NUMERICAL METHODS

ERROR ANALYSIS

An error refers to the degree of deviation from the exact value.
ie Error, e = Actual value − Approximate value
∴ Actual value = Approximate value + error

Types of errors

① Random errors

These are errors that occur due to either machine or human failure. Such errors can not be treated numerically.

② Rounding errors

These are errors that arrise when a numerical value is approximated. ie when rounded off or truncated to a given number of decimal places.

Examples

① Truncate the following to 2 decimal places.

(a) 0·384 (b) 0·3294

[0·38, 0·32]

② Round off the following to 2 decimal places.

(a) 1·23500
(b) 1·23400
(c) 1·25312
(d) 1·255

[1·24, 1 23, 1·25, 1 25]

③ Find the approximate value of $e^{0.6}$.

Soln

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$= 1 + 0.6 + \frac{0.6^2}{2} + \frac{0.6^3}{6} + \dots$$

$\simeq 1.816$ (3dps)

Basic terms used

(i) Absolute error

This is the magnitude of the error in a given function.

i.e $|\delta x| = |$Actual value $-$ Approximate value$|$

If X is approximated as x, then $\delta x = X - x$.

(ii) Relative error

This is the ratio of the absolute error to the exact value.

i.e Relative error $= \dfrac{|\delta x|}{x}$

(iii) Percentage error

This is the relative error expressed as a percentage.

ie Percentage error $= \dfrac{|\delta x|}{x} \times 100\%$

(iv) Maximum absolute error (MAE)

This is the maximum possible value of the magnitude of the error in a given function. It can also be termed as the tolerance error abbreviated as Tol.

__NB__:

For a number rounded off to n decimal places, MAE $= 0.5 \times 10^{-n}$

For any function, MAE $= \dfrac{(\text{Maximum value} - \text{Minimum value})}{2}$

(v) Limits (Range) of accuracy

These are the values that restrict the region beyond which the exact value of the function can not lie. These values bind the range of values for the actual value of a function.

The minimum value is called the

lower limit and the maximum value is called the upper limit.

Minimum value = Working value − MAE

Maximum value = Working value + MAE

Range of accuracy = [Minimum value , Maximum value]

NB:

If the actual value lies between a and b, then the range of accuracy can be described as [a, b] or ($a \le W{\cdot}V \le b$) where $a = W{\cdot}V - MAE$ and $b = W{\cdot}V + MAE$.

Propagation of errors

① Addition

let $z = x + y$, where x, y and z are approximations of X, Y and Z respectively.

$\Rightarrow X = x + \delta x$, $Y = y + \delta y$ and $Z = z + \delta z$

$$(z + \delta z) = (x + \delta x) + (y + \delta y)$$

$$\delta z = \delta x + \delta y$$

$$|\delta z| = |\delta x + \delta y|$$

from the triangle of inequalities,

$$|\delta x + \delta y| \le |\delta x| + |\delta y|$$

$$\therefore |\delta z| \le |\delta x| + |\delta y|$$

⑪ Subtraction

let $z = x - y$, where x, y and z are approximations of X, Y and Z respectively.

$\Rightarrow X = x + \delta x$. $Y = y + \delta y$ and $Z = z + \delta z$

$$(z + \delta z) = (x + \delta x) - (y + \delta y)$$

$$\delta z = \delta x - \delta y$$

$$|\delta z| = |\delta x - \delta y|$$

from the triangle of inequalities,

$$|\delta x - \delta y| \le |\delta x| + |-\delta y| \le |\delta x| + |\delta y|$$

$$\therefore |\delta z| \le |\delta x| + |\delta y|$$

(III) Multiplication

Let $z = xy$, where x, y and z are approximations of X, Y and Z respectively.

$\Rightarrow X = x + \delta x$, $Y = y + \delta y$ and $Z = z + \delta z$

$$(z + \delta z) = (y + \delta y)(x + \delta x)$$

$$= xy + x\delta y + y\delta x + \delta x \delta y$$

Since $\delta x \simeq 0$ and $\delta y \simeq 0$, $\delta x \delta y \simeq 0$

$$\Rightarrow z + \delta z = xy + x\delta y + y\delta x$$

$$\delta z = x\delta y + y\delta x$$

$$|\delta z| = |x\delta y + y\delta x|$$

From the triangle of inequalities,

$$|x\delta y + y\delta x| \leq x|\delta y| + y|\delta x|$$

$$\therefore |\delta z| \leq x|\delta y| + y|\delta x|$$

$$\text{Relative error} = \frac{|\delta z|}{z}$$

$$= \frac{x|\delta y| + y|\delta x|}{xy}$$

$$= \frac{|\delta x|}{x} + \frac{|\delta y|}{y}$$

(IV) Division

Let $z = \frac{x}{y}$, where x, y and z are approximations of X, Y and Z respectively.

$\Rightarrow X = x + \delta x$, $Y = y + \delta y$ and $Z = z + \delta z$

$$z + \delta z = \frac{x + \delta x}{y + \delta y}$$

$$= \frac{(x + \delta x)(y - \delta y)}{(y + \delta y)(y - \delta y)}$$

$$= \frac{xy + y\delta x + x\delta y + \delta x \delta y}{y^2 - y\delta y + y\delta y - (\delta y)^2}$$

Since $\delta x \simeq 0$ and $\delta y \simeq 0$, $\delta x \delta y \simeq 0$ and $(\delta y)^2 \simeq 0$

$$\Rightarrow z + \delta z = \frac{xy + y\delta x + x\delta y}{y^2}$$

$$= \frac{x}{y} + \frac{\delta x}{y} - \frac{x\delta y}{y^2}$$

$$\delta z = \frac{\delta x}{y} - \frac{x\,\delta y}{y^2}$$

$$|\delta z| = \left|\frac{\delta x}{y} - \frac{x\,\delta y}{y^2}\right|$$

from the triangle of inequalities,

$$\left|\frac{\delta x}{y} - \frac{x\,\delta y}{y^2}\right| \le \frac{|\delta x|}{y} + \frac{x}{y^2}|\delta y|$$

$$\therefore \quad |\delta z| \le \frac{|\delta x|}{y} + \frac{x|\delta y|}{y^2}$$

$$\text{Relative error} = \frac{|\delta z|}{z}$$

$$= \frac{\left(\frac{|\delta x|}{y} + \frac{x|\delta y|}{y^2}\right)}{x/y}$$

$$= \frac{|\delta x|}{x} + \frac{|\delta y|}{y}$$

(V) Error in a differentiable function

Let $y = f(x)$, where y and x are approximations of Y and X respectively.

$$\Rightarrow \quad Y = y + \delta y \quad \text{and} \quad X = x + \delta x$$

$$y + \delta y = f(x + \delta x)$$

From Taylor's series,

$$f(x+\delta x) = f(x) + \delta x f'(x) + \frac{(\delta x)^2 f''(x)}{2!} + \cdots$$

$$\Rightarrow \quad y + \delta y = f(x) + \delta x f'(x) + \frac{(\delta x)^2 f''(x)}{2!} + \cdots$$

Since $\delta x \approx 0$, $(\delta x)^2$ and higher powers also approximate to zero.

$$\Rightarrow \quad y + \delta y = f(x) + \delta x f'(x)$$

$$\delta y = \delta x f'(x)$$

$$|\delta y| = |\delta x| f'(x)$$

$$\text{Relative error} = \frac{|\delta y|}{y}$$

$$= \frac{|\delta x| f'(x)}{f(x)}$$

Examples

① Given that $x = 0.184$. Determine the lower and upper limit of the x values

Solution

$$MAE = 0.5 \times 10^{-3}$$
$$= 0.0005.$$

$\Rightarrow$ Lower limit $= 0.184 - 0.0005$
$= 0.1835$

Upper Limit $= 0.184 + 0.0005$
$= 0.1845$

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② If $x = 5.356$ and $y = 6.81$ where both numbers are rounded, find the Minimum and Maximum values of ;

(a) $y - x$ (b) $\frac{y}{x}$

③ Given that $M = 2.12$, $y = 3.8$ and $Z = 0.31$ where all numbers are round, state the range of accuracy for $\frac{M^2 + y}{y + z}$

④ Given that the error in measuring an angle is $0.5°$, find the maximum possible error in $\frac{\sin x}{\cos x}$ for $x = 30°$

Soln

Let $a = \sin x$ and $b = \cos x$

$\Rightarrow |\delta a| = |\delta x| \cos x$ and $|\delta b| = -|\delta x| \sin x$

Maximum possible error $= \frac{|\delta a|}{a} + \frac{|\delta b|}{b}$

$$= \frac{|\delta x| \cos x}{\sin x} + \frac{|-\delta x| \sin x}{\cos x}$$

$$= |\delta x| [\cot x + \tan x]$$

$$= \frac{\pi}{360} [\cot 30 + \tan 30]$$

$$= 0.02015$$

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⑤ Given that $y = x^n$, find the expression for the relative absolute error in y.

