-t} > 1$, for $t<0$.

   B. $\left(\frac{x}{y}\right)^t > 1$, for $t<0$.

   C. $(xy)^t > 1$, for $t<0$.

   D. $\left(\frac{y}{x}\right)^t < 1$, for $t<0$.

4. If $\log_2 9 = x$ and $2^y = \frac{\left(2^{\sqrt{3}}\right)^2 \sqrt{12}}{\sqrt[3]{16}}$, then y equals:

   A. $\frac{32+3x}{12}$ B. $\frac{64+16x}{12}$ C. $\frac{26+3x}{6}$ D. None of the above

5. If $\log 3.54 = 0.549$, then the mantissa of $\log 35{,}400$ is

   A. 0.549 B. 4.549 C. 0.051 D. 4

6. From the following statements, which one is true about an exponential expression a^x?

   A. If $a>0$, then for a positive real number x the value of $a^{\sqrt{x}}$ cannot be a real number.

   B. If $a>0$ and $x, y \in \mathbb{Q}$, then $a^x \in \mathbb{R}$ is found between a^m and a^n, where m and n are integers and $m<x<n$.

   C. If $a>0$ and $x = \frac{m}{n}$ is a positive rational number with $n>0$, then there is $b \in \mathbb{R}$, such that $b = a^x$ and $a^n = b^m$.

   D. For $a \neq 0$ and $x, y \in \mathbb{Q}$, if $a^x = b^y$ and $x \neq y$, then a can be any real number that is different fro 1.

7. Given $\log 5 = b,$ then $\log_5 4$ is equal to

   A. $2\left(\frac{b}{1-b}\right)$ B. $2\left(\frac{b}{b-1}\right)$ C. $2\left(\frac{1}{b}-1\right)$ D. $2\left(1-\frac{1}{b}\right)$

8. Which of the following is the solution of the equation $\log_2(x-2)+\log_2 x = 3$?

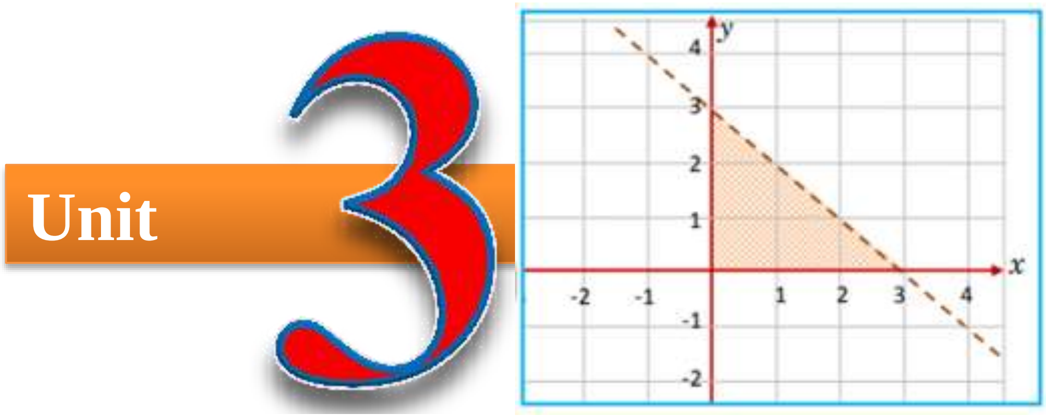
   A. -2 and 4 B. 4 C. $1+\sqrt{7}$ D. $1-\sqrt{7}$ and $1+\sqrt{7}$

9. The solution set of the equation $\log_x 5 + \frac{1}{2}\log_{\sqrt{5}} x + 8\log_{25}\left(\frac{1}{x}\right) = 2$ is:

   A. $\{5\}$ B. $\left\{\frac{1}{5}, \sqrt[3]{5}\right\}$ C. $\{125\}$ D. $\left\{\frac{1}{3}, -1\right\}$

10. Suppose $\log_3 2 = a$ and $\log_3 5 = b.$ Then $\log_3 0.0002$ is equal to:

   A. $-3a-4b$ B. $3a+4b$ C. $a-b$ D. $b-3a$



SOLVING INEQUALITIES

Unit Outcomes:

After completing this unit, you should be able to:

- *Know and apply methods and procedure in solving problems on inequalities involving absolute value.*
- *Solve quadratic inequalities.*

Main Contents:

3.1. Solving Linear inequalities in one variuable

3.2. Inequalities involving absolute value

3.3. Quadratic inequalities

3.1. SOLVING LINEAR INEQUALITIES IN ONE VARIABLE

We know that a linear equation in one variable can be expressed as $ax + b = 0$. A linear inequality in one variable can be written in one of the following forms $ax+b<0$, $ax+b>0$, $ax+b\leq 0$ or $ax+b\geq 0$ in each form $a \neq 0$.

Properties of inequalities:

For any three real numbers $a,\ b$ and c:

1. **The addition property of inequalities**
   - if $a < b$, then $a + c < b + c$.
   - if $a < b$, then $a - c < b - c$.
2. **The positive multiplication property of inequalities**
   - if $a < b$ *and c is positive*, then $a \times c < b \times c$.
   - if $a < b$ *and c is positive*, then $\frac{a}{c} < \frac{b}{c}$.
3. **The negative multiplication property of inequalities**
   - if $a < b$ *and c is negative*, then $a \times c > b \times c$.
   - if $a < b$ *and c is negative*, then $\frac{a}{c} > \frac{b}{c}$.

Example 1: Solve the following inequalities:

a. $4(x + 1) + 2 \geq 3x + 6$

b. $8x + 3 > 3(2x + 1) + x + 5$

c. $2x - 11 < -3(x + 2)$

Solution:

a. $4(x + 1) + 2 \geq 3x + 6$

$4x + 4 + 2 \geq 3x + 6$

$4x + 6 \geq 3x + 6$

$4x - 3x \geq 6 - 6$

$x \geq 0$

Therefore $s.s = \{x: x \geq 0\} = [0, \infty)$

b. $8x + 3 > 3(2x + 1) + x + 5$

$8x + 3 > 6x + 3 + x + 5$

$8x + 3 > 7x + 8$

$8x - 7x > 8 - 3$

$x > 5$

Hence the $s.s = \{x: x > 5\} = (5, \infty)$

c. $2x - 11 < -3(x + 2)$

$2x - 11 < -3x - 6$

$2x + 3x < -6 + 11$

$5x < 5$

$x < 1$

Therefore $s.s = \{x: x < 1\} = (-\infty, 1)$

3.2. INEQUALITIES INVOLVING ABSOLUTE VALUE

Definition 3.1:
If x is a real number, then the absolute value of x, denoted by $|x|$, is defined by

$$|x| = \begin{cases} x, if\ x \geq 0 \\ -x, if\ x < 0 \end{cases}$$

Example 2:

a. $|21| = 21$ because $21 > 0$

b. $\left|\frac{-2}{3}\right| = -\left(\frac{-2}{3}\right) = \frac{2}{3}$ because $-\frac{2}{3} < 0$

Theorem 3.1: Solution of the equation $|x| = a$

For any real number a, the equation $|x| = a$ has:

i. Two solutions $x = a$ and $x = -a, if\ a > 0$

ii. One solution ,$x = 0, if\ a = 0$ and

iii. No solution, if $a < 0$.

Example 3: Solve each of the following absolute value equation

a. $|2x + 1| = x + 5$

b. $3|x + 4| - 2 = 7$

Solution:

a. $2x + 1 = x + 5$ or $2x + 1 = -(x + 5)$

$\Rightarrow 2x - x = 5 - 1$ or $2x + 1 = -x - 5$

$\Rightarrow x = 4$ or $2x + x = -5 - 1$

$\Rightarrow x = 4$ or $3x = -6 \Rightarrow x = -2$

Therefore $s.s = \{4, -2\}$

b. $3|x + 4| - 2 = 7$

$\Rightarrow 3|x + 4| - 2 + 2 = 7 + 2$ . . . .add 2 to each side

$\Rightarrow 3|x + 4| = 9$

$\Rightarrow \frac{3|x+4|}{3} = \frac{9}{3}$ . . . . .divide both sides by 3

$\Rightarrow |x + 4| = 3$

$\Rightarrow x + 4 = 3$ or $x + 4 = -3$

$\Rightarrow x = 3 - 4$ or $x = -3 - 4$

$\Rightarrow x = -1$ or $x = -7$

Therefore $s.s = \{-1, -7\}$

3.3. QUADRATIC INEQUALITIES

In grade 9 mathematics, you have learned how to solve quadratic equations of the form $ax^2 + bx + c = 0$, $a \neq 0$ and a, b and $c \in \mathbb{R}$.

Definition 3.2:
An inequality that can be reduced to any one of the following forms
$ax^2 + bx + c \leq 0$ or $ax^2 + bx + c < 0$ or
$ax^2 + bx + c \geq$ or $ax^2 + bx + c > 0$
where a, b and c are constants and $a \neq 0$, is called a quadratic inequality.

Solving quadratic inequalities using product properties

Product properties:

1. $m \times n > 0$ *if and only if*
   i. $m > 0$ *and* $n > 0$ or ii. $m < 0$ *and* $n < 0$
2. $m \times n < 0$, *if and only if*
   i. $m > 0$ *and* $n < 0$ or ii. $m < 0$ *and* $n > 0$

Examples 4: Solve each of the following inequalities

a. $(2x + 7)(3x - 2) > 0$

b. $3x^2 + 4x + 1 \geq 0$

Solution:

a. $(2x + 7)(3x - 2) > 0$

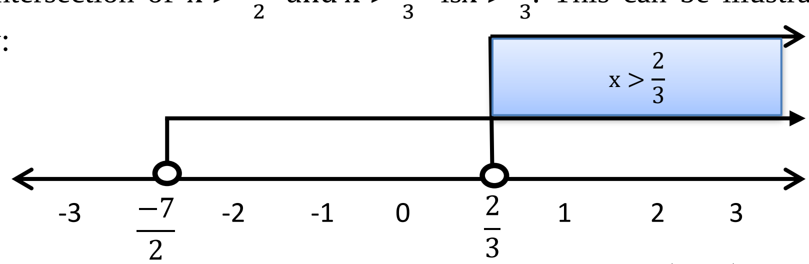
By product property, $(2x + 7)(3x - 2) > 0$ is positive if either both factors are positive or both factors are negative.

