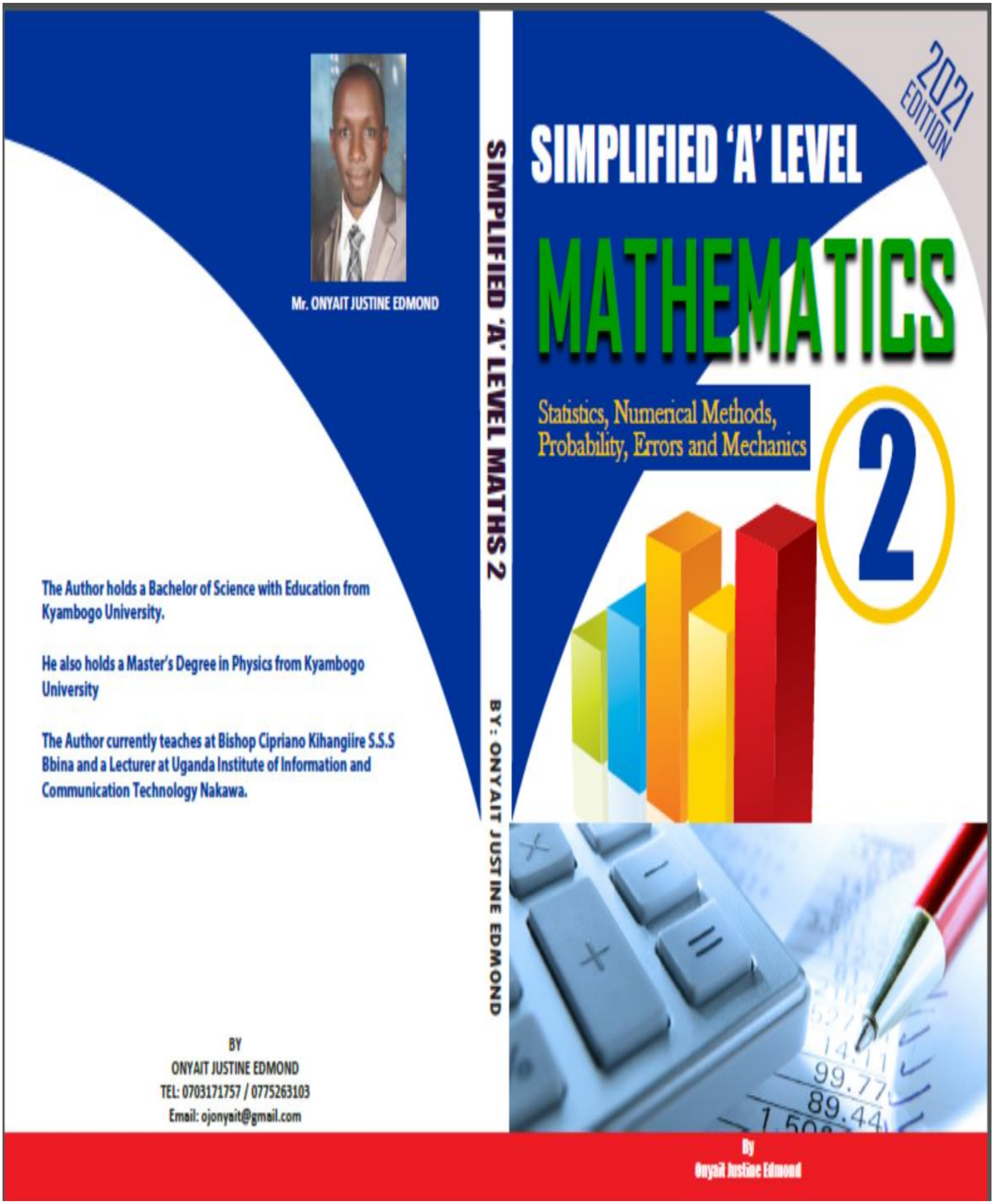
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SIMPLIFIED 'A' LEVEL MATHS 2
BY: ONYAIT JUSTINE EDMOND
2021 EDITION
SIMPLIFIED 'A' LEVEL
MATHEMATICS
Statistics, Numerical Methods, Probability, Errors and Mechanics
2
By
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PREFACE

The book covers Mechanics, Statistics, Probability, Errors and Numerical method. The content in this book is presented in a brief, concise and summarised manner with all the technical aspects of the paper captured.

Plenty of examples are presented in the book to enable a student easily understand applied mathematics with minimum help from a teacher. At the end of every chapter, you will find numerous exercises with their respective answers in highlighted brackets. You will also find UNEB questions and their answers dating from 1998 to 2020.

The book has been embraced and used by students from a number of schools such as Seeta high school, St Marys college Kitende, Kings College Budo and Bishop Cipriano Kihangiire S.S -Biina.

CONTENTS

STATISTICS

This is a branch of mathematics dealing with collection, presentation, analysis and interpretation of data.

DISCRETE OR UNGROUPED DATA

(i) Measure of central tendency

These are values of the distribution that tend to locate the central value. They include mean, median and mode

(a) Mean or average of a sample

It is denoted by $\bar{x}$ and defined as $\boxed{\bar{x} = \frac{\sum x}{n}}$

Where x is variable given and n total number of the variables

If assumed mean (working mean) A is given then

$\boxed{\bar{x} = A + \frac{\sum d}{n}}$ Where $d = x - A$

If the frequency, f is given then $\boxed{\bar{x} = \frac{\sum fx}{\sum f}}$ Or $\boxed{\bar{x} = A + \frac{\sum fd}{\sum f}}$ Where $d = x - A$

(b) Mode

This is the value of the distribution that appears most

(c) Median

This is the middle value of the distribution obtained after the values have been arranged either in ascending or descending order.

Median= $\left(\frac{N}{2}\right)^{th}$ $value$

Examples

1. Given the following sets of values

2, 1, 3, 4, 5, 6, 7, 8, 9, 10, 3, 4, 6, 7, 6, 8, 9, 6, 3, 2

(a) Form a frequency table of ungrouped data

(b) Use your table to find the mean and mode

(c) Find the median value

Solution

(a)

x	f	fx	C.f
1	1	1	1
2	2	4	3
3	3	9	6
4	2	8	8
5	1	5	9
6	4	24	13
7	2	14	15
8	2	16	17
9	2	18	19
10	1	10	20
	Σf=20	Σfx =109	

(b) Mean, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{109}{20} = 5.45$

Mode= 6 (appear most)

(c) Median= $\left(\frac{N}{2}\right)^{th} = \left(\frac{20}{2}\right)^{th} = 10^{th} value$ from c.f

Median= 6

2. Given the information below

x	10	11	12	13	14	15	16	17	18
f	4	2	6	3	7	2	1	2	2

Find;

(a) Mean value

(b) Modal value

(c) Median

Solution

x	f	fx	C.f
10	4	40	4
11	2	22	6
12	6	72	12
13	3	39	15
14	7	98	22
15	2	30	24
16	1	16	25
17	2	34	27
18	2	36	29
	Σf=29	Σfx =387	

(b) Mean, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{387}{29} = 13.34$

Modal value= 14 (appear most)

(c) Median= $\left(\frac{N}{2}\right)^{th} = \left(\frac{29}{2}\right)^{th} = 14.5^{th} value$ from c.f

Median= 13

(ii) Measure of dispersion

This a measure of find how the observations are spread out from the mean

(a) Variance of a sample

The variance of x denoted by $Var(X)$ is defined as

$$Var(X) = \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2$$

or

$$Var(X) = \frac{\sum x^2}{n} - \bar{x}^2$$

if the frequency is given then;

$$Var(X) = \frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2$$

or

$$Var(X) = \frac{\sum fx^2}{\sum f} - \bar{x}^2$$

(b) Standard deviation

$$S.d = \sqrt{var(X)}, \quad s.d = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2} \quad \text{Or} \quad s.d = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$$

Examples

1. Find the variance and standard deviation of the following; 45, 54, 64, 76, 86

Solution

$$Var(x) = \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2$$

$$Var(X) = \frac{22209}{5} - \left(\frac{325}{5}\right)^2 = 216.8$$

$$s.d = \sqrt{Var(x)} = \sqrt{216.8} = 14.72$$

2. The frequency distribution table shows the marks of some students from a certain school

x	55	63	65	66	70	72	75	80	90
f	2	2	3	1	2	2	4	3	1

Calculated the standard deviation

Solution

Using assumed mean to get variance and standard deviation

$$Var(X) = \frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2 \quad \Bigg| \quad s.d = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2} \quad \Bigg| \quad where;\ d = x - A$$

Example

1. The frequency distribution table shows the heights of some students at a certain school

Height	154	155	160	164	171	180
Frequency	4	6	8	5	4	3

Determine the variance and standard deviation of the data using a working mean of 160

Solution

(c) Quartiles

A quartile is a value that divides given values into four equal parts

q_1 is the lower quartile and it's defined by;

$q_1 = \left(\frac{1}{4}N\right)^{th}$ $value$ Where N is the sum of all the variables

q_3 is the upper quartile and it's defined by;

$q_3 = \left(\frac{3}{4}N\right)^{th}$ $value$ Where N is the sum of all the variables

(d) Percentiles

A percentile is a value that divides given values into 100 equal parts

P_{10} is the 10th percentile and it's defined by;

$P_{10} = \left(\frac{10}{100}N\right)^{th}$ $value$ Where N is the sum of all the variables

P_{85} is the 85th percentile and it's defined by;

$P_{85} = \left(\frac{85}{100}N\right)^{th}$ $value$ Where N is the sum of all the variables

(e) Deciles

A decile is a value that divides given values into 10 equal parts

D_7 is the 7th decile and it's defined by;

$D_7 = \left(\frac{7}{10}N\right)^{th}$ $value$ Where N is the sum of all the variables

Examples

1. The table below shows the marks obtained by 20 students in a test marked out of 20

Marks	10	11	12	13	14	15	16	17	18	19	20
Number of students	1	2	2	2	2	4	2	1	2	1	1

Find;

(a) Mean mark
(b) Standard deviation
(c) 60th percentile
(d) Interquartile range

Solution

CONTINUOUS OR GROUPED DATA

This is data whose scores or values are said to be continuous and take interval values

Example

class	20 - 29	30 - 39	40 - 49	50 - 59	60 - 69	70 - 79	80 -89
Number of students	4	5	7	3	6	4	1

Draw a frequency table

Terms used

(a) **Class**

These are the class limits of the distribution. In the table above, the classes are
$20-29, 30-39, 40-49, 50-59, 60-69, 70-79, 80-89.$

(b) **Class mark (mark)**

This is the mid-point value of the class. It is normally denoted by x. in the above table. Class mark is $24.5, 34.5, 44.5 \ldots$

(c) **Class boundary**

These are continuous class limits. In the above table the first class boundary is $(20-0.5)-(29+0.5)=19.5-29.5$. In this case the lower class boundary is 19.5 and upper class boundary is 29.5

For class interval of $2.0-2.9$, the class boundary is $(2.0-0.05)-(2.9+0.05)=1.95-2.95$

(d) **Class width or class interval**

This is the width of each class boundary. It is given by

$$class\ width = upper\ class\ boundary - lower\ class\ boundary$$

Solution

class	Frequency	Class width	Class mark, x	Class boundary
20-29	4	10	24.5	19.5-29.5
30-39	5	10	34.5	29.5-39.5
40-49	7	10	44.5	39.5-49.5
50-59	3	10	54.5	49.5-59.5
60-69	6	10	64.5	59.5-69.5
70-79	4	10	74.5	69.5-79.5
80-89	1	10	84.5	79.5-89.5

Examples

1. The data below shows the heights in centimeters of 70 students

Height (cm)	130-135	135-140	140-145	145-150	150-160	160-170	170-180
Number of students	10	12	8	9	11	15	5

Construct a frequency distribution for the above data.

Solution

Height	Class boundary	Width	x	f
130-135	130-135	5	132.5	10
135-140	135-140	5	137.5	12
140-145	140-145	5	142.5	8
145-150	145-150	5	147.5	9
150-160	150-160	10	155	11
160-170	160-170	10	165	15
170-180	170-180	10	175	5

2. Use the data below to construct a frequency distribution table

Marks	20-<30	30-<40	40-<50	50-<60	60-<70	70-<80	80-<90
Number of students	10	14	9	18	4	3	2

Solution

marks	Class boundary	Width	x	f
20-<30	20-30	10	25	10
30-<40	30-40	10	35	14
40-<50	40-50	10	45	9
50-<60	50-60	10	55	18
60-<70	60-70	10	65	4
70-<80	70-80	10	75	3
80-<90	80-90	10	85	2

3. The table below shows the ages of 35 people

Age	0-	5-	10-	15-	20-	30-	40-
Frequency	4	6	3	5	7	2	8

Draw a frequency table for the data

Solution

Age	Class boundary	Width	x	f
0-	0-5	5	2.5	4
5-	5-10	5	7.5	6
10-	10-15	5	12.5	3
15-	15-20	5	17.5	5
20-	20-30	10	25	7
30-	30-40	10	35	2
40-	40-45	5	42.5	8

Note: The last class width is 5 since it's the most common

4. The table below shows the marks of 40 people

marks	-20	-30	-40	-50	-60	-65	-70
Frequency	8	4	7	10	2	2	7

Draw a frequency table for the data

Solution

marks	Class boundary	Width	x	f
-20	10-20	10	15	8
-30	20-30	10	25	4
-40	30-40	10	35	7
-50	40-50	10	45	10
-60	50-60	10	55	2
-65	60-65	5	62.5	2
-70	65-70	5	67.5	7

Note: (i) The first class width is 10 because it's the most common

(ii) It is also acceptable for the first class to start from zero ie $(0 - 20)$

5. The data below shows the length in centimeter of different phone calls made by Airtel clients

Length (minutes)	<20	<30	<35	<40	<50	<60
Cumulative frequency	4	20	32	42	48	50

Construct a frequency distribution table

Solution

Length	Class boundary	Width	x	f
<20	10-20	10	15	4
<30	20-30	10	25	16
<35	30-35	5	32.5	12
<40	35-40	5	37.5	10
<50	40-50	10	45	6
<60	50-60	10	55	2

Note: (i) The first class width is 10 because it's the most common

(ii) It is also acceptable for the first class to start from zero ie $(0 - 20)$

Measure of central tendencies

(a) Measure of Mean or average of a sample, $\bar{x}$

The mean of grouped data is given by;

$$\bar{x} = \frac{\sum fx}{\sum f}$$

Where f –frequency

x –Mid point value

Or $\bar{x} = A + \frac{\sum fd}{\sum f}$

Where $d = x - A$

(b) Median

Median of grouped data is defined by

$$median = L_b + \left(\frac{\frac{£f}{2} - c.f_b}{f}\right)C$$

L_b = Lower class boundary of the median class

C = Class width of the median class
f = Frequency of the median class
$c.f_b$ = Cumulative frequency before that one of the median class

(c) Mode

Mode of grouped data with equal class width is defined as

$$mode = L_b + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right)C$$

L_b = Lower class boundary of the modal class

C = Class width of the modal class
Δ_1 = Modal frequency –Pre modal frequency
Δ_2 = Modal frequency –Post modal frequency

Examples

The table below shows the weight of 250 students at Kennedy secondary school

Weight (kg)	44.0 – 47.9	48.0 – 51.9	52.0 – 55.9	56.0 – 59.9	60.0 – 63.9	64.0 – 67.9	68.0 – 71.9	72.0 – 75.9
Frequency	3	17	50	45	46	57	23	9

Find;
(i) Average weight | (ii) Median weight | (iii) Modal weight

Solution

Mode of un-equal class width

Mode of grouped data with un-equal class width is defined as

$$mode = L_b + \left(\frac{\Delta f.d_1}{\Delta f.d_1 + \Delta f.d_2}\right)C$$

Modal class is determined from the highest frequency density

$$frequency\ density = \frac{frequency}{class\ width}$$

L_b = Lower class boundary of the modal class
C = Class width of the modal class
$\Delta f.d_1$ = Modal frequency density –Pre frequency density
$\Delta f.d_2$ = Modal frequency density –Post frequency density

Examples

Given the data below

Marks (x)	10-19	20-24	25-34	35-39	40-54	55-64	65-79
Frequency (f)	4	6	7	3	8	6	6

Find the mode

Solution

$$mode = L_b + \left(\frac{\Delta f.d_1}{\Delta f.d_1 + \Delta f.d_2}\right)C$$

$$mode = 19.5 + \left(\frac{1.2 - 0.4}{[1.2 - 0.4] + [1.2 - 0.7]}\right)5$$

$$mode = 22.58$$

Measure of dispersion

(a) Variance of a sample

The variance of x of grouped data denoted by Var(x) is defined as

$$Var(x) = \frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2$$

or

$$Var(x) = \frac{\sum fx^2}{\sum f} - \bar{x}^2$$

or

$$Var(x) = \frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2$$

$where\ d = x - A$

(b) Standard deviation

$$S.d = \sqrt{var(x)}$$

Percentiles and Quartiles

(a) Percentiles

A percentile is a value that divides a given distribution into 100 equal parts.

The 60th percentile denoted by P_{60} is defined as

$$P_{60} = L_b + \left(\frac{\frac{60}{100}\text{£}f - c.f_b}{f}\right)C$$

L_b = Lower class boundary of the 60th class

C = Class width of the 60th class
f = Frequency of the 60th class
$c.f_b$ = Cumulative frequency before that one of the 60th class

(b) Quartiles

A quartile is a value that divides a given distribution into4 equal parts.

The lower quartile denoted by q_1 is defined as

$$q_1 = L_b + \left(\frac{\frac{1}{4}\text{£}f - c.f_b}{f}\right)C$$

L_b = Lower class boundary of the q_1 class

C = Class width of the q_1 class
f = Frequency of the q_1 class
$c.f_b$ = Cumulative frequency before that one of the q_1 class

The upper quartile denoted by q_3 is defined as

$$q_3 = L_b + \left(\frac{\frac{3}{4}\text{£}f - c.f_b}{f}\right)C$$

L_b = Lower class boundary of the q_3 class

C = Class width of the q_3 class
f = Frequency of the q_3 class
$c.f_b$ = Cum.frequency before that of q_3

$$Inter\ quartile\ range = q_3 - q_1$$

$$semi - Inter\ quartile\ range = \frac{q_3 - q_1}{2}$$

Examples

1. The following table shows the marks obtained by 40 students in a physics test marked out of 100

Marks (%)	20-29	30-39	40-49	50-59	60-69	70-79	80-89	90 - 99
Number of students	4	6	2	5	7	8	5	3

Find
(i) Mean
(ii) Standard deviation
(iii) Median and mode
(iv) Semi interquartile range
(v) 40th and 85th percentile range

Solution

Class boundary	x	f	fx	fx²	C.f
19.5 – 29.5	24.5	4	98	2401	4
29.5 – 39.5	34.5	6	207	7142	10
39.5 – 49.5	44.5	2	89	3961	12
49.5 – 59.5	54.5	5	272.5	14851	17
59.5 – 69.5	64.5	7	451.5	29122	24
69.5 – 79.5	74.5	8	596	44402	32
79.5 – 89.5	84.5	5	422.5	35701	37
89.5 – 99.5	94.5	3	283.5	26791	40
		Σf=40	Σfx=2420	Σfx²=164370	

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{2420}{40} = 60.5\%$$

(a) $s.d = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$

$$s.d = \sqrt{\frac{164370}{40} - (60.5)^2} = 21.19\%$$

(b) $mode = L_b + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right)C$

$$mode = 69.5 + \left(\frac{8-7}{[8-7]+[8-5]}\right)10 = 72\%$$

$$median = L_b + \left(\frac{\frac{\text{£}f}{2} - c.f_b}{f}\right)C$$

$$\frac{\text{£}f}{2} = \frac{40}{2} = 20$$

Median class boundary is $60 - 69$, $f = 7$ and $C = 10$

$$median = 59.5 + \left(\frac{20-17}{7}\right)10 = 63.786\%$$

(c) $q_1 = L_b + \left(\frac{\frac{1}{4}\text{£}f - c.f_b}{f}\right)C$

$$\frac{\text{£}f}{4} = \frac{40}{4} = 10$$

$q_1\ class\ boundary\ is\ 30 - 39, f = 6\ and\ C = 10$

$q_1 = 29.5 + \left(\frac{10-4}{6}\right)10 = 39.5\%$

$q_3 = L_b + \left(\frac{\frac{3}{4}\pounds f - c.f_b}{f}\right)C$

$\frac{3\pounds f}{4} = \frac{3}{4}x40 = 30$

q_3 *class boundary is* $70 - 79$, $f = 8$ *and* $C = 10$

$q_3 = 69.5 + \left(\frac{30-24}{8}\right)10 = 77\%$

$semi - Inter\ quartile\ range = \frac{q_3 - q_1}{2}$

$S.I.R = \frac{77 - 39.5}{2} = 18.75\%$

(d) $P_{40} = L_b + \left(\frac{\frac{40}{100}\pounds f - c.f_b}{f}\right)C$

$\frac{40}{100}\pounds f = \frac{40}{100}x40 = 16$

P_{40} *class boundary is* $50 - 59$, $f = 5$ *and* $C = 10$

$P_{40} = 49.5 + \left(\frac{16-12}{5}\right)10 = 57.5\%$

$P_{85} = L_b + \left(\frac{\frac{85}{100}\pounds f - c.f_b}{f}\right)C$

$\frac{85}{100}\pounds f = \frac{85}{100}x40 = 34$

P_{85} *class boundary is* $80 - 89$, $f = 5$ *and* $C = 10$

$P_{85} = 79.5 + \left(\frac{34-32}{5}\right)10 = 83.5\%$

40th and 85th range$= 83.5 - 57.5 = 26\%$

Graphs

(a) Grouped data with equal class width

(i) Histogram

This is a graph consisting of vertical bars. It is a graph of frequency against class boundary. Histogram is used to determine the mode

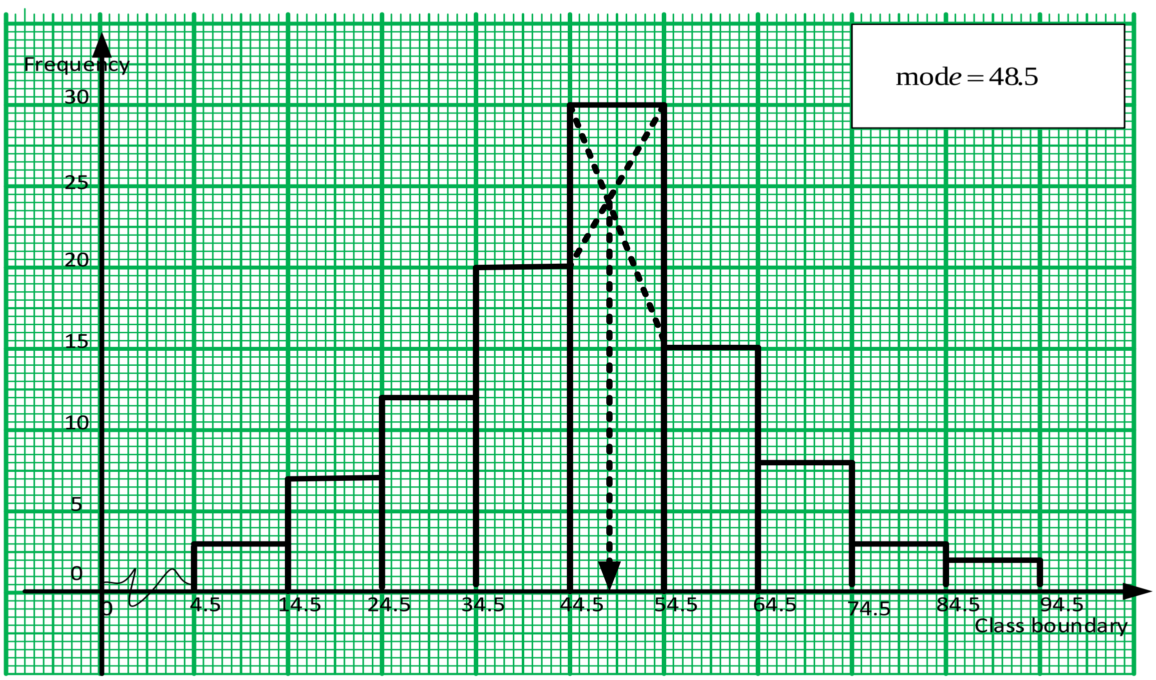
Examples

Given the data below

Marks	5-14	15-24	25-34	35-44	45-54	55-64	65-74	75-84	85-94
Frequency	3	7	12	20	30	15	8	3	2

Draw a histogram and use it to determine the mode

Solution



(ii) Cumulative frequency curve(ogive)

This a curve of cumulative frequency against class boundary. The ogive is used to determine median, quartiles, percentiles and deciles

Note The values of cumulative frequency must be plotted against upper class boundary and first value of the lower class boundary must be plotted against cumulative frequency zero.