Soln

Let $y = f(x)$; $f'(x) = 4x^3$

$$\Rightarrow \delta y = 4|\delta x|x^3$$

$\therefore$ Relative error $= \dfrac{4|\delta x|x^3}{x^4}$

$$= 4\left|\frac{\delta x}{x}\right|$$

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⑥ The area of a triangle of sides $a = 2.4 \pm 0.2$ and $b = 3.4 \pm 0.1$ is given by $|A| = |a||b|\sin\theta$ where $\theta = 30 \pm 0.5$. Find the percentage error in the area of the triangle.

Soln

Actual area, $A = ab\sin\theta$

$= 2.4 \times 3.4 \times \sin 30$

$= 4.0800$ sq. units

Minimum area, $A_{min} = a_{min} \times b_{min} \sin\theta_{min}$

$= 2.2 \times 3.3 \sin 29.5$

$= 3.5750$ sq. units

$MAE = \dfrac{|\delta a|}{a} + \dfrac{|\delta b|}{b} + \cot\theta|\delta\theta|$

Maximum area, $A_{max} = a_{max} \times b_{max} \sin\theta_{max}$

$= 2.6 \times 3.5 \times \sin 30.5$

$= 4.6186$ sq. units

Maximum absolute error $= \dfrac{A_{max} - A_{min}}{2}$

$= \frac{1}{2}(4.6186 - 3.5750)$

$= 0.5218$

%tage error $=$ Relative error $\times 100\%$

$= \dfrac{0.5218}{4.08} \times 100\%$

$= 12.79\%$

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⑦ The quantities X and Y are approximated as x and y with errors δx and δy respectively. Show that the

Relative error in using $x\sqrt{y}$ to approximate $X\sqrt{Y}$ is given by:

$$\left|\frac{\delta x}{x}\right| + \frac{1}{2}\left|\frac{\delta}{y}\right|$$

Soln

let $Z = X\sqrt{Y}$

$\Rightarrow \quad Z + \delta Z = (x+\delta x)(y+\delta y)^{\frac{1}{2}}$

$= (x+\delta x)\left(1+\frac{\delta y}{y}\right)^{\frac{1}{2}} y^{\frac{1}{2}}$

$= (x+\delta x)\left(1+\frac{1}{2}\left(\frac{\delta y}{y}\right) - \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\delta y}{y}\right)^2 + \cdots\right) y^{\frac{1}{2}}$

Since $\delta y \simeq 0$, $(\delta y)^2$ and higher powers also approximate zero.

$\Rightarrow \quad Z + \delta Z = (x+\delta x)\left(1+\frac{\delta y}{y}\right) y^{\frac{1}{2}}$

$$= x y^{\frac{1}{2}} + \frac{x y^{\frac{1}{2}} \delta y}{2y} + y^{\frac{1}{2}} \delta x + \frac{\delta x \delta y\, y^{\frac{1}{2}}}{2y}$$

Also $\delta x \delta y \simeq 0$

$\Rightarrow \quad Z + \delta Z = x y^{\frac{1}{2}} + \frac{x y^{\frac{1}{2}} \delta y}{2y} + y^{\frac{1}{2}} \delta x$

$$\delta Z = \frac{x\sqrt{y}\,\delta y}{2y} + \sqrt{y}\,\delta x$$

$$|\delta Z| = \frac{x\sqrt{y}|\delta y|}{2y} + \sqrt{y}|\delta x|$$

Relative error $= \frac{|\delta Z|}{Z}$

$$= \frac{\left(\frac{x\sqrt{y}|\delta y|}{2y} + \sqrt{y}|\delta x|\right)}{x\sqrt{y}}$$

$$= \frac{|\delta y|}{2y} + \frac{|\delta x|}{x}$$

$\therefore$ Relative error $= \left|\frac{\delta x}{x}\right| + \frac{1}{2}\left|\frac{\delta y}{y}\right|$

Alternatively;

let $Z = X\sqrt{Y}$

$\Rightarrow \quad Z + \delta Z = (x+\delta x)(y+\delta y)^{\frac{1}{2}}$

$(Z+\delta Z)^2 = (x+\delta x)^2(y+\delta y)$

$Z^2 + 2Z\delta Z + \delta Z^2 = (x^2 + 2x\delta x + \delta x^2)(y+\delta y)$

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Since $\delta z \simeq 0$ and $\delta x \simeq 0$, $\delta z^2 \simeq 0$ and $\delta x^2 \simeq 0$

$$\Rightarrow z^2 + 2z\delta z = (x^2 + 2x\delta x)(y + \delta y)$$
$$= x^2 y + 2xy\delta x + x^2 \delta y + 2x\delta y\delta x$$

Also $\delta y \delta x \simeq 0$

$$\Rightarrow z + 2z\delta z = x^2 y + 2xy\delta x + x^2 \delta y$$
$$\delta z = \frac{xy^{\frac{1}{2}}}{2} + \sqrt{y}\delta x + \frac{x}{2\sqrt{y}}\delta y - \frac{xy^{\frac{1}{2}}}{2}$$
$$= \sqrt{y}\delta x + \frac{x\delta y}{2\sqrt{y}}$$
$$|\delta z| = \sqrt{y}|\delta x| + \frac{x|\delta y|}{2\sqrt{y}}$$

$$\text{Relative error} = \frac{|\delta z|}{z}$$
$$= \frac{\left(\sqrt{y}|\delta x| + \frac{x|\delta y|}{2\sqrt{y}}\right)}{x\sqrt{y}}$$
$$= \left|\frac{\delta x}{x}\right| + \frac{1}{2}\left|\frac{\delta y}{y}\right|$$

Exercise

① The numbers A and B are rounded off to a and b with errors e_1 and e_2 respectively.

(i) Show that the maximum relative error made in the approximation of $\frac{A}{B}$ by a/b is given by $\left|\frac{e_1}{a}\right| + \left|\frac{e_2}{b}\right|$

(ii) If also the number C is rounded off to c with an error e_3, deduce the expression for the maximum relative error in taking the approximation of $\frac{A}{B+C}$ as $\frac{a}{b+c}$ in terms of e_1, e_2, e_3, a, b and c

$$\left(MAE = \left|\frac{e_1}{a}\right| + \left|\frac{|e_2| + |e_3|}{b+c}\right|\right)$$

(iii) Given that $a = 42.326$, $b = 27.26$ and $c = -12.19$ are rounded off to the given number of decimal places, find the range of values within which the exact value of the expression $\frac{A}{B+C}$ lies.

(Exact value ± 0.0021)

② The numbers $a = 23.037$ and $b = 8.4658$ are rounded off to the number of decimal places indicated.

(i) State the maximum possible errors in a and b.

(ii) Determine the absolute error in a/b

(iii) Find the range of values within which $\frac{a}{b}$ lies correct to 4 decimal places.

③ Given that $|\delta x| = E_1$ and $|\delta y| = E_2$, show that the maximum relative error made in approximating x^2y as X^2Y is given by;

$$2\left|\frac{E_1}{X}\right| + \left|\frac{E_2}{Y}\right|$$

④ If the errors in each of the values e^{x} and e^{-x} is ± 0.0005, find the minimum and maximum values of the quotient e^{x}/e^{-x} when $x = 0.04$ giving your answer to 5dps.

⑤ Given that $x = 2.4$, $y = 5.4$ and $z = 1.8$ where all numbers are rounded, find the maximum absolute error in $\frac{\sqrt{z}}{x^2y^3}$

(0.000123)

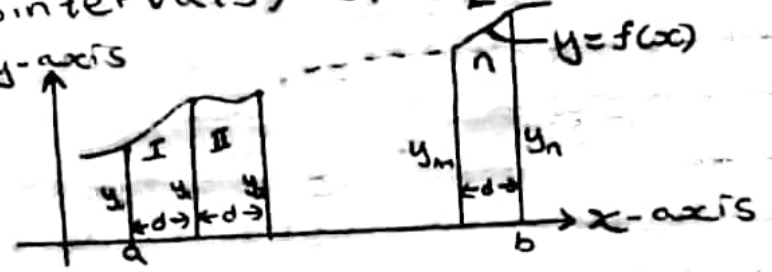
⑥ Given that $A = |x||y|\sin\theta$, deduce that the maximum possible relative error in A is given by $\left|\frac{\delta x}{x}\right| + \left|\frac{\delta y}{y}\right| + |\delta\theta|\cot\theta$

⑦ Given that $Y = 10^{2X}$ and $X = 0.9$. Find the range of values within which the actual value of y lies.

THE TRAPEZIUM RULE

This is a numerical method that can be used to calculate the approximate area under the curve $y = f(x)$ between $x = a$, $x = b$ and the x-axis.