Now, consider the following cases:

Case 1: When both factors are positive

$\Rightarrow 2x + 7 > 0$ and $3x - 2 > 0$
$\Rightarrow 2x > -7$ and $3x > 2$
$\Rightarrow x > \frac{-7}{2}$ and $x > \frac{2}{3}$

The intersection of $x > \frac{-7}{2}$ and $x > \frac{2}{3}$ is $x > \frac{2}{3}$. This can be illustrated on the number line as shown below:



The solution set for this first case is $s.s_1 = \{x: x > \frac{2}{3}\} = \left(\frac{2}{3}, \infty\right)$

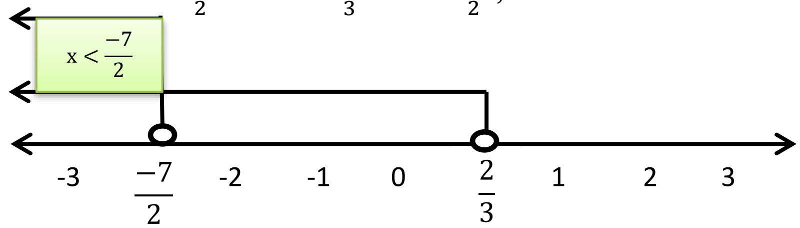
Case 2: When both factors are negative

$\Rightarrow 2x + 7 < 0$ and $3x - 2 < 0$

$\Rightarrow 2x < -7$ and $3x < 2$

$\Rightarrow x < \frac{-7}{2}$ and $x < \frac{2}{3}$

The intersection of $x < \frac{-7}{2}$ and $x < \frac{2}{3}$ is $x < \frac{-7}{2}$,this can be illustrated on the number line as below:



The solution set for the second case is $s.s_2 = \{x: x < -\frac{7}{2}\} = \left(-\infty, -\frac{7}{2}\right)$

Therefore, the solution set of $(2x + 7)(3x - 2) > 0$ is $s.s = s.s_1 \cup s.s_2 = \left(-\infty, -\frac{7}{2}\right) \cup \left(\frac{2}{3}, \infty\right)$

b. $3x^2 + 4x + 1 \geq 0$

First factorize $3x^2 + 4x + 1$ as $(3x + 1)(x + 1)$. So $3x^2 + 4x + 1 = (3x+1)(x+1) \geq 0$.

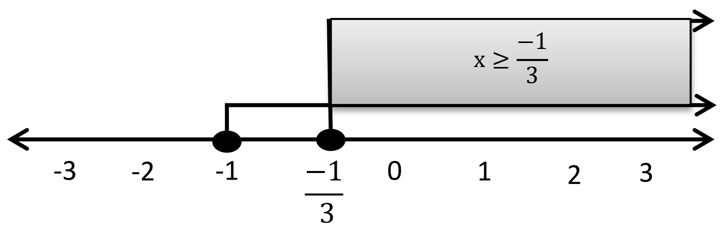
Case 1: When both factors are positive

$\Rightarrow 3x + 1 \geq 0$ and $x + 1 \geq 0$

$\Rightarrow 3x \geq -1$ and $x \geq -1$

$\Rightarrow x \geq \frac{-1}{3}$ and $x \geq -1$

The intersection of $x \geq \frac{-1}{3}$ and $x \geq -1$ is $x \geq -\frac{1}{3}$.this can be illustrated on the number line as below.



The solution set for this first case ıs $s.s_1 = \{x: x \geq -\frac{1}{3}\} = [\frac{-1}{3}, \infty)$.

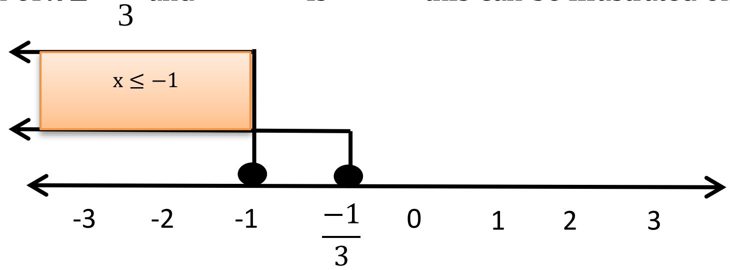
Case 2: When both factors are negative

$\Rightarrow 3x + 1 \leq 0$ and $x + 1 \leq 0$

$\Rightarrow 3x \leq -1$ and $x \leq -1$

$\Rightarrow x \leq \frac{-1}{3}$ and $x \leq -1$

The intersection of $x \leq -\frac{1}{3}$ and $x < -1$ is $x \leq -1$ this can be illustrated on the number line as below.



The solution set for the second case is $s.s_2 = \{x: x \leq -1\} = (-\infty, -1]$.

Therefore, the solution set of $3x^2 + 4x + 1 \geq 0$ is $s.s = s.s_1 \cup s.s_2 = (-\infty, -1] \cup [-\frac{1}{3}, \infty)$.

PRACTICE QUESTIONS ON UNIT 3

CHOOSE THE BEST ANSWER FROM THE GIVEN ALTERNATIVES

1. The solution of the inequality $|2x-5|>3$?

   A. $\{x:1\le x\le 4\}$ C. $\{x:x<1 \text{ or } x>4\}$
   B. $\{x:2\le x\le 8\}$ D. $\{x:x<2 \text{ or } x>8\}$

2. Which of the following is the solution set of the inequality $\frac{37-2x}{3}+x\le\frac{3x-8}{4}-9$?

   A. $\{x\in\Re : x\le 56\}$ C. ∅
   B. $\{x\in\Re : x\ge -56\}$ D. $\{x\in\Re : 56\le x\}$

3. The solution set of $-6<11x+3\le 3$ is

   A. $\left\{x:-\frac{9}{11}\le x<\frac{6}{11}\right\}$ C. $\left\{x:-\frac{9}{11}\le x\le 0\right\}$
   B. $\left\{x:-\frac{9}{11}\le x<0\right\}$ D. $\left\{x:-\frac{9}{11}\le x\le\frac{6}{11}\right\}$

4. The solution set of the inequality $x^2+7x+12\ge 0$ is:

   A. $\{x:-3\le x\le -4\}$ C. $\{x:x\le 3 \text{ or } x\ge 4\}$
   B. $\{x:3\le x\le 4\}$ D. $\{x:x\le -4 \text{ or } x\ge -3\}$

5. What is the solution set of the inequality $4x^2+4x+1>0$?

   A. $\left(-\frac{1}{2},\frac{1}{2}\right)$ B. $\mathbb{R}\setminus\left(-\frac{1}{2}\right)$ C. $\mathbb{R}$ D. $\left(-\infty,-\frac{1}{2}\right)\cup\left(\frac{1}{2},\infty\right)$

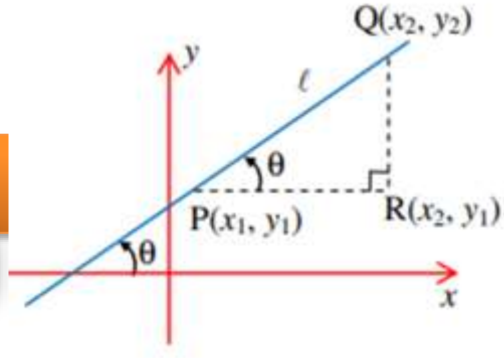
6. If $|3x+2|<1$, then x belongs to the interval:

   A. $\left(-1,-\frac{1}{3}\right)$ B. $(-\infty,-1)$ C. $\left[-1,-\frac{1}{3}\right]$ D. $\left(-\frac{1}{3},\infty\right)$

7. Which of the following is the least integral value of k such that $(k-2)x^2+8x+k+4\ge 0$ for all $x\in\mathbb{R}$?

   A. 5 B. 4 C. 3 D. k

Unit 4



COORDINATE GEOMETRY

Unit Outcomes:

After completing this unit, you should be able to:

- *apply the distance formula to find the distance between any two given points in the coordinate plane.*
- *formulate and apply the section formula to find a point that divides a given line segment in a given ratio.*
- *write different forms of equations of a line and understand related terms.*
- *describe parallel or perpendicular lines in terms of their slopes.*

Main Contents: (Practice on Questions at the end)

4.1. Division of a line segment

4.1. DIVISION OF A LINE SEGMENT

The point $R(x_0, y_0)$ dividing the line segment PQ internally in the ratio $m:n$ is given by
$R(x_0, y_0) = \left(\frac{nx_1+mx_2}{n+m}, \frac{ny_1+my_2}{n+m}\right)$.
This is called the **section formula**.

Example 1: Find the coordinate of a point R that divides the line segments with end-points $A(-3,3)\ and\ B(12,-7)$ in the ratio$2:3$.

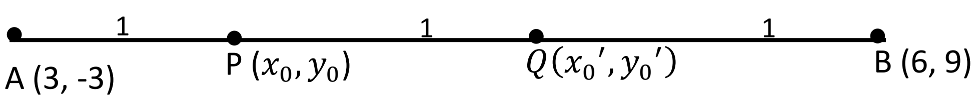
Solution: Put$(x_1, y_1) = (-3,3)$, $(x_2, y_2) = (12,-7)$, $m = 2\ and\ n = 3$, using the section formula ,you have:

$R(x_0, y_0) = \left(\frac{nx_1+mx_2}{n+m}, \frac{ny_1+my_2}{n+m}\right) = \left(\frac{3\times-3+2\times12}{3+2}, \frac{3\times3+2\times-7}{3+2}\right) = \left(\frac{-9+24}{5}, \frac{9-14}{5}\right) = (3,-1)$

Therefore,$R\ is\ (3,-1)$

Example 2: A line segment has end- points $(3,-3)$ and $(6,9)$ and it is divided into three equal parts. Find the coordinate of the points that trisect the segment.

Solution: Let $P(x_0, y_o)\ and\ Q(x_0{}', y_0{}')$ be points which trisect the line segment joining the points $(3,-3)\ and\ (6,9)$



The first point $P(x_0, y_0)$ divides the line segment in the ratio $1:2$ and hence

$(x_1, y_1) = (3,-3), (x_2, y_2) = (6,9), m = 1\ and\ n = 2$

$P(x_0, y_0) = \left(\frac{nx_1+mx_2}{n+m}, \frac{ny_1+my_2}{n+m}\right) = \left(\frac{2\times3+1\times6}{2+1}, \frac{2\times-3+1\times9}{2+1}\right) = \left(\frac{12}{3}, \frac{3}{3}\right) = (4,1)$

Therefore the first point $P(x_0, y_0) = (4,1)$

The second point $Q(x_0{}', y_0{}')$ divides the line segment in the ratio$2:1$.

Thus

$(x_1, y_1) = (3,-3), (x_2, y_2) = (6,9), m = 2\ and\ n = 1$

$Q(x_0{}', y_0{}') = \left(\frac{nx_1+mx_2}{n+m}, \frac{ny_1+my_2}{n+m}\right) = \left(\frac{1\times3+2\times6}{2+1}, \frac{1\times-3+2\times9}{2+1}\right) = \left(\frac{15}{3}, \frac{15}{3}\right) = (5,5)$

Therefore the second point $Q(x_0{}', y_0{}') = (5,5)$

PRACTICE QUESTIONS ON UNIT 4

CHOOSE THE BEST ANSWER FROM THE GIVEN ALTERNATIVES

1. If line l_1 passes through the points $(5, x)$ and $(-1,3)$ and line l_2 contains the points $(x,6)$ and $(2,0)$, then the value of x for which the two lines are perpendicular is:

   A. $\frac{2}{5}$ B. $\frac{5}{2}$ C. 5 D. $\frac{1}{2}$

2. If a line passes through $(2,8)$ and $(-5,15)$, then what is the degree measure of the angle of inclination that this line makes with positive $x-$axis?