Examples

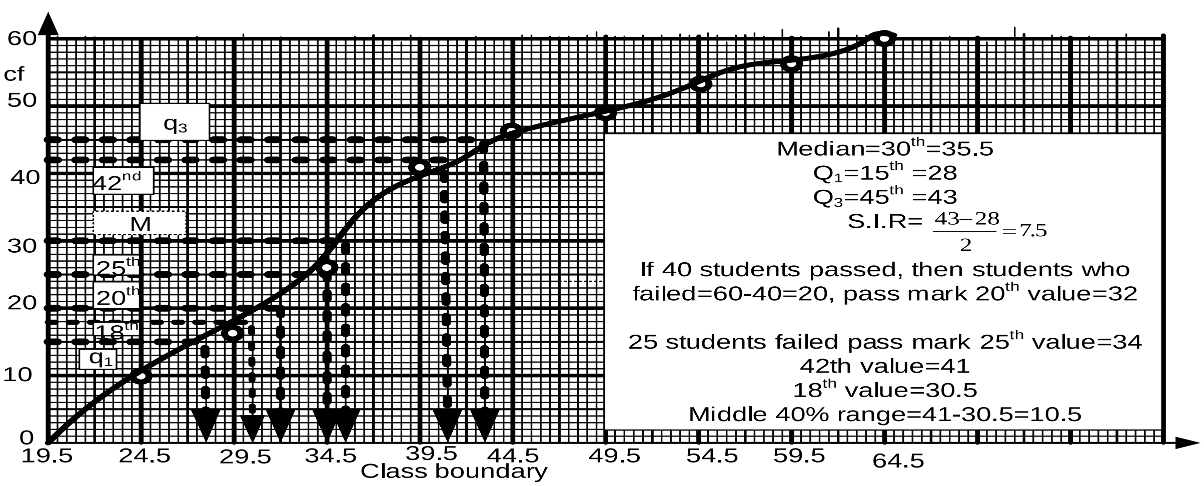
1. The table below shows the marks obtained by 60 students in a mathematics test of a certain school

Marks	20 - 24	25 - 29	30 - 34	35 - 39	40 - 44	45 - 49	50 - 54	55 - 59	60 - 64
Number of students	10	6	10	15	5	3	4	3	4

Draw a cumulative frequency curve and use it to estimate

(i) Median
(ii) Interquartile range
(iii) 60^{th} percentile
(iv) Pass mark if 40 students passed
(v) The pass mark if 25 students failed
(vi) The middle range of 40% of the students who did the test

Solution



if 40% of students are in the middle, then 60% are not in the middle

$\therefore$ Remainder of the students= $\frac{60\%}{2} = 30\%$ on either side

30%	Middle 40%	30%
$P_{30} = 30\% \, of \, 60 = 18^{th}$	$30 + 40 = 70\% \longrightarrow$	$P_{70} = 70\% \, of \, 60 = 42^{th}$

GROUPED DATA WITH UN EQUAL CLASS WIDTH

(i) Histogram

This is a graph of frequency density against class boundary

Note:

Frequency density= $\frac{frequency}{class \; width}$

(ii) Ogive

This a graph of cumulative frequency against class boundary

Examples

1. The data shows the length in centimeters for different calendars produced by a printing press. A cumulative frequency distribution was formed

Length (cm)	< 20	< 30	< 35	< 40	<50	< 60
Cumulative frequency	4	20	32	42	48	50

(a) Construct a frequency distribution table
(b) Find the mean length of the calendar
(c) Draw a histogram and use it to estimate the modal length. **An(33.5)**

Solution

Class boundary	x	f	fx	Frequency density
$0-20$	10	4	40	0.2
$20-30$	25	16	400	1.6
$30-35$	32.5	12	390	2.4
$35-40$	37.5	10	375	2
$40-50$	45	6	270	0.6
$50-60$	55	2	110	0.2
		$\sum f = 50$	$\sum fx = 1585$	

Mean $= \frac{1585}{50} = 31.7$

Exercise 1

1. The frequency distribution below shows the ages of 240 students admitted to a certain University. **Uneb 2018 No.9**

Age (years)	18-<19	19 - <20	20 - <24	24 - <26	26 - <30	30 - <32
Number of students	24	70	76	48	16	6

(a) Calculate the mean age of the students. **An(22.1458years)**
(b) (i) Draw a histogram for the given data.
(ii) Use the histogram to estimate the modal age. **An(19.58years)**

2. The table below shows the masses of bolts bought by a carpenter

Mass (grams)	98	99	100	101	102	103	104
Number of bolts	8	11	14	20	17	6	4

Find the; **Uneb 2019 No.1**
(i) Median mass (ii) Mean mass of the bolt
An((i)=101g, (ii)=100.7625g)

3. The table below shows the marks obtained in a mathematics test by a group of students

Marks	5 - < 15	15 - < 25	25 - < 35	35 - < 45	45 - < 55	55 - < 65	65 - < 75	75 - < 85
Number of students	5	7	19	17	7	4	2	3

(a) Construct a cumulative frequency curve (ogive) for the data
(b) Use your ogive to find; **Uneb 2019 No.10**
(i) Range between the 10^{th} and 70^{th} percentile
(ii) Probability that student selected at random scored below 50 marks
An((i)=26, (ii)=0.8125)

4. The table below shows the marks obtained by 100 students in a mathematics test **Uneb 2020 No.9**

Marks	20 - < 40	40 - < 50	50 - < 55	55 - < 60	60 - < 70	70 - < 90	90 - < 100
Number of students	5	15	10	15	25	25	5

(a) Calculate the mean mark
(b) Construct a cumulative frequency curve (ogive) and use it to find the;
(i) Median mark
(ii) Range of the middle 40% of the mark
An((a)=63.125, (i)=61.5, (ii)=15)

INDEX NUMBERS

This is the percentage ratio of one quantity to the other eg price index

Simple index numbers

The simple index numbers include;

(i) Price index or price relative

$$Price\ relative = \frac{P_1}{P_0}x100$$

$P_1 - price\ the\ current\ year$ and $P_0 - price\ in\ base\ year$

(ii) Wage index

$$Wage\ index = \frac{W_1}{W_0}x100$$

(iii) Quantity (quantum) index

$$Quantum\ index = \frac{Q_1}{Q_0}x100$$

Examples

1. A loaf of bread costed $sh. 1200/=$ in 2008 and $sh. 1800/=$ in 2014. Taking 2008 as the base year, find the price relative in 2014.

Solution

$$price\ index = \frac{P_1}{P_0}x100 = \frac{1800}{1200}x100 = 150$$

2. In 2020, the price index of a commodity using 2019 as the base was 180. In 2021, the price index using 2020 as the base year was 150. What is the price index in 2021 using 2019 as the base year.

Solution

Price indices

Price indices are divided into:

(a) Simple price index (b) Simple Aggregate price index (c) Weighted price index

(a) Simple price index

This is the average of the price relatives

It's given by $simple\ \ price\ index = \frac{\Sigma\left(\frac{P_1}{P_0}\right)}{n}x100$

Where $n - number\ of\ items$

(b) Simple aggregate price index

It's given by $simple\ aggregate\ price\ index = \left(\frac{\sum P_1}{\sum P_0}\right)x100$

Examples

1. The table below shows the price of beans and meat per kg in 2000 and 2008

	Year	
Item	2000	2008
Beans	700	1200
Meat	2500	4500

Using 2000 as the base year, find,

(a) Price relatives of each commodity
(b) Simple price index
(c) Simple aggregate price index

Solution

Weighted price index (Composite index)

If the weight or quantity in the base year and current year are the same, we use

(i) Weighted aggregate price index (ii) Average weighted price index

(i) Weighted aggregate price indices / Laspyre's theory

$$Weighted\ aggregate\ price\ index = \left(\frac{\sum P_1 w}{\sum P_0\, w}\right)x100$$

Examples

1. The table below shows the prices(shs) and amounts of item bought for making a cake in 2005 and 2006

Item	Price (shs)		amount
	2008	2009	
Flour per kg	6000	7800	3
Sugar per kg	5000	4000	1
Milk per litre	1000	1500	2
Eggs per egg	200	300	8

(a) Calculate the weighted aggregate price index taking 2008 as the base year

(b) In 2009, the cost of making a cake was 80,000/=. Using the weighted aggregate price index above, find the cost of the cake in 2008

Solution

(a) $W.A.P.I = \left(\frac{\sum P_1 w}{\sum P_0 w}\right)x100 = \left(\frac{7800x3+4000x1+1500x2+300x8}{6000x3+5000x1+1000x2+200x8}\right)x100 = 123.3083$

(b) $\frac{P_1}{P_0}x100 = 123.3083$

$$\frac{80{,}000}{P_0}x100 = 123.3083$$

$$P_0 = 64{,}878.033/=$$

(ii) Average Weighted price index/ cost of living index

$$Average\ \text{weighted}\ price\ index = \frac{\sum\left(\frac{P_1}{P_0}w\right)}{\sum w}x100$$

When the price relative (P.R) is given then: $Average\ \text{weighted}\ price\ index = \frac{\sum(\text{P.R x } w)}{\sum w}$

Examples

1. The table below shows the expenditure (Ug shs) if a student during the first and second terms

Item	Expenditure		amount
	1st term	2nd term	
Clothing	46,500	49,350	5
Pocket money	55,200	37,500	3
Books	80,000	97,500	8

Using the first terms expenditure as the base, find the average weighted price index

Solution

Weighted price index (Composite index)

If the weight or quantity in the base year and current year are different, we use

Weighted aggregate price indices / Paache's theory/ value index

$$Weighted\ aggregate\ price\ index = \frac{\sum P_1 W_1}{\sum P_0\, W_0}x100$$

Examples

1. The table below shows the prices of items per kg in the year 2001 and 2002

Item	2001=100		2002	
	Price (shs)	Quantity (kg)	Price(shs)	Quantity (kg)
Rice	2800	20	3200	30
Millet	1500	10	1900	10
Beans	2000	5	2500	70

Calculate for 2002

(i) Price index
(ii) Simple aggregate price index
(iii) Simple aggregate quantity index
(iv) Weighted aggregate price index

Solution

Exercise 2

1. The table below shows the price indices of beans, maize, rice and meat with corresponding weights.

Item	Price index 2008 (2007=100%)	Weight
Beans	105	4
Maize	x	7
Rice	104	2
Meat	113	5

Calculate the; **Uneb 2020 No.3**

(a) Value of x given that the price indices of maize in 2007 and 2008 using 2006 as the base year are 112 and 130 respectively

(b) weighted price index for 2008 using 2007 as the base year. **An(116.0714, 111.4167)**

2. The price index of an article in 2000 based on 1998 was 130. The price index for the article in 2005 based on 2000 was 80. Calculate the: **Uneb 2018 No.5**
   (a) price index of the particle in 2005 based on 1998. **An(104, 43,269.231/=)**
   (b) price of the article in 1998 if the price of the article was 45,000 in 2005.

3. The table below shows the prices(shs) and amounts of item bought weekly by a restaurant in 2002 and 2003 **Uneb 2017 No.7**

Item	Price (shs)		amount
	2002	2003	
Milk per litre	400	500	200
Eggs per tray	2500	3000	18
Cookig oil per litre	2400	2100	2
Flour per packet	2000	2200	15

(c) Calculate the weighted aggregate price index taking 2002 as the base year

(d) In 2003, the restaurant spent 450,000/=. Using the weighted aggregate price index above, find how the restaurant could have spent in 2002 **An(119.65, 376096.95)**

CORRELATIONS AND SCATTER DIAGRAMS

RANK CORRELATIONS

This is the approach used to determine the degree of the relationship between two variables by ranking them.

Rank correlation is determined using

1. Spearman rank correlation coefficient (ρ)
2. Kendall's rank correlation coefficient (τ)

Commenting on the rank correlation coefficient

The table below is used

Correlation coefficient	Interpretation
0 – 0.19	Very low correlation
0.2 – 0.39	Low correlation
0.4 – 0.59	Moderate correlation
0.6 – 0.79	High correlation
0.8 – 1.0	Very high correlation

Note:

The sign associated with the correlation coefficient will be the one responsible for the type of correlation

Eg, -0.84 is a very high negative correlation?

Spearman rank correlation coefficient (ρ)

Its given by $\rho = 1 - \frac{6\sum d^2}{n(n^2-1)}$

Where $d -$ is the difference between the ranks?

$n -$ Total number of pairs

Examples

1. Two examiners marked the scripts of 8 candidates. The table shows the marks awarded by two examiners x and y.

x	72	60	56	76	68	52	80	64
y	56	44	60	74	66	38	68	52

Calculate the ranks correlation coefficient and comment on your result

Solution

R_X	R_Y	d^2
3	5	4
6	7	1
7	4	9
2	1	1
4	3	1
8	8	0
1	2	1
5	6	1
		$\Sigma d^2=18$

$$\rho = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6x18}{8(8^2-1)} = 0.786$$

There is a high positive correlation between x and y

SIGNIFICANCE OF RANKS CORRELATION COEFFICIENT

- If the $|\rho_c| > |\rho_T|$, a significant relation exists
- If the $|\rho_c| < |\rho_T|$, no significant relation exists

Where $\rho_c -$ calculated spearman correlation coefficient

$\rho_T -$ table spearman correlation coefficient at either 1% or 5% level

Examples

1. The following shows the marks obtained by 8 students in mathematics and physics exams

mathematics	65	65	70	75	75	80	85	85
Physics	50	55	58	55	65	58	61	65

Calculate the ranks correlation coefficient and comment on the significance of your result at 5% level

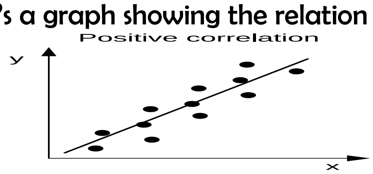
Solution

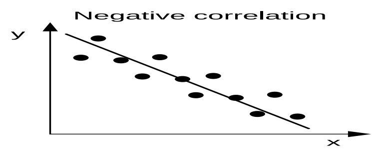
$$\rho = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6x21}{8(8^2-1)} = 0.75$$

Since $\rho_c(0.75) > \rho_T(0.71)$, a significant relation exists

Scatter graphs

It's a graph showing the relation between two variables





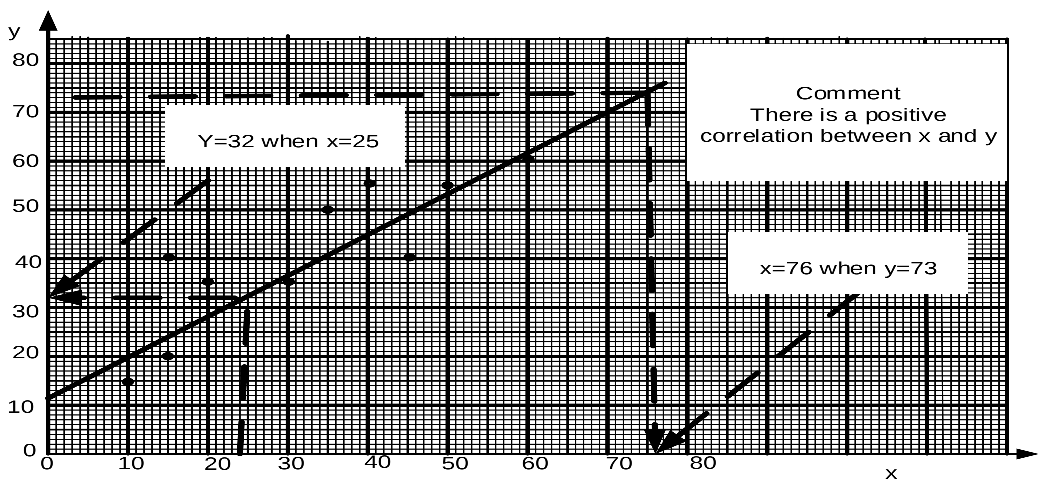
Examples

1. Given the information in the table below

x	10	15	20	15	30	35	40	45	50	60
y	15	20	35	40	35	50	55	40	55	60

   (a) Represent the above information on a scatter diagram and comment on the relationship between x and y
   (b) Draw the line of best fit
   (c) Use your graph to find
      (i) The value of y when $x = 25$ (ii) The value of x when $y = 73$

Solution



Exercise 3

1. The heights and ages of ten farmers are given in the table below **UNEB 2013 No9**

Height (cm)	156	151	152	160	146	157	149	142	158	140
Age (years)	47	38	44	55	46	49	45	30	45	30

   (a) Plot the data on a scatter diagram.
   (b) Draw the line of best fit on your graph and use it to estimate
      (i) Y when $x = 147$ **An(37)** (ii) X when $y = 43$ **An(151)**
   (c) Calculate the rank correlation coefficient for the data. Comment on your results **An(**$\rho = 0.752, \tau = 0.6$**)**

2. The table below gives the points awarded to eight schools by three judges J_1, J_2 and J_3 during a music competition. J_1 was the chief judge **UNEB 2015 No12**

J_1	72	50	50	55	35	38	82	72
J_2	60	55	70	50	50	50	73	70
J_3	50	40	62	70	40	48	67	67

   (a) Determine the rank correlation between the judgments of
      (i) J_1 and J_2 **An(**$\rho = 0.744$**)** (ii) J_1 and J_3 **An(**$\rho = 0.702$**)**
   (b) Who of the two judges had a better correlation with the chief judge?. Give a reason

PROBABILITY THEORY

Probability is the measure of the chance of an event occurring or not occurring

Union of events

For any two events A and B, the probability that either A or B or both occur r is $P(A\ U\ B)$

$$P(A\ U\ B) = P(A) + P(B) - P(A\ n\ B)$$

$$P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)$$

Complement of events

A^1 Denotes event A does not occur

For events A and B

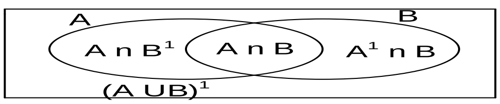
(i) $P(A) + P(A^1) = 1$

(ii) $P(B) + P(B^1) = 1$

(iii) $P(A\ n\ B) + P(An\ B)^1 = 1$

(iv) $P(AU\ B) + P(AU\ B)^1 = 1$

Venn diagram



(i) $P(A) = P(A\ n\ B^1) + P(A\ n\ B)$

(ii) $P(B) = P(B\ n\ A^1) + P(A\ n\ B)$

Contingency table

	B	B^1	
A	P(AnB)	P(AnB^1)	P(A)
A^1	P(A^{1}nB)	P(A^{1}nB^1)	P(A^1)
	P(B)	P(B^1)	1

(i) $P(A) = P(A\ n\ B^1) + P(A\ n\ B)$

(ii) $P(A^1) = P(A^1\ n\ B) + P(A^1\ n\ B^1)$

(iii) $P(B) = P(B\ n\ A^1) + P(A\ n\ B)$

(iv) $P(B^1) = P(A\ n\ B^1) + P(A^1\ n\ B^1)$

Demorgan's rule

(i) $P(A^1\ n\ B^1) = P(A\ u\ B)^1 = 1 - P(Au\ B)$

$P(neither\ A\ nor\ B) = 1 - P(A\ or\ B)$

(ii) $P(A^1\ u\ B^1) = P(A\ n\ B)^1 = 1 - P(An\ B)$

Undefined events

For undefined events, there is no restriction on $P(An\ B)$

Examples

1. Events A and B are such that $P(A) = \frac{19}{30}$, $P(B) = \frac{2}{5}$ and $P(A\ U\ B) = \frac{4}{5}$. Find

(i) $P(A\ n\ B)$

(ii) $P(A^1\ n\ B^1)$

(iii) $P(A^1\ n\ B)$

(iv) $P(A^1\ U\ B)$

(v) $P(A\ n\ B^1)$

Solution

(i) $P(A\ U\ B) = P(A) + P(B) - P(A\ n\ B)$

$$\frac{4}{5} = \frac{19}{30} + \frac{2}{5} - P(A\ n\ B)$$

$$P(A\ n\ B) = \frac{7}{30}$$

(ii) $P(A^1\ n\ B^1) = P(A\ u\ B)^1 = 1 - P(Au\ B)$

$$P(A^1\ n\ B^1) = 1 - \frac{4}{5} = \frac{1}{5}$$

(iii) $P(A^1\ n\ B) = P(B) - P(A\ n\ B)$

$$P(A^1\ n\ B) = \frac{2}{5} - \frac{7}{30} = \frac{1}{6}$$

(iv) $P(A^1\ U\ B) = P(A^1) + P(B) - P(A^1\ n\ B)$

$$P(A^1\ U\ B) = \left(1 - \frac{19}{30}\right) + \frac{2}{5} - \frac{1}{6} = \frac{3}{5}$$

(v) $P(A\ n\ B^1) = P(A) - P(A\ n\ B)$

$$P(A\ n\ B^1) = \frac{19}{30} - \frac{7}{30} = \frac{2}{5}$$

2. Events A and B are such that $P(A\ ) = 0.7$, $P(A\ n\ B) = 0.45$ and $P(A^1\ n\ B^1) = 0.18$. Find **UNEB 2010**

(i) $P(B^1\ )$

(ii) $P(A\ or\ B, but\ not\ both\ A\ an\ B)$

Solution

3. The probability that Anne reads the New-vision is 0.75 and the probability that she reads the New-vision and not the Daily-monitor is 0.65. The probability that she reads neither of the papers is 0.15. find the probability that she reads daily monitor **UNEB 2008 No.1**
Solution

4. Events A and B are such that $P(A^1 \; n \; B) = 3x$, $P(\; n \; B^1) = 2x$, $P(A^1 \; n \; B^1) = x$ and , $P(B) = \frac{4}{7}$. Use a venn diagram to find the value of **UNEB 2011 No.4**
   (i) x (ii) $P(A \; n \; B)$
Solution

Mutually exclusive events

Two events A and Bare mutually exclusive if they do not occur together ie. $P(A \; n \; B) = 0$

$$\boxed{P(A \quad B) = P(A) + P(B)}$$

Examples

1. Events A and B are mutually exclusive such that $P(B) = \frac{1}{2}$, $P(A) = \frac{2}{5}$ find
   (i) $P(A \; U \; B)$ (ii) $P(A^1 \; n \; B)$ (iii) $P(A^1 \; n \; B^1)$
   Solution

2. In an athletics competition in which there are no dead heats, the probability that Kiplimo wins is 0.5, the probability that Bekele wins is 0.2, the probability that Cheptegei wins is 0.1. Find the probability that;
   (i) Bekele or Kiplimo wins (ii) Neither Kiplimo nor Cheptegei wins
Solution

(i) $P(B \; U \; K) = P(B) + P(K)$
$P(B \; U \; K) = 0.2 + 0.5 = 0.7$

(ii) $P(K^1 \; n \; C^1) = P(K \; U \; C)^1 = 1 - P(K \; U \; C)$
$= 1 - (0.5 + 0.1) = 0.4$

Exercise 4b

Independent events

Two events A and B are independent if the occurrence of one does not affect the other

(i) $P(A \; n \; B) = P(A)x \; P(B)$
(ii) $P(A^1 \; n \; B) = P(A^1)x \; P(B)$
(iii) $P(A \; n \; B^1) = P(A)x \; P(B^1)$
(iv) $P(A^1 \; n \; B^1) = P(A^1)x \; P(B^1)$

Examples

1. Events A and B are independent **UNEB 2009 No.9**
   (i) Show that the events A and B^1 are also independent
   (ii) Find $P(B \;)$ given that $P(A \;) = 0.4$ and $P(A \; U \; B) = 0.8$
   Solution

(i) $P(A \; n \; B^1) = P(A) - P(A \; n \; B)$
$P(A \; n \; B^1) = P(A) - P(A)xP(B)$
$P(A \; n \; B^1) = P(A)x[1 - P(B)]$
$P(A \; n \; B^1) = P(A)xP(B^1)$

(ii) $P(A \; U \; B) = P(A) + P(B) - P(A)xP(B)$
$0.8 = 0.4 + \mathrm{y} - 0.4y$
$y = 0.667 \quad \therefore P(A) = 0.667$

2. The probability of two independent events A and B occurring together $\frac{1}{8}$. The probability that either or both events occur is $\frac{5}{8}$. Find **UNEB 2004 No.2**

(i) $P(A)$ (ii) $P(B)$

Solution

3. Abel, Bob and Charles applied for the same job in a certain company. The probability that Abel will take the job is $\frac{3}{4}$, the probability that Bob will take it is $\frac{1}{2}$, while the probability that Charles will take the job is $\frac{2}{3}$, what is the probability that **UNEB 2004 No.9a**

(i) None of them will take the job (ii) One of them will take job

Solution

CONDITIONAL PROBABILITY

If A and B are two events, then the conditional probability that A occurs given that B has already occurred is $P(A/B)$

$$P(A/B) = \frac{P(A\,n\,B)}{P(B)}$$

Examples

1. Events A and B are independent. Given that $P(A\,n\,B^1) = \frac{1}{4}$ and $P\left(A^1/B\right) = \frac{1}{6}$, find **UNEB 2004 No.9**

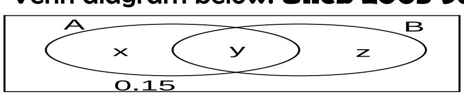
(i) $P(A)$ (iii) $P(A\,nB)$ (iv) $P(A\,u\,B)^1$

(ii) $P(B)$

Solution

Exercise 4d

1. A and B are intersecting sets as shown in the Venn diagram below. **Uneb 2005 9a**



Given that $P(A) = 0.6$, $P\left(A^1/B\right) = \frac{5}{7}$ and $P(AUB) = 0.85$. Find

(i) the value of x, y and z

(ii) $P(A/B)$ $(iii)\ P\left(A/B^1\right)$

An $(i)\ x = 0.5, y = 0.1, z = 0.25\ (ii) = \frac{2}{7}\ (ii) = \frac{10}{13}$

2. Two events A and B are such $P(B/A) = 1/3$, $P(B) = \frac{1}{8}$ and $P(AnB) = \frac{1}{10}$. Find: **UNEB 2019 No.13**

(a) $P(A)$ (b) $P(AuB)$ $(c)\ P\left(A/\bar{B}\right)$

An $(\boldsymbol{a}) = \mathbf{0.3}\ (\boldsymbol{b}) = \mathbf{0.325}\ (\boldsymbol{c}) = \mathbf{0.2286}$

3. Two events A and B are such $P(A/B) = 0.1$, $P(A) = 0.7$ and $P(B) = 0.2$. Find: **UNEB 2020 No.6**

(b) $P(AuB)$ (b) $P(AnB')$

An $(\boldsymbol{a}) = \mathbf{0.88}\ (\boldsymbol{b}) = \mathbf{0.68}$

COMBINATIONS

The number of combinations of r objects from n unlike objects is $\boldsymbol{n_{C_r}}$ where

$$n_{C_r} = \frac{n!}{(n-r)!\,r!}$$

Examples

1. A bag contains 5 Pepsi and 4 Mirinda bottle tops. Three bottle tops are picked at random from the bag one after the other without replacement. Find the probability that the bottle tops picked are of the same type. **Uneb 2016 No.8**

Solution

$n\ (s) = 3\ tops\ from\ the\ 9 = 9_{C_3} = 84ways$
$n\ (E) = 3\ pepsi\ from\ 5 + 3\ mirinda\ from\ 4$
$= 5_{C_3}x4_{C_0} + 5_{C_0}x4_{C_3} = 10 + 4 = 4ways$

$P(same\ type) = \frac{14}{84} = \frac{1}{6}$

2. In a group of 12 international referees, there are 3 from Africa, 4 from Asia and 5 from Europe. To officiate at a tournament 3 referees are chosen at random from the group find the probability that;
(i) A referee is chosen from each continent
(ii) Exactly 2 referees are chosen from Asia
(iii) 3 referees are chosen from the same continent

Solution

$(s) = 3\ refs\ from\ the\ 12 = 12_{C_3} = 220ways$
$n\ (E) = 3_{C_1}x4_{C_1}x5_{C_1} = 60ways$
$P(1\ from\ each) = \frac{60}{220} = \frac{3}{11}$
$(ii)\ n\ (E) = 4_{C_2}x3_{C_1}x5_{C_0} + 4_{C_2}x3_{C_0}x5_{C_1}$
$= 18 + 30 = 48ways$

$(2\ from\ Asia) = \frac{48}{220} = \frac{12}{55}$
$(iii)\ n\ (E) = 4_{C_3}x3_{C_0}x5_{C_0} + 4_{C_0}x3_{C_3}x5_{C_0}$
$+4_{C_0}x3_{C_0}x5_{C_3}$
$= 4 + 1 + 10 = 15ways$
$P(3\ from\ same) = \frac{15}{220} = \frac{3}{44}$

3. Box P contains 4 red and 3 green sweets and box Q contains 7 red and 4 green sweets. A box is randomly selected and 2 sweets are randomly picked from it, one at a time without replacement. If P is twice as likely to be picked as Q, find the probability that both sweets are
(i) Same colour
(ii) of different colours,
(iii) from P given that they are of different colours.