To obtain the area under the curve using the trapezium rule, the area under the curve is divided into stripes (subintervals) of equal width d.



When the tops of the stripes are joined by straight lines, they approximate to trapeziums.

Area of trapezium I $= \frac{1}{2}d(y_0 + y_1)$

The total area of the trapeziums is therefore given by;

$$A \approx \frac{1}{2}d\left[(y_0+y_1) + (y_1+y_2) + \cdots + (y_{n-1}+y_n)\right]$$

$$= \frac{1}{2}d\left[(y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-1})\right]$$

$\therefore$ In general;

$$A = \int_a^b f(x)\,dx \approx \frac{1}{2}d\left[(y_0+y_n) + 2(y_1+y_2+\cdots y_{n-1})\right]$$

The expression above is the trapezium rule.

<u>NB</u>:

(1) For n ordinates, there are $(n-1)$ sub intervals.

(II) For n subintervals, the width d is given by ; $d = \frac{b-a}{n}$.

(III) For n ordinates, the width d is given by ; $d = \frac{b-a}{n-1}$ ie $\begin{pmatrix}\text{No of}\\ \text{Subintervals}\end{pmatrix} = \begin{pmatrix}\text{No of}\\ \text{ordinates}\end{pmatrix} - 1$

(IV) The accuracy depends on the number of subintervals used ie the higher the number of sub intervals, the greater the accuracy.

(V) The actual (exact) area under the curve is obtained from Calculus. ie $A = \int_a^b y\,dx = \int_a^b f(x)\,dx$

Examples

① Use the trapezium rule, with 5 subintervals to workout $\int_0^1 x^2 e^x\,dx$

Soln

$d = \frac{1-0}{5}$

$= 0.2$

x	y	
0.0	0.00000	
0.2		0.04886
0.4		0.23869
0.6		0.65596
0.8		1.42435
1.0	2.71828	
Total	2.71828	2.36786

$\int_a^b f(x)\,dx \simeq \frac{1}{2}d\left[(y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-1})\right]$

$\therefore \int_0^1 x^2 e^x\,dx \simeq \frac{1}{2}\times 0.2\left[2.71828 + 2(2.36786)\right]$

$= 0.74540$

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Exercise

① Use the trapezium rule with 6 ordinates to workout:

(i) $\int_0^4 \frac{dx}{1+x}$ (ii) $\int_0^1 e^{-x^2}dx$

(iii) $\int_0^{\frac{1}{4}\pi} \sec x\, dx$ (iv) $\int_0^{\pi/2} \cos x\, dx$

②(i) Use the trapezium rule with 5 sub intervals to estimate the area of $y = 3^x$ between the x-axis and the lines $x = 1$ and $x = 2$.

(ii) What is the exact value of $\int_1^2 3^x dx$?

(iii) Find the percentage error in the calculations in (i) and (ii) above and state how it can be reduced.

③(a) Use the trapezium rule with 5 sub intervals to estimate the area of $y = 5^{2x}$ between the x-axis and the lines $x = 0$ and $x = 1$.

(b) Find the exact value of $\int_0^1 5^{2x} dx$.

(c) Determine the percentage error in the two calculations in (a) and (b) above.

(d) How can the error be reduced?

④(a) Use the trapezium rule with 8 subintervals to estimate $\int_2^4 \frac{10}{2x+1}dx$ correct to 4 decimal places.

(b) Determine the percentage error in the estimation and state how it can be reduced.

⑤(a) Use the trapezium rule with 5 sub intervals to estimate $\int_0^4 \frac{1}{1+x^2}dx$ to 3 dps.

(b) Determine the percentage error in your estimate. How can it be reduced?

* Talk about Simpson's rule ie $\int_a^b f(x)dx \approx \frac{1}{3}h(y_0 + 4y_1 + y_2)$, where $h = \frac{1}{2}(b-a)$

APPROXIMATE NUMERICAL METHODS

These are numerical methods used to locate the position of the real root of a function in the form $f(x)=0$.

They include graphical and analytical methods.

(a) Analytical method

If $y=f(x)$ and $f(a)\cdot f(b)<0$ (i.e negative), then there exists a real root for $y=f(x)=0$ between $x=a$ and $x=b$.

NB:

a and b are values of x not y.

Examples

① Show that there exists a real root for the function $2x^2+3x-3=0$ between $x=-3$ and $x=-2$

Soln

Let $f(x)=2x^2+3x-3$; $f(x)=0$

$f(-3)=2(-3)^2+3(-3)-3 = 6\ (+ve)$

$f(-2)=2(-2)^2+3(-2)-3 = -1\ (-ve)$

$f(-2)\cdot f(-3) = -6$ i.e negative.

Since $f(-3)\cdot f(-2)<0$, there exists a real root for the function given by $2x^2+3x-3=0$ between $x=-3$ and $x=-2$.

Exercise

① Show that $x+\log_e x=0.5$ has a real root between $x=0.5$ and $x=1$.

② Show that there exists a real root for $2x^2=6x+3$ in the range $[3,4]$

③ Show that $x+e^x=0$ has a real root between $x=0$ and $x=-1$.

(b) Graphical method

When a graph of $y = f(x)$ where $f(x) = 0$ is plotted, it will cut the x-axis between two x values between which a real root of the function lies.

When the function $f(x)$ is split up into two functions which are then plotted on the same axes, their graphs will intersect between two x values between which a real root of the function lies.

Examples

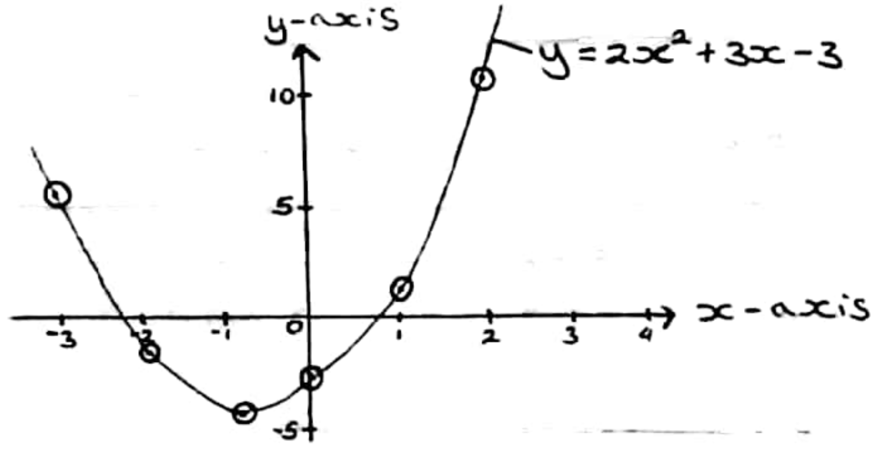
① Use a graphical method to locate the intervals within which the real roots of the equation $2x^2 + 3x - 3 = 0$ lie for the range $-3 \le x \le 3$.

Soln

let $y = 2x^2 + 3x - 3$; $y = 0$

Table of values

x	-3	-2	-1	0	1	2	3
y	6	-1	-4	-3	2	11	24



$\therefore$ There exists a real root for the equation $2x^2 + 3x - 3 = 0$ between $x = 0$ and $x = 1$ and between $x = -3$ and $x = -2$.

Exercise

① Use a graphical method for $-4 \le x \le 4$ to locate the positions of the real roots of;

(i) $2x^2 - 6x - 3 = 0$ (ii) $xe^x + 5x = 10$ (iii) $e^x + x^3 = 4x$

ITERATIVE NUMERICAL METHODS

These are numerical methods used to find a better approximation of the equation in the form $f(x)=0$.

In such methods, a sequence of approximations $x_0, x_1, x_2, x_3, \ldots, x_n$ is found, each subsequent one being closer to the real root of $f(x)=0$ than the previous one.

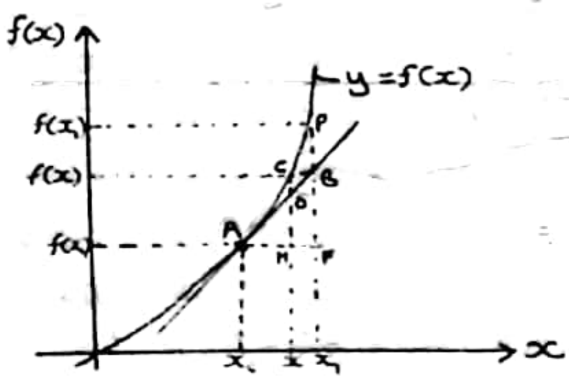
(a) LINEAR INTERPOLATION / EXTRAPOLATION

This is a numerical method used to find an approximate value of the function which lies either between two known values (interpolation) or adjacent to two known values (extrapolation).

Consider the table below showing x values and their corresponding values of $f(x)$.