   A. 30^0 B. 45^0 C. 135^0 D. 225^0

3. If the line passing through points $(2,8)$ and $(-7, t+4)$ is parallel to the line passing through points $(1,t)$ and $(4,-2)$, then what is the value of t?

   A. $-\frac{1}{2}$ B. $\frac{5}{2}$ C. -5 D. 1

4. Which one of the following is the equation of a line that is perpendicular to the line with equation $2x+3y+4=0$?

   A. $3y-2x+4=0$ C. $3x-2y+4=0$

   B. $-3x-2y-4=0$ D. $2x+3y-4=0$

5. What are the co-ordinates of appoint that divides the line segment joining points A (2, 3) and B (5, -7) in the ratio 3:4?

   A. $\left(\frac{23}{7},\frac{9}{7}\right)$ B. $\left(\frac{2}{7},-\frac{9}{7}\right)$ C. $\left(-\frac{23}{7},-\frac{2}{7}\right)$ D. $\left(\frac{23}{7},-\frac{9}{7}\right)$

6. If a line with x-intercept 4 and y-intercept -6 is given, then its slope is equal to______________.

   A. $-\frac{2}{3}$ B. $-\frac{3}{2}$ C. $\frac{2}{3}$ D. $\frac{3}{2}$

7. The distance between P (2,3) and Q (1, -1) is:

   A. 17 units B. 16 units C. $\sqrt{17}$ units D. 9 units

8. Which one of the following pairs of equations represents perpendicular lines?

   A. $x+y=0$ and $-x+y=1$ C. $x+y=1$ and $y-2x=2$

   B. $2x+y=1$ and $-2x-y=1$ D. $3x-2y=0$ and $3x+2y=2$

9. Which one of the following lines is parallel to the line $5x-2y=0$?

   A. $y=-\frac{5}{2}$ C. $-5x-2y=1$

   B. $2x+5y=-4$ D. $-5x+2y=6$

10. Which one of the following is **true** about a second quadrant angle θ in standard position whose terminal side lies on the line $2x+y=0$?

    A. $\sin\theta=\frac{1}{\sqrt{5}}$ B. $\cos\theta=\frac{2}{\sqrt{5}}$ C. $\sin\theta=\frac{2}{\sqrt{5}}$ D. $\cos\theta=\frac{1}{\sqrt{5}}$

11. The equation of the line that passes through the point (2, -1) and perpendicular to the line $2x+4y+3=0$ is given by:
   A. $x+2y=0$
   C. $x-2y+2=0$
   B. $-2x+y-4=0$
   D. $-2x+y+5=0$
12. Which one of the following is the equation of a line whose x-intercept and y-intercept are 1 and -2 respectively?
   A. $-2x+y-1=0$ B. $x-y+1=0$ C. $2x-y-2=0$ D. $x-2y-1=0$
13. What is the equation of the line passing through mid-point of the line segment with end points (-1, 3) and (3, 1), and perpendicular to the line whose angle of inclination is double the angle of inclination of the line $2x+y=7$?
   A. $3y+4x=10$
   C. $2y-x=3$
   B. $4y-5x=3$
   D. $4y+3x=11$
14. What is the coordinate of the point R on the line segment with end points P(-3, 0) and Q(0, -3) such that $\frac{PR}{RQ}=\frac{2}{3}$?
   A. $(-1,-1)$ B. $\left(-1,\frac{7}{5}\right)$ C. $\left(-\frac{9}{5},-\frac{6}{5}\right)$ D. $\left(-\frac{7}{5},0\right)$
15. Let (a,b) and (b,a) be points such that $a\neq b$. Which of the following is the equation of the line through (a,b) and perpendicular to the line containing the given points?
   A. $y=x+b-a$
   C. $y=-x+b+a$
   B. $y=x-a-b$
   D. $y=-x-a+b$
16. The equation straight line passing through P (-2, 1) and perpendicular to the line with equation $6x+5y=10$ is:
   A. $6y+5x+16=0$
   C. $y=5x-10$
   B. $6y-5x+16=0$
   D. $6y-5x-16=0$
17. If P (2, -1) and Q (-3, 5) are points on the coordinates plane, so that, P is the midpoint of $\overline{QR}$, then what are the coordinates of R?
   A. (7, 7) B. (7, -7) C. (6, 7) D. (10, 11)
18. Line l passes through $(0,\ 5)$ and $(-5,\ 0)$. What is the angle between the $y-axis$ and l in radian measure?
   A. $\frac{\pi}{4}$ B. $\frac{\pi}{3}$ C. $\frac{\pi}{2}$ D. $\frac{3}{4}\pi$
19. Suppose l_1 and l_2 are perpendicular lines intersecting at $(2,\ 1)$. If the angle of inclination of l_2 is 45^0, what is the equation of l_1 ?
   A. $y=-x+2$ B. $y=-x+3$ C. $y=x-1$ D. $y=-2x+5$
20. If a line with angle of inclination of $\frac{3}{4}\pi$ passes through $(0,\ 1)$, which one of the following is the equation of the line?
   A. $y=-x+1$ B. $y=x+1$ C. $y=-x-1$ D. $y=x-1$

Unit 5

Leonardo da Vinci obtained the "Mona Lisa" smile by tilting the lips so that the ends lie on a circle which touches the outer corners of the eyes. The outline of the top of the head is the arc of another circle exactly twice as large as the first.

PLANE GEOMETRY

Unit Outcomes:

After completing this unit, you should be able to:

- *Know more theorems special to triangles.*
- *Know basic theorems specific to quadrilaterals.*
- *Know theorems about circles and angles inside, on and outside a circle.*
- *Solve geometrical problems involving quadrilaterals, circles and regular polygons.*

Main Contents:

5.1. Theorems on triangles

5.2. Special quadrilaterals

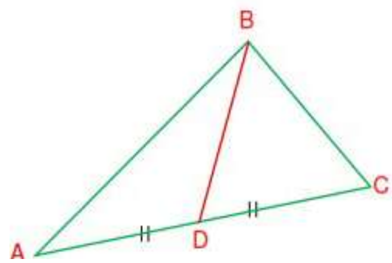
5.3. More on circles

5.4. Regular polygons

5.1. THEOREMS ON TRIANGLES

1. Median of a triangle

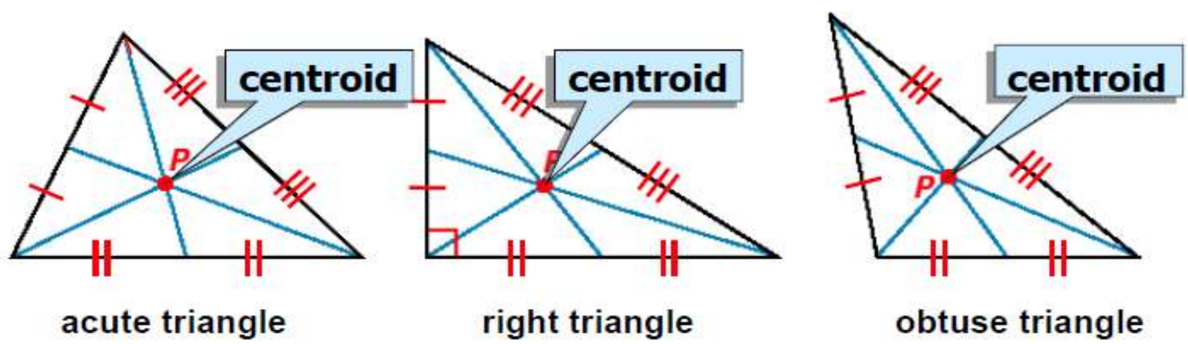
A median of a triangle is a line segment drawn from any vertex to the mid-point of the opposite side.



$\overline{BD}$ is the median of $\Delta ABC \implies \overline{AD} = \overline{CD}$

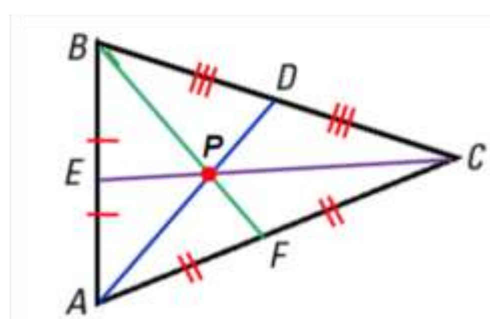
Theorem 5.1
The median of a triangle are concurrent at a point $\frac{2}{3}$ of the distance from each vertex to the mid-point of the opposite side

Note : The three medians of a triangle are concurrency is called the **centroid** and is always inside the triangle.



Concurrency of medians of a triangle
The medians of a triangle intersect at a point that is two-thirds of the distance from each vertex to the mid-point of the opposite side

Illustration
If p is the centroid of $\triangle ABC$, then



- $AP = \frac{2}{3}AD$
- $BP = \frac{2}{3}BF$
- $CP = \frac{2}{3}CE$
- $DP = \frac{1}{3}AD, EP = \frac{1}{3}CE, FP = \frac{1}{3}BF$

Example 1: In the figure 6.7, $\overline{AN}, \overline{CM}$ and $\overline{BL}$ are medians of ΔABC. If $AN = 12cm, OM = 5cm$ and $BO = 6cm$, find BL, ON and OL.

Solution: By theorem 6.1 $BO = \frac{2}{3}BL$ and $AO = \frac{2}{3}AN$

Substituting $6 = \frac{2}{3}BL$ and $AO = \frac{2}{3} \times 12$

So $BL = 9cm$ and $AO = 8cm$

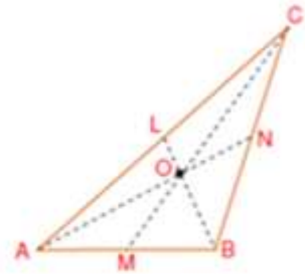
Since $BL = BO + OL$

$\Rightarrow OL = BL - BO = 9 - 6 = 3cm$

Now $AN = AO + ON$

$\Rightarrow ON = AN - AO = 12 - 8 = 4cm$

$\therefore BL = 9cm, OL = 3cm$ and $ON = 4cm$



Theorem 5.2
The perpendicular bisectors of the sides of any triangle are concurrent at a point which is equidistant from the vertices of the triangle.

Theorem 6.3
The altitudes of a triangle are concurrent.

2. Angle bisector of a triangle

Theorem 5.4
The angle bisectors of any triangle are concurrent at a point which is equidistant from the sides of the triangle.