Solution

(i) $\boldsymbol{P(both}\ alls\ same\ colour) = \frac{2}{3}\left[\frac{4_{C_2}x3_{C_0}}{7_{C_2}}\right] + \frac{2}{3}\left[\frac{4_{C_0}x3_{C_2}}{7_{C_2}}\right] + \frac{1}{3}\left[\frac{7_{C_2}x4_{C_0}}{11_{C_2}}\right] + \frac{1}{3}\left[\frac{7_{C_0}x4_{C_2}}{11_{C_2}}\right] = 0.4494$

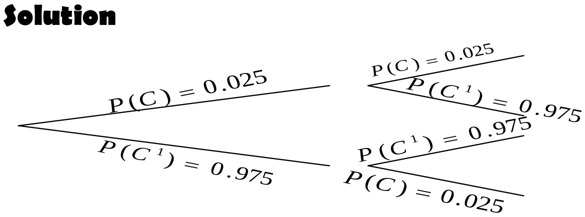
(ii) $\boldsymbol{P(oth}\ balls\ different\ colour) = \frac{2}{3}\left[\frac{4_{C_1}x3_{C_1}}{7_{C_2}}\right] + \frac{1}{3}\left[\frac{7_{C_1}x4_{C_1}}{11_{C_2}}\right] = 0.5506$

(iii) $\boldsymbol{P\left(^{from\ P}/_{different\ colour}\right)} = \frac{\frac{2}{3}\left[\frac{4_{C_1}x3_{C_1}}{7_{C_2}}\right]}{0.5506} = 0.6919$

PROBABILITY TREE DIAGRAMS

1. A factory makes yoghurts. When an inspector tests a random sample of yoghurts, the probability of any yoghurt being contaminated is 0.025. if a student buys two of the yoghurts made the factory. Find the probability
(i) Both yogurts are contaminated
(ii) Only one is contaminated

Solution



(i) $\boldsymbol{P(both\ contaminated)} = P(CnC)$
$= 0.025x0.025 = 0.00063$
$\boldsymbol{P(One\ contaminated)} = P(CnC^1) + P(C^1nC)$
$= 0.025x0.975 + 0.975x0.025 = 0.0488$

2. A box contains 3 red balls and 4 blue balls. Two balls are randomly drawn one after the other without replacement. Find the probability that
(i) 1st ball is blue
(ii) 2nd ball is red
(iii) 2nd ball is red give that the 1st was blue
(iv) Both ball are of the same colour
(v) Different colour

Solution

Exercise 4e

1. (a) A box contains 7 red balls and 6 blue balls. Three balls are selected at random without replacement. Find the probability that;
   (i) They are of the same colour
   (ii) At most two are blue

   (b) Two boxes P and Q contain white and brown cards. P contains 6 white and 4 brown. Q contains 2 white and 3 brown. A box is selected at random and a card is selected. Find the probability that;
   (i) A brown card is selected
   (ii) Box Q is selected given that the card is white **UNEB 2007 No.15**

   An $(\boldsymbol{i}) = 0.1923$ $(\boldsymbol{ii}) = 0.9301, (b)(i) = 0.5, (ii) = 0.4$

2. A bag contains 30 white, 20 blue and 20 red balls. Three balls are selected at random without replacement. Find the probability that the first ball is white and the third ball is also white. **UNEB 2014 No.9a An** $= \mathbf{0.18}$
3. A box A contains 4 white and 2 red balls. Box B contains 3 white and 2 red balls. A box is selected at random and two balls are picked one after the other without replacement.
   (i) Find the probability that the two balls picked are red
   (ii) Given that two white balls are picked, what is the probability that they are from box B **UNEB 2015 No.16**

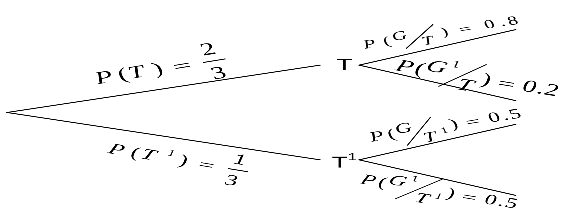
An $(\boldsymbol{i}) = 0.1333$ $(\boldsymbol{ii}) = 0.3333$

CONDITIONAL PROBABILITY USING A TREE DIAGRAM / BAYE'S RULE

Examples

1. During planting season a farmer treats $\frac{2}{3}$ of his seeds and $\frac{1}{3}$ of the seeds are left untreated. The seeds which are treated have a probability of germinating of 0.8 while the untreated seeds have a probability of germination of 0.5. find the probability that a seed selected at random
   (i) will germinate
   (ii) had been treated, given that it had germinated

Solution



(i) $\boldsymbol{P(G)} = P(TnG) + P(T^1 n\ G)$

$$= \frac{2}{3}x0.8 + \frac{1}{3}x0.5 = 0.7$$

(ii) $\boldsymbol{P(^T/_G)} = \frac{(TnG)}{P(G)} = \frac{\frac{2}{3}x0.8}{0.7} = 0.762$

Exercise 4f

1. At a bus park, 60% of the buses are of Teso coaches, 25% are Kakise buses and the rest are Y.Y buses. Of the Teso coaches 50% have TVs, while for the Kakise and Y.Y buses only 5% and 1% have TVs respectively. If a bus is selected at random form the park, determine the probability that **UNEB 1999 NO.7**
   (i) It has a TV
   (ii) Kakise bus is selected given that it has a TV **An**$(i) = 0.0315$ $(ii) = 0.0398$
2. On a certain day, fresh fish from lakes, Kyoga, Victoria, Albert and George were supplied to a market in ratio a 30%, 40%, 20% and 10% respectively. Each lake had an estimated ratios of poisoned fish of 2%, 3%, 3% and 1% respectively. If a health inspector picked a fish at random. **UNEB 2002 NO.1**
   (i) What is the probability that the fish was poisoned?
   (ii) Given that the fish was poisoned, what is the probability that it was from lake Albert. **An**$(\boldsymbol{i}) = 0.025$ $(ii) = 0.24$
4. A mobile phone dealer imports Nokia and Motorola phones. In a given consignment, 55% were Nokia and 45% were Motorola phones. The probability that a Nokia phone is defective is 4%. The probability that a Motorola phone is defective is 6%. A phone is picked at random from the consignment. Determine that it is; **UNEB 2020 No.8**
   (i) defective
   (ii) a Motorola given that it is defective

An $(i) = \mathbf{0.049}$ $(ii) = \mathbf{0.551}$

DISCRETE PROBABILITY DISTRIBUTIONS

A probability density function (p.d.f) is discrete if it takes on specific values

Properties of discrete probability density functions

(i) $\sum P(X = x) = 1$ OR $\sum f(x) = 1$

(ii) $P(X = x) \geq 0$

Examples

1. A discrete random variable has a probability function $P(X = x) = \begin{cases} cx^2, & x = 0,1,2,3,4 \\ 0, & otherwise \end{cases}$

   Find the value of c and draw the graph of $P(X = x)$

   Solution

$$\sum P(X = x) = 1$$

$$c(0^2) + c(1)^2 + c(2)^2 + c(3)^2 + c(4)^2 = 1$$

$$c + 4c + 9c + 16c = 1$$

$$c = \frac{1}{30}$$

2. A discrete random variable has a probability function

$$P(X = x) = \begin{cases} k\left(\frac{2}{3}\right)^x, & x = 0,1,2,3 \dots\dots\dots\dots.. \\ 0, & otherwise \end{cases}$$

   Find the value of k

   Solution

Finding probabilities

1. A discrete random variable X has a probability distribution

x	1	2	3	4	5
P(X=x)	0.15	0.20	0.15	c	0.1

Find

(i) The value of c

(ii) $P(X < 4)$

(iii) $P(X \leq 4)$

(iv) $P(2 \leq X \leq 4)$

(v) $P(X > 2 / X \leq 4)$

(vi) Mode

Solution

Exercise 5a

1. A discrete random variable X has a probability distribution

x	1	2	3	4	5
P(X=x)	0.2	0.25	0.4	a	0.05

Find

(i) The value of a

(ii) $P(1 \leq X \leq 3)$

(iii) $P(X > 2)$

(iv) $P(2 < X < 5)$

(v) Mode

An$(\boldsymbol{i = 0.1, ii = 0.85, iii = 0.55, iv = 0.3, v = 3})$

2. Lisa plays a game in which she throws 2 dice. If she gets two sixes, she wins 20,000/=, if she gets one six she wins 10,000/=, otherwise she wins nothing. She pays 5,000/= to enter the game. Write down the probability distribution of X, the amount she gains in one turn

An

x	-5000	5000	15000
P(X=x)	$\frac{25}{36}$	$\frac{5}{18}$	$\frac{1}{36}$

EXPECTATION OF X, E(X) OR MEAN

The expected value of x is given by

$$E(X) = \sum xP(X = x)$$

Examples

1. At a cinema, movie night follows a discrete random variable. Three girls, Jolly, Cathy and Sheila watch with probabilities of $\frac{1}{4}$, $\frac{1}{3}$, $\frac{2}{5}$ respectively. Write out the probability distribution X, for the number of girls who watch the movie and determine the number of girls who watched.

Solution

$$P(X = 3) = P(JnCn) = \frac{1}{4}x\frac{1}{3}x\frac{2}{5} = \frac{1}{30}$$

$$P(X = 2) = P(JnCn\bar{S}) + P(Jn\bar{C}nS) + P(\bar{J}nCnS)$$

$$= \left(\frac{1}{4}x\frac{1}{3}x\frac{3}{5}\right) + \left(\frac{1}{4}x\frac{2}{3}x\frac{2}{5}\right) + \left(\frac{3}{4}x\frac{1}{3}x\frac{2}{5}\right) = \frac{13}{60}$$

$$P(X = 1) = P(J\bar{C}n\bar{S}) + P(\bar{J}nCn\bar{S}) + P(\bar{J}n\bar{C}nS)$$

$$= \left(\frac{1}{4}x\frac{2}{3}x\frac{3}{5}\right) + \left(\frac{3}{4}x\frac{1}{3}x\frac{3}{5}\right) + \left(\frac{3}{4}x\frac{2}{3}x\frac{2}{5}\right) = \frac{9}{20}$$

$$P(X = 0) = P(\bar{J}n\bar{C}n\bar{S}) = \frac{3}{4}x\frac{2}{3}x\frac{3}{5} = \frac{3}{10}$$

x	0	1	2	3
P(X=x)	$\frac{3}{10}$	$\frac{9}{20}$	$\frac{13}{60}$	$\frac{1}{30}$

$$E(X) = \sum xP(X = x)$$

$$= \left(0x\frac{3}{10}\right) + \left(1x\frac{9}{20}\right) + \left(2x\frac{13}{60}\right) + \left(3x\frac{1}{30}\right) = 0.983$$

2. A gambling game consists of tossing 3 coins. A participant is paid 5,000/= if he gets either all head or all tail, otherwise he pays out 3,000/=. What is the participants expected gain per toss

Solution

An$= -1000/=$

3. Box P contains 4 red and 3 green sweets and box Q contains 5 red and 6 green sweets. A box is randomly selected and 2 sweets are randomly picked from it, one at a time without replacement. If P is twice as likely to be picked as Q, find the probability that both sweets are **UNEB 2011 NO15**
   (i) of the same colours,
   (ii) from P given that they are of same colours.
   (iii) expected number of red sweets removed

Exercise 5b

1. A discrete random variable X has a probability distribution. **UNEB 2019 No.4**

x	0	1	2	3	4	5
P(X=x)	0.11	0.17	0.2	0.13	p	0.09

Find
   (i) The value of the p
   (ii) Expected value of X

   An $p = 0.3,\ ,\mu = 2.6$

2. A certain football team has three matches to play. The probabilities of winning the first, second and third matches are $\frac{3}{5}$, $\frac{2}{5}$ and $\frac{1}{5}$ respectively. **UNEB 2020 No.15**
   (a) Find the probability that the team wins;
      (i) Exactly two matches
      (ii) All matches
      (iii) No match
   (b) If a random variable X is defined as "the number of matches won"
      (i) Construct a probability distribution table for X
      (ii) Calculate the expectation of X, E(X)

   An$(\boldsymbol{a}) = \mathbf{0.08, 0.19, 0.464}, (\boldsymbol{b}) = \mathbf{1.2}$

Properties of the mean

(i) $E(a) = a$

(ii) $E(ax) = a\,E(x)$

(iii) $E(ax + b) = aE(x) + b$

(iv) $E(ax - b) = aE(x) - b$

Where a and b are constants

Examples

1. A random variable X of a discrete probability distribution is given by

x	1	2	3
P(X=x)	0.1	0.2	0.3

Find

(i) $E(x)$ (ii) $E(5x)$ (iii) $E(4X+6)$

Solution

(i) $E(X) = \sum xP(X=x)$

$E(X) = (1x0.1) + (2x0.2) + (3x0.3) = 2.2$

(ii) $E(5X) = 5E(X) = 5x2.2 = 11$

(iii) $E(4X+6) = 4E(X) + 6 = 4x2.2 + 6 = 14.8$

Variance, Var(X)

$$\boldsymbol{Var(X)} = E(X^2) - [E(X)]^2$$

Where $E(X^2) = \sum x^2 P(X=x)$

Examples

1. The discrete random variable Y has a probability distribution is given by

$$P(Y=y) = c|y|, \quad y = -3, -2, -1, 0, 1, 2, 3,$$

Find

(i) The value of c (ii) $mean, \mu$ (iii) Standard deviation

Solution

y	-3	-2	-1	0	1	2	3
P(Y=y)	3c	2c	c	0	c	2c	3c

(i) $\sum P(Y=y) = 1$

$3c + 2c + c + 0 + c + 2c + 3c = 1$

$$c = \frac{1}{12}$$

(ii) $E(X) = \sum xP(X=x)$

$= (-3x3c) + (-2x2c) + (-1xc) + (1xc) + (2x2c) + (3x3c)$

$E(X) = 0$

$\boldsymbol{Var(X)} = E(X^2) - [E(X)]^2$

$E(X^2) = \sum x^2 P(X=x)$

$= ((-3)^2 x3c) + ((-2)^2 x2c) + ((-1)^2 xc) + ((1)^2 xc) + ((2)^2 x2c) + ((2)^2 x2c)$

$$E(X^2) = 72c = 72x\frac{1}{12} = 6$$

$\boldsymbol{Var(X)} = 6 - [0]^2 = 6$

S.D$= \sqrt{6} = 2.45$

2. Two marbles are drawn without replacement from a box containing 3 red marbles and 4 white marbles. The marbles are drawn at random. If X is the random variable for the number of red marbles drawn, find

(i) Expected number of red marbles **(ii)** The standard deviation of X

Solution

(i) $E(X) = \frac{6}{7}$

$$\boldsymbol{Var(X)} = \frac{20}{49}$$

3. A vendor stocks 12 copies of a magazine each week and the probability for each possible total number of copies sold is shown below

Number of copies	9	10	11	12
Probability	0.20	0.35	0.30	0.15

(a) Estimate the mean and variance of the number of copies sold

(b) The vendor buys the magazine at 8,500/= and sells at 14,500/=. Any copies not sold are destroyed. Construct a probability distribution table for the vendors weekly profit and hence find the expected weekly profit

Solution

$$E(X) = 10.4$$

$$\boldsymbol{Var(X)} = 0.94$$

$$E(Y) = 48000$$

4. The table below shows the number of red and green balls put in three identical boxes A, B and C.

UNEB 2018 No.15

Boxes	A	B	C
Red balls	4	6	3
Green balls	2	7	5

A box is chosen at random and two balls are then drawn from it successively without replacement. If the random variable X is "the number of green balls drawn",

(a) draw a probability distribution table for X.
(b) calculate the mean and variance of X

Solution

$$\boldsymbol{P(X=0)} = \frac{1}{3}\left[\frac{4_{C_2}x2_{C_0}}{6_{C_2}} + \frac{6_{C_2}x7_{C_0}}{13_{C_2}} + \frac{3_{C_2}x5_{C_0}}{8_{C_2}}\right] = 0.2332$$

$$\boldsymbol{P(X=1)} = \frac{1}{3}\left[\frac{4_{C_1}x2_{C_1}}{6_{C_2}} + \frac{6_{C_1}x7_{C_1}}{13_{C_2}} + \frac{3_{C_1}x5_{C_1}}{8_{C_2}}\right] = 0.5358$$

$$\boldsymbol{P(X=2)} = \frac{1}{3}\left[\frac{4_{C_0}x2_{C_2}}{6_{C_2}} + \frac{6_{C_0}x7_{C_2}}{13_{C_2}} + \frac{3_{C_0}x5_{C_2}}{8_{C_2}}\right] = 0.2310$$

x	0	1	2
P(X=x)	0.2332	0.5358	0.2310

(i) $E(X) = \sum xP(X=x)$

$= (0x0.2332) + (1x0.5358) + (2x0.231)$

$E(X) = 0.9978$

(ii) $\boldsymbol{Var(X)} = E(X^2) - [E(X)]^2$

$E(X^2) = \sum x^2 P(X=x)$

$= (0^2x0.2332) + (1^2x0.5358) + (2^2x0.231)$

$E(X^2) = 1.4598$

$\boldsymbol{Var(X)} = 0.9978 - [1.4598]^2 = 0.4642$

Properties of the Variance

(i) $Var(a) = 0$
(ii) $Var(aX) = a^2\, Var(X)$
(iii) $Var(aX + b) = a^2 Var(X)$
(iv) $Var(aX - b) = a^2 Var(X)$

Where a and b are constants

Examples

1. A discrete random variable X has a probability distribution

x	1	2	3	4	5
P(X=x)	0.2	0.25	0.4	0.1	0.05

Find
(i) The mean (ii) The variance (iii) Var $(3X - 2)$

Solution

2. A random variable X of a discrete probability distribution is given by

x	10	20	30
P(X=x)	0.1	0.6	0.3

Find (i) The mean (ii) The variance (iii)Var $(4X + 3)$

Solution

BINOMIAL DISTRIBUTION

A binomial distribution is a special type of a discrete random variable in which an experiment gives rise to only two out comes either success or failure

Formula for a binomial distribution

If $X\sim B\ (n\ ,p)$, the probability of obtaining, r successes in n trials $P(X=r)$ where

$$\boxed{P(X=r)=\ ^nC_r p^r q^{n-r}} \text{ for } r=0\ ,1\ ,2\ ,3\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..n$$

Where $q=1-p$

Examples

1. At freedom city supermarket, 60% of the customer's shop on Saturday. Find the probability that in a randomly selected sample of 10 customers
   (i) Exactly 2 shop on Saturday
   (ii) More than 7 shop on Saturday

   Solution

   (i) $n\ =10, p\ =0.6, q=1-0.6=0.4$

   $P(X=2)=\ ^{10}C_2\,0.6^2 0.4^8=0.011$

   (ii) $P(X>7)=P(X=8)+P(=9)+P(X=10)$

   $P(X>7)=\ ^{10}C_8\,x0.6^8x0.4^2+\ ^{10}C_9\,x0.6^9x0.4^1+\ ^{10}C_{10}\,x0.6^{10}x0.4^0$

   $P(X>7)=0.121+0.040+0.006=0.167$

2. A biased coin is such that the chance of a head appearing uppermost when tossed is twice the chance of a tail appearing uppermost. If the coin is tossed 10 times. Find the probability that
   (i) Exactly 6 heads will appear
   (ii) Between 5 and 8 heads will appear

   Solution

   (i) $P(H)+P(T)=1$

   $2x+x=1$

   $x=\dfrac{1}{3}$

   $n\ =10, p\ =\dfrac{2}{3}, q=\dfrac{1}{3}$

   $P(X=6)=\ ^{10}C_6\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^4=0.228$

   (ii) $P(5\ <X<\ 8\ )=\ P(X=6)+P(X=7)$

   $=\ ^{10}C_6\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^4+\ ^{10}C_7\left(\frac{2}{3}\right)^7\left(\frac{1}{3}\right)^3$

   $=0.228+0.260=0.488$

3. A box contains a large number of pens. The probability that a pen is faulty is 0.1. How many pens would you need to select to be more than 95% certain of picking at least one faulty pen

   Solution

   Least value of n is 29

Using cumulative binomial probability table

The tables give values $P(X\geq x)$ for values of n and P

For values of

(i) $P(X\leq x)=1-\ P(X\geq x+1)$

(ii) $P(X=x)=P(X\geq x)-\ P(X\geq x+1)$

Examples

1. The random variable X is distributed $B(5{,}0.3)$. Find ;
   (i) $P(X\geq 3)$
   (ii) $P(X>1)$
   (iii) $P(X\leq 4)$
   (iv) $\boldsymbol{P(X<3)}$
   (v) $P(X=2)$

Solution

Exercise 6a

1. A biased coin is such that a head is three times as likely to occur as a tail. The coin is tossed 5 times. Find the probability that at most two tails occur. **UNEB 2018 No.8 An 0.8965**

2. Tom's chance of passing an examination is $^2/_3$. If he sits for four examinations, calculate the probability that; **UNEB 2014 No.4**
   (i) Only two examinations
   (ii) More than half of the examinations
   An (i) 0.2963 (ii) 0.5926
3. Usain Bolt makes 5 practice runs in the 100m sprint. A run is successful if he runs it in less than 11 seconds. There are 8 chances out of 10 that he is successful. Find the probability that;
   (i) He records at least no success at all
   (ii) He records at least 2 success
   (iii) If he is successful in 5 practice runs, he makes two additional runs. The probability of success in either of the additional runs is 0.7. Determine the probability that Bolt will make 7 successful runs in total. **An (i) 0.0003 (ii) 0.9933 (iii) 0.1606 UNEB 1993 No.13**
4. In a test there are 10 objective questions each with a choice of five possible alternatives out of which only one is correct. If a student guesses each of the answers, find the probability that he gets at least two answers correct **UNEB 1992 No.11 An 0.6242**

Expectation and variance of a binomial distribution

If X~B(n, p) then

$\boxed{E(x) = np}$ and $\boxed{Var\ (x) = npq}$ where $q = 1 - p$

Examples

1. The random variable X is B$(4,0.8)$. Find the mean and the variance

Solution

$E(x) = np$
$E(x) = 4x0.8 = 3.2$
Mean $= 3.2$
$Var\ (x) = npq$
$Var\ (x) = 4x0.8x0.2 = 0.64$
Variance$= 0.64$

Mode of the binomial distribution

The mode is the value of X that is **most likely** to occur. The value of X with the highest probability and its close to the mean gives the mode.

Examples

1. The probability that a student is awarded a distinction in the mathematics examination is 0.15. In a randomly selected group of 15 students, what is the most likely number of students awarded a distinction.

Solution

Exercise 6c

1. In a certain a family, the probability that they will have a baby boy is 0.6. If there are 5 children in a family determine; **UNEB 1988 No.10**
   (a) The expected number of girls
   (b) The probability that there are at least three girls
   (c) The probability that they are all boys
   An (i) 2 **(ii) 0.317 (iii) 0.0778**
2. The probability of winning a game is 0.8. Ten games are played. What is the; **UNEB 1997** No.7
   (a) Mean number of success and variance
   (b) Probability of at least 8 success in the ten games **An (a) 8, 1.6 (b) 0.6778**
3. In a test there are 10 multiples choice questions. Each question has got four possible alternatives out of which only one is correct. If a student guesses each of the answers, find the;
   (i) Probability that at least four answers are correct **UNEB 1998 mar** No.12
   (ii) Most likely number of correct answers
   An (i) 0.2241 (ii) 2
4. A biased coin is such that a head is twice as likely to occur as a tail. The coin is tossed 15 times. Find the **UNEB 2001 No.**4
   (i) The expected number of heads
   (ii) probability that at most two tails occur.
   An (i) 10 (ii) 0.0793

CONTINUOUS PROBABILITY DISTRIBUTION

A probability density function (p.d.f) is continuous, if it takes on values between an interval

Properties of a continuous probability density functions

(i) $\int f(x)dx = 1$ | (ii) $f(x) \geq 0$

Examples

1. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \leq x \leq 2 \\ 2k(x-1), & 2 \leq\leq 4 \\ 0, & else\ where \end{cases}$

Find the value of the constant k

Solution

$$\int_0^2 kx\ d + \int_2^4 2k(2x-1)\ dx = 1$$

$$k\left[\frac{x^2}{2}\right]_0^2 + 2k\left[\frac{x^2}{2} - x\right]_2^4 = 1$$

$$k\left(\frac{2^2}{2} - \frac{0^2}{2}\right) + 2k\left\{\left(\frac{4^2}{2} - 4\right) - \left(\frac{2^2}{2} - 2\right)\right\} = 1$$

$$2k + 8k = 1$$

$$k = \frac{1}{10}$$

Sketching f(x)

- Find the initial and final points of f(x)
- Join the initial and final points of f(x) using a line or a curve

Note

- A line is in the form of $y = mx + c$
- A curve has a power of x being 2 and above or fractional power eg $y = x^2$
- A curve with a positive coefficient of x^2 has a minimum turning point while a curve with a negative coefficient x^2 has a maximum turning point

Examples

1. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k\ (x+2)\ , & -2 \leq x \leq 0 \\ k(2-x), & 0 \leq x \leq 2 \\ 0, & else\ where \end{cases}$

Find the value of the constant k and sketch f(x)

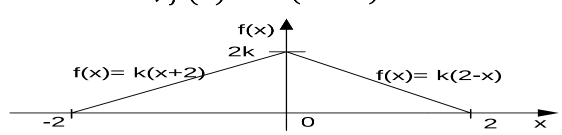
Solution

$$\int_{-2}^{0} k(x+2)\ dx + \int_0^2 k(2-x)dx = 1$$

$$k\left[\frac{x^2}{2} + 2x\right]_{-2}^{0} + k\left[2x - \frac{x^2}{2}\right]_0^2 = 1$$

$$k = \frac{1}{4}$$

When $x = -2$, $f(x) = k(-2+2) = 0$
When $x = 3$, $f(x) = k(0+2) = 2k$
When $x = 2$, $f(x) = k(2-2) = 0$



Finding probabilities

Examples

1. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \leq x \leq 6 \\ 0, & else\ where \end{cases}$

Find

(i) the value of the constant k | (ii) $P(X > 4)$

(iii) $P(X < 3)$

(iv) $P(1 < X < 3)$

(v) $P(X > 2/X \leq 4)$

Solution

(i) $\int_0^6 kx\,dx = 1$

$$k\left[\frac{x^2}{2}\right]_0^6 = 1$$

$$k\left(\frac{6^2}{2} - \frac{0^2}{2}\right) = 1$$

$$k = \frac{1}{18}$$

(i) $P(X > 4) = \frac{1}{18}\int_4^6 x\,dx = \frac{1}{18}\left[\frac{x^2}{2}\right]_4^6$

$$= \frac{1}{18}\left(\frac{6^2}{2} - \frac{4^2}{2}\right) = \frac{5}{9} = 0.5556$$

(ii) $P(X < 3) = \frac{1}{18}\int_0^3 x\,dx = \frac{1}{18}\left[\frac{x^2}{2}\right]_0^3$

$$= \frac{1}{18}\left(\frac{3^2}{2} - \frac{0^2}{2}\right) = \frac{1}{4} = 0.25$$

(iii) $P(1 < X < 3) = \frac{1}{18}\int_1^3 x\,dx = \frac{1}{18}\left[\frac{x^2}{2}\right]_1^3$

$$= \frac{1}{18}\left(\frac{3^2}{2} - \frac{1^2}{2}\right) = \frac{2}{9} = 0.2222$$

(vi) $P(X > 2/X \leq 4) = \frac{P(X>2 \text{ n } X\leq 4)}{P(X\leq 4)}$

$$= \frac{P(2 < X \leq 4)}{P(X \leq 4)} = \frac{\frac{1}{18}\int_2^4 x\,dx)}{\frac{1}{18}\int_0^4 x\,dx}$$

$$= \frac{3}{4}$$

2. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k, & 0 \leq x \leq 2 \\ k(2x-3), & 2 \leq x \leq 3 \\ 0, & else\ where \end{cases}$

Find

(i) the value of the constant k and sketch f(x)

(ii) $P(X < 1)$

(iii) $P(X > 2.5)$

(iv) $P(0 \leq X \leq 2/X \geq 1)$

Solution

(i) $\int_0^2 k\,dx + \int_2^3 k(2x-3)\,dx$

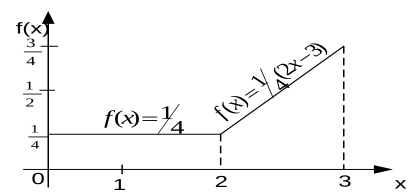
$$k[x]_0^2 + k[x^2 - 3x]_2^3 = 1$$

$$k = \frac{1}{4}$$

When $x = 0$, $f(x) = k = \frac{1}{4}$

When $x = 2$, $f(x) = k = \frac{1}{4}$

When $x = 3$, $f(x) = \frac{1}{4}x(2x3 - 3) = 4$



3. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(x+2)^2, & -2 \leq x \leq 0 \\ 4k, & 0 \leq x \leq 4/3 \\ 0, & else\ where \end{cases}$

Find

(i) the value of the constant k and sketch f(x)

(ii) $P(-1 < X < 1)$

(iii) $P(X > 1)$

Solution

(i) $\int_{-2}^0 k\,(x+2)^2 dx + \int_0^2 4k\,dx$

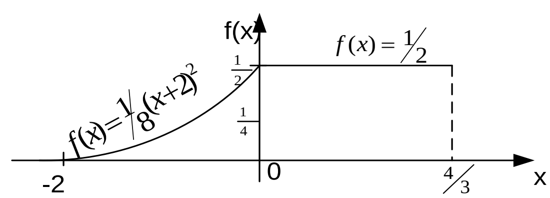
$$k\left[\frac{(x+2)^3}{3}\right]_{-2}^0 + 4k[x]_0^2 = 1$$

$$k = \frac{1}{8}$$

When $x = -2$, $f(x) = k(-2+2)^2 = 0$

When $x = 0$, $f(x) = k(0+2)^2 = \frac{1}{2}$

When $x = 4/3$, $f(x) = 4k = \frac{1}{2}$



Finding the constant from a sketch graph

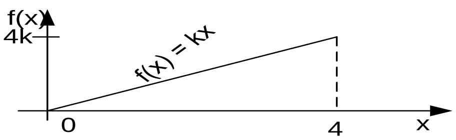
1. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \le x \le 4 \\ 0, & else\ where \end{cases}$