Value of x	x_0	x	x_1
Value of $f(x)$	$f(x_0)$	$f(x)$	$f(x_1)$

A graph of $f(x)$ against x may be assumed to have the shape below.



If we let the line ADB approximate the curve ACP, then D will be close to C and B will be close to P. It follows that HD ≃ HC and FB ≃ FP

Triangles AHD and ABF are similar,

$$\Rightarrow \frac{AH}{AF} = \frac{HD}{FP} \approx \frac{HC}{FP}$$

$$\therefore \frac{x - x_0}{x_1 - x_0} = \frac{f(x) - f(x_0)}{f(x_1) - f(x_0)}$$

The expression above is the expression used when linear interpolating or extrapolating.

Examples

① The table below shows the values of temperature θ at different times T.

T (s)	0	120	240	360	480	600
θ (°c)	100	80	75	65	56	48

Use linear interpolation or extrapolation to determine;

(i) θ when $T = 370$s (ii) θ when $T = 720$s

(iii) T when $\theta = 70$°c (iii) T when $\theta = 38$°C

Soln

(i) Extract

T (s)	360	370	480
θ (°c)	65	θ	56

From, $\dfrac{f(x_1) - f(x_0)}{f(x) - f(x_0)} = \dfrac{x_1 - x_0}{x - x_0}$

$$\frac{480 - 360}{370 - 360} = \frac{56 - 65}{\theta - 65}$$

$$\theta = 64.25°c$$

Exercise

① The table below shows the values of x and their corresponding values of $f(x)$

x	2	3	4	5	6
$f(x)$	3.88	5.11	8.14	11.94	12.23

Use linear interpolation or extrapolation to determine the value of ;
(i) $f(x)$ when $x = 2.15$ (ii) x when $f(x) = 10.72$

② In the table below is an extract of $\sec x$.

$x = 60°$	$0'$	$12'$	$24'$	$36'$	$48'$
$\sec x$	2.0000	2.0122	2.0242	2.0371	2.0498

Use linear interpolation to estimate the;
(i) Value of $\sec 60°15'$
(ii) angle whose secant is 2.0436.

③ A physics - Mathematics teacher is confident that there is a linear relationship between his class' performance in physics and mathematics. He marks all papers of physics and only two mathematics papers. On realising that a student who scored 59% in physics scored 72% in Mathematics while the one who scored 76% in phsics had 81% in Mathematics, he mathematically predicted the results of the rest of the students. Find the teacher's prediction for 34% and 91%.

Soln

PHYSICS	59	76
MATHS	72	81

④ Show that the equation $e^x - 2x = 1$ has a real root between $x = 1$ and $x = 2$ and hence use linear interpolation to estimate the root to two decimal places

⑤ Show that $x + e^x = 0$ has a real root between $x = 0$ and $x = -1$ hence use linear interpolation to estimate the root correct to 3 decimal places.

(b) THE GENERAL ITERATIVE METHOD

This is the method used to determine the root of a differentiable function $f(x) = 0$ by expressing $f(x) = 0$ into several functions of the form $x = g(x)$ and considering the one whose successive roots tend to converge.

NB:

The successive roots of the function will converge if and only if $|g'(x)| < 1$ otherwise the roots will diverge and the function under consideration will not be appropriate for finding the root.

Examples

① Determine the general iterative formula for determining the root of the equation $x^3 - 3x - 12 = 0$.

Soln

Let $x^3 - 3x - 12 = f(x)$

$$\Rightarrow f(x) = 0$$

$$3x = x^3 - 12$$

$$\Rightarrow x_{n+1} = \frac{x_{n+1}^3 - 12}{3}$$

OR

$$x_{n+1} = \sqrt[3]{3x_n + 12}$$

OR

$$x_{n+1} = \frac{12}{x_n^2 - 3}$$

OR

$$x_{n+1} = \sqrt{3 + \frac{12}{x_n}}$$

OR

$$x_{n+1} = \frac{3x_n + 12}{x_n^2}, \quad n = 0, 1, 2, 3, \ldots$$

② Show that the general iterative formula for solving the equation $x^3 - x - 1 = 0$ is given by $x_{n+1} = \sqrt{\left(1 + \frac{1}{x_n}\right)}$ where $n = 0, 1, 2, \ldots$

<u>Soln</u>

$$x^3 - x - 1 = 0$$

$$x^3 = x + 1$$

$$x^2 = 1 + \frac{1}{x}$$

$$\Rightarrow x = \sqrt{1 + \frac{1}{x}}$$

$$\therefore \quad x_{n+1} = \sqrt{1 + \frac{1}{x_n}}, \quad n = 0, 1, 2, \ldots$$

---

③ Determine the general iterative formula for finding the root of the equation $3x^2 - e^x = 0$ that lies between 0 and 1.

<u>Soln</u>

$$x_{n+1} = \left(\frac{e^{x_n}}{3}\right)^{\frac{1}{2}} \quad \text{or} \quad x_{n+1} = \ln(3x_n^2), \quad n = 0, 1, 2, \ldots$$

$$x_0 = \frac{0+1}{2} = 0.5000$$

For $g(x) = \ln(3x^2)$

$$g'(x) = \frac{6x}{3x^2} = \frac{2}{x}$$

$$g'(0.5) = 4$$

$\Rightarrow |g'(x)| > 1$ (Discard)

For $g(x) = \left(\frac{e^x}{3}\right)^{\frac{1}{2}}$

$$g'(x) = \frac{1}{2\sqrt{3}} e^{\frac{1}{2}x}$$

$$g'(0.5) = 0.371$$

$\Rightarrow |g'(x)| < 1$

$\therefore$ The general iterative formula for finding the root of the equation $3x^2 - e^x = 0$ is $x_{n+1} = \left(\frac{e^{x_n}}{3}\right)^{\frac{1}{2}}, \quad n = 0, 1, 2, \ldots$

<u>Exercise</u>

① The positive root of the equation $e^x - 2x - 1 = 0$ lies between 1.5 and 1.8. Use each of the formulae below twice to find a better root.

Formula (I) $x_{n+1} = \frac{1}{2}(e^{x_n} - 1)$

(II) $x_{n+1} = \dfrac{e^{x_n}(x_n - 1) + 1}{e^{x_n} - 2}$

State, with a reason, which of the two formulae is appropriate for find the root.

(c) BISECTION ALOGARITHM (INTERVAL BISECTION)

This is an alogarithm that is used to determine a better root of the equation in the form $f(x) = 0$ by taking the average of the limits of the interval within which the root lies.

i.e if $f(x) = 0$ and $f(a)\cdot f(b) < 0$ then $f(x) = 0$ has a root between $x = a$ and $x = b$. An estimate of the root using bisection alogarithm is given by $x_0 = \frac{a+b}{2}$. The intervals are rearranged the the process of bisection is continued until the best estimate to the root is obtained.

Examples

① Show that $x^3 - x - 2 = 0$ has a root between $x = 1$ and $x = 2$. Use the bisection alogarithm to estimate the root correct to 2 d.ps.

Soln

$f(x) = x^3 - x - 2 = 0$

$f(1) = 1^3 - 1 - 2 = -2$

$f(2) = 2^3 - 2 - 2 = 4$

$f(1)f(2) < 0$ i.e $-ve$

$\Rightarrow x^3 - x - 2 = 0$ has a root between $x = 1$ and $x = 2$.

let $x_0 = \frac{1}{2}(1+2) = 1.5$

$f(1.5) = 1.5^3 - 1.5 - 2 = -0.125$

$f(1.5)f(2) < 0$

$\Rightarrow$ The root lies between $x = 1.5$ and $x = 2$

$x_1 = \frac{1}{2}(1.5+2) = 1.75$; $|error| = 0.25$

$f(1.75) = 1.75^3 - 1.75 - 2 = 1.6094$

$f(1.5)f(1.75) < 0$

$\Rightarrow$ The root lies between $x = 1.5$ and $x = 1.75$

$x_2 = \frac{1}{2}(1.5 + 1.75) = 1.625$; $|error| = 0.125$

$f(1.625) = 1.625^3 - 1.625 - 2 = 0.666$

$f(1.5)f(1.625) < 0$

$\Rightarrow$ The root lies between $x = 1.5$ and $x = 1.625$

$x_3 = \frac{1}{2}(1.5 + 1.625) = 1.5625$; $|error| = 0.06$

$f(1.5625) = 1.5625^3 - 1.5625 - 2 = 0.252$

$f(1.5)f(1.5625) < 0$

$\Rightarrow$ The root lies between $x = 1.5$ and $x = 1.5625$

$x_4 = \frac{1}{2}(1.5 + 1.5625) = 1.5312$; $|error| = 0.03$

$f(1.5312) = 1.5312^3 - 1.5312 - 2 = 0.059$

$f(1.5)f(1.5312) < 0$

$\Rightarrow$ The root lies between $x = 1.5$ and $x = 1.5312$

$x_5 = \frac{1}{2}(1.5 + 1.5312) = 1.5156$; $|error| = 0.015$

$f(1.5156) = 1.5156^3 - 1.5156 - 2 = -0.034$

$\Rightarrow$ The root lies between $x = 1.5156$ and $x = 1.5312$

$x_6 = \frac{1}{2}(1.5156 + 1.5312) = 1.5234$; $|error| = 0.008$

$f(1.5234) = 1.5234^3 - 1.5234 - 2 = 0.012$

$f(1.5234)f(1.5156) < 0$

$\Rightarrow$ The root lies between $x = 1.5156$ and $x = 1.5234$

$x_7 = \frac{1}{2}(1.5156 + 1.5234) = 1.5195$; $|error| = 0.0039$

Since $|error| < 0.005$, 1.5195 is a better root.