3. Altitude theorem

The altitude theorem is stated here for a right angled triangle. It relates the length of the altitude to the hypotenuse of a right angled triangle, to the lengths of the segments of the hypotenuse.

Theorem 5.5 Altitude theorem
In a right angled triangle ABC with altitude $\overline{CD}$ to the hypotenuse$\overline{AB}$, $\frac{AD}{DC} = \frac{CD}{DB} \Rightarrow (CD)^2 = (AD)(DB)$.

Proof: consider ΔABC as shown in the figure 6.13 $\triangle ABC \sim \triangle ACD$ . . . . AA similarity

So $\angle ABC \cong \angle ACD$

Similarly, $\triangle ABC \sim \triangle CBD$ . . . . AA similarity.

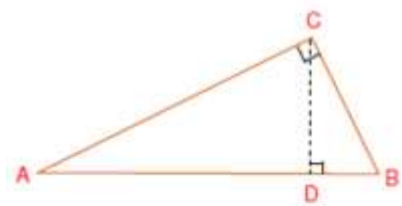
So $\angle ABC \cong \angle CBD$

It follows that $\angle ACD \cong \angle CBD$

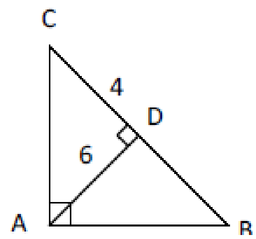
By AA similarity , $\triangle ACD \sim \triangle CBD$

Hence $\frac{AD}{CD} = \frac{CD}{BD}$ . . . .*

Equivalently ,$\frac{AD}{DC} = \frac{CD}{DB}$



The following are some forms of the altitude theorem from $*$,
$(CD)^2 = (AD)(BD)\ or\ (CD)(CD) = (AD)(DB)$
This can be stated as the square of the length of the altitude is the product of the length of the segments of the hypotenuse.
Example 2: Find the lengths of all sides that are not given in the following figures



Solution:

By altitude theorem
$(AD)^2 = (CD)(BD)$
$\Rightarrow (6)^2 = (4)(BD)$
$\Rightarrow BD = \frac{36}{4} = 9\ units$
By Pythagoras theorem $(for\ \triangle ADC)$
$(AC)^2 = (AD)^2 + (CD)^2$
$\Rightarrow (AC)^2 = (6)^2 + (4)^2 = 36 + 14 = 52$
$\Rightarrow (AC)^2 = 52 \Rightarrow AC = \sqrt{52} = 2\sqrt{13}$ Units
Again by Pythagoras theorem $(for\ \triangle ABC)$
$(BC)^2 = (AC)^2 + (AB)^2$
$\Rightarrow (BD + DC)^2 = \left(\sqrt{52}\right)^2 + (AB)^2$
$\Rightarrow (9 + 4)^2 = 52 + (AB)^2$
$\Rightarrow (13)^2 = 52 + (AB)^2$
$\Rightarrow 169 - 52 = (AB)^2$
$\Rightarrow (AB)^2 = 117$
$\therefore AB = \sqrt{117} = \sqrt{9 \times 13} = 3\sqrt{13}$ units

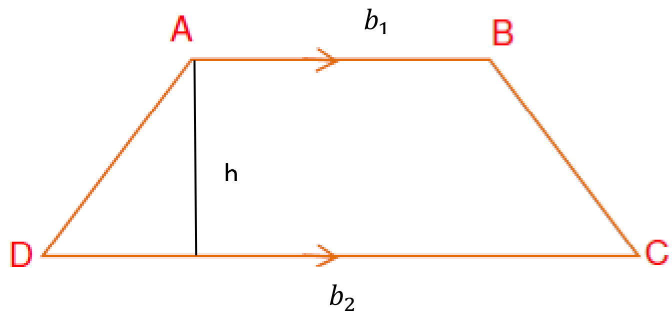
4. Menelaus' theorem

Theorem 5.6 Menelaus' theorem
If points D, E and F on the sides $\overline{BC}, \overline{CA}\ and\ \overline{AB}$ respectively of $\triangle ABC$ (on their extensions) are collinear, then $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = -1$. conversely, if $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = -1$,then the points D, E and F are collinear.

5.2. SPECIAL QUADRILATERALS

1. TRAPEZIUM

Definition 5.1: A trapezium is a quadrilateral where only two of the sides are parallel.



$\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \nparallel \overline{BC}$

$\overline{AB}$ *and* $\overline{DC}$ are its bases

If the two non- parallel sides of a trapezium are congruent is called isosceles trapezium.

In the above isosceles trapezium $\overline{AD} \cong \overline{BC}$ and $\angle D \cong \angle C$

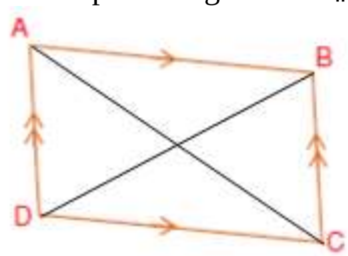
Area of trapezium

The area of a trapezium with bases b_1 *and* b_2 and altitude h is given by $A = \frac{1}{2}(b_1 + b_2)h$.

2. PARALLELOGRAM

Definition 5.2: A parallelogram is a quadrilateral in which both pairs of opposite sides are parallel

In Figure 6.23, the quadrilateral *ABCD* is a parallelogram. *AB* ∥ *DC* and *AD*∥ *BC*



Properties of a parallelogram and tests for a quadrilateral to be a parallelogram are stated in the following theorem:

Theorem 5.7

a. The opposite sides of a parallelogram are congruent i.e $\overline{AB} \cong \overline{DC}$ and $\overline{AD} \cong \overline{BC}$

b. The opposite angles of a parallelogram are congruent i.e $\angle A \cong \angle C$ and $\angle B \cong \angle D$

c. The diagonals of a parallelogram bisect each other i.e $\overline{AO} \cong \overline{CO}$ *and* $\overline{BO} \cong \overline{DO}$

d. If the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

e. If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

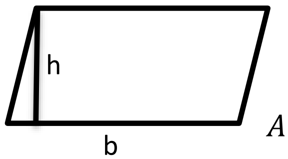
f. If the opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

g. consecutive angles in a parallelogram are supplementary

i.e $\angle A + \angle B = 180^\circ, \angle B + \angle C = 180^\circ, \angle C + \angle D = 180^\circ$ *and* $\angle D + \angle A = 180^\circ$

Area of a parallelogram

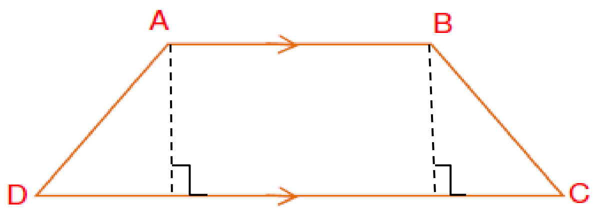
The area of a parallelogram with base b and altitude h is given by



$A = bh$

Example 3: The shorter base of an isosceles trapezium is 12cm long and the non-parallel bases are each 10cm. Find the area of this trapezium if its altitude is 6cm

Solution:



In ΔDEA ,By Pythagoras theorem

$(AD)^2 = (AE)^2 + (DE)^2$

$\Rightarrow (10)^2 = (6)^2 + (DE)^2$

$\Rightarrow 100 - 36 = (DE)^2$

$\Rightarrow 64 = (DE)^2$

$\Rightarrow DE = \sqrt{64}$ = 8cm

$DE \cong FC$, and $AB \cong EF$

$\Rightarrow DC = DE + EF + FC$

$= 8cm + 12cm + 8cm = 28cm$

$A = \frac{1}{2}(b_1 + b_2)h$

$= \frac{1}{2}(12cm + 28cm) \times 6cm$

$= 120cm^2$

Example 4: One of the sides of the parallelogram is 8cm long and the perimeter of this parallelogram is 28cm.if the altitude to the longer base is 4cm, what the altitude to the shorter base is and what is the area of the parallelogram

Solution:

Perimeter = $2(s_1 + s_2)$ where $s_1 and\ s_2$ are the sides of parallelogram.

$28cm = 2(8cm + s_2)$

$\Rightarrow 14cm = 8cm + s_2$

$\Rightarrow s_2 = 6cm$

Hence the longer base is 8 cm and the area of the parallelogram becomes

Area = $bh = 8cm \times 4cm = 32\ cm^2$

Now the altitude to the shorter base can be found from:

$Area = bh$

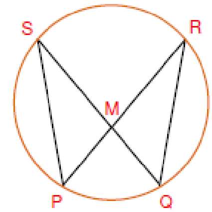
$\Rightarrow 32cm^2 = 6cm \times h$

$\Rightarrow h = \frac{32cm^2}{6cm} = \frac{16}{3}cm$

5.3. MORE ON CIRCLES MORE ON CIRCLES

Theorem 5.8
The measure of an angle formed by two chords intersecting inside a circle is half the sum of the measures of the arc subtending the angle and its vertically opposite angle.

Example 5: In the figure beow, $m(\angle MRQ) = 30^\circ$, and $m(\angle MQR) = 40^\circ$.Write down the measure of all the other angles in the two triangles, $\triangle PSM$ and $\triangle QMR$. What do you notice about the two triangles?



Solution: $m(\angle QMR) = 180^\circ - (30^\circ + 40^\circ)$(why?)
$= 180^\circ - 70^\circ = 110^\circ$

$m(\angle RQS) = \frac{1}{2}m(\widehat{RS})$

So, $40^\circ = \frac{1}{2}m(\widehat{RS})$

$\therefore\ m(\widehat{RS}) = 80^\circ$

$m(\angle PRQ) = \frac{1}{2}m(\widehat{PQ})$

Hence $30^\circ = \frac{1}{2}m(\widehat{PQ})$

$\therefore\ m(\widehat{PQ}) = 60^\circ$

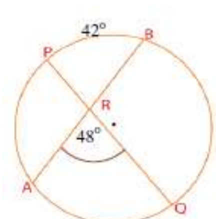
$m(\angle PSQ) = \frac{1}{2}m(\widehat{PQ}) = \frac{1}{2}(60^\circ) = 30^\circ$

$m(\angle RPS) = \frac{1}{2}m(\widehat{RS}) = \frac{1}{2}(80^\circ) = 40^\circ$

The two triangles are similar by **AA** similarity theorem.

Example 6: An angle formed by two chords intersecting within a circle is 48°, and one of the intercepted arcs measures 42°. Find the measures of the other intercepted arc.

Solution: Consider the the following figure.