(a) sketch f(x) and find the value of constant k
(b) Find ;
(i) $P(X \le 1)$
(ii) $P(1 < X < 2)$

Solution

(a)

When $x = 0, f(x) = kx0 = 0$
When $x = 4, f(x) = kx4 = 4k$



$\frac{1}{2}x4x4k = 1$

$8k = 1$

$k = \frac{1}{8}$

(i) $P(X \le 1) = \frac{1}{8}\int_0^1 x\,dx = \frac{1}{8}\left[\frac{x^2}{2}\right]_0^1$

$= \frac{1}{8}\left(\frac{1^2}{2} - \frac{0^2}{2}\right) = \frac{1}{16}$

(ii) $P(1 < X < 2) = \frac{1}{8}\int_1^2 x\,dx = \frac{1}{8}\left[\frac{x^2}{2}\right]_1^2$

$= \frac{1}{8}\left(\frac{2^2}{2} - \frac{1^2}{2}\right) = \frac{3}{16}$

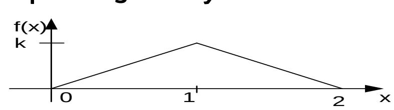
2. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} kx, & 0 \le x \le 2 \\ k(4-x), & 2 \le x \le 4 \\ 0, & else\ where \end{cases}$

(a) sketch f(x) and find the value of the constant k
(b) find;
(i) $P(X < 1)$
(ii) $P(X > 3)$
(iii) $P(1 \le X \le 3)$
(iv) $P({X \ge 1}/{X \le 3})$

Solution

Finding p.d.f from a sketch graph

1. A random variable X of a continuous p.d.f is given by



Find ; **Uneb 2012 No.12**
(a) the value of k
(b) Find f(x)
(c) $P\left({X < 1.5}/{X > 0.5}\right)$
(d) Mean of X

"Expectation or mean of X"

For a continuous random variable with p.d.f, f(x): $E(X) = \int x\,f(x)\,dx$

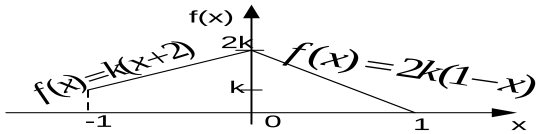
Examples

1. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k\,(x+2), & -1 \le x \le 0 \\ 2k(1-x), & 0 \le x \le 1 \\ 0, & else\ where \end{cases}$

(i) sketch f(x)
(ii) find the value of k
(iii) $P\left({0 < X < 0.5}/{X > 0}\right)$
(iv) find E(X) **Uneb 1997 No.10**

Solution

(i) When $x = -1, f(x) = k(-1+2) = k$
When $x = 0, f(x) = k(0+2) = 2k$
When $x = 1, f(x) = 2k(1-1) = 0$



$$\frac{1}{2}x1x(k+2k)+\frac{1}{2}x1x2k=1$$

$$k=\frac{2}{5}$$

$(iii)\ P\left(0<X<0.5/_{X>0}\right)=\dfrac{P(0<X<0.5)}{P(X>0)}$

$$=\frac{\frac{4}{5}\int_0^{0.5}(1-x)\,dx}{\frac{4}{5}\int_0^{1}(1-x)\,dx}=\frac{\left[x-\frac{x^2}{2}\right]_0^{0.5}}{\left[x-\frac{x^2}{2}\right]_0^{1}}=\frac{3}{8}\Big/\frac{1}{2}=0.75$$

$$(iv)\ E(X)=\frac{2}{5}\int_{-1}^{0}x(x+2)\,dx+\frac{4}{5}\int_{0}^{1}x(1-x)\,dx$$

$$=\frac{2}{5}\left[\frac{x^3}{3}+x^2\right]_{-1}^{0}+\frac{4}{5}\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1=-\frac{2}{15}$$

Properties of the mean

- **(i)** $E(a)=a$
- **(ii)** $E(ax)=a\,E(x)$
- **(iii)** $E(ax+b)=aE(x)+b$
- **(iv)** $E(ax-b)=aE(x)-b$

Where a and b are constants

Examples

1. A random variable X of a continuous p.d.f is given by $f(x)=\begin{cases}\frac{1}{20}(x+3), & 0\le x\le 4\\ 0, & else\ where\end{cases}$
   - (i) Sketch f(x)
   - (ii) Find $E(X)$
   - (iii) Find $E(2X+5)$

   Solution

Exercise 7b

2. A random variable X of a continuous p.d.f is given by $f(x)=\begin{cases}\alpha(1-cosx), & 0\le x\le\frac{\pi}{2}\\ \alpha\ sinx, & \frac{\pi}{2}\le x\le\pi\\ 0, & else\ where\end{cases}$

   Find the **Uneb 2008 No.12**
   - (i) Value of α
   - (ii) Mean, μ
   - $(iii)\ P\left(\frac{\pi}{3}<X<\frac{3\pi}{4}\right)$

   An. $(i)\ \boldsymbol{\alpha=\frac{2}{\pi}}$ $\quad(\boldsymbol{ii})=\boldsymbol{1+\frac{\pi}{4}}$ $\quad(\boldsymbol{iii})=\boldsymbol{0.6982}$

3. A random variable X of a continuous p.d.f is given by $f(x)=\begin{cases}k_1x, & 1\le x\le 3\\ k_2(4-x), & 3\le x\le 4\\ 0, & ele\ where\end{cases}$
   - (a) Show that $k_2=3k_1$ **Uneb 2011 No.9**
   - (b) Find
     - (i) Value of k_1 and k_2
     - (ii) Mean, μ

   An. $(\boldsymbol{i})\ k_1=,k_2=\quad(\boldsymbol{ii})=$

4. A random variable Y of a continuous p.d.f is given by $f(x)=\begin{cases}\frac{y+1}{4}, & 1\le y\le \mathrm{k}\\ 0, & else\ where\end{cases}$

   Find: **Uneb 2015 No.9**
   - (i) Values of k
   - (ii) Expectation of Y
   - (iii) $(1\le Y\le 1.5)$

   An. $(\boldsymbol{i})\ \boldsymbol{k=2},\ (\boldsymbol{ii})=\boldsymbol{1.6667},(\mathbf{iii})=\boldsymbol{0.2813}$

Variance of X

For a continuous random variable with p.d.f, f(x)

$$\boxed{Var(X)=E(X^2)-[E(X)]^2}\quad \textbf{Or}\quad \boxed{Var(X)=E(X^2)-\mu^2}$$

Where $E(X^2)=\int x^2\,f(x)\,dx$ and μ –Mean

Properties of the Variance

- **(i)** $Var(a)=0$
- **(ii)** $Var(ax)=a^2\,Var(x)$
- **(iii)** $Var(ax+b)=a^2Var(x)$
- **(iv)** $Var(ax-b)=a^2Var(x)$

Where a and b are constants

Examples

1. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(1-x^2), & 0 \le x \le 1 \\ 0, & else\ where \end{cases}$

Find **Uneb 1993 No.10a**

(i) the value of the constant k

(ii) $E\ (X)$

(iii) Var(X)

Solution

(i) $\int_0^1 k(1-x^2)dx = 1$

$$k\left[x - \frac{x^3}{3}\right]_0^1 = 1$$

$$k = 1.5$$

(ii) $E(X) = 1.5\int_0^1 x(1-x^2)dx = 1.5\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$

$$= \frac{3}{8}$$

$$E(X^2) = 1.5\int_0^1 x^2(1-x^2)dx$$

$$= 1.5\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{1}{5}$$

$$Var(X) = \frac{1}{5} - \left(\frac{3}{8}\right)^2 = \frac{19}{320}$$

Mode

This is the value of X for which f(x) is maximum in the f(x) given range of X

(i) The mode is obtained from $\frac{d}{dx}f(x) = 0$

The maximum value is confirmed if $\frac{d^2}{dx^2}f(x) = negative$

(ii) When a sketch of f(x) is drawn, the value of X for which f(x) is maximum gives the mode

Note: For any line the mode can only be determine from a sketch of f(x)

Examples

1. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} k(2+x)(4-x), & 0 \le x \le 4 \\ 0, & lse\ where \end{cases}$

Find

(i) The value of k and sketch f(x)

(ii) $mode$

Solution

Median

This is the value of X for which : $\int_a^m f(x)dx = 0.5$

Where is the median and a is the lower limit

Examples

1. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{8}x, & 0 \le\le 4 \\ 0, & else\ where \end{cases}$

Find the median

Solution

$$\int_0^m \frac{1}{8}x\ dx = 0.5$$

$$\left[\frac{1}{16}x^2\right]_0^m = 0.5$$

$$\frac{1}{16}m^2 = \mathbf{0.5}$$

$$m = \pm 2.828$$

Median is 2.828 since it lies in the range

2. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{2}{5}(x+2), & -1 \le x \le 0 \\ \frac{4}{5}(1-x), & 0 \le x \le 10 \\ 0, & else\ where \end{cases}$

Find the median

Solution

Exercise 7d

1. The continuous random variable X has a probability density function

$$f(x) = \begin{cases} kx(4-x^2) & 0 \le x \le 2 \\ 0 & else\ where \end{cases}$$

Find **Uneb 1998 No.11**

(i) Constant k (i) Median

(iii) Mean (iv) Standard deviation

An $(\boldsymbol{i})\boldsymbol{k} = \frac{1}{4}$ (ii) $median = 2.6131$

(iii) $E(x) = 1.0667,$ $(iv)\delta = 0.4422$

2. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \alpha, & 2 \le x \le 3 \\ \alpha(x-2), & 3 \le x \le 4 \\ 0, & else\ where \end{cases}$

(i) Sketch f(x) (ii) Find Constant α

(iii) Median m (iv) $P(2.5 < X < 3.5)$

Uneb 2006 No.15

An. $(\boldsymbol{ii})\ \boldsymbol{\alpha} = \frac{2}{5}$ $(\boldsymbol{iii}) = \mathbf{3.225}$ $(\boldsymbol{iv}) = \mathbf{0.65}$

CUMULATIVE DISTRIBUTION FUNCTION, F(x)

The cumulative distribution function F(x) is defined by $\boldsymbol{F(x) = \int_a^x f(x)dx}$

Steps in finding F(X)

- For each interval, integrate its function from a lower limit to x with respect to x
- Substitute the upper limit in the integral and carry it forward to the next interval
- Continue the process until when the last upper limit has been substituted to get a 1

Examples

1. A random variable X of a continuous p.d.f is given by $f(x) = \begin{cases} \frac{1}{6}(x+1), & 1 \le x \le 3 \\ 0, & else\ where \end{cases}$

Find F(x)

Solution

Finding the median, quartiles and probabilities from F(x)

The median is the value of m for which $F(m) = 0.5$

The lower quartile is the value of q_1 for which $F(q_1) = 0.25$

The upper quartile is the value of q_3 for which $F(q_3) = 0.75$

Examples

1. The continuous random variable X has a cumulative distribution function given below

$$F(x) = \begin{cases} 0, & x \le 0 \\ \frac{x^2}{16} & 0 \le x \le 4 \\ 1, & x \ge 4 \end{cases}$$

Find;

(i) $P(0.3 \leq x \leq 1.8)$

(ii) Median m

(iii) Interquartile range

(iv) Sketch F(x)

Solution

Exercise 7e

1. A continuous random variable X has a probability density function

$$f(x) = \begin{cases} kx & 0 \leq x \leq 1 \\ k(4 - x^2) & 1 \leq x \leq 2 \\ 0 & ese\ where \end{cases}$$

Find **Uneb 1998 No.13**

(i) Value of k

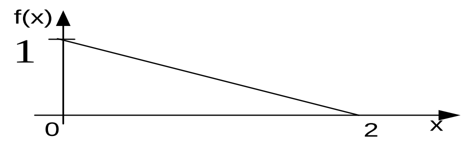
(ii) E(X) and var (X)

(iii) F(x), cumulative distribution function

An $F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{3}{13}x^2 & 0 \leq x \leq 1 \\ \frac{1}{13}(24x - 2x^3 - 19) & 1 \leq x \leq 2 \\ 1 & x \geq 2 \end{cases}$

$(i) = \frac{6}{13}, (ii) = 1.1923, \quad (\text{iii}) = 0.1399$

2. The probability density function f(x) of a random variable X takes on the form shown in the diagram below



Find **Uneb 2000 No.14b**

(i) Expression for f(x)

(ii) F(x), cumulative distribution function

(iii) Mean and variance of X

An $(iii)\mu = \frac{2}{3}, var(X) = \frac{2}{9}$

Finding f(x) from F(x)

f(x) can be obtained from ; $f(x) = \frac{d}{dx}F(x)$

Examples

1. The continuous random variable X has a c.d.f $F(x) = \begin{cases} 0, & x \leq 0 \\ \frac{x^3}{27} & 0 \leq x \leq 3 \\ 1, & x \geq 3 \end{cases}$

Find the probability density function, f(x) and sketch f(x)

Solution

Exercise 7f

1. The continuous random variable X has a cumulative distribution function F(x) where

$$F(x) = \begin{cases} 0, & x \leq 1 \\ \frac{x^2 - 1}{2} - \text{x} & 1 \leq x \leq 2 \\ 3x - \frac{x^2}{2} & 2 \leq x \leq 3 \\ 1, & x \geq 3 \end{cases}$$

Find the **Uneb 2003 NO.10**

(i) Probability density function, f(x) and sketch it

(ii) $P(1.2 < X < 2.4)$

(iii) Mean of X

An $(ii) \quad = 0.8 \quad (iii)\ \mu = 2,$

2. A continuous random variable X has a cumulative function given by **Uneb 2020 NO.12**

$$F(x) = \begin{cases} 0 & ; x \leq 0 \\ \frac{k}{2}x^2 ; & 0 \leq x \leq 2 \\ k(6x - x^2 - 6); & 2 \leq x \leq 3 \\ 1 & ; x \geq 3 \end{cases}$$

(a) Determine the value of k
Hence sketch the graph of $F(x)$

(b) Find the Probability density function, $f(x)$

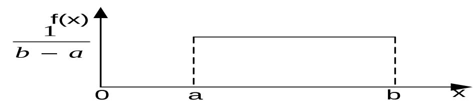
An $(a) \quad = {}^1/_3$

UNIFORM OR RECTANGULAR DISTRIBUTION

A continuous random variable X is said to be uniformly distributed over the interval a and b, if the p.d.f is given by

$$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & else\ where \end{cases}$$

Graph of f(x)



Expectation of X, E(x)

$$E(X) = \int_a^b xf(x)\,dx$$

$$\int_a^b x\left(\frac{1}{b-a}\right)dx$$

$$= \frac{1}{2(b-a)}[x^2]_a^b$$

$$= \frac{1}{2(b-a)}(b^2 - a^2) = \frac{(b-a)(b+a)}{2(b-a)}$$

$$E(x) = \frac{(b+a)}{2}$$

Variance of X, var(x)

$$var(x) = \int_a^b x^2 f(x)\,dx - [E(x)]^2$$

$$\int_a^b x^2\left(\frac{1}{b-a}\right)dx - \left[\frac{(b+a)}{2}\right]^2$$

$$= \frac{1}{3(b-a)}[x^3]_a^b - \left[\frac{(b+a)}{2}\right]^2$$

$$= \frac{1}{3(b-a)}(b^3 - a^3) - \left[\frac{(b+a)}{2}\right]^2$$

$$= \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} - \left[\frac{(b+a)}{2}\right]^2$$

$$= \frac{(b^2+ab+a^2)}{3} - \frac{b^2+2ab+a^2}{4}$$

$$= \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12} = \frac{b^2-2ab+a^2}{12}$$

$$Var(x) = \frac{(b-a)^2}{12}$$

Examples

1. X is a rectangular distribution between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.
   (i) Write the probability density function
   (ii) Sketch f(x)
   (iii) Find the mean and variance

Solution

(i) $f(x) = \begin{cases} \frac{1}{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)} & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 0 & else\ where \end{cases}$

(ii) $E(x) = \frac{(b+a)}{2} = \frac{\left(\frac{\pi}{2}+\frac{\pi}{2}\right)}{2} = 0$

(iii) $ar(x) = \frac{(b-a)^2}{12} = \frac{\left(\frac{\pi}{2}-\frac{\pi}{2}\right)^2}{12} = \frac{\pi^2}{12}$

Exercise 7g

1. The number of patients visiting a certain hospital is uniformly distributed between 150 and 210 ; **Uneb 2014 NO.1**
(i) Write down the probability density function (p.d.f) of the number of patients
(ii) $P(170 < X < 194)$ **An (ii) 0.4**

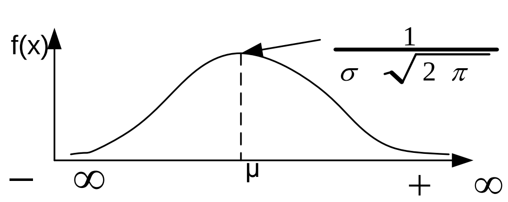
NORMAL DISTRIBUTION

A continuous random variable, X follows a normal distribution with mean μ and variance, σ^2, if

$$\boxed{X \sim N(\mu, \sigma^2)}$$

Its p.d.f is given by $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$

Sketch of $f(x)$ gives a normal curve



Properties of the curve

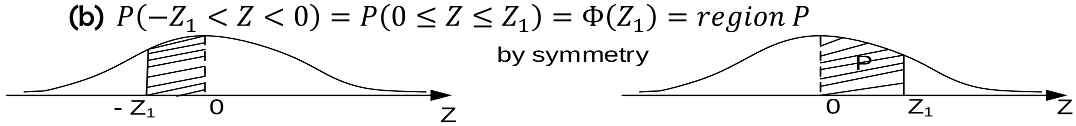
- It is bell shaped
- It is symmetrical about μ
- It extends from $-\infty$ to ∞
- The maximum value of f(x) is $\frac{1}{\sigma\sqrt{2\pi}}$
- The total are under the curve is 1

How to read the cumulative normal distribution table

(i) Between 0 and any Z value

(a) $P(0 \le Z \le Z_1) = \Phi(Z_1) = region\ P$

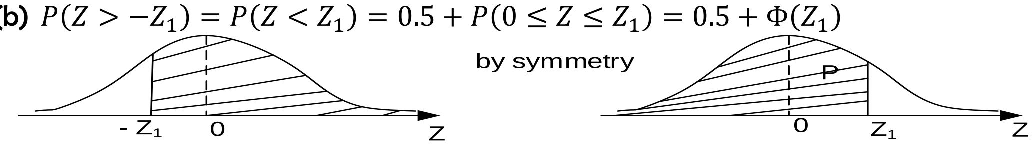
(b) $P(-Z_1 < Z < 0) = P(0 \le Z \le Z_1) = \Phi(Z_1) = region\ P$



(ii) Less than any positive Z value

(a) $P(Z < Z_1) = 0.5 + P(0 \le Z \le Z_1) = 0.5 + \Phi(Z_1)$

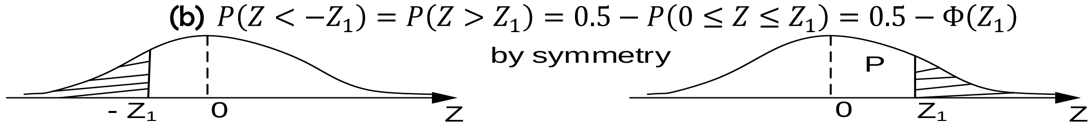
(b) $P(Z > -Z_1) = P(Z < Z_1) = 0.5 + P(0 \le Z \le Z_1) = 0.5 + \Phi(Z_1)$



(iii) Greater than any positive Z value

(a) $P(Z > Z_1) = 0.5 - P(0 \le Z \le Z_1) = 0.5 - \Phi(Z_1)$

(b) $P(Z < -Z_1) = P(Z > Z_1) = 0.5 - P(0 \le Z \le Z_1) = 0.5 - \Phi(Z_1)$



Examples

1. Find

(i) $P(Z < 2)$ (ii) $P(< 0.345)$ (iii) $P(> 0.85)$

Solution

(i) $P(Z < 2) = 0.5 + \Phi(2) = 0.5 + 0.4772 = 0.9772$

(ii) $P(Z < 0.345) = 0.5 + \Phi(0.345) = 0.5 + 0.1331 + 0.0019 = 0.6350$

(iii) $P(Z > 0.85) = 0.5 - \Phi(0.85) = 0.5 - 0.3023 = 0.1977$

2. Find

(i) $P(Z < -0.25)$ **(ii)** $P(Z > -1.377)$ **(iii)** $P(Z < -1.377)$

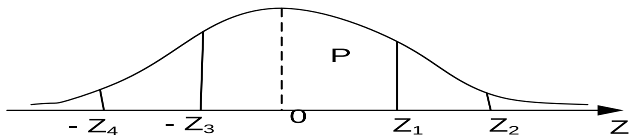
Solution

(i) $P(Z < -0.25) = P(Z > 0.25) = 0.5 - \Phi(0.25) = 0.5 - 0.0987 = 0.4013$

(ii) $P(Z > -1.377) = P(Z < 1.377) = 0.5 + \Phi(1.377) = 0.5 + 0.4147 + 0.0011 = 0.9158$

(iii) $P(Z < -1.377) = P(Z > 1.377) = 0.5 - \Phi(1.377) = 0.5 - (0.4147 + 0.0011) = 0.0842$

Other important results



(i) Between two Z values on the same side of the mean

(a) $P(Z_1 < Z < Z_2) = P(0 < Z < Z_2) - P(0 < Z < Z_1) = \Phi(Z_2) - \Phi(Z_1)$

(b) $P(-Z_4 < Z < -Z_3) = P(0 < Z < Z_4) - P(0 < Z < Z_3) = \Phi(Z_4) - \Phi(Z_3)$

(ii) Between two Z values on the opposite side of the mean

(a) $P(-Z_3 < Z < Z_1) = P(0 < Z < Z_3) + P(0 < Z < Z_1) = \Phi(Z_3) + \Phi(Z_1)$

(b) $P(|Z| < Z_1) = P(-Z_1 < Z < Z_1) = 2xP(0 < Z < Z_1) = 2x\Phi(Z_1)$

(c) $P(|Z| > Z_1) = 1 - P(|Z| < Z_1) = 1 - 2xP(0 < Z < Z_1) = 1 - 2x\Phi(Z_1)$

Examples

Find

(i) $P(1.5 < Z < 1.88)$ **(iii)** $P(-2.696 < Z < 1.865)$ **(v)** $P(|Z| < 1.75)$

(ii) $P(-2.5 < Z < 1)$ **(iv)** $P(-1.4 < Z < -0.6)$ **(vi)** $P(|Z| > 1.433)$

Solution

(i) $P(1.5 < Z < 1.88) = \Phi(1.88) - \Phi(1.5) = 0.4699 - 0.4332 = 0.0367$

(ii) $P(-2.5 < Z < 1) = \Phi(1) + \Phi(2.5) = 0.3413 + 0.4938 = 0.8351$

(iii) $P(-2.696 < Z < 1.865) = \Phi(1.865) + \Phi(2.696) = 0.469 + 0.4964 = 0.9654$

(iv) $P(-1.4 < Z < -0.6) = \Phi(1.4) - \Phi(0.6) = 0.4192 - 0.2257 = 0.1935$

(v) $P(|Z| < 1.75) = P(-1.75 < Z < 1.75) = 2x\Phi(1.75) = 2x0.4625 = 0.925$

(vi) $P(|Z| > 1.433) = 1 - P(|Z| < 1.433) = 1 - 2x\Phi(1.433) = 1 - 2x0.424 = 0.152$

Standardizing a random variable X

If a random variable, X follows a normal distribution with mean μ and variance, σ^2, then $X \sim N(\mu, \sigma^2)$ and can be standardized using the equation below and read from a cumulative normal table

$$Z = \frac{X - \mu}{\sigma}$$

Examples

1. Given that the random variable X is $X \sim N(300,25)$. Find;
   (i) $P(X > 305)$
   (ii) $P(X < 291)$
   (iii) $P(X < 312)$
   (iv) $P(X > 286)$

Solution

(i) $P(X > 305) = P\left(Z > \frac{305-300}{5}\right) = P(Z > 1)$
$= 0.5 - \Phi(1) = 0.5 - 0.3413 = 0.1587$

(ii) $P(X < 291) = P\left(Z < \frac{291-300}{5}\right)$
$= P(Z < -1.8)$
$= P(Z > 1.8) = 0.5 - \Phi(1.8)$
$= 0.5 - 0.4641 = 0.0359$

(iii) $P(X < 312) = P\left(Z < \frac{312-300}{5}\right) = P(Z < 2.4)$
$= 0.5 + \Phi(2.4)$
$= 0.5 + 0.4918 = 0.9918$

(iv) $P(X > 286) = P\left(Z > \frac{286-300}{5}\right)$
$= P(Z > -2.8)$
$P(Z < 2.8) = 0.5 + \Phi(2.8)$
$= 0.5 + 0.4974 = 0.9974$

2. A bakery supplies bread to the shop every day. The taken to deliver bread to the shop is normally distributed with mean 12 minutes and a standard deviation of 2 minutes. Estimate the number of days during the year when he takes.
   (i) Longer than 17 minutes (ii) Less than 10 minutes (iii) Between 9 and 13 minutes

Solution

3. (a) In a certain athletics competition, points are awarded according to level of performance. The average grade was 82 points with a standard deviation of 5 points. All competitors whose grades ranged between 88 to 94 points received certificates. If the grades are normally distributed and 8 competitors received certificates. How many participants took part in the competition
   (b) If certificates were to be awarded to only those having between 90 and 94 points. What proportion of the participants would acquire certificates

Solution

Exercise 8a

1. The amount of meat sold by a butcher is normally distributed with mean 43kg and standard deviation 4kg. determine the probability that the amount of meat sold is between 40kg and 50kg **Uneb 2019 NO.7 (0.7333)**
2. In a school of 800 students their average weight is 54.5kg and standard deviation 6.8kg. Given that the weight of the students are normally distributed, find the **Uneb 2003 NO.13**
   (i) Probability that the weight of any student randomly selected is 52.8 kg or less
   (ii) Number of students who weigh over 75kg
   (iii) Weight of the middle 56% of the students

   An (i)=0.4013, (ii)=1 (iii) $49.251 < X < 59.750$
3. A certain maize firm sells maize in bags of mean weight 40kg and standard deviation 2kg. Given that the weight of the bags are normally distributed, find the **Uneb 1989 NO.14**
   (i) Probability that the weight of any bag of maize randomly selected lies between 41.0 and $42.5kg$
   (ii) Percentage of bags whose weight exceeds 43kg
   (iii) Number of bags that will be rejected out of 500 bags purchased for weighing below 38.5kg

   An (i)=0.2029, (ii)=6.68%, (iii)=113

How to obtain Z-values from a given probability

If you are interested in finding the Z-value whose probabilities are given, it's important to note here that the Z-value may be positive or negative.