$\therefore$ The root is 1.52

Exercise

① Show by plotting suitable graphs on the same coordinate axes or otherwise that the root of the equation $e^x - 2x + 1 = 0$ lies between $x = 1$ and $x = 1.5$. Use bisection alogarithm to find the root correct to 3 decimal places.

② By plotting graphs of $y = \sin x$ and $y = \frac{1}{2}$ on the same axes, show that the root of the equation $\sin x - \frac{1}{2} = 0$ lies between $x = 1.5$ and $x = 2$. Use bisection alogarithm to find the root correct to 2 decimal places.

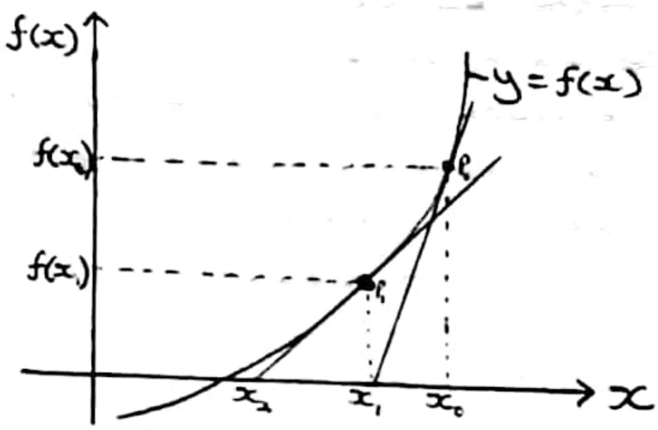
(d) NEWTON'S RAPHSON'S METHOD

If x_n is the initial approximation for the root of the equation $f(x)=0$, then a better approximation x_{n+1} can be obtained from the expression; $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

e.g $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ and $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

Geometrical derivation of Newton's Raphson's Method.

Consider the graph of $y=f(x)$ shown below.



Let $P_0(x_0, f(x_0))$ and $P_1(x_1, f(x_1))$ be two close points on the curve.

The tangent at P_0 cuts the x-axis at x_1, a better approximation to the root of the equation $f(x)=0$ than x_0.

Similarly the tangent at P_1 cuts the x-axis at x_2, also a better approximation to the root of the equation $f(x)=0$ than x_1.

Gradient of the tangent at $P_0 = \frac{f(x_0)-0}{x_0 - x_1}$

But $\frac{f(x_0)-0}{x_0 - x_1} = f'(x)$

$\Rightarrow \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ -------- (1)

Gradient of the tangent at $P_1 = \frac{f(x_1) - 0}{x_1 - x_2}$

But $\frac{f(x_1) - 0}{x_1 - x_2} = f'(x_1)$

$\Rightarrow \quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ ------------ (ii)

In general;

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, where $n = 0, 1, 2, 3, \ldots$

Examples

① Using the range $-2 \le x \le 5$, determine the interval within which the real roots of the equation $2x^2 - 6x - 3 = 0$ lie hence use NRM to find the biggest root correct to two decimal places.

Soln

x	-2	-1	0	1	2	3	4	5
$f(x)$	17	5	-3	-7	-7	-3	5	17

There is a root for the equation $2x^2 - 6x - 3 = 0$ between $x = -1$ and $x = 0$ and also between $x = 3$ and $x = 4$.

$\Rightarrow$ The biggest root lies between $x = 3$ and $x = 4$.

Let $2x^2 - 6x - 3 = f(x)$

$\Rightarrow f(x) = 0$ and $f'(x) = 4x - 6$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{2x_n^2 - 6x_n - 3}{4x_n - 6}$$

$$= \frac{2x_n^2 + 3}{4x_n - 6}$$

For the biggest root, $x_0 = \frac{3+4}{2}$

$= 3.50000$

x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
3.50000	3.43750	0.06250
3.43750	3.43649	0.00101

Since $\lvert 3.43649 - 3.43750 \rvert \leq 0.005$, 3.43649 is a better approximation to the root.

$\therefore$ The biggest root is 3.44

② Show that the iterative formula for finding the fourth root of a number N is given by $\frac{3}{4}\left(x_n + \frac{N}{3x_n^3}\right)$ hence estimate $18^{\frac{1}{4}}$ to 3dps

Exercise

① Show that the Newton's Raphson's formula for finding the root of the equation $xe^x + 5x - 10 = 0$ is given by:

$$x_{n+1} = \frac{x_n^2 e^{x_n} + 10}{e^{x_n}(x_n+1) + 5}$$

If the root lies between $x = 1$ and $x = 2$, use NRM to find that root to 3d.ps.

② Show that the root of the equation $e^x + x^3 = 4x$ lies between 1 and 2. Use NRM to find the root correct to 2d.ps.

③ Show that the NRF for approximating the k^{th} root of the number N is given by: $x_{n+1} = \frac{1}{k}\left[(k-1)x_n + \frac{N}{x_n^{k-1}}\right]$

Use your formula to find the positive square root of 67 correct to 4 s.f.

④ If α is an approximate root of the equation $x^2 = n$, show that the iterative formula for the root reduces to $\frac{1}{2}\left(\frac{n}{\alpha} + \alpha\right)$ hence taking $\alpha = 4$, estimate the square root of 17 correct to 3d.ps.

5. Use a gravical method to show that the equation $e^x + x - 4 = 0$ has only one real root. Use NRM to find the root correct to 3 s.fs

6. Show that the equation $x = \ln(8 - x)$ has a root between 1 and 2. Use NRM to find the approximate root correct to 3 d.Ps.

7. Show that the NRF for finding the root of the equation $x = N^{1/5}$ is given by:
$$x_{n+1} = \frac{4x_n^5 + N}{5x_n^4}, \quad n = 0, 1, 2, \ldots$$
Taking $N = 50$ and $x = 2.2$, find the root correct to 3 decimal places.

8. (a) If a is the first approximation to the root of the equation $x^5 - b = 0$, show that the second approximation is $\frac{1}{5}\left(4a + \frac{b}{a^4}\right)$.
(b) Show that the positive root of the equation $x^5 - 17 = 0$ lies between 1.5 and 1.8 hence use the expression in 8(a) above to find the root to 3d.Ps

9. Show that the equation $e^x - 2x - 1 = 0$ has a root between $x = 1$ and $x = 1.5$ hence use NRM to find the root to 3 d.Ps.

10. Find the simplest iterative formula based on Newton Raphson's method for approximating 2. Taking $x_0 = 1$ as the first approximation, find the second approximation.
Let $x = 2 \Rightarrow x^2 = 4 \Rightarrow x_{n+1} = \frac{x_n^2 + 4}{2x_n}$

11. Derive the simplest iterative formula based on NRM for approximating the root of the equation $e^{3x} - 3 = 0$. Starting with $x = \frac{1}{3}$, find the root to 4 s.f hence find $\log_e 3$.
$[\log_e 3 = 1.099]$

FLOW CHARTS

A flow chart is a diagramatic representation of an ordered step by step plan of an alogarithm for executing a given computational procedure.

The steps are represented by various geometrical shapes interconnected by lines called flow lines. A flow line is a straight line with an arrow to indicate the direction in which the computational procedure is executed.

NB:

Flow lines must not cross each other.

Stages in a flow chart

① The start stage

This is the first stage of the flow chart. It indicates the start of a computational procedure.

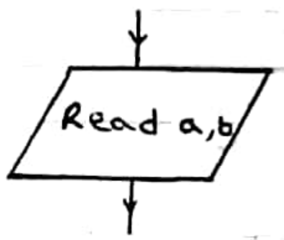
The start stage is represented by a circle.

② The read stage (Infusion stage)

This is where the values of the dummy input are fed into the memory.

The read stage is represented by a parallerogram.