$m(\angle PRB) = \frac{1}{2}m(\widehat{PB}) + \frac{1}{2}m(\widehat{AQ})$ (by theorem 6.11)

$\Rightarrow 48^\circ = \frac{1}{2}(42^\circ) + \frac{1}{2}m(\widehat{AQ}) = \frac{42^\circ + m(\widehat{AQ})}{2}$

$\Rightarrow 48^\circ \times 2 = 42^\circ + m(\widehat{AQ})$

$\Rightarrow 96^\circ - 42^\circ = m(\widehat{AQ})$

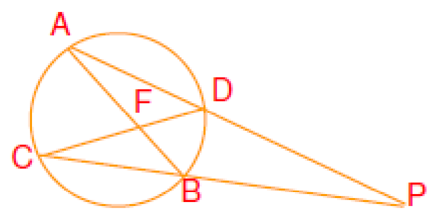
$\therefore m(AQ) = 54^\circ$

ANGLES AND ARCS DETERMINED BY LINES INTERSECTING OUTSIDE A CIRCLE

Theorem 5.9
The measure of the angle formed by the lines of two chords intersecting outside a circle is half the difference of the measure of the arcs they intercept

Theorem 5.10
The measure of an angle formed by a tangent and a secant drawn to a circle from a point outside the circle is equal to one-half the difference of the measures of the intercepted arcs.

Example 7: In figure below , from P secants $\overline{PA}$ and $\overline{PC}$ are drawn so that $m\,(\angle APC) = 30^\circ$; chords $\overline{AB}$ and $\overline{CD}$ intersect at F such that $m\,(A\angle FC\,) = 85^\circ$. Find the measure of arc AC, measure of arc BD and measure of $A\angle BC$.



Solution: Let $m(\widehat{AC}) = x \text{ and } m(\widehat{BD}) = y.$

Since $m(\angle AFC) = \frac{1}{2}m(\widehat{AC}) + \frac{1}{2}m(\widehat{BD})$

$\Rightarrow 85^\circ = \frac{1}{2}(x+y)$

$\Rightarrow x + y = 170^\circ. \ . \ . \ . \ .(1)$

Again as $m(\angle APC) = \frac{1}{2}m(\widehat{AC}) - \frac{1}{2}m(\widehat{BD})$

$\Rightarrow 30^\circ = \frac{1}{2}(x-y)$

$\Rightarrow x - y = 60^\circ. \ . \ . \ . \ .(2)$

Solving equation 1 and equation 2 simultaneously, we get

$$\begin{cases} x + y = 170^\circ \\ x - y = 60^\circ \end{cases}$$

$\Rightarrow 2x = 230^\circ \Longrightarrow x = 115^\circ$

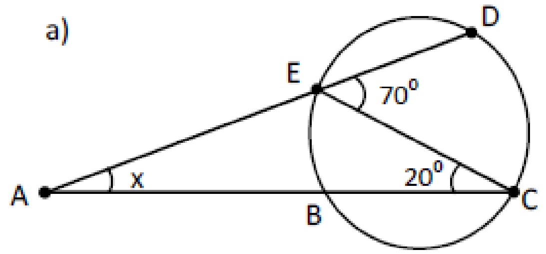
Substituting for x in equation 2

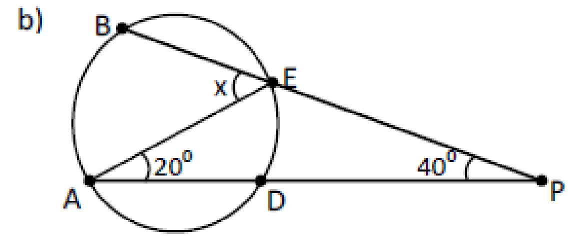
$\Rightarrow 115^\circ - y = 60^\circ \Longrightarrow y = 55^\circ$

Therefore, $m(\widehat{AC}) = 115^\circ$ and $m(\widehat{DB}) = 55^\circ$

$m(\angle ABC) = \frac{1}{2}m(\widehat{AC}) = \frac{1}{2}(115^\circ) = 57.5^\circ$

Example 8: Find the measures of the marked angles





Solution:

a. $70^\circ = \frac{1}{2}\left(m(\widehat{DC})\right) \Rightarrow 70^\circ \times 2 = \left(m(\widehat{DC})\right)$

$\Rightarrow 140^\circ = \left(m(\widehat{DC})\right)$

$\Rightarrow \angle ECB = \frac{1}{2}\left(m(\widehat{EB})\right)$

$\Rightarrow 20^\circ = \frac{1}{2}\left(m(\widehat{EB})\right)$

$\Rightarrow 20^\circ \times 2 = \left(m(\widehat{EB})\right)$

$\Rightarrow 40^\circ = \left(m(\widehat{EB})\right)$

$\therefore \angle DAC = \frac{1}{2}\left(m(\widehat{DC}) - m(\widehat{EB})\right)$ and $\angle x = \frac{1}{2}(140^\circ - 40^\circ) = 50^\circ$

b. $\angle EAD = \frac{1}{2}\left(m(\widehat{ED})\right) \Rightarrow 20^\circ = \frac{1}{2}\left(m(\widehat{ED})\right)$

$\Rightarrow 20^\circ \times 2 = \left(m(\widehat{ED})\right)$

$\Rightarrow 40^\circ = \left(m(\widehat{ED})\right)$

$\Rightarrow \angle BPA = \frac{1}{2}\left(m(\widehat{AB}) - m(\widehat{ED})\right)$

$\Rightarrow 40^\circ = \frac{1}{2}(m(\widehat{AB}) - 40^\circ)$

$\Rightarrow 40^\circ \times 2 = (m(\widehat{AB}) - 40^\circ)$

$\Rightarrow 80^\circ = (m(\widehat{AB}) - 40^\circ)$

$\Rightarrow 80^\circ + 40^\circ = m(\widehat{AB})$

$\Rightarrow \left(m(\widehat{AB})\right) = 120^\circ$

$\therefore \angle AEB = \frac{1}{2}\left(m(\widehat{AB})\right) = \frac{1}{2}(120^\circ)$ and $\angle x = \frac{1}{2}(120^\circ) = 60^\circ$.

5.4. REGULAR POLYGON

You may recall that a polygon all whose angles have equal measure and all of whose sides have equal length is called a regular polygon. i.e. a regular polygon is both equiangular and equilateral. In this section, we will study regular polygons by relating them to circles.

5.4.1. Perimeter of a Regular Polygon

You have studied how to find the length of a side (s) and perimeter (P) of a regular polygon with radius "*r*" and the number of sides "*n*" in Grade 9.

Theorem 5.11: Formulae for the length of side *s*, apothem *a*, perimeter *P* and area *A* of a regular polygon with *n* sides and radius *r*

1. $s = 2r\sin\frac{180^\circ}{n}$
2. $a = r\cos\frac{180^\circ}{n}$
3. $p = 2nr\sin\frac{180^\circ}{n}$
4. $A = \frac{1}{2}ap$

Example 9: Find the length of a side and perimeter of a regular quadrilateral with radius 6 units.

Solution: Given: $n = 4,\quad r = 6units$

Length of a side s:

$$s = 2r\sin\frac{180^\circ}{n}$$

$$\Rightarrow s = 2 \times 6 \times \sin\frac{180^\circ}{4}$$

$$\Rightarrow s = 12 \times \sin 45^\circ$$

$$\therefore s = 6\sqrt{2}\ units$$

perimeter p:

$$\Rightarrow p = 2nr\sin\frac{180^\circ}{n}$$

$$\Rightarrow p = 2 \times 4 \times 6\sin\frac{180^\circ}{4}$$

$$\Rightarrow p = 48 \times \sin 45^\circ$$

$$\therefore p = 24\sqrt{2}\ units$$

5.4.2. Area of a Regular Polygon

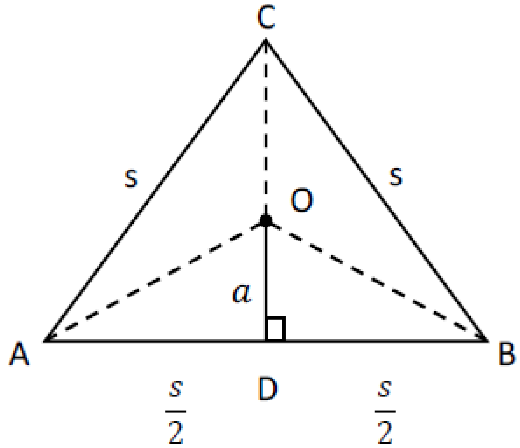
Theorem 5.12

The area *A* of a regular polygon with *n* sides and radius *r* is $A = \frac{1}{2}nr^2\sin\frac{360^\circ}{n}$.

This formula for the area of a regular polygon can be used to find the area of a circle. As the number of sides increases, the area of the polygon becomes closer to the area of the circle.

Example 10: Find the side, apothem, perimeter and area of an equilateral triangle of radius 6 units.

Solution: Given $n = 3 \quad and\ r = 6units$



1. $s = 2r\sin\frac{180^\circ}{n}$

$\Rightarrow s = 2 \times 6 \times \sin\frac{180^\circ}{3}$

$\Rightarrow s = 12 \times \sin 60^\circ$

$\Rightarrow s = 6 \times \frac{\sqrt{3}}{2}\ units$

$\Rightarrow s = 3\sqrt{3}\ units$

2. $p = 2nr\sin\frac{180^\circ}{n}$

$\Rightarrow p = 2 \times 3 \times 6\sin\frac{180^\circ}{3}$

$\Rightarrow p = 36 \times \sin 60^\circ$

$\Rightarrow p = 36 \times \frac{\sqrt{3}}{2}\ units = 18\sqrt{3}\ units$

3. $A = \frac{1}{2}nr^2\sin\frac{3600^\circ}{n}$

$\Rightarrow A = \frac{1}{2} \times 3 \times 6^2 \times \sin\frac{360^\circ}{3}$

$\Rightarrow A = \frac{1}{2} \times 3 \times 36 \times \sin 120^\circ$

$\Rightarrow A = 3 \times 18 \times \frac{\sqrt{3}}{2}\ units$

$\Rightarrow A = 27\sqrt{3}\ units^2$

4. $a = r\cos\frac{180^\circ}{n}$

$\Rightarrow a = 6 \times \sin\frac{180^\circ}{3}$

$\Rightarrow a = 6 \times \sin 60^\circ$

$\Rightarrow a = 6 \times \frac{1}{2}\ units = 3units$

Unit 6

The Pyramids at Giza in Egypt are among the best known pieces of architecture in the world. The Pyramid of Khafre was built as the final resting place of the Pharaoh Khafre and is about 136 m high.