Sign	Probability	Z-value
$\prec$	$\prec$ 0 . 5	-
$\succ$	$\succ$ 0 . 5	-
$\prec$	$\succ$ 0 . 5	+
$\succ$	$\prec$ 0 . 5	+

Note: For the above table, the probability given in the question always corresponds to Q in the critical table

Examples

1. $P(Z < Z_1) = 0.25$, find Z_1

Solution

$$P(Z < Z_1) = 0.25(\text{Q})$$

$Z_1 = -0.674$(negative since $0.25 < 0.5$) read directly from a critical table

2. $P(Z < Z_1) = 0.0968$, find Z_1

Solution

Since $0.096(\text{Q})$ is not on a critical table

$P(Z < Z_1) = 0.5 - 0.0968 = 0.4032(\text{P})$

$Z_1 = -1.3$(negative since $0.0968 < 0.5$) read directly from a critical table

3. $P(Z < a) = 0.787$, find a

Solution

Since $0.787(\text{Q})$ is not on a critical table

$P(Z < a) = 0.787 - 0.5 = 0.287$

using a cumulative normal distribution

$P(0 < Z < Z_1) = 0.287$

From the table 0.287 lies between 0.2852 and 0.2881. Since the extra information to the right hand side is **add.** So we consider the smallest value i.e 0.2852 but 0.2852 corresponds to 0.79. To get the next.

$$\begin{array}{r} 0.2870 \\ -\ 0.2852 \\ \hline 0.0018 \end{array}$$

So we look for 0.0018 in the add column which gives 6

$$\therefore \boldsymbol{a} = 0.79 + 0.006 = 0.796$$

Inverse process (De-standardizing Z)

It involves converting the Z-value to raw data (X) from:

$$Z = \frac{X-\mu}{\sigma}$$

$$\therefore \boxed{= Z\sigma + \mu}$$

Examples

1. If $X \sim N(100,36)$ and $P(X < a) = 0.8907$, find the value of a.

Solution

Since 0.8907(Q) is not on a critical table

$P\left(Z < \frac{a-100}{6}\right) = 0.8907 - 0.5 = 0.3907$ (P)

From table $\text{Z} = 1.23$

$a = Z\sigma + \mu$

$a = 1.23x6 + 100$

$a = 107.38$

2. The height of flowers in a farm is normally distributed with mean 169cm and standard deviation 9cm. If X stands for the height of flowers in cm. find X values for
   (a) $P(X < a) = 0.8$
   (b) $P(X > b) = 0.6$

Solution

3. The marks of 500 students in a mock examination for which the pass mark was 50%. Their marks are normally distributed with mean 45 marks and standard and deviation 20 marks.
   (a) Given that the pass mark is 41, estimate the number of candidates who passed the examination
   (b) If 5% of the candidates obtain a distinction by scoring X marks or more, estimate the value of X
   (c) Estimate the interquartile range of the distribution

Solution

3. If $X \sim N(70,25)$ and $P(|X - 70| < a) = 0.8$, find the value of a and hence the limit within which the central 80% of the distribution lies.

Solution

Exercise 8b

1. The length of type A rod is normally distributed with mean of 15cm and a standard deviation of 0.1cm. the length of another type B rod is also normally distributed with mean of 20cm and standard deviation 0.16cm. For type A rod to be acceptable, its length must be between 14.8cm and 15.2cm and type B rod, the length must be between 19.8cm and 20.2cm
   (i) What proportion of type A rod is of acceptable length?
   (ii) What is the probability that one of them is of acceptable length?

An (i) 95.44%, (ii) 0.7528, 0.2375

2. The marks of 1000 students in an examination were normally distributed with mean 55 marks and standard and deviation 8 marks.
   (i) If a mark of 71 or more is required for A-pass, estimate the number of A -passes awarded
   (ii) If 15% of the candidates failed, estimate the minimum mark required for a pass
   (iii) Calculate the probability that two candidates chosen at random both passed the examination

An (i) 23 (ii)47, (iii) 0.7225

3. The marks in an examination were found to be normally distributed with mean 53.9 and standard deviation 16.5. 20% of the candidates who sat this examination failed. Find the pass mark for this examination **UNEB 2015 No. 7**

An 40.007

Finding the values of μ OR σ OR BOTH

Recall $\boxed{X = Z\sigma + \mu}$

Examples

1. If $X \sim N(100, \sigma^2)$ and $P(X < 106) = 0.8849$. find the value of the standard deviation, σ

Solution

$$P(X < 106) = 0.8849(\text{Q} - \text{value})$$
$$P\left(Z < \frac{106-100}{\sigma}\right) = 0.8849 - 0.5 = 0.3849(\text{P} - \text{value})$$
From table $z = 1.2$

$$\frac{106 - 100}{\sigma} = 1.2$$
$$\sigma = 5$$

2. The masses of boxes of oranges are normally distributed such that 30% of them are greater than 4.00kg and 20% are greater than 4.53kg. estimate the mean and standard deviation of the masses

Solution

3. The masses of articles produced in a particular shop are normally distributed with mean μ and standard deviation σ. 5% of the articles have masses greater than 85g and 10% have masses less than 25g.
   (i) find the value of μ and σ
   (ii) Find the symmetrical limit, about the mean, within which 75% of the masses lie

   Solution

4. A total population of 700 students sat a mock examination for which the pass mark was 50%. Their marks were normally distributed. 28 students scored below 40% while 35 students scored above 60%.
   (a) Find the mean mark and standard deviation of the students **UNEB 1995 NO. 13**
   (b) What is the probability that a students chosen at random passed the exam
   (c) Suppose the pass mark is lowered by 2%, how many more students will pass

   Solution

5. A random variable X has a normal distribution when $P(X > 55) = 0.2$ and $P(35 < X < 55) = 0.5$. Find
   (a) the values of the mean, μ and standard deviation, σ.
   (b) The percentage of those with $P(X > 45)$

   Solution

Exercise 8c

1. The marks in an examination were normally distributed with mean μ and standard deviation σ. 10% of the candidates had more than 75 marks and 20% had less than 40 marks. Find the values of μ and σ. **Uneb 2014 No.6 An 53.87, 16.48**
2. The number of cows owned by residents in village is assumed to be normally distributed. 15% of the residents have less than 60 cows. 5%of the residents have over 90 cows. **Uneb 2017 No.15**
   (a) Determine the value of the mean and standard deviation of the cows
   (b) If there are 200 residents, find how many have more than 80 cows

   An $\mu = 71.5926$, $\sigma = 11.1899$, 45
3. A random variable X has a normal distribution when $P(X > 9) = 0.9192$ and $P(X < 11) = 0.7580$. Find: **Uneb 2018 No.12**
   (c) the values of the mean and standard deviation.
   (d) $P(X > 10)$ **An $\mu = 10.3333$, $\sigma = 0.9524$, 0.6386**

BINOMIAL APPROXIMATION TO A NORMAL DISTRIBUTION

Under certain conditions, the normal distribution can be used as an approximation to the binomial distribution.

Condition

(i) The number of trials of the binomial experiment should be large is $n > 20$

(ii) The probability of success not to small or too large ie P constant and very close to 0.5

$$X \sim N(np, npq)$$

The Z value is obtained from $Z = \frac{X \pm 0.5 - \mu}{\sigma} = \frac{X \pm 0.5 - np}{\sqrt{npq}}$

Where ± 0.5 is used to make the binomial distribution continuous

Examples

1. In a box containing different pens, the probability that a pen is red is 0.35. find the probability that in a random sample of 400 pens from the box.
   (i) Less than 120 are red pens
   (ii) More than 160 are red pens
   (iii) Between 120 and 150 inclusive are red pens

   Solution

   $n = 400, p = 0.35, q = 0.65$

   Mean $\mu = np = 400x0.35 = 140$

   $\sigma = \sqrt{npq} = \sqrt{400x0.35x0.65} = \sqrt{91}$

   (i) $P(X < 120) = P(X \leq 119)$

   $= P\left(Z \leq \frac{119.5 - 140}{\sqrt{91}}\right) = P(Z \leq -2.149)$

   $= P(Z \geq 2.149) = 0.5 - \Phi(2.149)$

   $= 0.5 - 0.4821 = 0.0158$

   (ii) $P(X > 160) = P(X \geq 161)$

   $= P\left(Z \geq \frac{160.5 - 140}{\sqrt{91}}\right) = P(Z \geq 2.149)$

   $= 0.5 - \Phi(2.149) = 0.5 - 0.4821 = 0.0158$

   (iii) $P(120 \leq X \leq 150)$

   $= P\left(\frac{119.5 - 140}{\sqrt{91}} \leq Z \leq \frac{150.5 - 140}{\sqrt{91}}\right)$

   $= P(-2.149 \leq Z \leq 1.101)$

   $= \Phi(2.149) + \Phi(1.101)$

   $= 0.4821 + 0.3645 = 0.8466$

2. Two players play a game in which each of them tosses a balanced coin. The game ends in a draw if both get the same result. Determine the probability that in 100 trials, the game ends in a draw.
   (i) At least 53 times
   (ii) At most 53 times

   Solution

3. In a certain book, the number of words per page follows a normal distribution with mean 800 words and standard deviation 40 words. Three pages are chosen at random, what is the probability that
   (i) None of them has between 830 and 845 words
   (ii) At let two pages have between 830 and 845 words

   Solution

Exercise 8d

1. On a certain farm, 20% of all the cows are infected by a tick disease. Find the probability that in a sample of 50 cows selected at random not more than 10% of the cows are infected.
   Uneb 2000 No.6 An(0.0558)
2. A pair of balanced dice, each numbered from 1 to 6 is tossed 180 times. Determine the probability that a sum of seven appears;
   (i) Exactly 40 times
   (ii) Between 25 and 35 inclusive times
   Uneb 2002 No.10 (i) 0.0108, (ii) 0.7286
3. A research station supplies three varieties of seeds S_1, S_2 and S_3 in the ratio of $4:2:1$. The probabilities of germination of S_1, S_2 and S_3 are 50%, 60% and 80% respectively.
   (i) Find the probability that a selected seed will germinate
   (ii) Given that 150 seeds are selected at random, find the probability that less 90 seeds will germinate. **Uneb 2013 No.16**
4. A biased die with faces labeleld 1, 2, 2, 3, 5 and 6 is tossed 45 times. Calculate the probability that 2 will appear; **Uneb 2019 No.16**
   (i) More than 18 times
   (ii) Exactly 11 times.
   An((i)0.1342, (ii) 0.0568)

DISTRIBUTION OF SAMPLE MEAN OF A NORMAL DISTRIBUTION POPULATION

If a population X in with $E(X) = \mu$ and $Var(X) = \sigma^2$.
Take n independent observations $X_1, X_2,\ X_3 \dots, X_n$ from X.
Since $E(X) = \mu$,
$E(X_1) = \mu,\ \ E(X_2) = \mu,\ \ E(X_3) = \mu, \dots, E(X_n) = \mu,$
Since $Var(X) = \sigma^2$,
$Var(X_1) = \sigma^2,\ \ Var(X_2) = \sigma^2,\ Var(X_3) = \sigma^2, \dots, Var(X_n) = \sigma^2,$

Sample mean, $\bar{X} = \frac{X_1 + X_2 + X_3 + \cdots, + X_n}{n}$

$$\bar{X} = \frac{1}{n}X_1 + \frac{1}{n}X_2 + \frac{1}{n}X_3 + \cdots + \frac{1}{n}X_n$$
$$E(\bar{X}) = E\left(\frac{1}{n}X_1 + \frac{1}{n}X_2 + \frac{1}{n}X_3 + \cdots + \frac{1}{n}X_n\right)$$
$$E(\bar{X}) = E\left(\frac{1}{n}X_1\right) + E\left(\frac{1}{n}X_2\right) + E\left(\frac{1}{n}X_3\right) + \cdots + E\left(\frac{1}{n}X_n\right)$$
$$E(\bar{X}) = \frac{1}{n}E(X_1) + \frac{1}{n}E(X_2) + \frac{1}{n}E(X_3) + \cdots + \frac{1}{n}E(X_n)$$
$$E(\bar{X}) = \frac{1}{n}\mu + \frac{1}{n}\mu + \frac{1}{n}\mu + \cdots + \frac{1}{n}\mu$$
$$E(\bar{X}) = n\frac{1}{n}\mu$$
$$\boxed{E(\bar{X}) = \mu}$$

Also $\bar{X} = \frac{1}{n}X_1 + \frac{1}{n}X_2 + \frac{1}{n}X_3 + \cdots + \frac{1}{n}X_n$

$$Var(\bar{X}) = Var\left(\frac{1}{n}X_1 + \frac{1}{n}X_2 + \frac{1}{n}X_3 + \cdots + \frac{1}{n}X_n\right)$$
$$= Var\left(\frac{1}{n}X_1\right) + Var\left(\frac{1}{n}X_2\right) + Var\left(\frac{1}{n}X_3\right) + \cdots + Var\left(\frac{1}{n}X_n\right)$$
$$= \frac{1}{n^2}Var(X_1) + \frac{1}{n^2}Var(X_2) + \frac{1}{n^2}Var(X_3) + \cdots + \frac{1}{n^2}Var(X_n)$$
$$= \frac{1}{n^2}\sigma^2 + \frac{1}{n^2}\sigma^2 + \frac{1}{n^2}\sigma^2 + \cdots + \frac{1}{n^2}\sigma^2$$
$$Var(\bar{X}) = n\frac{1}{n^2}\sigma^2$$
$$\boxed{Var(\bar{X}) = \frac{\sigma^2}{n}}$$

Therefore

$$\boxed{Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}}$$

Examples

1. At a certain school, the masses of students are normally distributed with mean 70kg and standard deviation 5kg. If 4 students are randomly selected, find the probability that their mean mass is less 65kg.

Solution

$$P(\bar{X} < 65) = P\left(Z < \frac{65 - 70}{\frac{5}{\sqrt{4}}}\right) = P(Z < -2)$$

$$P(Z > 2) = 0.5 - \Phi(2) = 0.5 - 0.4772$$
$$= 0.0228$$

2. The distribution of a random variable X is $X\sim N(25, 340)$ and the sample mean $\bar{X}$ for each random sample is calculated. If $P(\bar{X} > 28) = 0.005$, find the value of n

Solution

$$n = 250$$

Exercise 8e

1 If $X\sim N(200, 80)$ and a random sample of size 5 is taken from the distribution, find the probability that the sample mean,
   (i) Is greater than 207
   (ii) Lies between 201 and 209

   An((i) 0.0401, (ii)0.3891,)

3. The random variable is such that $X\sim N(\mu, 4)$. A random sample, size n is taken from the population. Find the least n such that $P(|\bar{X} - \mu| < 0.5) = 0.95$ **An(62)**

4. Boxes made in a factory have weights which are normally distributed with a mean of $4.5kg$ and a standard deviation of $2.0kg$. If a sample of 16 boxes is drawn at random, find the probability that their mean is; **Uneb 1998 nov No. 14**
   (i) Between 4.6 and 4.7 kg
   (ii) Between 4.3 and 4.7 kg

   An((i) 0.0761, (ii)0.3108)

5. The masses of soap powder in a certain packet are normally distributed with mean 842g and variance 225 g^2. Find the probability that a random sample of 120 packets has sample mean with mass. **Uneb 2009 No. 15**
   (i) Between 844 g and 846 g
   (ii) Less than 843g

   An((i) 0.0702, (ii)0.7673)

ESTIMATION OF POPULATION PARAMETERS

Statistical estimation is used to describe the unknown characteristics of the population (population parameter) by using sample characteristic.
A sample is a representation of a population parameter such as population mean, μ and population variance σ^2.

Types of parameter estimation

- Point estimates
- Interval estimates

(a) Point estimates

(i) The unbiased estimate of the population mean, μ is

$\bar{x} = \frac{\sum x}{n}$ or $\bar{x} = \frac{\sum fx}{\sum f}$ Where $\bar{x}$ is sample mean

(ii) The unbiased estimate of the population variance, σ^2 is $\hat{\sigma}^2$ where

$\hat{\sigma}^2 = \frac{n}{n-1}s^2$ Where s^2 is sample variance

OR $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2\right]$ or $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$

Examples

1. Find the best un biased estimate of the mean μ and variance σ^2 of the population from which each of the following sample is drawn
   (i) 46, 48, 50, 45, 53, 50, 48, 51
   (ii)

x	20	21	22	23	24	25
f	4	14	17	26	20	9

   (ii) $\sum x = 100, \sum x^2 = 1028, n = 10$

Solution

x	f	fx	fx^2
45	1	45	2025
46	1	46	2116
48	2	96	4608
50	2	100	5000
51	1	51	2601
53	1	53	2809
	Σf=8	Σfx=391	Σfx^2=19159

Unbiased estimate for the mean

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{391}{8} = 48.875$$

The unbiased estimate of the population variance,

$$\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$$

$$\hat{\sigma}^2 = \frac{8}{8-1}\left[\frac{19159}{8} - \left(\frac{391}{8}\right)^2\right] = 6.982$$

2. The fuel consumption of a new car model was being tested. In one trials 8 cars chosen at random were driven under identical condition and the distance $x\ km$ covered on one litre of petrol was recorded. The following results were obtained, $\sum x = 152.98, \sum x^2 = 2927.1$. Calculate the unbiased estimate of the mean and variance of the distance covered by the car
   Solution

(b) Interval estimate

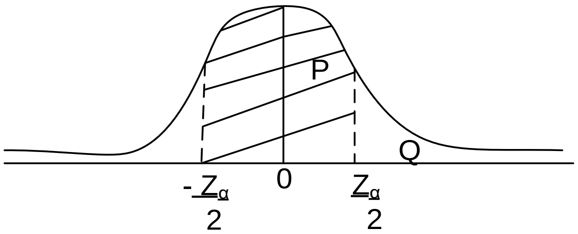
Here we are interested in obtaining the interval over which the true population lies (confidence interval)

Confidence intervals

The unbiased estimate of the population mean, μ is $\bar{x}$

$Z = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}$ Where n is sample size

Z is the area under the normal curve leaving an area of $\frac{\alpha}{2}$ on either side of the curve



$P\left(-Z\alpha_{/2} < Z < Z\alpha_{/2}\right) = {}^{\alpha}/_{2}$

$P\left(-Z\alpha_{/2} < \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} < Z\alpha_{/2}\right) = {}^{\alpha}/_{2}$

$P\left(\bar{x}-Z\alpha_{/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + Z\alpha_{/2}\frac{\sigma}{\sqrt{n}}\right) = {}^{\alpha}/_{2}$

Confidence intervals $\left[\bar{x}-Z\alpha_{/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + Z\alpha_{/2}\frac{\sigma}{\sqrt{n}}\right]$

Confidence limits $\left(\bar{x}-Z\alpha_{/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + Z\alpha_{/2}\frac{\sigma}{\sqrt{n}}\right)$

Or $\mu = \bar{x} \pm Z\alpha_{/2}\frac{\sigma}{\sqrt{n}}$

(i) Confidence interval for population mean, μ

- of a normal or non-normal population
- with known population variance σ^2 or standard deviation σ
- Using any sample size, n large or small

The confidence interval is obtained from

$\mu = \bar{x} \pm Z\alpha_{/2}\frac{\sigma}{\sqrt{n}}$ Where $\bar{x}$ is sample mean

Examples

1. The mass of vitamin E in a capsule is normally distributed with standard deviation 0.042mg. a random sample of 5 capsules was taken and the mean mass if vitamin E was found to be 5.12mg. calculate a symmetric 95% confidence interval for the population mean mass.

Solution

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$

$Z\alpha_{/2} = 1.96$

$\mu = \bar{x} \pm Z\alpha_{/2}\frac{\sigma}{\sqrt{n}}$

$\mu = 5.12 \pm 1.96x\frac{0.042}{\sqrt{5}}$

Lower limit $= 5.08$
Upper limit$= 5.16$

4. After a particular rainy night, 12 worms were picked and their length in cm were measured;
9.5, 9.5, 11.2, 10.6, 9.9, 11.1, 10.9, 9.8, 10.1, 10.2, 10.9, 11.0.
Assuming that this sample came from a normal population with standard deviation 2, find the 95% confidence interval for the mean length of all the worms.

Solution

5. A plant produces steel sheets whose weights are known to be normally distributed with a standard deviation of 2.4kg. A random sample of 36 sheets had a mean weight of 31.4kg.
   (i) find the 99% confidence limits for the population mean.
   (ii) Find the width of the 99% confidence limit

Solution

6. The marks scored by students are normally distributed with mean μ and standard deviation 1.3. It is required to have a 95% confidence interval for μ with width less than2. Find the least number of students that be sampled to achieve this.

Solution

(ii) Confidence interval for population mean, μ

- of a normal or non -normal population
- with un known population variance σ^2 or standard deviation σ
- Using a large sample size (n≥ 30)

If the population variance, σ^2 is not given or unknown, then the confidence interval is obtained from

$$\mu = \bar{x} \pm Z\alpha/_2 \frac{\hat{\sigma}}{\sqrt{n}}$$

Where $\hat{\sigma}^2 = \frac{n}{n-1}s^2$

$s^2 - sample\ variance$

$$\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2\right]$$

Examples

1. The fuel consumption of a new car model was being tested. In one trials 50 cars chosen at random were driven under identical condition and the distance $x\ km$ covered on one litre of petrol was recorded. The following results were obtained, $\sum x = 525$, $\sum x^2 = 5625$. Calculate the 95% confidence interval for the mean petrol consumption, in km per litre of cars of this type.

Solution

Unbiased estimate for the mean

$$\bar{x} = \frac{\sum x}{n} = \frac{525}{50} = 10.5$$

The unbiased estimate of the population variance,

$$\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2\right]$$

$$\hat{\sigma}^2 = \frac{50}{50-1}\left[\frac{5625}{50} - (10.5\,)^2\right] = 1.515$$

$$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$$

$$Z\alpha/_2 = 1.96$$

$$\mu = 10.5 \pm 1.96x\frac{1.515}{\sqrt{50}}$$

$$[10.08\text{km/litre}, 10.92\text{km/litre}]$$

2. The mean and standard deviation of a random sample of size 100 is 900 and 60 respectively. Given that the population is normally distributed, find a 96% confidence interval of the population mean.

Solution

Exercise 8f

1. The concentration, in mg per litre of a trace element in 7 randomly chosen samples of water from a spring were.
$240.8, 237.3, 236.7, 236.6, 234.2, 233.9, 232.5.$
Determine the unbiased estimate of the mean and the variance of the concentration of the trace element per litre of water from the spring. **An(236, 7.58)**
2. A factory produces cans of meat whose masses are normally distributed with standard deviation 18g. A random sample of 25 cans is found to have a mean of 458g. Find the 99% confidence interval for the population mean mass of a can of meat produced at the factory. **An(448.7, 467.3)**
3. A tennis ball is known to have a height of bounce which is normally distributed with standard deviation 2cm. A sample of 60 tennis balls is tested and the mean height of bounce of the sample is 140cm. Find
   (i) 95%
   (ii) 98% confidence interval for the mean height of bounce of the tennis ball
   An(139.5, 140.51) (139.4, 140.6)
4. The distribution of measurements of masses of a random sample of bags packed in a factory is shown below.

Mass (kg)	72.5	77.5	82.5	87.5	92.5	97.5	102.5	107.5
Frequency	6	18	32	57	102	51	25	9

   (i) Find the mean and standard deviation of the masses
   (ii) Find the 95% confidence limits **An((ii)90.5,92.2)**
5. The age , X in years of 250 mothers when their first child was born is given below

x	18 -	20 -	22 -	24 -	26 -	28 -	30 -	32 -	34 -	36 -	38 -
Number of mothers	14	36	42	57	48	26	17	7	2	0	1

   (i) Estimate the mean and standard deviation of X
   (ii) If 250 mothers are randomly selected form a large population of mothers, find the 95% confidence limits for the mean age of the total population **An((i)25.3, 3.6 (ii) (24.9, 25.8))**

NUMERICAL METHODS, ERRORS AND FLOW CHARTS

ERRORS

An error is the difference between exact and approximate value

Types of errors

(a) Rounding errors

These are errors that arise as a result of simply approximating the true value of different numbers

Examples

(b) Truncation errors

These occur when an infinite number is terminated at some point

Examples

Common terms used

(a) Error or absolute error

If x represents an approximate value to X and Δx is the error in approximation

$$|\boldsymbol{error}| = |\boldsymbol{exact\ value} - \boldsymbol{approximate\ value}|$$

$$|\Delta x| = |X - x|$$

Examples

Round off 32.5263 to 2dp and determine the absolute error

Solution

(b) Relative error

$$Relative\ error = \frac{absolute\ error}{exact\ value} = \frac{|\Delta x|}{X} = \frac{|X - x|}{X}$$

(c) Percentage error or percentage relative error

$$Percentage\ Relative\ error = \frac{absolute\ error}{exact\ value} x100\% = \frac{|\Delta x|}{X} x100\% = \frac{|X - x|}{X} x100\%$$

(d) **Maximum possible error in an approximated number**

This depends on the number of dp the number is rounded to

If the number is rounded off to n dp, then the maximum possible error in that number is $= 0.5x10^{-n}$

Examples

1. If a student weighs 50kg. Find the range where his weight lies

Solution

2. A value of $w = 150.58m$ was obtained when measuring the width of the football pitch. Given that the relative error in this value was 0.07%,find the limit within which the value of w lies

Solution

$$\%\ Relative\ error = \frac{|\Delta w|}{w} x100\%$$

$$0.07 = \frac{|\Delta w|}{150.58} x100\%$$

$$|\Delta w| = 0.105$$

$$lower\ limit = 150.58 - 0.105 = 150.475$$

$$upper\ limit = 150.58 + 0.105 = 150.685$$

Absolute error in an operation

When the minimum and maximum value is known then;

$$Absolute\ error\ = \frac{1}{2}[\boldsymbol{maximum\ value - minimum\ alue}\ ]$$

(i) Absolute error in addition

Given two number a and b with errors Δa and Δb

$(a+b)_{max} = a_{max} + b_{max} = (a+\Delta a) + (b+\Delta b)$ $\quad$ $(a+b)_{min} = a_{min} + b_{min} = (a-\Delta a) + (b-\Delta b)$

(ii) Absolute error in subtraction

Given two number a and b with errors Δa and Δb

$(a+b)_{max} = a_{max} - b_{min} = (a+\Delta a) - (b-\Delta b)$ $\quad$ $(a+b)_{min} = a_{min} - b_{max} = (a-\Delta a) - (b+\Delta b)$

Examples

1. Given that $a = 2.453$, $b = 6.79$. Find the limits and hence the absolute error of

(i) $a+b$ $\qquad$ (ii) $a-b$

Solution

$a = 2.453, \Delta a = 0.0005, b = 6.79, \Delta b = 0.005$

(i) $(a+b)_{maximum} = a_{max} + b_{max}$

$(a+b)_{max} = (2.453 + 0.0005) + (6.79 + 0.005)$
$= 9.2485$

$(a+b)_{minimum} = a_{min} + b_{min}$

$(a+b)_{min} = (2.453 - 0.0005) + (6.79 - 0.005)$
$= 9.2375$

$lower\ limit = 9.2375$

$upper\ limit = 9.2485$

$Absolute\ error\ = \frac{1}{2}[9.2485 - 9.2375\ ] = 0.0055$

(ii) $(a-b)_{maximum} = a_{max} - b_{min}$

$(a-b)_{max} = (2.453 + 0.0005) - (6.79 - 0.005)$
$= -4.3315$

$(a-b)_{miniimum} = a_{min} - b_{max}$

$(a-b)_{min} = (2.453 - 0.0005) - (6.79 + 0.005)$
$= -4.3425$

$lower\ limit = -4.3425$

$upper\ limit = -4.3315$

$Absolute\ error\ = \frac{1}{2}[-4.3315 - -4.3425\ ]$
$= 0.0055$

(iii) Absolute error in multiplication

Given two number a and b with errors Δa and Δb

$(ab)_{maximum} = a_{max}b_{max} = (a+\Delta a)(b+\Delta b)$ $\quad$ $(ab)_{minimum} = a_{min}b_{min} = (a-\Delta a)(b-\Delta b)$

Examples

1. Given that $a = 4.617$, $b = 3.65$. Find the;
   (i) the maximum possible error in a and b $\qquad$ (ii) the absolute error ab

Solution

2. Given that $a = 4.617$, $b = -3.65$. Find the;
   (i) the limits of values where ab lies
   (ii) the interval of values where ab lies $\qquad$ (iii) the absolute error in ab

Solution

(iv) Absolute error in division

Given two number a and b with errors Δa and Δb

$\left(\frac{a}{b}\right)_{maximum} = \frac{a_{max}}{b_{min}} = \frac{(a+\Delta a)}{(b-\Delta b)}$ $\quad$ $\left(\frac{a}{b}\right)_{minimum} = \frac{a_{min}}{b_{max}} = \frac{(a-\Delta a)}{(b+\Delta b)}$

Examples

1. Given that $a = 1.26, b = 0.435$. Find the absolute error of
   (i) Find the maximum possible errors in a and b
   (ii) Range of values where $\frac{a}{b}$ lies
   (iii) Absolute error in $\frac{a}{b}$

   Solution

2. The numbers 2.6754, 4.8006, 15.175 and 0.92 have been rounded off to the given number of dp. Find the range of values within which the exact value of $2.6754\left(4.8006 - \frac{15.175}{0.92}\right)$ can be expected to lie

   Solution

3. Obtain the interval of values within which the exact value of $\frac{15.36+27.1-1.672}{2.36x1.043}$ lies

   Solution

Other examples

1. The dimensions of a rectangle are $7.4cm$ and $6.25cm$,
   (i) State the maximum possible error in each dimension
   (ii) Find the range which the area of the rectangle lies

   Solution

2. The numbers $a = 24.57, b = 12.49$ and $c = 7.2$ are calculated with percentage errors of 5, 3 and 1 respectively. Find the limit to two decimal places within which the exact value of the expression $ab - \frac{b}{c}$ lies.