E.g

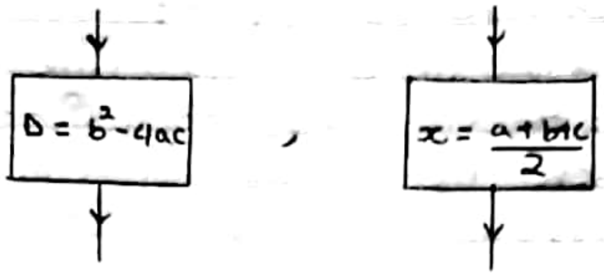


③ The assignment (Arithmetic/Computation) stage

This is where the arithmetic computations are performed.

The assignment stage is representated by a rectangle.

E.g

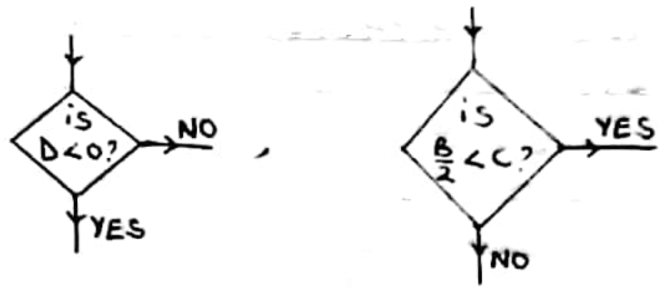


④ The decision (control statement) stage

This is a branching operation from which alternative paths are taken after a suitable decision has been considered basing on the control statement.

The decision stage is represented by a rhombus.

E.g

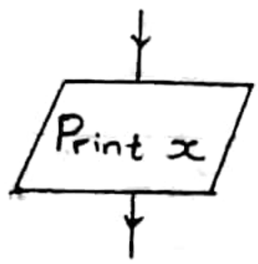


⑤ The print stage

This is where the output values of a computational procedure are displayed for viewing.

The print stage is represented by a parallerogram.

E.g



⑥ The Stop Stage

This is the last stage of a flow chart. It indicates termination of a computational procedure.

The stop stage is represented by a circle.

Dry run of a flow chart

This is a check up plan for a computational procedure that yields an output value from an input value in a finite number of steps.

NB:

If the output value is required to n decimal places, the dry run is operated to n+1 decimal places and the tolerance error (Tol) is 0.5×10^{-n}.

The tolerance error is the magnitude of the maximum acceptable difference between two consecutive output values.

ie $|x_{n+1} - x_n| \leq Tol$

Examples

① Study the flow chart below and answer the questions that follow.

(I) Perform a dry run for the flow chart above and state its purpose.
(II) Write down the relationship between n and A.

Soln

(I)

n	A
0	1
1	1
2	2
3	6
4	24
5	120
6	720
7	5040

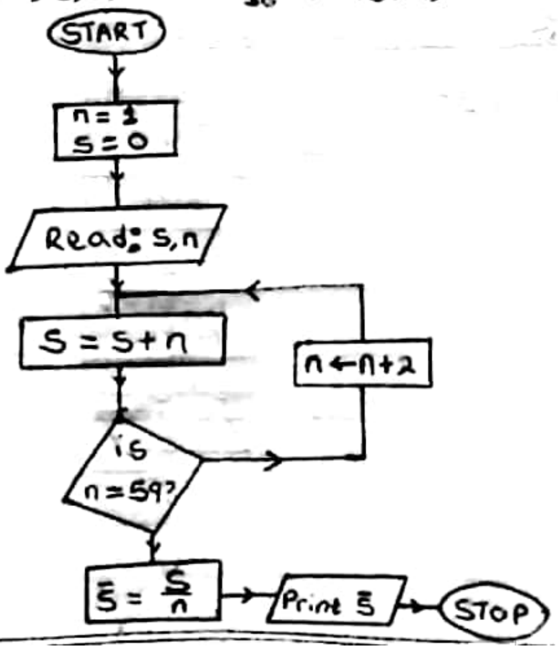
The purpose of the flow chart is to print natural numbers 0 to 7 and to compute and print their factorials.

(II) $A = n!$

② Draw a flow chart that computes and prints the average value of the first 30 odd numbers.

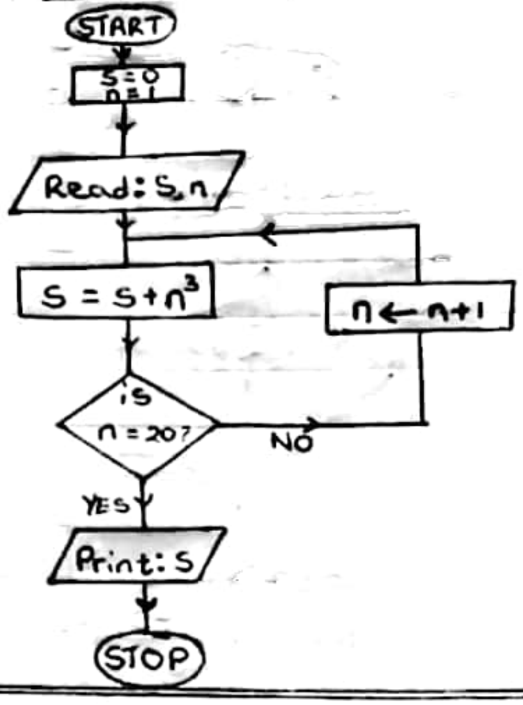
Soln

Odd numbers are 1, 3, ... $\Rightarrow t_{30} = 1 + (30-1) \times 2 = 59$



③ Draw a flow chart that computes and prints the cubes of the first 20 counting numbers