MEASUREMENT

Unit Outcomes:

After completing this unit, you should be able to:

- *solve problems involving surface area and volume of solid figures.*
- *know basic facts about frustums of cones and pyramids.*

Main Contents:

6.1. Revision on Surface Areas and Volumes of Prisms and Cylinders

6.2. Pyramids, Cones and Spheres

6.3. Frustums of Pyramids and Cones

6.4. Surface Areas and Volumes of Composite Solids

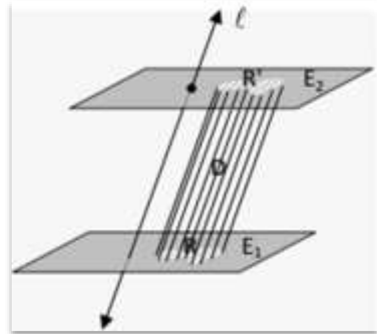
6.5. REVISION ON SURFACE AREAS AND VOLUMES OF PRISMS AND CYLINDERS

NOTE: Some important terms;

- For the cylinder D, the region R is called its lower base or simply base and R' is its upper base.
- The line ℓ is called its directrix and the perpendicular distance between E_1 and E_2 is the altitude of D.

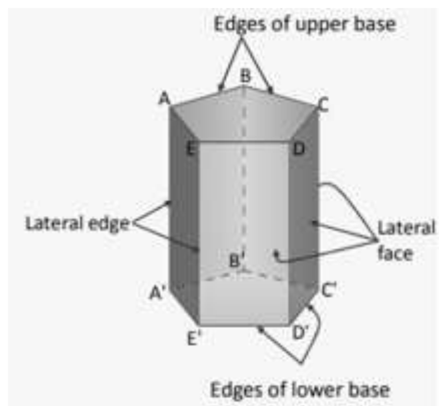
If ℓ is perpendicular to E_1, then D is called a right cylinder, otherwise it is an oblique cylinder.

If R is a circular region, then D is called a circular cylinder.



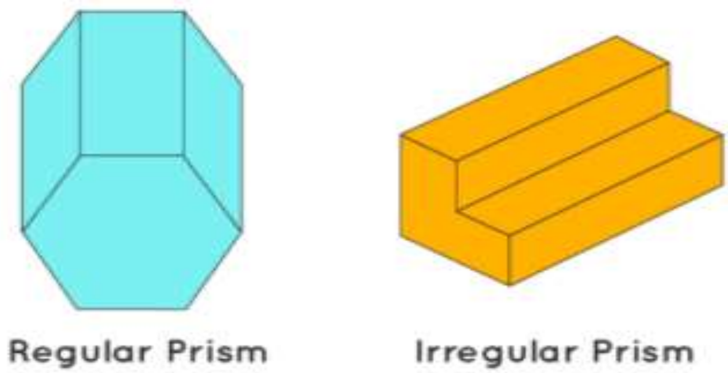
NOTE: In the prism shown below,

1. $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DE}, \overline{EA}$ are edges of the upper base.
   $\overline{A'B'}, \overline{B'C'}, \overline{C'D'}, \overline{D'E'}, \overline{E'A'}$ are edges of the lower base.
2. $\overline{AA'}, \overline{BB'}, \overline{CC'}, \overline{DD'}, \overline{EE'}$ are called lateral edges of the prism.
3. The parallelogram regions $ABB'A'$, $BCC'B'$, $AEE'A'$, $DCC'D'$, $EDD'E'$ are called lateral faces of the prism.
4. The union of the lateral faces of a prism is called its lateral surface.
5. The union of its lateral faces and its two bases is called its total surface or simply its surface.



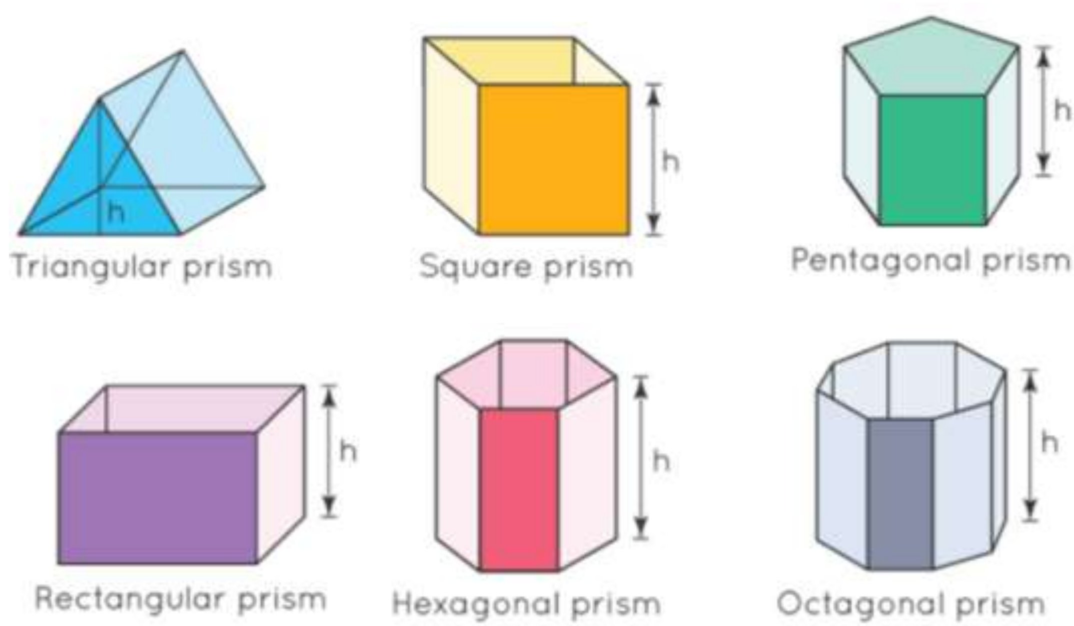
A prism is classified on the basis of the type of polygon base it has. There are two types of prisms in this category named as:

- Regular Prism: If the base of the prism is in the shape of a regular polygon, the prism is a regular prism.
- Irregular Prism: If the base of the prism is in the shape of an irregular polygon, the prism is an irregular prism.



A prism is named on the basis of the shape obtained by the cross-section of the prism. They are further classified as:

- Triangular Prism: A prism whose bases are triangle in shape is considered a triangular prism.
- Square Prism: A prism whose bases are square in shape is considered a square prism.
- Rectangular prism: A prism whose bases are rectangle in shape is considered a rectangular prism (a rectangular prism is cuboidal in shape).
- Pentagonal Prism: A prism whose bases are pentagon in shape is considered a pentagonal prism.
- Hexagonal Prisms: A prism whose bases are hexagon in shape is considered a hexagonal prism.
- Octagonal Prism: A prism whose bases are octagon in shape is considered an octagonal prism.
- Trapezoidal Prism: A prism whose bases are trapezoid in shape is considered a trapezoidal prism.



If we denote the lateral surface area of a prism by A_L, the area of the base by A_B, altitude h and the total surface area by A_T, then:

$A_L = Ph$; where P is the perimeters of the base and h is the height of the prism.

$A_T = 2A_B + A_L$

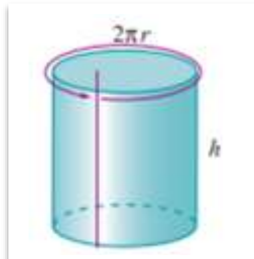
- Similarly, the lateral surface area (A_L) of a right circular cylinder is equal to the product of the circumference of the base and altitude (h) of the cylinder. That is,

  $A_L = 2\pi rh$, where r is the radius of the base of the cylinder.

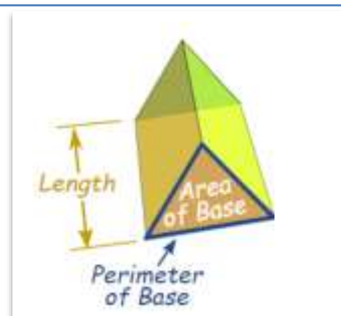
- The total surface area A_T is equal to the sum of the areas of the bases and the lateral surface area. That is,

  $A_T = A_L + 2A_B$

  $A_T = 2\pi rh + 2\pi r^2 = 2\pi r(h+r)$



- The volume (V) of any prism equals the product of its base area (A_B) and altitude (h). That is, $V = A_B h$

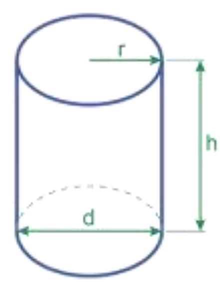


- Volume of a right circular cylinder

  The volume (V) of a circular cylinder is equal to the product of the base area (A_B) and its altitude (h). That is,

  $V = A_B h$

  $V = \pi r^2 h$, where r is the radius of the base.



Example1: Bontu has given a cylinder of surface area 1728π square units. Find the height of the cylinder if the radius of the base of the circle is 24 units.

Solution: The surface of the cylinder, $A_L = 1728\pi$

Using the total surface area, $A_T = 2\pi r(h+r)$:

$$1728\pi = 2\pi \times 24 \times (h+24)$$

$$\Rightarrow h+24 = \frac{1728\pi}{48\pi} = 36$$

$$\Rightarrow h = 12 \text{ units}$$

So, the height of the cylinder is 12 units .

6.2. PYRAMIDS, CONES AND SPHERES

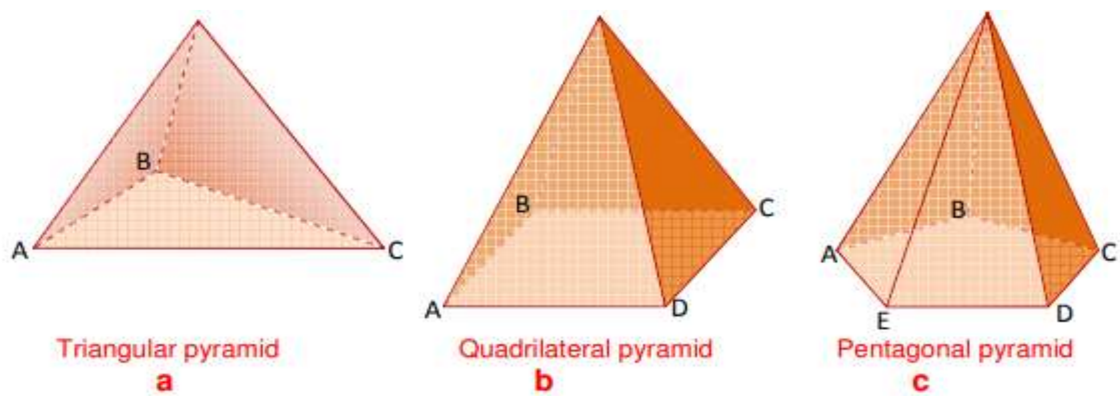
Do you remember what you learnt about pyramids, cones and spheres in your previous grades? Can you give some examples of pyramids, cones and spheres from real life?

1. PYRAMIDS

Definition 6.2:

A pyramid is a solid figure formed when each vertex of a polygon is joined to the same point not in the plane of the polygon.