   Solution

$$\Delta a = \frac{5x24.57}{100} = 1.229 \qquad \Delta b = \frac{3x12.49}{100} = 0.375 \qquad \Delta c = \frac{1x7.2}{100} = 0.072$$

$$lower\ limit = (24.57 - 1.229)(12.49 - 0.375) - \left(\frac{12.49 + 0.375}{7.2 - 0.072}\right) = 282.97$$

$$Upper\ limit = (24.57 + 1.229)(12.49 + 0.375) - \left(\frac{12.49 - 0.375}{7.2 + 0.072}\right) = 330.24$$

3. A mobile money business man makes an annual profit of 8 million with a margin error of $\pm 5\%$ and an annual loss of 2 million with a margin error of $\pm 3\%$
   (i) Find the range of values corresponding to his gross income
   (ii) If his annual income tax is 1.5 million, what percentage of the gross income goes to tax, giving your answer as a range of values.

   Solution

Exercise 9a

1. The numbers $X = 1.2, Y = 1.33$ and $Z = 2.245$ have been rounded off to the given decimal places. Find the maximum possible value of $\frac{Y}{Z-X}$ correct to three decimal places. **UNEB 2020 No.5 An(1.342)**
2. Given that $y = e^x$ and $x = 0.62$ correct to two decimal places, find the interval within which the exact value of y lies. **UNEB 2019 No.14a An [1.8497, 1.8682]**
3. Two numbers A and B have maximum possible errors e_a and e_b respectively. **UNEB 2018 No.6**
   (a) Write an expression for the maximum possible errors in their sum.
   (b) If $A = 2.03$ and $B = 1.547$, find the maximum possible error in $A + B$. **An (0.0055)**
4. Given that $y = \frac{1}{x} + x$ and $x = 2.4$ correct to one decimal places, find the limits within which y lies. **UNEB 2017 No.6 An (lower limit=2.8755, upper limit=2.7582)**

ERROR PROPAGATION

Triangular inequality

It states that $|a \pm b| \leq |\Delta a| + |\Delta b|$

(i) Addition

Consider two numbers X and Y are approximated by x and y with errors Δx and Δy

$$|e_{x+y}| = |[(x + \Delta x) + (y + \Delta y)] - (x + y)|$$
$$|e_{x+y}| = |\Delta x + \Delta y|$$
$$|e_{x+y}| \leq |\Delta x| + |\Delta y|$$
$$e_{max} = |\Delta x| + |\Delta y|$$

$$R.E_{max} = \left|\frac{\Delta x}{x+y}\right| + \left|\frac{\Delta y}{x+y}\right|$$

Alternatively

Examples

1. If $x = 4.95$ and $y = 2.2$ are each rounded off to the given number of decimal places. Calculate;
   (i) The percentage error in $x + y$
   (ii) the limit within which $(x + y)$ is expected to lies. Give your answer to two decimal places

Solution

(i) $\Delta x = 0.005$, $\Delta y = 0.05$

$$\%\ error = \left[\left|\frac{\Delta x}{x+y}\right| + \left|\frac{\Delta y}{x+y}\right|\right] x100\%$$

$$\%\ error = \left[\left|\frac{0.005}{4.95+2.2}\right| + \left|\frac{0.05}{4.95+2.2}\right|\right] x100\%$$

$$\%\ error = 0.769$$

Alternatively

Working value$= x + y = 4.95 + 2.2 = 7.15$

$$|e_{x+y}| = |\Delta x| + |\Delta y| = 0.005 + 0.05 = 0.055$$

$$\%\ error = \frac{0.055}{7.15} x100\% = 0.769$$

(ii) $upper\ limit = 7.15 + 0.055 = 7.21$

$lower\ limit = 7.15 - 0.055 = 7.10$

Alternatively

$$(x + y)_{max} = 4.955 + 2.25 = 7.21$$
$$(x + y)_{min} = 4.945 + 2.15 = 7.10$$

(ii) Subtraction

Consider two numbers X and Y are approximated by x and y with errors Δx and Δy

Alternatively

$$absolute\ error = \frac{1}{2}|max - min|$$

$$= 1/2([(x + \Delta x) - (y - \Delta y)] - [(x - \Delta x) - (y + \Delta y)])$$

$$|e_{x-y}| = |\Delta x + \Delta y|$$

$$|e_{x-y}| \leq |\Delta x| + |\Delta y|$$
$$e_{mx} = |\Delta x| + |\Delta y|$$
$$R.E_{max} = \left|\frac{\Delta x}{x-y}\right| + \left|\frac{\Delta y}{x-y}\right|$$

Examples

1. Given numbers $x = 6.375$ and $y = 4.46$ rounded off to the given number of decimal places. Find the limit within which $(x - y)$lies

Solution

$\Delta x = 0.0005$, $\Delta y = 0.005$

$$|e_{x-y}| = |\Delta x| + |\Delta y| = |0.0005| + |0.005| = 0.0055$$

Working value$= x - y = 6.375 - 4.46 = 1.915$

$$upper\ limit = 1.915 + 0.0055 = 1.9205$$
$$lower\ limit = 1.915 - 0.0055 = 1.9095$$

Alternatively

$$(x - y)_{max} = 6.3755 - 4.455 = 1.9205$$
$$(x - y)_{min} = 6.3745 - 4.465 = 1.9095$$

(iii) Multiplication

Consider two numbers X and Y are approximated by x and y with errors Δx and Δy

Alternatively

$$absolute\ error = \frac{1}{2}|max - min|$$

$$= \frac{1}{2}([(x + \Delta x)(y + \Delta y)] - [(x - \Delta x)(y - \Delta y)])$$

$$|e_{xy}| = |y\Delta x + x\Delta y|$$
$$|e_{xy}| \leq |y\Delta x| + |x\Delta y|$$
$$e_{max} = |y\Delta x| + |x\Delta y|$$

$$R.E_{max} = \left|\frac{y\Delta x}{xy}\right| + \left|\frac{x\Delta y}{xy}\right| \qquad R.E_{max} = \left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|$$

Examples

1. Given numbers $x = 6.375$ and $y = 4.46$ rounded off to the given number of decimal places. Find the limit within which (xy)lies

Solution

$\Delta x = 0.0005$, $\Delta y = 0.005$, $|e_{xy}| = |y\Delta x| + |x\Delta y|$

$|e_{xy}| = |4.46x0.0005| + |6.375x0.005| = 0.0341$

Working value= $xy = 6.375x4.46 = 28.4325$

$upper\ limit = 28.4325 + 0.0341 = 28.4666$

$lower\ limit = 28.4325 - 0.0341 = 28.3984$

Alternatively

$(xy)_{max} = 6.3755x4.465 = 28.4666$

$(xy)_{min} = 6.3745x4.455 = 28.3984$

(iv) Division

Consider two numbers X and Y are approximated by x and y with errors Δx and Δy

$$\left|e_{x/y}\right| = \left|\frac{(x+\Delta x)}{(y+\Delta y)} - \left(\frac{x}{y}\right)\right|$$

$$\left|e_{x/y}\right| = \left|\frac{xy + y\Delta x - x\Delta y - xy}{y^2 + y\Delta y}\right|$$

$$\left|e_{x/y}\right| = \left|\frac{y\Delta x - x\Delta y}{y^2\left(1 + \frac{\Delta y}{y}\right)}\right|$$

Since Δx and Δy are very small, then $\frac{\Delta y}{y} \approx 0$

$$\left|e_{x/y}\right| = \left|\frac{y\Delta x - x\Delta y}{y^2}\right|$$

$$\left|e_{x/y}\right| \le \frac{|y\Delta x| + |x\Delta y|}{|y^2|}$$

$$e_{max} = \frac{|y\Delta x| + |x\Delta y|}{|y^2|}$$

$$R.E_{max} = \frac{|y\Delta x| + |x\Delta y|}{|y^2|} \div \frac{x}{y}$$

$$R.E_{max} = \left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|$$

Alternatively

Examples

1. Given numbers $x = 5.794$ and $y = 0.28$ rounded off to the given number of decimal places. Find the limit within which $\left(\frac{x}{y}\right)$lies

Solution

$\Delta x = 0.0005$, $\Delta y = 0.005$

$$\left|e_{x/y}\right| = \frac{|y\Delta x| + |x\Delta y|}{|y^2|}$$

$$\left|e_{x/y}\right| = \frac{|0.28x0.0005| + |5.794x0.005|}{|0.28^2|} = 0.3713$$

Working value= $x/y = 5.794/0.28 = 20.6929$

$upper\ limit = 20.6929 + 0.3713 = 21.0642$

$ower\ limit = 20.6929 - 0.3713 = 20.3198$

Alternatively

$(x/y)_{max} = 5.7945/0.275 = 21.0709$

$(x/y)_{min} = 5.7935/0.285 = 20.3281$

2. (a) The numbers A, B and C are approximated by a, b and c with errors e_1, e_2 and e_3 respectively. Show that the maximum possible relative error in taking the approximation of $\frac{A}{B+C}$ as $\frac{a}{b+c}$ is $\left|\frac{e_1}{a}\right| + \left|\frac{e_2}{b+c}\right| + \left|\frac{e_3}{b+c}\right|$

(b) Given that $a = 40.235$, $b = 14.15$ and $c = 2.45$ are rounded off to the given decimal places. Find the range within which the approximation $\frac{A}{B+C}$ lies

Solution

ERROR IN FUNCTIONS

Given a function $f(x)$ with a maximum possible error Δx.

Absolute error, $|e| = |\Delta x| f^1(x)$

Maximum possible relative error, $R.E = \frac{|\Delta x| f^1(x)}{f(x)}$

Examples

1. Find the absolute error and maximum relative error in each of the functions

(i) $y = x^4$ (ii) $y = x^{\frac{3}{2}}$ (iii) $y = sinx$

Solution

(i) $|e| = |\Delta x| f^1(x) = 4x^3|\Delta x|$

$R.E = \frac{4x^3|\Delta x|}{x^4} = \frac{4|\Delta x|}{x}$

(ii) $|e| = |\Delta x| f^1(x) = \frac{3}{2}x^{\frac{1}{2}}|\Delta x|$

$R.E = \frac{\frac{3}{2}x^{\frac{1}{2}}|\Delta x|}{x^{\frac{3}{2}}} = \frac{3}{2}\frac{|\Delta x|}{x}$

(iii) $|e| = |\Delta x| f^1(x) = cosx\,|\Delta x|$

$R.E = \frac{cosx\,|\Delta x|}{sinx} = |\Delta x||cotx|$

2. Given that the error in measuring an angle is $0.4°$. Find the maximum possible error and relative error in $tanx$ if $x = 60°$

Solution

Error in a function that has more than one variable

Given a function $f(x, y)$ with a maximum possible error Δx and Δy respectively .

Absolute error, $|e| = |\Delta x| f^1(x) + |\Delta y| f^1(y)$

Maximum possible relative error, $R.E = \frac{|\Delta x| f^1(x) + |\Delta y| f^1(y)}{f(x,y)}$

Examples

1. If x and y are recorded off to give X and Y with errors Δx and Δy respectively. Show that the maximum relative error recorded is approximating $x^{\frac{1}{2}}y^3$ by $X^{\frac{1}{2}}Y^3$ is given by

$$\frac{1}{2}\left|\frac{\Delta x}{X}\right| + 3\left|\frac{\Delta y}{Y}\right|$$

Solution

2. The volume of a cylinder is given by $v = \pi r^2 h$. The radius r and height h of the cylinder are measured with corresponding errors Δr and Δh respectively. Find the absolute error and range of values within which v lies if $r = 4.6cm$ and height $h = 15.8cm$

Solution

Exercise 9b

1. The numbers X and Y are approximated with possible errors of Δx and Δy respectively.
   (a) Show that the maximum absolute error in the quotient $\frac{X}{Y}$ is given by $\frac{|Y|\,|\Delta x| + |X|\,|\Delta y|}{Y^2}$
   (b) Given that $x = 2.68$ and $y = 0.9$ are rounded to the given number of decimal places. Find the interval within which the exact value of $\frac{X}{Y}$ is expected to lie. **An** $[\mathbf{2.8158, 3.1588}]$

(b) A cylindrical pipe has a radius of 2.5cm measured to the nearest unit. If the relative absolute error made in calculating its volume is 0.125, find the absolute relative error made in measuring its height

ANS $((a)\ \mathbf{0.32}, (b)\mathbf{0.000211})$

2. The volume of the sphere increases by 2%. Find the corresponding percentage increase in the;
   (i) Radius (ii) Surface area

An($(i) = \frac{2}{3}, (ii) = \frac{4}{3}$)

3. The period of a simple pendulum (T) is given by $T = 2\pi\sqrt{\frac{l}{g}}$ where π and . If the percentage increase in the length (l) is 4%. Find the percentage increase in the period (T) **An**(2)
4. The volume of a cone is given by $v = \frac{1}{3}\pi r^3 h$. The radius r and height h of the cone are measured with corresponding errors Δr and Δh respectively show that the maximum possible relative error in the volume is $3\left|\frac{\Delta r}{r}\right| + \left|\frac{\Delta h}{h}\right|$
5. Given that $A = \ |x||y|\boldsymbol{sin\theta}$

(a) Deduce that the maximum possible relative error in A is given by $\left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right| + cot\theta|\Delta\theta|$ where $\Delta x, \Delta y\ and\ \Delta\theta$ are small numbers compared to x, y and θ respectively

(b) find the error made in the area, if x and y are measured with errors of ± 0.05 and angle with an error of $\pm 0.5^0$ given that $x = 2.5cm, y = 3.4cm$ and $\theta = 30°$

ANS $(\mathbf{0.212})$

6. Show that the maximum possible relative error $\boldsymbol{ysin^2x}$ is $\left|\frac{\Delta y}{y}\right| + 2cotx|\Delta x|$ where Δx and Δy are errors in x and y respectively. Hence find the percentage error in calculating $\boldsymbol{ysin^2x}$ **if** $y = 5.2 \pm 0.05$ and $x = \frac{\pi}{6} \pm \frac{\pi}{360}$ **UNEB 2019. No 14b**

ANS $(\mathbf{3.9845})$

LINEAR INTERPOLATION AND EXTRAPOLATION

(a) Linear interpolation

This deals with computation of values that lie within given values with gradient formula

Examples

1. Given the table below

x	9	10	11	12
f(x)	2.66	2.42	2.18	1.92

Using linear interpolation find

(i) $f(x)$ when $x = 10.15$ (ii) $f^{-1}(2.02)$

Solution

2. Given the table below

x^0	40.0^0	40.4^0	40.8^0	50.4^0
$sinx^0$	0.6428	0.6481	0.6534	0.7705

Find

(i) $sin\,40.5°$ (ii) $sin^{-1}\,0.6445$

Solution

(b) Linear extrapolation

This deals with computation of values that lie outside given values

Given the table below

x	2.2	2.6	3.1
x^3	10.648	17.576	29.791

Find 3.4^3

Solution

Exercise 10a

1. The table below is an extract from the table of $cosx$. **Uneb 1991 No.2 b**

	0^1	10^1	20^1	30^1	40^1	50^1
80^0	0.1736	0.1708	0.1679	0.1650	0.1622	0.1593

Use linear interpolation to determine

(i) $cos80°36'$ **(ii)** $cos^{-1}(0.1685)$. **An 0.1633,** $80°18'$

2. The table below shows the commuter bus fares from stage A to stages B, C, D and E. **Uneb 2019 No.6**

Stage	A	B	C	D	E
Distances (km)	0	12	16	19	23
Fares (shs)	0	1300	1700	2200	2500

(a) Jane boarded from A and stopped at a place 2km after E. How much did she pay

(b) Okello paid Shs 2000. How far from A did the bus leave him? **An ((i) 2650/=, (ii) 17.8km)**

3. The table below shows the value of x and the corresponding values of a function $f(x)$ **Uneb 2020 No.2**

x	0.3	0.6	0.9	1.2
f(x)	3.00	3.22	3.69	4.06

Use linear interpolation to find;

(a) f(x) when $x = 0.4$ (b) x when $f(x) = 3.82$

An ((i)=3.0733 , (ii)=1.0054)

TRAPEZIUM RULE

It is used for estimating an integral area under a curve of a continuous function over a given interval $[a, b]$

If $y = f(x)$

$$A = \int_a^b y dx$$

Using several strips between $x = a$ and $x = b$ of equal width, trapezium rule can be used to determine the area

$$A \approx \frac{1}{2}h[(first + last\ ordnate) + 2(sum\ of\ middle\ ordinates)]$$

Where $h = \frac{b-a}{subinterval}$

Note:

(i) Sub-interval, sub-division and strips are the same

(ii) Sub-interval =(ordinates-1)

(iii) When dealing with a trigonometric function, calculator must be in radian mode

(iv) When the final answer is required to a specific number of the d.p's, then workings before should be at least a d.p higher but the final answer rounded to the d.p's required

Examples

1. Use trapezium rule with 4 sub-intervals to estimate to 2 decimal places $\int_{0.2}^{1.0} \frac{2x+1}{x^2+x} dx$

Solution UNEB 2014 No.3

$$h = \frac{1 - 0.2}{4} = 0.2$$

x	$\frac{2x+1}{x^2+x}$	
0.2	5.833	
0.4		3.214
0.6		2.292
0.8		1.806
1.0	1.500	
sum	7.333	7.312

$$\int_{0.2}^{1.0} \frac{2x+1}{x^2+x} dx \approx \frac{1}{2}x0.2[7.333 + 2(7.312)]$$

$$\approx 2.20$$

2. Use trapezium rule with 4 sub-intervals to estimate to 3 decimal places $\int_0^{\frac{\pi}{2}} cos\, x\, dx$

Solution

3. (a) Use trapezium rule to estimate the integral value of $\int_0^1 x^2 e^x dx$ using five sub-intervals correct to 3dp.

(b) (i) Find the exact value of $\int_0^1 x^2e^x\,dx$

(ii) Find the percentage error in your estimation

Solution

Exercise 10b

1. Use trapezium rule with six strips to estimate $\int_0^\pi xsinx\,dx$ correct to 2dp. Determine the percentage relative error in your estimation. **An(3.07,3.14,2%) UNEB 1999 No.9b**
2. Use trapezium rule to estimate the approximate value of $\int_0^1 \frac{1}{1+x^2}dx$ using 6 ordinates correct to 3 decimal places **An[0.784] UNEB 2000 No.5**
3. Use trapezium rule with seven ordinates to estimate $\int_0^3((1.2)^x-1)^{1/2}\,dx$. Correct to 2 decimal places **UNEB 2019 No.3 An(1.58)**
4. (a) A use the trapezium rule with 6 -ordinates to estimate $\int_{0.1}^{0.5}\frac{1}{2x+1}dx$ correct to 3 significant figures. **UNEB 2020 No.11**

   (b) Evaluate $\int_{0.1}^{0.5}\frac{1}{2x+1}dx$ correct to 3 significant figures.

   (c) (i) Determine the percentage error in the estimation in (a) above, correct to two decimal places

   (ii) Suggest how the percentage error may be reduced**. An((a)=0.256, (b)=0.255,(c)=0.39)**

LOCATION OF REAL ROOTS

The range where the root of an equation lies can be located using the following methods

(i) Graphical method (ii) Sign change method

(i) Sign change

Examples

1. Show that the equation $x^3-6x^2+9x+2=\mathbf{0}$ has a root between -1 and 0

Solution

$$f(x)=x^3-6x^2+9x+2$$
$$f(-1)=(-1)^3-6(-1)^2+9(-1)+2=-14$$
$$f(0)=(0)^3-6(0)^2+9(0)+2=2$$

Since there is a sign change, then the root lies between -1 and 0

2. Show that the equation $e^{2x}sinx-1=0$ has a root between 0 and 1.

Solution

(ii) Graphical method

One or more graphs can be drawn to locate the root

(a) Single graph method

When one graph is drawn, then the root lies between the two points where the curve crosses the x-axis

Examples

(b) Double graph method

When two graph are drawn, then the root lies between the two points where the two curves meet

Note:

(i) Both axes must have a consistent scale and should all be labelled
(ii) A line must be drawn using a ruler while a curve must be drawn using a free hand
(iii) Both graphs must be labeled
(iv) The initial approximation of the root must be located and indicated in the graph

Examples

1. Use a graphical method to show that the equation $e^x - x - 2 = 0$ has only one real root by drawing two graphs of $y = e^x$ and $y = x + 2$
Solution

2. Given the equation $y = sin - \frac{x}{3}$, show by plotting two suitable graphs on the same axes that the positive root lies between $\frac{2\pi}{3}$ and $\frac{5\pi}{6}$
Solution

Exercise 10c

1. By sketching the graphs of 2x and $tan\,x$ show that the equation $2x = tan\,x$ has only one root between $x = 1.1$ and 1.2. Use linear interpolation to find the value of the root correct to 2dp **UNEB 1995 No.1b** **An=1.17,**
2. Show graphically that the equation $x + log_e\,x = 0.5$ has only one real root that lies between 0.5 and 1 **UNEB 1999 No.3**
3. Show graphically that the positive real root of the equation $2x^2 + 3x - 3 = 0$,lies between 0 and 1. **An**$= 0.7$
4. On the same axes, draw graphs of $y = 3 - 3x$ and $y = 2x^2$ to show that the root of the equation $2x^2 + 3x - 3 = 0$ lies between -3 and -2. **An**$= -2.2$
5. Show graphically that the positive real root of the equation $x^3 - 3x - 1 = 0$, lies between 1 and 2 **An**$= \mathbf{1.6}$
6. On the same axes, draw graphs of $y = 3x - 1$ and $y = x^3$ to show that the root of the equation $x^3 - 3x - 1 = 0$ lies between 0 and 1. **An**$= 0.35$
7. Using suitable graphs and plotting them on the same axes, find the real root of the equation $e^{2x}sinx - 1 = 0$, in the interval $x = 0.1$ and $x = 0.8$ **An**$= 0.44$
8. Show graphically that equation $e^{-x} = x$ has only one real root between 0.5 and 1.0. **An**$= 0.56$

METHOD OF SOLVING FOR ROOTS

The following methods can be used

(a) Interpolation

Examples

1. Show that the equation $\mathbf{x^4 - 12x^2 + 12 = 0}$ has a root between 1 and 2. Hence use linear interpolation to get the first approximation of the root
Solution

$$f(x) = x^4 - 12x^2 + 12$$
$$f(1) = 1^4 - 12(1)^2 + 12 = 1$$
$$f(2) = 2^4 - 12(2)^2 + 12 = -20$$

Since there is a sign change, then the root lies between 1 and 2

x	1	x_0	2
f(x)	1	0	-20

$$\frac{x_0 - 1}{0 - 1} = \frac{2 - 1}{-20 - 1}$$
$$x_0 = 1.05$$

2. Show that the equation $2x - 3\cos\left(\frac{x}{2}\right) = 0$ has a root between 1 and 2. Hence use linear interpolation twice to get the approximation of the root of the root
Solution

(b) General iterative method

This involves generating equation by splitting the original equation into several equations by making x the subject

Examples

1. Given $x^2 + 4x - 2 = 0$. Find the possible equations for estimating the roots

Solution

Test for convergence

From the several iterative equations obtained, the equation whose $|f^1(x_n)| < 1$ is the one which converges and gives the correct root

Examples

1. Given the two iterative formulas

(i) $x_{n+1} = \frac{{x_n}^3 - 1}{5}$ **(ii)** $x_{n+1} = \sqrt{\left(5 + \frac{1}{x_n}\right)}$

Using $x_0 = 2$ deduce a more suitable formula for solving the equation. Hence find the root correct to 2dp

Solution

Exercise 10d

1. Show that the iterative formula for solving the equation $x^3 - x - 1 = 0$ is $x_{n+1} = \sqrt{\left(1 + \frac{1}{x_n}\right)}$ hence starting $x_0 = 1$ find the root of the equation correct to 3s.f **UNEB 1993 No2b An(1.33)**

2. **(a) (i)** Show that the equation $e^x - 2x - 1 = 0$ has a root between $x = 1$ and $x = 1.5$

(ii) Use linear interpolation to obtain an approximation for the root

(b) (i) Solve the equation in (a)(i), using each formula below twice. Take the approximation in (a) (ii) as the initial value.

Formula 1: $x_{n+1} = {}^1/_2\,(e^{x_n} + 1)$

Formula 2: $x_{n+1} = \frac{e^{x_n}(x_n - 1) + 1}{e^{x_n} - 2}$

(ii) Deduce with a reason which of the two formula is appropriate for solving the given equation in (a)(i). Hence write down a better approximate root, correct to 2dp. **UNEB 2012 No11 An(1.18, 1.26)**

(c) Newton Raphson Method

Its given by

$$x_{+1} = x_n - \left[\frac{f(x_n)}{f^1(x_n)}\right] \quad n = 0, 1,, \ldots$$

Examples

1. Use newton raphson method to find the root of the equation $x^3 + x - 1 = 0$ using $x_0 = 0.5$ as the initial approximation, correct your answer to 2 decimal places.