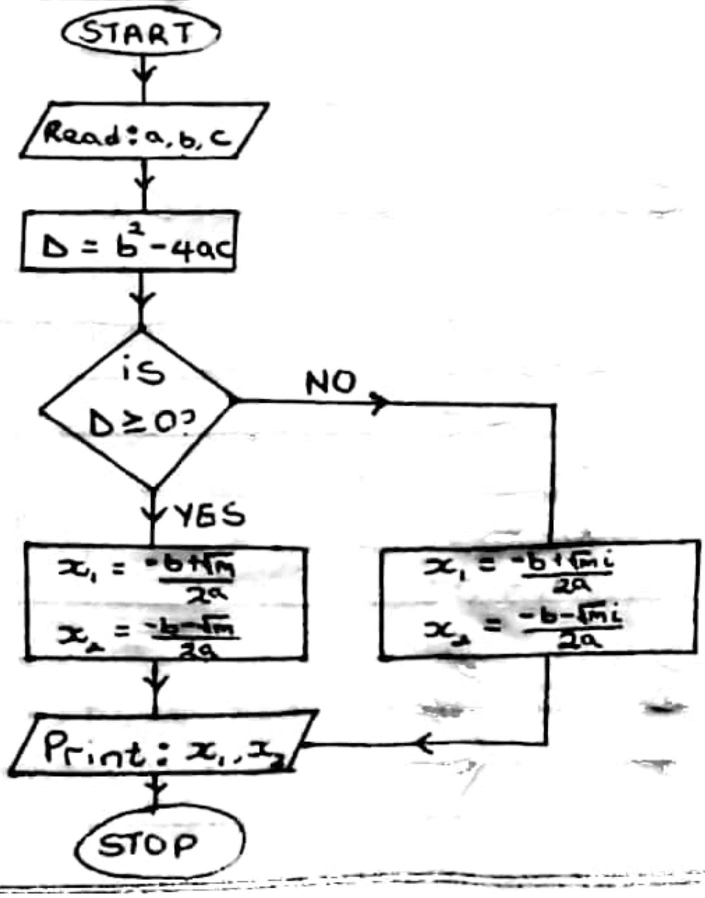
Soln



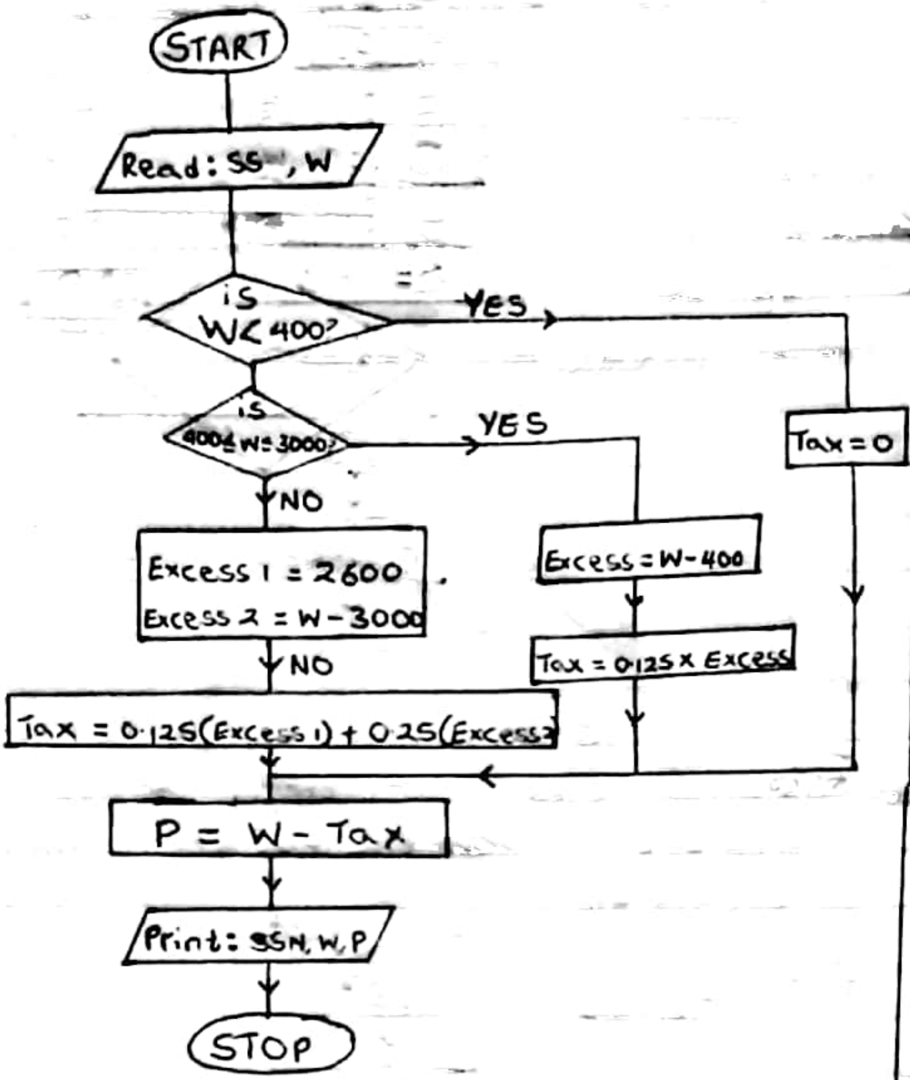
④ Draw a flow chart that computes and prints the roots of the equation $ax^2+bx+c=0$ where $a \neq 0$.

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Soln



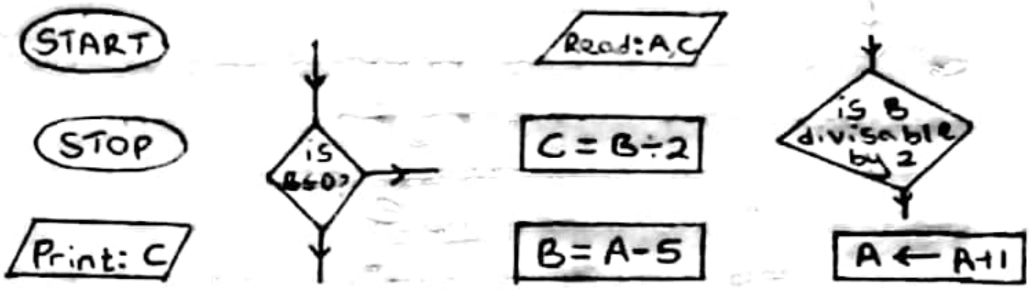
⑤ The flow chart below shows the social security number (SSN) and the monthly wage (W shillings) of an employee whose net pay is P.



Copy and complete the table below.

SSN	W	T	P
280-04	300	0	300
180 - 34	840	55	785
179 - 93	4500	700	3800
80 - 66	5550	970	4610
385 - 03	8000	1575	6425

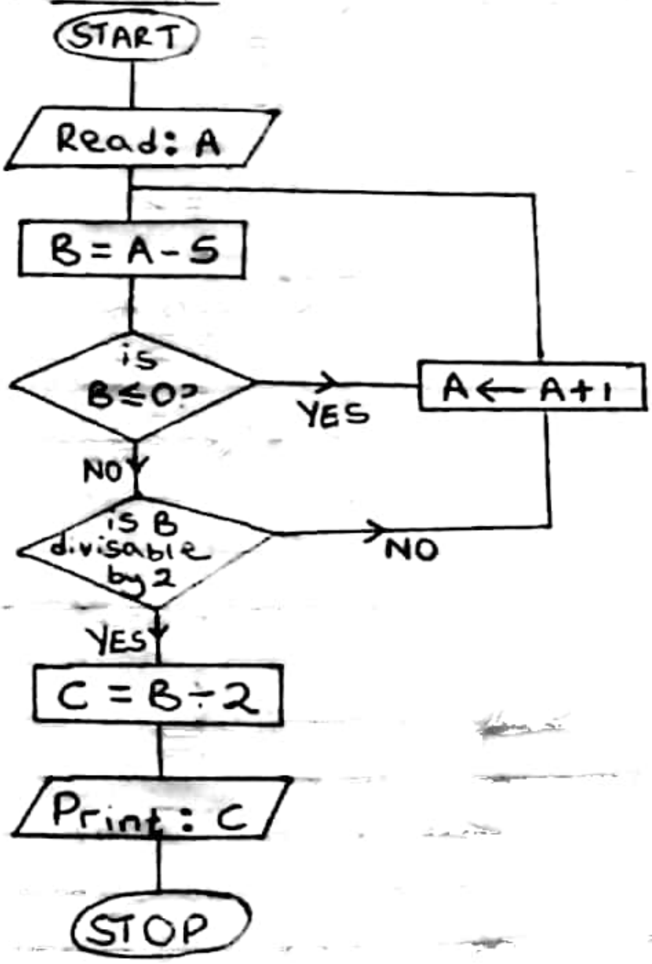
⑥ Given below are stages of a flow chart



(a) Arrange the stages to construct a complete logical flow chart.
(b) State the purposes of the flow chart.
(c) Perform a dry run for your flow chart by copying and completing the table below.

A	B	C
46	____	____
77	____	____
120	____	____
177	____	____

Soln



(b) To compute and print positive natural numbers divisable by 2.

(c):

A	B	C
46	41	-
47	42	21
77	72	36
120	115	-
121	116	58
177	172	86

7. The iterative formula for finding the cube root of a number N is given by;

$$x_{n+1} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right), \quad n = 0, 1, 2, \ldots$$

(a) Draw a flow chart that;

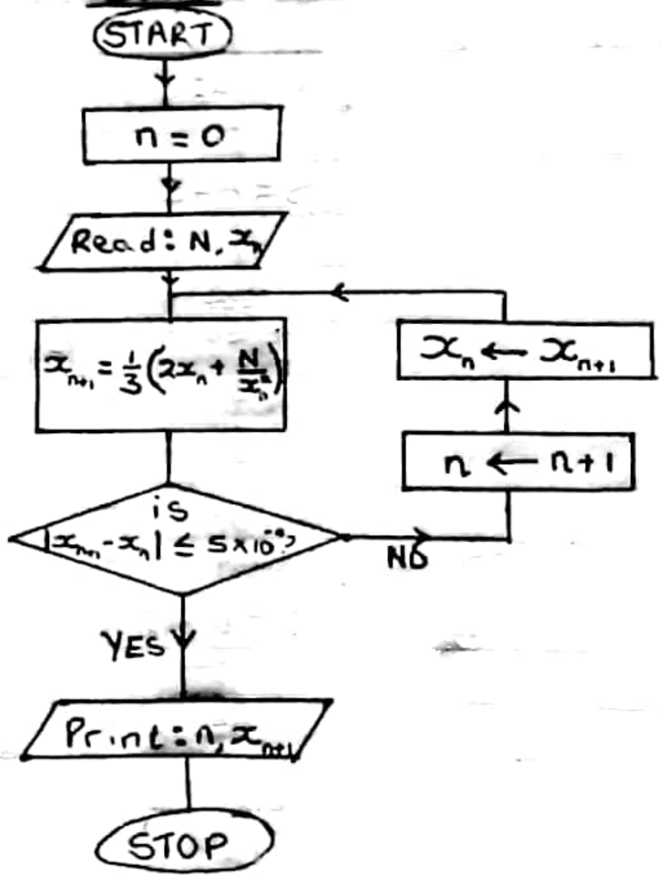
(i) reads the values of N and x_n.

(ii) Computes and prints values of x_{n+1} to 3d.Ps.

(b) Perform a dry run for N = 66 and $x_0 = 4$.