Examples:



NOTE:

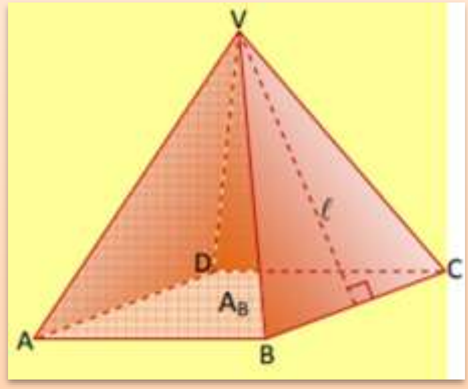
a. The altitude of a pyramid is the length of the perpendicular from the vertex to the plane containing the base.

b. The slant height of a regular pyramid is the altitude of any of its lateral faces.

c. A regular pyramid is a pyramid whose base is a regular polygon and whose altitude passes through the center of the base.

NOTE:

The lateral surface area of a regular pyramid is equal to half the product of its slant height and the perimeter of the base. That is,



$$A_L = \frac{1}{2}P\ell,$$

where A_L denotes the lateral surface area;

P denotes of the perimeter of the base;

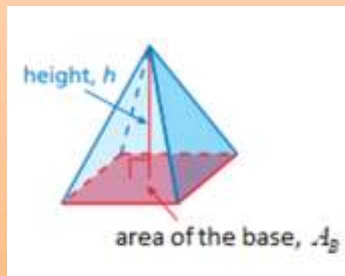
ℓ denotes the slant height.

The total surface area (A_T) of a pyramid is given by $A_T = A_B + A_L = A_B + \frac{1}{2}P\ell$,

where A_B is area of the base.

The volume V of a pyramid is one-third the product of the area of the base and the height of the pyramid.

That is, $V = \frac{1}{3}A_B h$

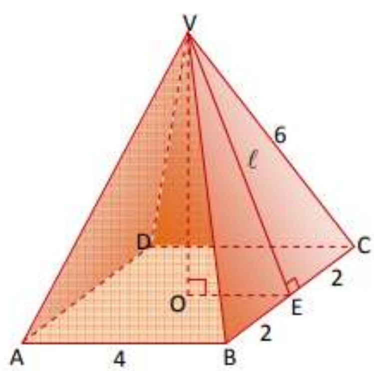


Example 2: A regular pyramid has a square base whose side is $4cm$ long. The lateral edges are $6cm$ each.

a. What is its slant height?
b. What is the lateral surface area?
c. What is the total surface area?
d. What is the volume of the pyramid?

Solution:

Consider the following figure,



a. $(VE)^2 + (EC)^2 = (VC)^2$

$\Rightarrow \ell^2 + 2^2 = 6^2$

$\Rightarrow \ell^2 = 32$

$\Rightarrow \ell = 4\sqrt{2}cm$

Therefore, the slant height is $4\sqrt{2}cm$.

b. There are 4 isosceles triangles.

Therefore,

$$A_L = 4 \times \frac{1}{2} BC \times VE$$

$$= 4 \times \left(\frac{1}{2} \times 4 \times 4\sqrt{2}\right) = 32\sqrt{2}cm^2 \text{ or}$$

$$A_L = \frac{1}{2} P\ell = \frac{1}{2}(4+4+4+4) \times 4\sqrt{2} = 8 \times 4\sqrt{2} = 32\sqrt{2}cm^2$$

c. $A_T = A_B + A_L = 32\sqrt{2} + 4 \times 4 = 16\left(2\sqrt{2}+1\right)cm^2$.

d. $(VO)^2 + (OE)^2 = (VE)^2$

$\Rightarrow h^2 + 2^2 = \left(4\sqrt{2}\right)^2$

$\Rightarrow h^2 + 4 = 32$

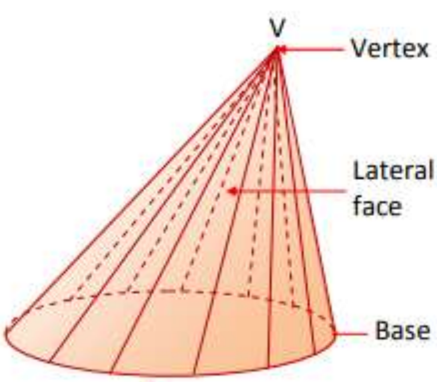
$\Rightarrow h^2 = 28 \Rightarrow h = 2\sqrt{7}cm$

Therefore, $V = \frac{1}{3} A_B h = \frac{1}{3} \times (4 \times 4) \times 2\sqrt{7} = \frac{32}{3}\sqrt{7}cm^3$

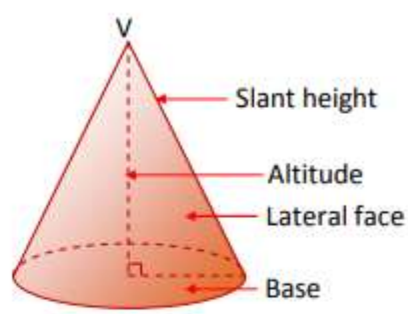
2. CONES

Definition 6.3:

The solid figure formed by joining all points of a circle to a point not on the plane of the circle is called a cone.



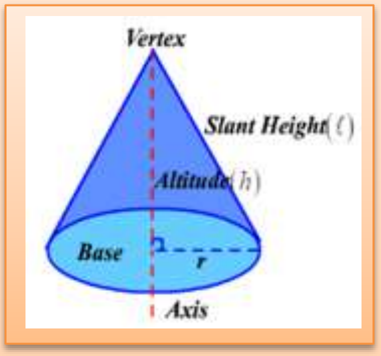
a



b

NOTE:

- The lateral surface area of a right circular cone is equal to half the product of its slant height and the circumference of the base. That is,



$$A_L = \frac{1}{2}P\ell = \frac{1}{2}(2\pi r)\ell = \pi r\ell \; ;$$

$$\ell = \sqrt{h^2 + r^2}$$

where P denotes the lateral surface area;

r stands for the base radius;

l denotes the slant height;

h for the altitude.

- The total surface area is equal to the sum of the area of the base and the lateral surface area. That is, $A_T = A_B + A_L = \pi r\ell + \pi r^2 = \pi r(\ell + r)$, where A_B is area of the base.
- The volume V of a circular cone is one-third the product of its base area and its altitude. That is, $V = \frac{1}{3}A_B h = \frac{1}{3}\pi r^2 h$

Example 3: The base radius and height of a right circular cone is 7cm and 24cm. Find its curved surface area, total surface area and volume.

Solution:

Here, $r = 7cm$ and $h = 24cm$

✓ So, slant height $\ell = \sqrt{r^2 + h^2}$

$\Rightarrow \ell = \sqrt{7^2 + 24^2} = 25cm$

✓ Thus, curved surface area $A_L = \pi r\ell = \pi \times 7cm \times 24cm = 168\pi cm^2$

✓ Total surface area $A_T = \pi r(\ell + r)$

$$\Rightarrow A_T = \pi \times 7cm \times (25cm + 7cm) = 224\pi cm^2$$

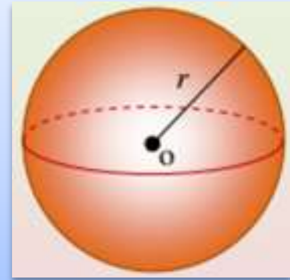
✓ The volume $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times (7cm)^2 \times 24cm = 392\pi cm^3$

3. SPHERES

Definition 6.4:

A sphere is a closed surface, all points of which are equidistant from a point called the centre.

Most familiar examples of a sphere are baseball, tennis ball, bowling, and so forth.

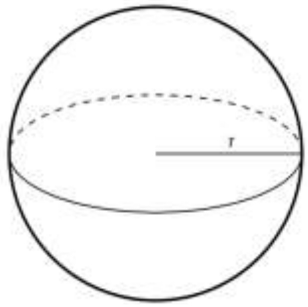


The surface area (A) and the volume (V) of

a sphere of radius r are given by

$$A = 4\pi r^2$$

$$V = \frac{4}{3}\pi r^3$$



Example 4: The diameter of a sphere is 13.5m. Find its surface area and volume.

Solution:

Here, $d = 13.5m$

- Surface area $A = 4\pi r^2 = 4\pi\left(\frac{d}{2}\right)^2$

  $$\Rightarrow A = \pi d^2 = \pi(13.5m)^2 = 182.25\pi m^2$$

- Volume of sphere $V = \frac{4}{3}\pi r^3 = \frac{\pi}{6}d^3$, **(Why?)**

  $$\Rightarrow V = \frac{\pi}{6}(13.5m)^3 = 410.0625\pi m^3$$

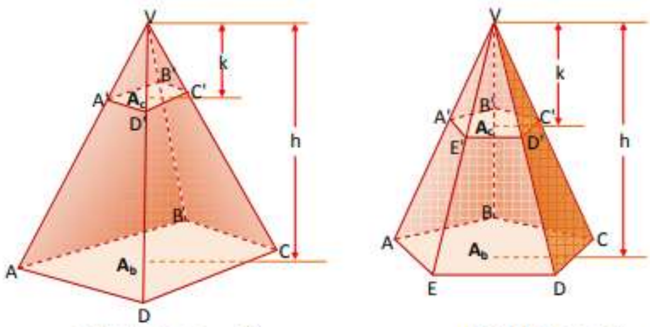
6.3. FRUSTUMS OF PYRAMIDS AND CONES

Definition 6.5:

If a pyramid or a cone is cut by a plane parallel to the base, the intersection of the plane and the pyramid (or the cone) is called a horizontal cross section of the pyramid (or the cone).

Theorem 6.1:

In any pyramid, the ratio of the area of a cross-section to the area of the base is $\frac{k^2}{h^2}$, where h is the altitude of the pyramid and k is the distance from the vertex to the plane of the cross-section.



$$\frac{A_c}{A_b} = \frac{\text{area}(A'B'C'D')}{\text{area}(ABCD)} = \frac{k^2}{h^2}$$

$$\frac{\text{area}(A'B'C'D'E')}{\text{area}(ABCD)} = \frac{k^2}{h^2}$$

Example 5: The area of the base of a pyramid is $90cm^2$. The altitude of the pyramid is $12cm$. What is the area of a horizontal cross-section $4cm$ from the vertex?

Solution:

Let A_c be the area of the cross-section, and A_b the base area.

Then, $\frac{A_c}{A_b} = \frac{k^2}{h^2} \Rightarrow \frac{A_c}{90} = \frac{4^2}{12^2}$

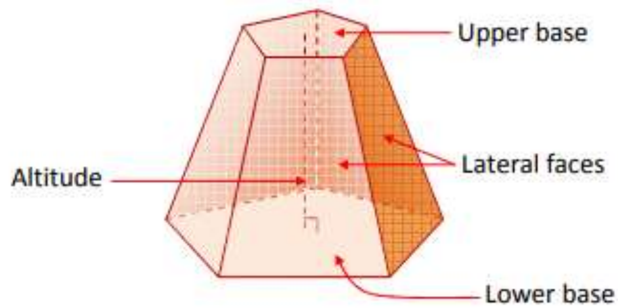
$$\therefore A_c = \frac{90 \times 16}{144} cm^2 = 10cm^2$$

6.3.1. Frustum of a pyramid

Definition 6.6:

A frustum of a pyramid is a part of the pyramid included between the base and a plane parallel to the base.