Solution

$f(x) = x^3 + x - 1, \quad f^1(x) = 3x^2 + 1$

$$x_{n+1} = x_n - \left(\frac{{x_n}^3 + x_n - 1}{3{x_n}^2 + 1}\right)$$

$$x_{n+1} = \frac{x_n(3{x_n}^2 + 1) - ({x_n}^3 + x_n - 1)}{3{x_n}^2 + 1}$$

$$x_{n+1} = \frac{2{x_n}^3 + 1}{3{x_n}^2 + 1}$$

$x_0 = 0.5, \quad |e| = 0.005$

$$x_1 = \frac{2x0.5^3 + 1}{3x0.5^2 + 1} = 0.7142$$

$$x_2 = \frac{2x0.7142^3 + 1}{3x0.7142^2 + 1} = 0.6831$$

$$x_3 = \frac{2x0.6831^3 + 1}{3x0.6831^2 + 1} = 0.6824$$

$|0.6824 - 0.6831| = 0.0007 < 0.005$

Root is 0.68

2. Show that the equation $5x - 3\,cos2x = 0$ has a root between 0 and 1. Hence use Newtons Raphson method to find the root of the equation correct to 2 decimal places

Solution

A student should be able to use sign change to show that the root lies between 0 and 1

Note: For a trigonometric function the calculator must be strictly in radians

$f(x) = 5x - 3\,cos2x,\quad f^1(x) = 5 + 6\,sin2x$

$$x_{n+1} = x_n - \left(\frac{5x_n - 3\,cos2x_n}{5 + 6\,sin2x_n}\right)$$

$$x_{n+1} = \frac{x_n(5 + 6\,sin2x_n) - (5x_n - 3\,cos2x_n)}{5 + 6\,sin2x_n}$$

$$x_{n+1} = \frac{6\,x_n sin2x_n + 3\,cos2x_n}{5 + 6\,sin2x_n}$$

$x_0 = 0.5,\qquad |e| = 0.005$

$$x_1 = \frac{6x0.5\,sin2(0.5) + 3\,cs2(0.5)}{5 + 6\,sin2(0.5)} = 0.4125$$

$$x_2 = \frac{6x0.4125\,sin2(0.4125) + 3\,cos2(0.4125)}{5 + 6\,sin2(0.4125)} = 0.4096$$

$$x_3 = \frac{6x0.4096\,sin2(0.4096) + 3\,cos2(0.4096)}{5 + 6\,sin2(0.4096)} = 0.4096$$

$|0.4096 - 0.4096| = 0 < 0.005$

Root is 0.41

3. Use newton raphson iterative formula to show that the cube root of a number N is given $\frac{1}{3}\left(2x_n + \frac{N}{x_n{}^2}\right)$ Hence taking $x_0 = 2.5$ determine $\sqrt[3]{10}$ correct to 3 decimal places

Solution

EXERCISE 10e

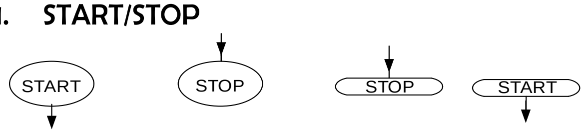
1. Using the iterative formula NRM, show that the reciprocal of a number N is $x_n(2 - Nx_n)$. **UNEB 1991 No.1 a**
2. Use newton raphson iterative formula to show that the cube root of a number N is given $\frac{1}{3}\left(2x_n + \frac{N}{x_n{}^2}\right)$ Hence use the iterative formula to find $\sqrt[3]{96}$ correct to 3 decimal places **UNEB 1992 No.1 An(4.579**
3. Show that the equation $3x^3 + x - 5 = 0$ has a real root between $x = 1$ and $x = 2$ **UNEB 1993 No.1**
   (i) Using linear interpolation, find the first approximation for this root to 2dp
   (ii) using NRM twice find the value of this root correct to 2dp. **Ans [i)=1.045, ii)=1.09]**
4. (a) Show that the equation $x - 3sinx = 0$ has a root between 2 and 3
   (b) Use Newton – Raphson's iterative formula for estimating the root of the equation in (a) is given by $x_{n+1} = \frac{3(\sin x_n - x_n \cos x_n)}{1 - 32\cos x_n}$ n=0,1,2...
   Hence find the root of the equation correct to 2 decimal places. **UNEB 2019 No11 An(2.28)**
5. (a) (i) Draw on the same axes the graphs of equation $y = xsinx$ and $y = e^x - 2$ for $0.5 \leq x \leq 1.5$
   (ii) Use your graphs to find an approximate root of the equation $2 - e^x + xsinx = 0$
   (b) Using Newton Raphson iterative formula and your approximate root in a (ii) above as the initial value, calculate the root of the given equation correct to three decimal places.
   UNEB 2020 No14 An((ii)=1.1, (b)=1.085)

FLOW CHARTS

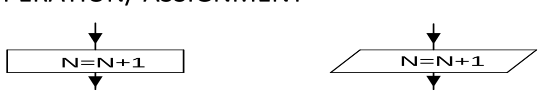
A flow chart is a diagram comprising of systematics steps followed in order to solve a problem.

Shapes used

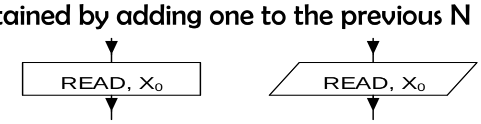
1. START/STOP



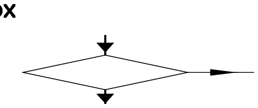
2. OPERATION/ ASSIGNMENT



This indicates that the new number N is obtained by adding one to the previous N



3. Decision box

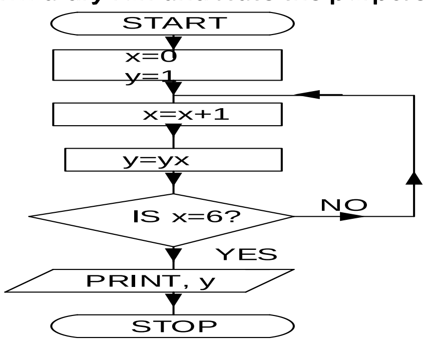


Note:
All other shapes can be interchanged except for the decision box

DRY RUN OR TRACE
This is the method of predicting the outcome of a given flow chart using a table

Examples

1. Perform a dry run and state the purpose of the flow chart. **UNEB 2006 No.10a**



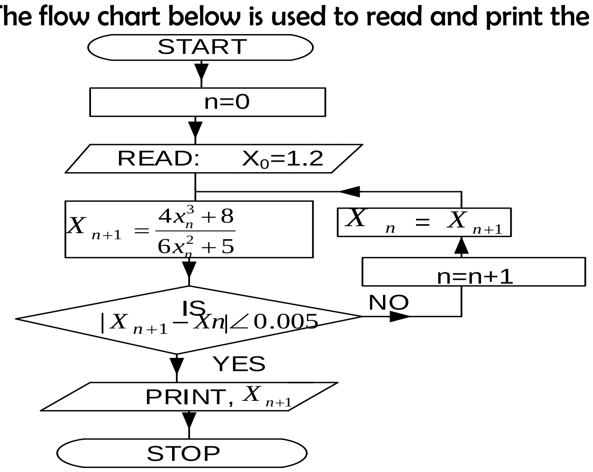
Solution

Dry run

x	y
0	1
1	1
2	2
3	6
4	24
5	120
6	720

Purpose is to compute and print $6!$
Relationship is $y = x!$

2. The flow chart below is used to read and print the root of the equation $2x^3 + 5x - 8 = 0$



Carry out a dry run of the flow chart and obtain the root of with an error of 0.005

Solution

n	X_n	X_{n+1}	$\lvert X_{n+1} - X_n \rvert$
0	1.2	1.0933	0.1067
1	1.0933	1.0867	0.0066
2	1.0867	1.0866	0.001

Root is $= 1.087$

Constructing flow charts

1. Draw a flow char that reads and prints the mean of the first ten counting numbers
Solution

2. Draw a flow chart for computing and printing the mean of the square roots of the first 20 natural numbers
Solution

Newton Raphson and Flow charts

1. (a) Show that the iterative formula base on Newton Raphson's method for approximating the sixth root of a number N is given by $x_{n+1} = {}^1/_6\left(5x_n + \frac{N}{x_n^5}\right)$
(b) Draw a flow chart that;
(i) Reads N and the initial approximation x_0 of the root
(ii) Computes and prints the root to three decimal places
(a) Taking $N = 60$, $x_0 = 1.9$, perform a dry run for the flow chart, give your root correct to three decimal places

Solution

2. (a) Show that the iterative formula base on Newton Raphson's method for approximating the fourth root of a number N is given by $x_{n+1} = {}^{3}/_{4}\left(x_n + \frac{N}{3x_n^3}\right)$

(b) Draw a flow chart that;

(i) Reads N and the initial approximation x_0 of the root

(ii) Computes and prints the root after four iterations

(c) Taking $N = 39.0$, $x_0 = 2.0$, perform a dry run for the flow chart, give your root correct to three decimal places

Solution

(a)

3. (a) Show that the iterative formula based on Newton Raphson's method for finding the natural logarithm of a number N is given by $x_{n+1} = \frac{e^{x_n}(x_n - 1) + N}{e^{x_n}}$, $n = 0, 1, 2 \dots \dots \dots \dots \dots \dots$

(b) Draw a flow chart that;

(i) Reads N and the initial approximation x_0 of the root

(ii) Computes and prints the natural logarithm after four iterations and gives the natural logarithm to three decimal places

(c) Taking, $N = 10, x_0 = 2$, perform a dry run for the flow chart, give your root correct to three decimal places

Solution

Exercise 10f

1. (a) Show that the iterative formula based on Newton Raphson's method for approximating the fourth root of a number N is given by $x_{n+1} = {}^{3}/_{4}\left(x_n + \frac{N}{3x_n^3}\right)$ $n = 0, 1, 2 \dots \dots \dots \dots$

(b) Draw a flow chart that;

(i) Reads N and the initial approximation x_0 of the root

(ii) Computes and prints the root to two decimal places

(c) Taking $N = 35$, $x_0 = 2.3$, perform a dry run for the flow chart, give your root correct to two decimal places **An(2.43)**

2. (a) Show that the iterative formula based on Newton Raphson's method for finding the root of the $2\,Inx - x + 1 = 0$ is given by $x_{n+1} = x_n\left(\frac{2\,In\,x_n - 1}{x_n - 2}\right)$, $n = 0, 1, 2 \dots \dots \dots \dots$

(b) Draw a flow chart that;

(i) Reads the initial approximation x_0 of the root

(ii) Computes and prints the root to two decimal places

(c) Taking, $x_0 = 3.4$, perform a dry run for the flow chart

3. (a) Show that the iterative formula based on Newton Raphson's method for finding the root of the $Inx + x - 2 = 0$ is given by $x_{n+1} = x_n\left(\frac{3 - In\,x_n}{1 + x_n}\right)$, $n = 0, 1, 2 \dots \dots \dots \dots$

(b) Draw a flow chart that;

(i) Reads the initial approximation r of the root

(ii) Computes and prints the root of the equation, when the error is less than $1.0x10^{-4}$

(c) Taking, $r = 1.6$, perform a dry run for the flow chart.

4. (a) Show that the iterative formula based on Newton Raphson's method for finding the root of the $2x^3 + 5x - 8 = 0$ is given by $x_{n+1} = \left(\frac{4x_n^3 + 8}{6x_n^2 + 5}\right)$, $n = 0, 1, 2 \dots \dots \dots \dots \dots \dots$

(b) Draw a flow chart that;

(i) Reads the initial approximation α of the root

(ii) Computes and prints the root of the equation, when the error is less than 0.001

(c) Taking, $\alpha = 1.1$, perform a dry run for the flow chart, give your root correct to three decimal places **An(1.087)**

5. A shop offers a 25% discount on all items in their store and a second discount of 5% for paying cash.

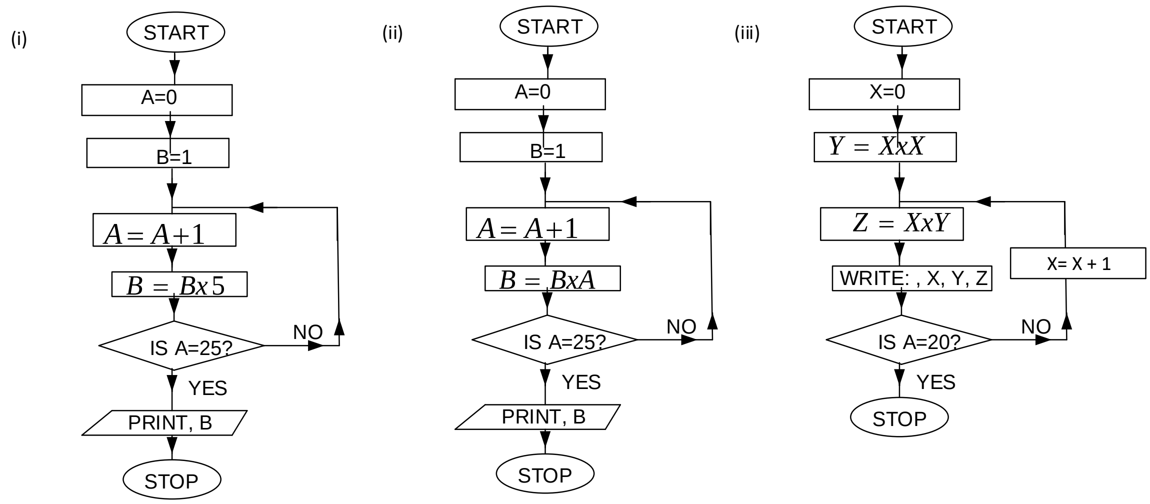
(a) Construct a flow chart for the above information

(b) Perform a dry run for;

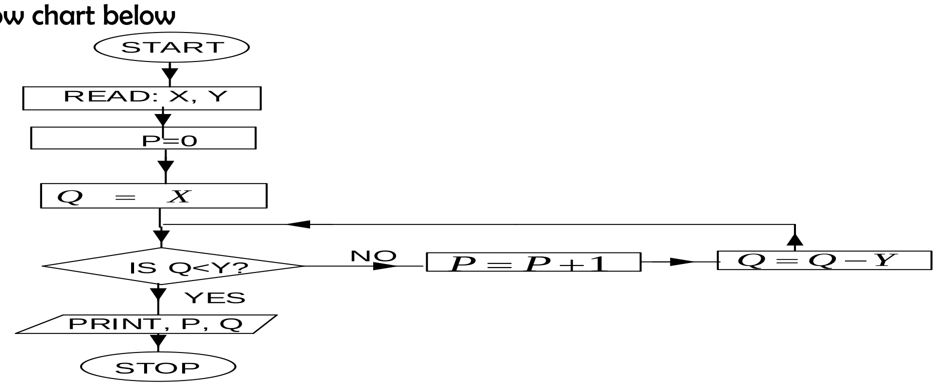
(i) A radio 125,000/= cash

(ii) A T.V 340,000/= credit | **An((i)=89.062.50, (ii)=255,000)**

6. Study the flow charts below and perform a dry run of each flow chart and state the purpose of each flow chart



7. Given the flow chart below



Perform a dry run by completing the tables below

(i) X=17, Y=13

P	Q
0	17

(ii) X=50, Y=7

P	Q

(iii) X=9, Y=2

P	Q

MECHANICS

a) Kinematics
- Motion in a straight line
- Motion under gravity
- Motion on a smooth plane
- Projectile motion
- Relative motion
- Variable acceleration

(b) Dynamics
- Force and newton's laws
- Connected particles
- Work, energy and power
- Linear Momentum
- Simple harmonic motion
- Elasticity
- Circular motion

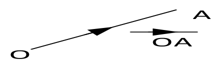
(C) Statics
- Particles in equilibrium
- Resolution and composition
- Moments
- Friction
- Coplanar forces
- Centre of gravity

CHAPTER 1: VECTORS

A vector is a quantity which has both magnitude and direction. Examples include Force, displacement, velocity, acceleration and momentum

Representation of a vector

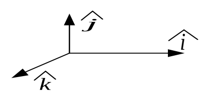
A vector is represented by a line with an arrow to indicate the direction of the vector.



Where order of the letters shows the direction

Vectors in three dimensions

Consider a three-dimensional rectangular co-ordinate system with $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ being unit vectors along the x, y and z- axes



RESULTANT OF VECTORS

When several vectors ($V_1, V_2, \cdots, V_N$) are acting on an object of mass m, the net vector, R is calculated as the vector sum

$$\boldsymbol{R} = V_1 + V_2 + V_3 \dots\dots\dots\dots\dots\dots.. + V_N = \sum_{r=1}^{N} V_r$$

Examples

1. Find the resultant of each of the following forces
   (a) $(2\hat{\imath} + 3\hat{\jmath} + 3\hat{k})N, (2\hat{\imath} + 4\hat{\jmath} - 8\hat{k})N$ | **(b)** $(7\hat{\imath} - 4\hat{\jmath} + 3\hat{k})N, (5\hat{\imath} - 2\hat{\jmath} + 8\hat{k})N, (\hat{\imath} - \hat{k})N,$
   Solution

2 The resultant of the forces $(5\hat{\imath} - 2\hat{\jmath})N$, $(7\hat{\imath} + 4\hat{\jmath})N$, $(a\hat{\imath} + b\hat{\jmath})N$, and $(-3\hat{\imath} + 2\hat{\jmath})N$ is a force $(5\hat{\imath} + 5\hat{\jmath})N$. Find a and b. **An**$(a = -4, b = 1)$

3 The resultant of the forces $3\hat{\imath} + (a - c)\hat{\jmath}$, $(2a + 3c)\hat{\imath} + 5\hat{\jmath}$, $(4\hat{\imath} + 6\hat{\jmath})N$ acting on a particle is a force $(10\hat{\imath} + 12\hat{\jmath})N$. Find .**Uneb 2006 No.4**
   (i) Values of a and c. | (ii) Magnitude of $(2a + 3c)\hat{\imath} + 5\hat{\jmath}$,
An$(\boldsymbol{a = 1.2, c = 0.2, 5.83N})$

MAGNITUDE OR MODULUS OF A VECTOR

This is the length of a vector

(i) Given that $\vec{R} = x\hat{\imath} + y\hat{\jmath}$

$$|\vec{R}| = \sqrt{x^2 + y^2}$$

(ii) Given that $\vec{R} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$|\vec{R}| = \sqrt{x^2 + y^2 + z^2}$$

Examples

Find the magnitude of the following vectors

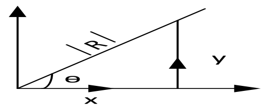
(i) $a = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ | (ii) $b = 2\hat{\imath} + 3\hat{\jmath} - 6\hat{k}$

Solution

$|\boldsymbol{a}| = \sqrt{4^2 + 3^2} = 5$ | $|\boldsymbol{b}| = \sqrt{2^2 + 3^2 + (-6)^2} = 7$

DIRECTION OF THE VECTOR

Consider $\vec{R} = x\hat{\imath} + y\hat{\jmath}$



$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Examples

1. Find the magnitude and direction of the resultant of each of the following

(a) $(2\hat{\imath} + 3\hat{\jmath})N$, $(5\hat{\imath} - 2\hat{\jmath})N$, $(-3\hat{\imath} + 3\hat{\jmath})N$

(b) $\begin{pmatrix}2\\4\end{pmatrix}N$, $\begin{pmatrix}-6\\-5\end{pmatrix}$ and $\begin{pmatrix}2\\1\end{pmatrix}$ | **(c)** $\begin{pmatrix}4\\3\end{pmatrix}N$, $\begin{pmatrix}-1\\-5\end{pmatrix}$

Solution

2. Four forces $a\hat{\imath} + (a-1)\hat{\jmath}$, $3\hat{\imath} + 2a\hat{\jmath}$, $(5\hat{\imath} - 6\hat{\jmath})N$ and $-\hat{\imath} - 2\hat{\jmath}$ act on a particle. The resultant of the forces makes an angle of 45° with the horizontal. Find the value of a and hence determine the magnitude of the resultant force .**Uneb 1999 No.2. An(8, 21.2N)**

UNIT VECTOR

This is a vector whose magnitude is unit (1).

Unit vector of r denoted by $\hat{r}$ is given by $\hat{r} = \frac{r}{|r|}$

Example

Find the unit vector of $\boldsymbol{a} = 6\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$

Solution

Parallel vectors

If a vector a and b are parallel, then one of them is a scalar multiple of the other

If a vector r of magnitude $|r|$ moves in direction $x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ then, $r = |r|\left(\frac{x\hat{\imath}+y\hat{\jmath}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}\right)$

Examples

1. The force A of magnitude 5N acts in the direction with unit vector $^1/_5\,(3\hat{\imath} + 4\hat{\jmath})$ and force B of magnitude 13N acts in the direction with unit vector $^1/_{13}\,(5\hat{\imath} - 12\hat{\jmath})$. Find the resultant of forces A and B. **Uneb 2008 No.3**

Solution

$|\boldsymbol{F}| = 11.3137N$

2. A particle P moves through a displacement of 2m when acted upon by two forces F_1 and F_2. Find the work done by the resultant force, if $F_1 = \hat{\imath} - \hat{\jmath}$ and $F_2 = 10N$ and acts in the direction $4\hat{\imath} + 3\hat{\jmath}$

Solution

$\boldsymbol{W} = 20.5912J$

Exercise: 11A

3. A force of magnitude 50N acts on a body in the direction $24\hat{\imath} + 7\hat{\jmath}$. Find the force. **An**$(48\hat{\imath} + 14\hat{\jmath})$.
4. Two forces F_1 and F_2 have magnitudes αN and βN and act in the direction $\hat{\imath} - 2\hat{\jmath}$ and $4\hat{\imath} + 3\hat{\jmath}$ respectively. Given that the resultant of F_1 and F_2 is $(48\hat{\imath} + 14\hat{\jmath})$.Find αN and βN
An($\alpha = 8\sqrt{5}N$ and $50N$).
5. If $a = 2\hat{\imath} + 7\hat{\jmath} + 7\hat{k}$, $b = 6 - 3\hat{\jmath} + 2\hat{k}$ and $c = -4\hat{\jmath} - 3\hat{k}$. Find the;
   - (i) Resultant of a and b
   - (ii) Resultant of a and c
   - (iii) $|b|$
   - (iv) $|a + b + c|$
   - (v) Vector is parallel to $(a + b + c)$ and has a magnitude of 50units

An (i) $(8\hat{\imath} + 4\hat{\jmath} + 9\hat{k})$, (ii) $(2\hat{\imath} + 3\hat{\jmath} + 4\hat{k})$, (iii) 7units , (iv) 10 units, (v) $(40\hat{\imath} + 30\hat{k})$,

SCALAR PRODUCTS OR DOT PRODUCTS

The dot product of two vectors a and b inclined at an angle θ to each other is given by

$$a \bullet b = |a||b|\cos\theta$$

Note

If two vectors are perpendicular then the angle between them is $90°$ and $a \bullet b = 0$

Examples

1. If $p = \hat{\imath} - 2\hat{k}$ and $q = 3\hat{\imath} - 3\hat{\jmath} + \hat{k}$. Find;
   (i) $p.q$ (ii) The angle between p and q

   Solution
2. If $a = 2\hat{\imath} - \hat{\jmath} + 3\hat{k}$ and $b = \hat{\imath} + 4\hat{\jmath} + 3\hat{k}$. Find the angle between a and b

   Solution
3. If the angle between two vectors $a = x\hat{\imath} + 2\hat{\jmath}$ and $b = 3\hat{\imath} + \hat{\jmath}$ is $45°$. Find the two possible values of constant x.

   Solution

4. If $a = 2\alpha\hat{\imath} + 7\hat{\jmath} - \hat{k}$ and $b = 3\alpha\hat{\imath} + \alpha\hat{\jmath} + 3\hat{k}$. Find the value of the scalar α if the vectors are perpendicular

   Solution

Exercise: 11B

1. If $a = 4\hat{\imath} + 5\hat{\jmath}$ and $b = \alpha\hat{\imath} - 8\hat{\jmath}$. Find the value of the scalar α if the vectors are perpendicular. **An**(10)
2. If $a = 6\hat{\imath} - \hat{\jmath}$ and $b = 2\hat{\imath} + \alpha\hat{k}$. Find the value of the scalar α if the vectors are perpendicular **An**(12)
3. Given that $\begin{pmatrix} \lambda \\ 2+\lambda \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \\ 4-\lambda \end{pmatrix}$ are perpendicular vectors. Find the value of the constant **An**(18)
4. If $a = \lambda\hat{\imath} + 8\hat{\jmath} + (3\lambda + 1)\hat{k}$ and $b = (\lambda + 1)\hat{\imath} + (\lambda - 1)\hat{\jmath} - 2\hat{k}$. Find the value of the possible values of constant λ if the vectors are perpendicular. **An**$(2\ or -3)$

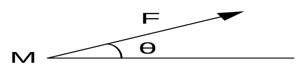
FORCES

A force is anything which can change a body's state of rest or uniform motion in a straight line.
Examples include weight, tension, reaction, friction, resistance force

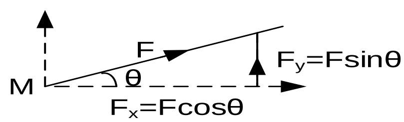
RESOLUTION OF FORCES

The component of a vector is the effective value of a vector a long a particular direction. The component along any direction is the magnitude of a vector multiplied by the **cosine of the angle** between its direction and the direction of the component.

Suppose a force F pulls a body of mass m along a truck at an angle θ to the horizontal as shown below;



The effective force that makes the body move along the horizontal is the component of F along the horizontal



$Cos\theta = \frac{F_x}{F}$

$\boxed{F_x = Fcos\theta}$

$Sin\theta = \frac{F_y}{F}$

$\boxed{F_y = Fsin\theta}$

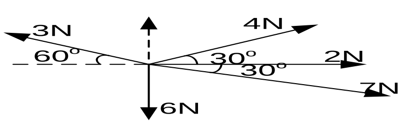
$\boxed{\text{Resultant vector } \boldsymbol{F_R} = \sqrt{\boldsymbol{F_x^2 + F_y^2}}}$

$\boxed{\text{Direction}\alpha = \boldsymbol{tan^{-}}\left(\frac{\boldsymbol{F_y}}{\boldsymbol{F_x}}\right)}$

Examples

1. Find the resultant of the system of forces below

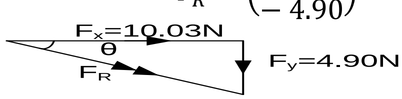
(a)



Solution

$$F_R = \begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}4cos30\\4sin30\end{pmatrix} + \begin{pmatrix}7cos30\\-7sin30\end{pmatrix} + \begin{pmatrix}-3cos60\\3sin60\end{pmatrix} + \begin{pmatrix}0\\-6\end{pmatrix}$$

$$F_R = \begin{pmatrix}10.03\\-4.90\end{pmatrix}$$

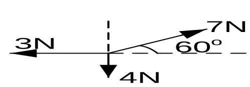


$$F_R = \sqrt{10.03^2 + 4.90^2} = 11.16N$$

$\theta = tan^{-1}\frac{F_y}{F_x} = tan^{-1}\frac{4.90}{10.03}$ =**26.04°**

The resultant force is 11.16N at 26.04° below the horizontal.

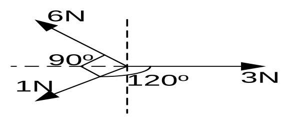
2. A body of mass 1kg is acted upon by the forces shown below. Find;
   - (i) Magnitude of the resultant force
   - (ii) Acceleration of the body
   - (iii) Distance moved in 2s



Solution

EXERCISE 11D

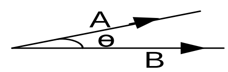
1. The resultant of two forces PN and 3N is 7N. If the 3N force is reversed, the resultant if $\sqrt{17}N$. Find the value of P and the angle between the two forces. **An(**$2\sqrt{6}N$, **57.02**° **)**
2. Four forces $(a\hat{\imath} - 1\hat{\jmath})$, $(3\hat{\imath} + 3a\hat{\jmath})N$, $(5\hat{\imath} - 6\hat{\jmath})N$ and $(-\hat{\imath} - 2\hat{\jmath})N$ act on a particle. The resultant of the forces make an angle of 45° with horizontal. Find the value of a and hence determine the magnitude of the resultant. **An(**$a = 8, \mathrm{R} = 15\sqrt{3}$ **)**
3. Three forces act on a body of mass 0.5kg as shown is the diagram. Find the position of the particle after 4 seconds.