Soln

(a)



(b)

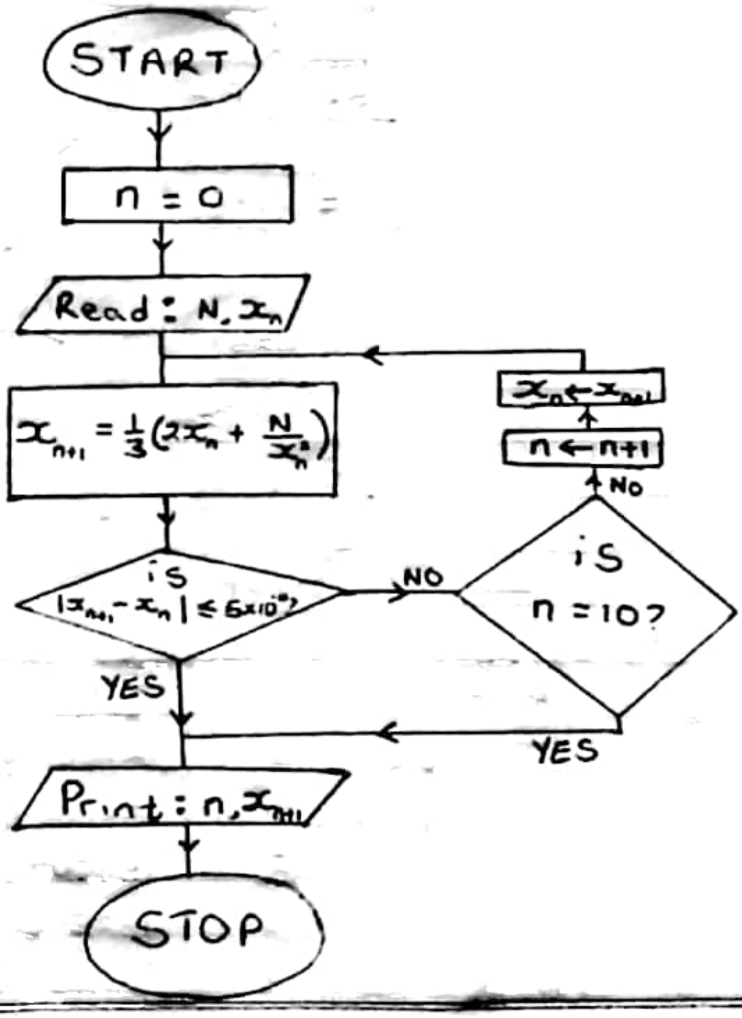
n	x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	4.0000	4.0417	0.0414
1	4.0417	4.0412	0.0005
2	4.0412	4.0412	0.0000

$\therefore\ 66^{1/3} = 4.041$

NB:

The flow chart above can be modified so that only a particular maximum number of iterations say 10 iterations is performed.

In such a case, a counter which increases by a specific quantity is used and it is checked after each iteration is performed.



(8) A building society gives a compound interest of A shillings on P shillings worth of shares at a rate of r% per annum.

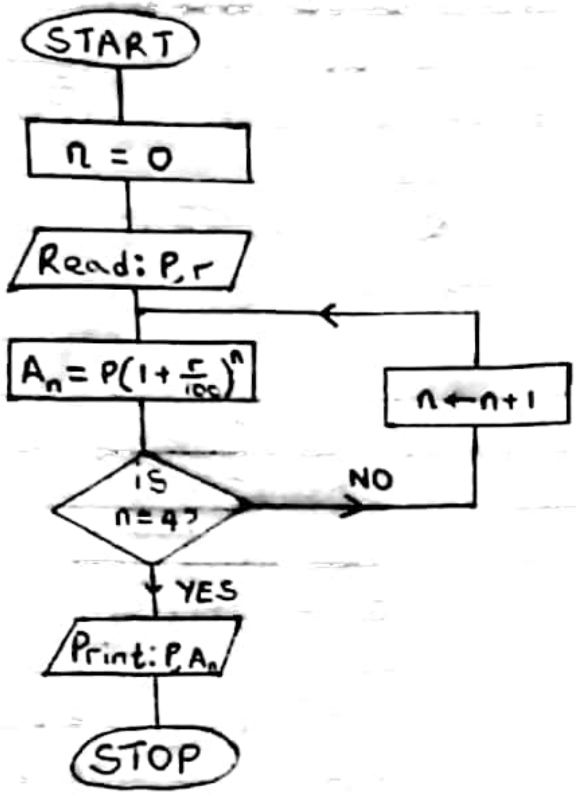
(i) Write down an alogarithm for computing

the amount accumulated on shares worth P shillings for the first n years
(ii) Given that P = 120000, draw a flow chart that computes and prints the amount of money accumulated at a compound interest rate of 15% per annum after 4 years.
(iii) Perform a dry run for your flow chart.

Soln

(i) $A_n = P(1+r\%)^n$

(ii)



(iii)

P	n	A_n
120000	0	120000.00
	1	138000.00
	2	158700.00
	3	182505.00
	4	209880.75

Exercise

① Draw a flow chart that computes and prints the mean of the first 10 counting numbers and perform a dry run for your flow chart.

② Draw flow chart that computes and prints the sum S of the first 20 even numbers. Perform a dry run for your flow chart.

③ (a) Show that the Newton Raphson formula for finding the root of the equation $2x^3 + 5x - 8 = 0$ is given by:

$$x_{n+1} = \frac{4x_n^2 + 8}{6x_n^3 + 5}, \quad \text{where } n = 0, 1, 2, \ldots$$

(b) Draw a flow chart that computes and prints the number of iterations and the root of the equation to two decimal places. By taking the initial approximation as 1.2, perform a dry run for your flow chart.

④ (a) Show that the Newton Raphson formula for finding the square root of a positive number N is given by:

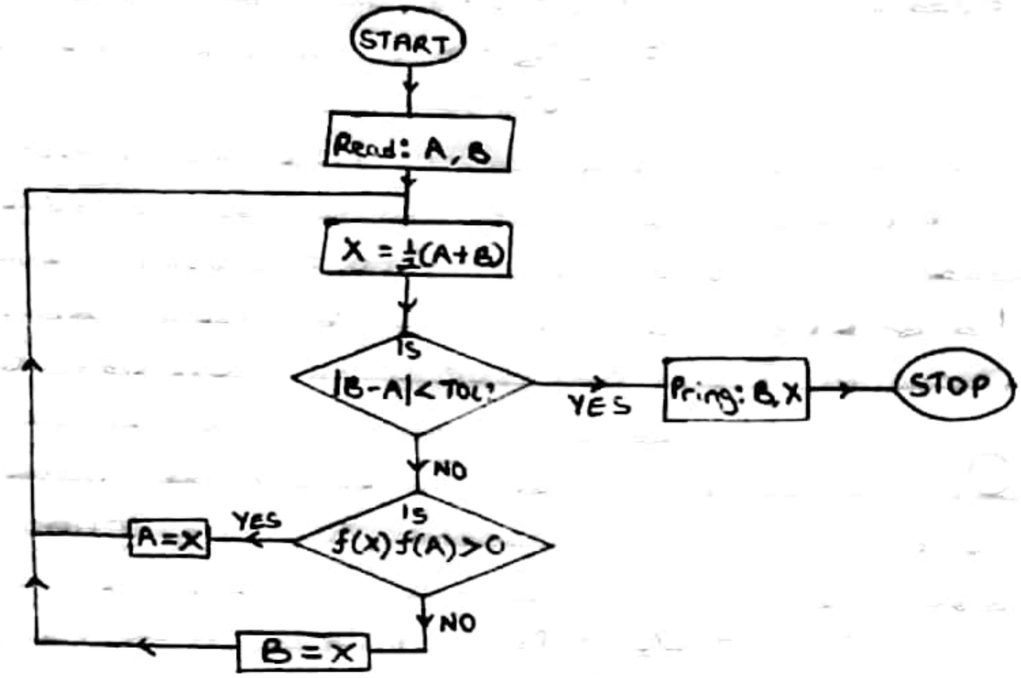
$$x_{n+1} = \frac{x_n^2 + N}{2x_n}, \quad \text{Where } n = 0, 1, 2, \ldots$$

(b) Draw a flow chart that computes and prints the square root to 4 decimal places. Taking the initial approximation as 6.71, perform a dry run for your flow chart for N = 45.

⑤ (a) A retail shop gives a 15% trade discount and an additional 5% cash discount on any item bought from the shop. Each customer is entitled to a trade discount however only customers who pay cash are entitled to a cash discount. Constcuct a flow chart that computes and prints the amount of money that Muddu pays for a television set with a market price D.

(b) Use your flow chart to calculate the amount of money that Muddu pays for a television set with a market price of Shs 350,000 if he pays cash.

6 The iterative formula for approximating the root of the equation $f(x)=0$ is described by the flow chart below.



Given that $A = 1.6875$, $B = 1.8750$ and $Tol = 10^{-2}$, perform a dry run for the flow chart. Calculate $5^{1/3}$ tabulating the values of A, B and X at each stage.

$$\left[\,5^{1/3} = 1.71\,\right]$$

7(a) Determine the iterative formula based on NRM for finding the fourth root of a given number N.

(b)(i) Draw a flow chart that reads N and the initial approximation x_0, computes and prints the fourth root of N to 3 decimal places.

(ii) Perform a dry run for N = 150 and $x_0 = 3.2$