The altitude of a frustum of a pyramid is the perpendicular distance between the bases.



NOTE:

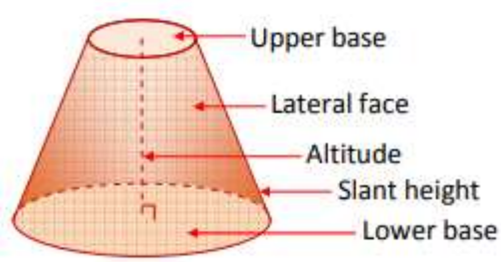
1. The altitude of a frustum of a pyramid is the perpendicular distance between the bases.
2. The lateral faces of a frustum of a pyramid are trapeziums.
3. The lateral faces of a frustum of a regular pyramid are congruent isosceles trapeziums.
4. The slant height of a frustum of a regular pyramid is the altitude of any one of the lateral faces.
5. The lateral surface area of a frustum of a pyramid is the sum of the areas of the lateral faces.

6.3.2. Frustum of a cone

Definition 6.7:

A frustum of a cone is a part of the cone included between the base and a horizontal cross-section made by a plane parallel to the base.

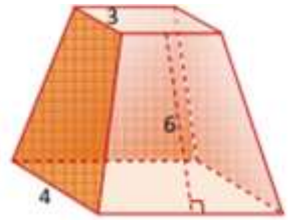
A frustum of a cone is a part of the cone included between the base and a horizontal cross-section made by a plane parallel to the base.



Example 6: The lower base of the frustum of a regular pyramid is a square $4cm$ long, the upper base is $3cm$ long. If the slant height is $6cm$, find its lateral surface area.

Solution:

As shown in figure below, each lateral face is a trapezium,



the area of each lateral face is

$$A_L = \frac{1}{2}\times h(b_1+b_2) = \frac{1}{2}\times 6(3+4) = 21cm^2$$

Theorem6.2:

The lateral surface area (A_L) of a frustum of a regular pyramid is equal to half the product of the slant height (ℓ) and the sum of the perimeter (P) of the lower base and the perimeter (P') of the upper base.

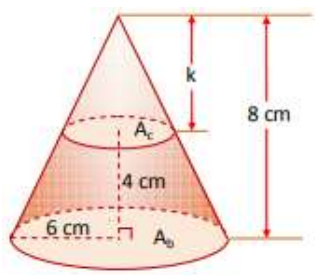
That is, $A_L = \frac{1}{2}\ell(P+P')$

Theorem 6.3:

For a frustum of a right circular cone with altitude h and slant height ℓ, if the circumferences of the bases are c and c', then the lateral surface area of the frustum is given by

$$A_L = \frac{1}{2}\ell(c+c') = \frac{1}{2}\ell(2\pi r + 2\pi r') = \ell\pi(r+r')$$

Example 7: A frustum of height $4cm$ is formed from a right circular cone of height $8cm$ and base radius $6cm$ as shown in below. Calculate the lateral surface area of the frustum.



Solution:

Let A_b, A_c and A_L stand for area of the base of the cone, area of the cross-section and lateral surface area of the frustum, respectively.

$$\frac{\text{Area of cross-section}}{\text{Area of the base}} = \left(\frac{k}{h}\right)^2$$

$$\Rightarrow \frac{A_c}{A_b} = \left(\frac{4}{8}\right)^2, \text{ since } k = 8cm - 4cm = 4cm$$

$$\Rightarrow \frac{A_c}{36\pi} = \frac{1}{4}\left(\text{area of the base} = \pi r^2 = \pi \times 6^{\,2} = 36\pi\right)$$

$$\Rightarrow A_c = \frac{1}{4} \times 36\pi = 9\pi cm^2$$

$$\Rightarrow A_c = \pi\left(r'\right)^2$$

$$\Rightarrow \pi\left(r'\right)^2 = 9\pi cm^2 \Rightarrow r' = 3cm$$

Slant height of the bigger cone is: $\ell = \sqrt{h^2 + r^2} = \sqrt{8^2 + 6^2} = 10cm$

Slant height of the smaller cone is: $\ell = \sqrt{k^2 + \left(r\right)^2} = \sqrt{4^2 + 3^2} = 5cm$

Now the lateral surface area of:

the smaller cone $= \pi r'\ell' = \pi\left(3cm\right) \times 5cm = 15\pi cm^2$

the bigger cone $= \pi r\ell = \pi\left(6cm\right) \times 10cm = 60\pi cm^2$

Hence, the area of the lateral surface of the frustum is:

$A_L = 60\pi cm^2 - 15\pi cm^2 = 45\pi cm^2$.

The lateral surface (curved surface) of a frustum of a circular cone is a trapezium whose parallel sides (bases) have lengths equal to the circumference of the bases of the frustum and whose height is equal to the height of the frustum.

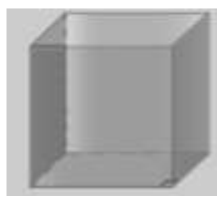
Summary

1. Prism

$$A_L = Ph$$

$$A_T = 2A_b + A_L$$

$$V = A_b h$$

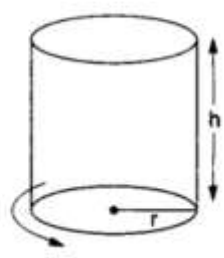


2. Right circular cylinder

$$A_L = 2\pi rh$$

$$A_T = 2\pi r^2 + 2\pi rh = 2\pi r\left(r + h\right)$$

$$V = \pi r^2 h$$

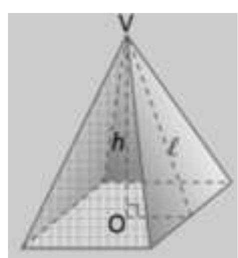


3. Regular pyramid

$$A_L = \frac{1}{2}P\ell$$

$$A_T = A_b + \frac{1}{2}P\ell$$

$$V = \frac{1}{3}A_b h$$

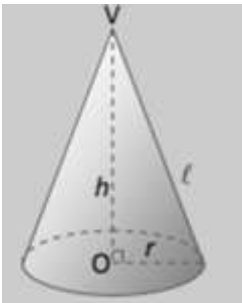


4. Right circular cone

$$A_L = \pi r \ell$$

$$A_T = \pi r^2 + \pi r \ell = \pi r (r + \ell)$$

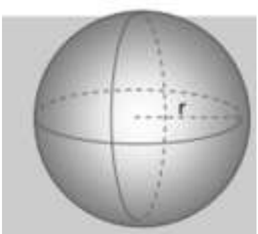
$$V = \frac{1}{3}\pi r^2 h$$



5. Sphere

$$A = 4\pi r^2$$

$$V = \frac{4}{3}\pi r^3$$

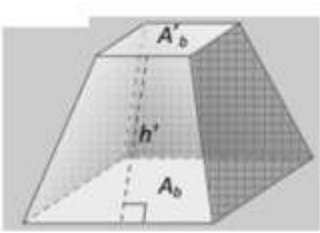


6. Frustum of a pyramid

$$A_L = \frac{1}{2}\ell (P + P')$$

$$A_T = \frac{1}{2}\ell = \pi r (P + P') + A_b + A'_b$$

$$V = \frac{1}{3}h'\left(A_b + A'_b + \sqrt{A_b A'_b}\right)$$



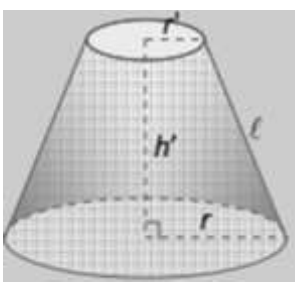
7. Frustum of a pyramid

$$A_L = \frac{1}{2}\ell (2\pi r + 2\pi r') = \ell \pi (r + r')$$

$$A_T = \frac{1}{2}\ell (2\pi r + 2\pi r') + \pi r^2 + \pi (r')^2$$

$$= \ell \pi (r + r') + \pi \left(r^2 + (r')^2\right)$$

$$V = \frac{1}{3}h' \pi \left(r^2 + (r')^2 + rr'\right)$$



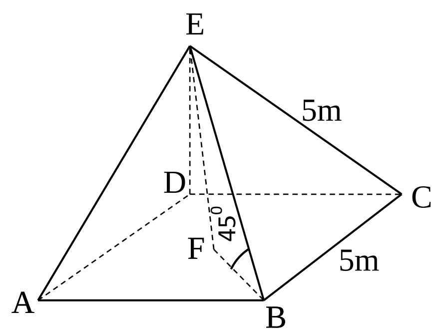
PRACTICE QUESTIONS ON UNIT 6

CHOOSE THE BEST ANSWER FROM THE GIVEN ALTERNATIVES

1. The slant height of a right circular cone is $8cm$. If the angle between the slant height and the height at the vertex of the cone is 30^0, then what is the volume of the cone?

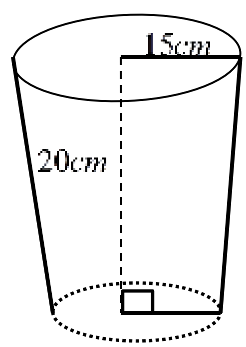
   A. $\frac{128\pi}{3}cm^3$ B. $64\pi cm^3$ $\frac{64\sqrt{3}\pi}{3}cm^3$ C. $128\pi cm^3$ D. $\frac{64\sqrt{3}\pi}{3}cm^3$

2. The right pyramid in the figure below has square base of dimension 5m by 5m. If the edge $\overline{EB}$ makes angle 45^0 with the diagonal $\overline{DB}$, what s the volume of the pyramid?



   A. $\frac{125\sqrt{3}}{6}$ m^3
   B. $\frac{125\sqrt{2}}{6}$ m^3
   C. 125 m^3
   D. 25 m^3

3. The figure shown below is a container made from an inverted frustum a right circular cone. The radius of its lower base is $10\ cm$ and that of the upper base is $15\ cm$.



   If the height of this container is $20\ cm$, then which one of the following is its volume?

   A. $5000\pi\ cm^3$ B. $\frac{9500}{3}\pi\ cm^3$ C. $7500\pi\ cm^3$ D. $\frac{2500}{3}\pi\ cm^3$

4. The volume of a pyramid that has height of 8 in and a rectangular base of dimension 6 in by 4in is

   A. 576 in^3 B. 192 in^3 C. 96 in^3 D. 64 in^3

5. The diameter of the base and the height of a circular cone are found to be *a* and *2b* units long respectively. What is the formula for the volume V of the cone?

   A. $V=\frac{2}{3}\pi a^2 b$ B. $V=\frac{1}{3}\pi a^2 b$ C. $V=\frac{1}{6}\pi a^2 b$ D. $V=\frac{4}{3}\pi a^2 b$