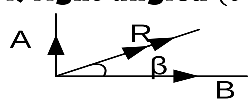
An[3.44N, 55.2m]

Resultant of two forces

Consider two forces A and B inclined to each other at an angle θ



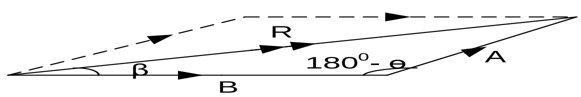
(i) $\boldsymbol{\theta}$ **is right angled** $(\boldsymbol{\theta} = \mathbf{90}°)$



Resultant R is obtained from, $R^2 = A^2 + B^2$

Direction of resultant, $\beta = tan^{-1}\left(\frac{B}{A}\right)$

(ii) $\boldsymbol{\theta}$ **is acute** $(0° \leq \boldsymbol{\theta} \leq \mathbf{90}°)$



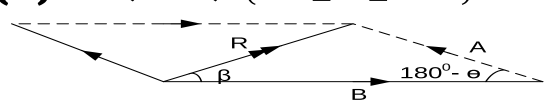
Resultant R is obtained from,

$R^2 = A^2 + B^2 - 2AB\, Cos(180 - \theta)$

Direction of resultant, $\frac{sin\,\beta}{A} = \frac{sin(180-\theta)}{R}$

$$\beta = sin^{-1}\left(\frac{A\, sin(180 - \theta)}{R}\right)$$

(iii) $\boldsymbol{\theta}$ **is obtuse** $(\mathbf{90}° \leq \boldsymbol{\theta} \leq \mathbf{180}°)$



Resultant R is obtained from,

$R^2 = A^2 + B^2 - 2AB\, Cos(180 - \theta)$

Direction of resultant, $\frac{sin\,\beta}{A} = \frac{sin(180-\theta)}{R}$

$$\beta = sin^{-1}\left(\frac{A\, sin(180 - \theta)}{R}\right)$$

Examples

1. Two forces of magnitude 5N and 12N act on a particle with their direction inclined at 90°. Find the magnitude and direction of the resultant.

Solution

$$R^2 = 5^2 + 12^2$$
$$R = 13N$$

$$\beta = tan^{-1}\left(\frac{5}{12}\right)$$

$\beta = 22.6°$ to $12N$ force

2. Forces of 7N and 9N act on a particle at an angle of 60° between them. Find the magnitude and direction of the resultant

Solution

3. Forces of 3N and 2N act on a particle at an angle of 150° between them. Find the magnitude of the resultant force and its direction

Solution

Exercise 11C

RESOLUTIONS OF FORCES ACTING ON A POLYGON

For any regular polygon;

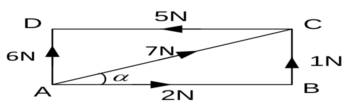
- All sides are equal
- All interior angles are equal
- All exterior angles are equal

$A\ exterior\ angle = \frac{360}{n}$ where n is number of sides

Examples

1. ABCD is a rectangle with $AB = 4cm$, and $BC = 3cm$. Forces of magnitude 2N, 1N, 5N, 6N, and 7N act along the line AB, BC, CD, AD, and AC respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine
   (i) The magnitude of the resultant force (ii) Direction of the resultant with AB

Solution



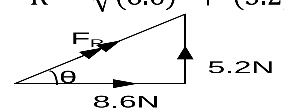
$\tan\alpha = \frac{3}{4}$

$\alpha = 36.87°$

$$R = \begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}-5\\0\end{pmatrix} + \begin{pmatrix}6\\0\end{pmatrix} + \begin{pmatrix}7cos36.87\\7sin36.87\end{pmatrix}$$

$$R= \begin{pmatrix}8.60\\5.2\end{pmatrix}$$

$$R = \sqrt{(8.6)^2 + (5.2)^2} = 10.0499N$$

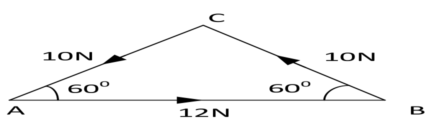


Direction $\theta = tan^{-1}\left(\frac{5.2}{8.6}\right)$ $\theta = 31.16°$

Direction $\mathrm{E}1.16°\mathrm{N}$

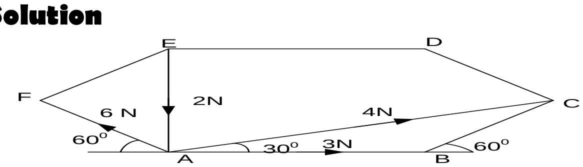
2. ABC is an equilateral triangle. Forces of magnitude 12N, 10N and 10N act along the line AB, BC and CA, respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine
   (i) The magnitude of the resultant force (ii) Direction of the resultant with AB

Solution



3. ABCDEF is a regular hexagon. Forces of magnitude 3N, 4N, 2N, and 6N act along the line AB, AC, EA, and AF respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine the magnitude of the resultant force and direction of the resultant with AB

Solution



Exercise 11E

1. ABCD is a square. Forces of magnitude 6N, 4N and $2\sqrt{2}$N act along the line AD, AB and AC respectively in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force. **Uneb 1998 nov/dec No.3 An(10N at 53.1° with AB)**
2. In a square ABCD, three forces of magnitude 4N, 10N and 7N act along the line AB, AD and CA respectively in each case the direction of the force being given by the order of the letters. Determine the magnitude of the resultant force. **Uneb 2017 No.4 An(5.1388N)**
3. In an equilateral triangle PQR, three forces of magnitude 5N, 10N and 8N act along the sides PQ, QR and PR respectively. Their directions are in the order of the letters. Find the magnitude of the resultant forces. **Uneb 2018 No.7 An**(16.0935N)

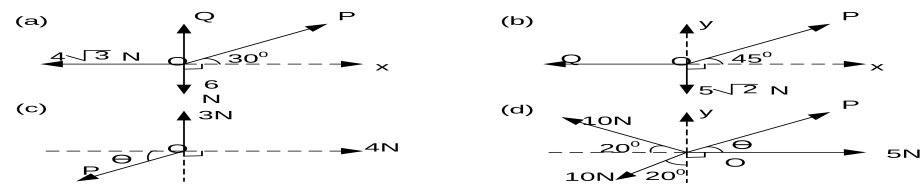
EQUILIBRIUM OF FORCES

if there are several forces acting on a particle in equilibrium, then their resultant is equal to zero.

ie $F_R = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Examples

1. For each of the diagrams below, the particle is in equilibrium under the forces shown below. Find the unknown forces and angles



Solution

2. Each of the following diagrams shows a particle in equilibrium under the forces shows

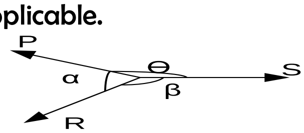
Prove Ptanθ=W

Prove than tanθ= 2√3

Exercise 11D

EQUILIRIUM OF THREE FORCES (LAMI'S THEOREM)

For any <u>three forces</u> acting on a particle in equilibrium where none of them is <u>parallel</u> to each other, lami's theorem is applicable.



$$\frac{P}{sin\beta} = \frac{s}{sin\alpha} = \frac{R}{sin\theta}$$

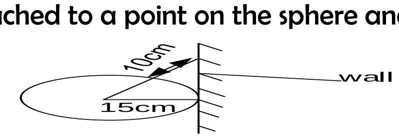
Examples

1. One end of a light in extensible string of length 75cm is fixed to a point on a rigid pole. The particle of weight 12N is attached to the other end of the string. The particle is held 21cm away from the pole by horizontal force. Find the magnitude of this force and the tension of the sting so that that the particle is equilibrium

   Solution Uneb 1996 No.1

4. A light inextensible string AB whose end A is fixed has end B attached to a particle of mass 5kg. A force P acting perpendicular to the string is applied on the particle keeping it in equilibrium with the string inclined at 60° to the vertical. Find the value of P and the tension in the string

5. A sphere of weight 20N and radius 15cm rests against a smooth vertical wall. A sphere is supported in its position by a string of length 10cm attached to a point on the sphere and to a point on the wall as shown.



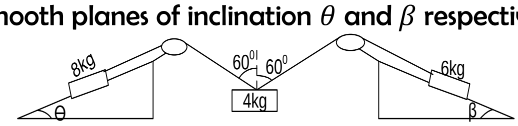
i) Calculate the reaction on the sphere due to the wall.
ii) Find the tension in the string

Solution

6. A light inextensible string passes over a smooth fixed pulley at the top of a smooth plane inclined at 30° to the horizontal. A particle of mass 2kg is attached to one end of the string and rests vertically in equilibrium when a particle of mass m resting on the surface of the plane is attached to the other end of the. Find;
   (i) The normal reaction between m and the plane.
   (ii) The tension in the string and the value of m
   Solution

Exercise 11E

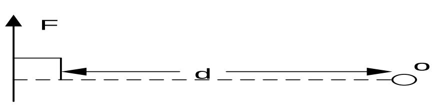
1. A particle of weight 8N is attached to a point B of a light inextensible string AB. It hangs in equilibrium with point A fixed and AB at an angle of 30° to the downward vertical. A force F at B acting at right angles to AB, keeps the particle in equilibrium. Find the magnitude of the force F and the tension in the string .**Uneb 1998 Mar No.4. An(4N, 4**$\sqrt{3}N$)
2. A particle of mass 5kg is held at equilibrium on smooth plane inclined at $tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ to the horizontal by a horizontal force P. Find the value of P and the reaction between the particle and the plane. **Uneb 2001 No.5**
   An ($= 28.3 = 56.6N,$)
3. (a) A particle of mass of 3kg is attached to the lower end B of a light inextensible string. The upper end A of the string is fixed to a point on the ceiling of a roof. A horizontal force of 22N and upward vertical force of 4.9N act upon the particle making it to be in equilibrium, with the string making an angle α with the vertical. Find the value of α and the tension in the string.
   (b) A non- uniform rod of mass 9kg rests horizontally in equilibrium supported by two light inextensible strings tied to the ends of the rod. The strings make angles of 50° and 60° with the rod. Calculate the tension in the strings **Uneb 2002, No.12. An((a)41.9°, 33N, (b) 60.33N, 46.93N)**
4. The diagram below shows masses of 8kg and 6kg lying on smooth planes of inclination θ and β respectively



Light inextensible strings attached to these masses pass along the lines of greatest slopes over smooth pulleys and are connected to a 4kg mass hanging freely. The strings both make an angle of 60° with the upward vertical as shown above. If the system rest in equilibrium find θ and β. **An(**$\theta = 30°, \beta = 41.8°$**)**

CHAPTER 2: MOMENT OF A FORCE

This is the product of a force and perpendicular distance from the pivot to the line of action of the force
The unit of moments is Nm



The moment of F about point O is$= Fxd$

MATRIX APPROACH OF FINDING SUM OF MOMENTS ABOUT THE ORIGIN

If forces $(a_1\hat{\imath} + b_1\hat{\jmath})N$, $(a_2\hat{\imath} + b_2\hat{\jmath})N$,… … … … … … … . $(a_n\hat{\imath} + b_n\hat{\jmath})N$ act on the body at points (x_1, y_1), (x_2, y_2), … … … … … … … (x_n, y_n). The sum of the moments about the origin is;

$$G = \begin{vmatrix} x_1 & a_1 \\ y_1 & b_1 \end{vmatrix} + \begin{vmatrix} x_2 & a_2 \\ y_2 & b_2 \end{vmatrix} + \cdots \dots\dots\dots\dots\dots\dots . . \begin{vmatrix} x_n & a_n \\ y_n & b_n \end{vmatrix}$$

$$G = (b_1x_1 - a_1y_1) + (b_2x_2 - a_2y_2) + \cdots \dots\dots\dots (b_nx_n - a_ny_n)$$

Note

If G is positive, the sum of moments will be anticlockwise and if G is negative then the sum of moments will be clockwise

Examples

1. Find the moment about the origin of a force of $4\tilde{j}N$ acting at a point which has position vector $5\tilde{\imath}m$

Solution

$$G = \begin{vmatrix} 5 & 0 \\ 0 & 4 \end{vmatrix} = 20Nm\ anticlockwise$$

2. Forces of $(2\hat{\imath} - 3\hat{\jmath})N$, $(5\hat{\imath} + \hat{\jmath})N$ and $(-4\hat{\imath} + 4\hat{\jmath})N$ act on a body at points with position vectors $(\hat{\imath} + \hat{\jmath})$, $(-2\hat{\imath} + 2\hat{\jmath})$ and $(3\hat{\imath} - 4\hat{\jmath})$ respectively. Find the sum of moments of the forces about the
   (i) origin
   (ii) point with position vector$(\hat{\imath} - \hat{\jmath})$

Solution

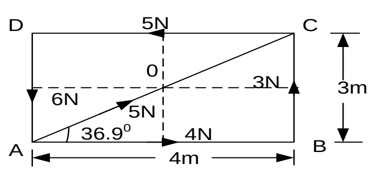
Exercise 12A

MOMENT OF FORCES ACTING ON A POLYGON

Examples

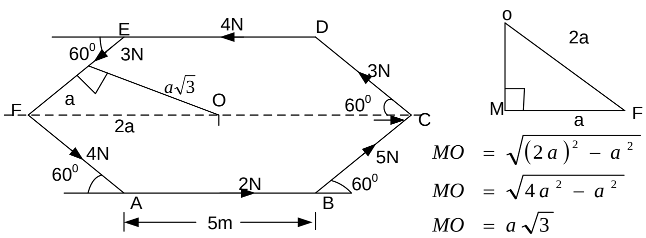
1. ABCD is a rectangle with $AB = 4m$, and $BC = 3m$. Forces of magnitude 4N, 3N, 5N, 6N, and 5N act along the line AB, BC, CD, DA, and AC respectively, in each case the direction of the force being given by the order of the letters. . Find the sum of moments of the forces about
   (i) Centre O of the square | (ii) Point A

Solution



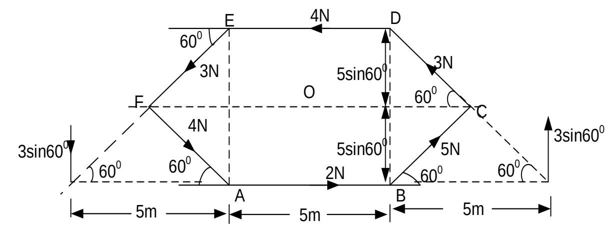
2. ABCDEF is a regular hexagon of side 5m. Forces of magnitude 2N, 5N, 3N, 4N, 3N and 4N act along the line AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, Find the sum of moments of the forces about
   (i) Centre O of the hexagon | (ii) Point A

Solution



O $G = (2+5+3+4+3+4)a\sqrt{3} = 21x2.5\sqrt{3}$

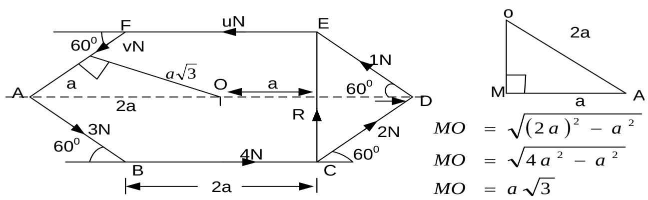
$G = 52.5\sqrt{3}\ Nm = 90.933Nm\ anticlockwise$



A $G = (5sin60°x5) + (3sin60°x10) +$
$(4x10sin60°) + (3sin60°x5)$
$= 55\sqrt{3}\ Nm = 95.26Nm\ anticlockwise$

3. ABCDEF is a regular hexagon of side 2a. Forces of magnitude 3N, 4N, 2N, 1N, uN and vN act along the line AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters. Find the resultant of the value of u and v if the resultant of the six forces acts along CE

Solution



Exercise 12B

1. Forces of magnitude 3N, 4N, 5N and 6N act on a rectangle along the line AB, BC, CD and DA respectively in each case the direction of the force being given by the order of the letters. Given that BC is horizontal, Find **Uneb 2005 No.7**
   (i) Magnitude and direction of the resultant force.
   (ii) couple at the centre of the rectangle of sides 2m by 4m. **An**($2\sqrt{2}N, 135°\ to\ BC, 26Nm$
2. Forces of 2N, 3N, 4N, and 5N act along the sides of a square ABCD of side 4m in the direction AB, BC, CD, and AD respectively. Find the sum of moments of the forces about
   (i) The center of square.
   (ii) Point A **An(8Nm, 28Nm)**

COUPLE

These are equal forces acting in opposite direction

Conditions for forces to form a couple

Force reduce to a couple if;

- Resultant force is zero
- If the sum of moments about a point is not zero

Examples

1. Forces of $(-5\tilde{i} - \tilde{j})\ N$, $-3\tilde{j}$ N and $(5\tilde{i} + 4\tilde{j})\ N$ act a body at a point with position vectors $(\tilde{i} - \tilde{j})m$, $(2\tilde{i} + \tilde{j})\ m$ and $(4\tilde{i} - 5\tilde{j})m$ respectively. Show that these forces reduce to a couple

Solution

$$R = \begin{pmatrix}-5\\-1\end{pmatrix} + \begin{pmatrix}0\\-2\end{pmatrix} + \begin{pmatrix}5\\4\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

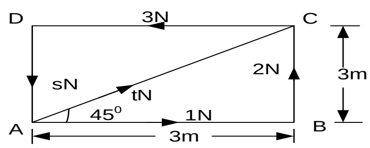
(a) $G = \begin{vmatrix}1 & -5\\-1 & -1\end{vmatrix} + \begin{vmatrix}2 & 0\\1 & -3\end{vmatrix} + \begin{vmatrix}4 & 5\\-5 & 4\end{vmatrix}$

$$G = (1x-1 - -5x-1) + (-3x2 - 1x0) + (4x4 - 5x-5) = 29\ Nm$$

Since the resultant force is zero and $G \neq 0$, then the forces reduce to a couple

2. ABCD is a square of side 3m. Forces of magnitude 1N, 2N, 3N, sN, and tN act along the line AB, BC, CD, DA, and AC respectively, in each case the direction of the force being given by the order of the letters. Taking AB as horizontal and BC as vertical, find the values of s and t so that the resultant of the forces is a couple.

Solution



$$R = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\2\end{pmatrix} + \begin{pmatrix}-3\\0\end{pmatrix} + \begin{pmatrix}0\\-s\end{pmatrix} + \begin{pmatrix}t\cos 45\\t\sin 45\end{pmatrix}$$

$$R = \begin{pmatrix}0\\0\end{pmatrix}$$

$(\rightarrow)$ $t\cos 45 = 2$

$$t = \frac{2}{\cos 45} = 2\sqrt{2}N$$

$(\uparrow)$ $2 - s + t\sin 45 = 0$

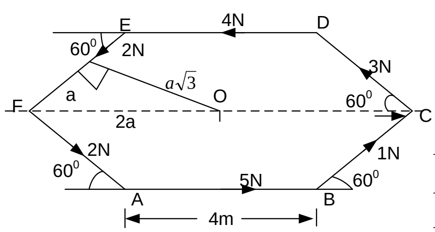
$$s = 2 + 2\sqrt{2}\sin 45 = 4N$$

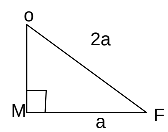
It must also be shown that $G \neq 0$

$G = 2x3 + 3x3 = 15\text{Nm}$

3. ABCDEF is a regular hexagon of side 4m. Forces of magnitude 5N, 1N, 3N, 4N, 2N and 2N act along the line AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, Show that these forces reduce to a couple

Solution





$MO = \sqrt{(2a)^2 - a^2}$

$MO = \sqrt{4a^2 - a^2}$

$MO = a\sqrt{3}$

Exercise 12C

LINE OF ACTION OF THE RESULTANT FORCE

The equation of the line of action is given by

$$G = \begin{vmatrix}x & F_x\\ y & F_y\end{vmatrix} = xF_y - yF_x$$

$$\boxed{G - xF_y + yF_x = 0}$$

Note

The line of action cuts the horizontal when $y = 0$ and cuts the vertical axis when $x = 0$

Examples

1. Five Forces of magnitude 3N, 4N, 4N, 3N and 5N act along AB, BC, CD, DA and AC respectively of a square of side 1m. The direction of the forces being in the order of the letters. Taking AB and AD as horizontal and vertical axis respectively. Find **Uneb 2016 No.9**
   (i) the magnitude and the direction of the resultant force
   (ii) Equation of the line of action
   (iii) Point where the line of action cuts AB.

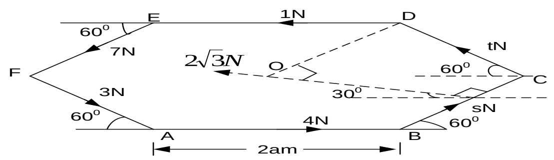
Solution

2. The centre of a regular hexagon ABCDEF of side 2a is O. Forces of magnitude 4N, sN, tN, 1N, 7N and 3N act along the sides AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters.; **Uneb 2010 No.13**
   (i) Given that the resultant of these six forces is of magnitude $2\sqrt{3}N$ acting in a direction perpendicular to BC, determine the value of s and t

(ii) Show that the sum of moments of the forces about O is $27a\sqrt{3}Nm$

(iii) If the mid point of BC is M, find the equation of the line of action of the resultant, refer to OM as x-axis and OD as y-axis

Solution



$$\begin{pmatrix}-2\sqrt{3}\cos 30\\ 2\sqrt{3}\sin 30\end{pmatrix} = \begin{pmatrix}4\\0\end{pmatrix} + \begin{pmatrix}s\cos60\\ s\sin60\end{pmatrix} + \begin{pmatrix}-t\cos60\\ t\sin60\end{pmatrix}$$

$$+\begin{pmatrix}-1\\0\end{pmatrix} + \begin{pmatrix}-7\cos60\\ -7\sin60\end{pmatrix} + \begin{pmatrix}3\cos60\\ -3\sin60\end{pmatrix}$$

$$\begin{pmatrix}-2\sqrt{3}x\dfrac{\sqrt{3}}{2}\\ 2\sqrt{3}x\dfrac{1}{2}\end{pmatrix} = \begin{pmatrix}4+\dfrac{s}{2}-\dfrac{t}{2}-1-3.5+1.5\\ \dfrac{s\sqrt{3}}{2}+\dfrac{t\sqrt{3}}{2}-\dfrac{10\sqrt{3}}{2}\end{pmatrix}$$

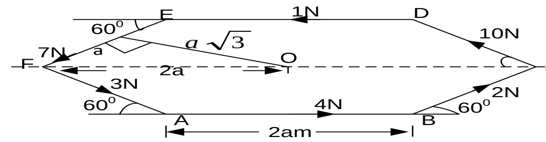
$$-2\sqrt{3}x\frac{\sqrt{3}}{2} = \frac{s}{2}-\frac{t}{2}+1$$

$$s = t-8 \dots\dots (i)$$

$$2\sqrt{3}x\frac{1}{2} = \frac{s\sqrt{3}}{2}+\frac{t\sqrt{3}}{2}-\frac{10\sqrt{3}}{2}$$

$$2 = s+t-10 \dots\dots (ii)$$

$$t = 10, s = 2$$



$$G = (4+2+10+1+7+3)a\sqrt{3}$$

$$G = 27a\sqrt{3}\ Nm$$

Since the resultant acts along the direction of OM, then

$X = 2\sqrt{3}, Y = 0$

$$G = \begin{vmatrix}x & X\\ y & Y\end{vmatrix} = \begin{vmatrix}x & 2\sqrt{3}\\ y & 0\end{vmatrix}$$

$$27a\sqrt{3} = -2\sqrt{3}\,y$$

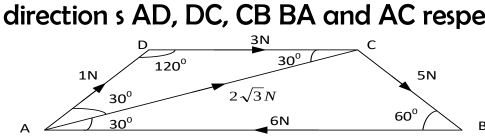
Line cuts AB when $y = -13.5a$

Exercise 12D

1. PQRSTUP is a regular hexagon 2m. Forces of magnitude 9N, 5N, 7N, 3N, 1N and 4N act along the line PQ, QR, RS, ST, TU and UP respectively, in each case the direction of the force being given by the order of the letters. Given that PQ is horizontal, Find the; **Uneb 2000 No.16**
   (i) Magnitude and direction of the resultant force
   (ii) Point where the line of action of the resultant cuts PQ

   An ($8.9N$ at $43°$ to PQ, $7.43m$)

2. The diagram below shows a trapezium ABCD. $AD = DC = CB = 1m$, and $AB = 2m$. Forces of magnitude 1N, 3N, 5N, 6N and $2\sqrt{3}N$ act in the direction s AD, DC, CB BA and AC respectively



   (a) Calculate the magnitude of the resultant force and the angle it makes with side AB
   (b) Given that the line of action of the resultant force meets AB at X, find the distance X. **Uneb 2019 No.15. An** (3.464N at 30°, 6.5m)

PARALLEL FORCES IN EQUILIBRIUM

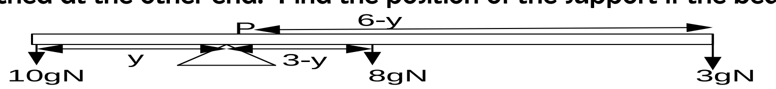
Condition for a body to be in equilibrium

When a system of parallel forces act on a body then it will be in equilibrium when;

1. Sum of the forces acting in one direction are equal to the sum of forces acting in opposite direction.
2. Sum of the clock wise moments about a point are equal to the sum of the anti clock wise moment about the same point.

Examples

1. A uniform beam AB of length 6m and mass 8kg has a mass of 10kg attached at one end and a mass of 3kg attached at the other end. Find the position of the support if the beam rests horizontally



Solution:

2. A uniform beam of mass 50kg and length 4m rests horizontally on two supports pivoted at each end. A load of mass 20kg is placed 1m from one end. Find the reaction on each support.

   Solution

3. A non- uniform beam AB of length 4m rests in horizontal position on vertical supports at A and B. The centre of gravity is at 1.5m from end A. the reaction at B is 37.5N. Find the ;
   (i) Mass of the beam
   (ii) reaction at A.

   Solution

Exercise: 12E

CHAPTER 3: MOTION IN A STRAIGHT LINE

Distance and displacement

Distance is the length between 2 fixed point **Displacement** is the distance covered in a specific direction

Speed and velocity

Speed is the rate of change of distance with time

Velocity is the rate of change of displacement with time

Average speed and average velocity

$$verage\ speed = \frac{total\ distance}{total\ time}$$

$$average\ velocity = \frac{total\ displacement}{total\ time}$$

Examples

1. A, B and C are three points in that order on a straight line with $AB = 5km$ and $BC = 4km$. A man runs from A to B at $20kmh^{-1}$ and then walks from B to C at $8kmh^{-1}$. Find
   (i) Total time taken to travel from A to C
   (ii) The average speed of the man for the journey from A to C

   Solution

   (i) $constant\ speed = \frac{distance}{time}$

   $t_{AB} = \frac{5}{20} = 0.25h$ and $t_{BC} = \frac{4}{8} = 0.5h$

   Total time$= 0.25 + 0.5 = 0.75h$

   (ii) $average\ speed = \frac{Total\ distance}{Total\ time}$

   $$\text{average speed} = \frac{5+4}{(0.75)} = 12\ \text{kmh}^{-1}$$

2. A, B and C are three points in that order on a straight line with $AB = 60m$ and $AC = 80m$. A man walks from A to B at an average speed of $10ms^{-1}$ and then walks from B to C in a time of $4s$ and then returns to B. The average speed for the whole journey is $5ms^{-1}$. Find
   (i) Average speed of the man in the second stage of the motion $(i.e\ B \to C)$. **An** $5ms^{-1}$
   (ii) The average speed of the man in moving from A to C **An** $8ms^{-1}$
   (iii) The time taken for the third stage 0f the motion $(i.e\ C \to B)$. **An** $10s$
   (iv) Average velocity for the complete motion **An** $3ms^{-1}$

ACCELERATION

This is the rate of change of velocity

Acceleration $= \frac{change\ in\ velocity}{time}$

$$a = \frac{v-u}{t}$$

UNIFORM ACCELERATION

This is when the rate of change of velocity is constant

Equations of uniform acceleration

1st equation

Suppose a body moving in a straight line with uniform acceleration a, increases its velocity from u to v in a time t, then from definition of acceleration

$$a = \frac{v-u}{t}$$

$$at = v - u$$

$$\boxed{v = u + a\,t} \text{.................1}$$

2nd equation

Suppose an object with velocity u moves with uniform acceleration for a time t and attains a velocity v, the distance s travelled by the object is given by S = average velocity x time

$S = \left(\frac{v+u}{2}\right)t$ But $v = u + at$

$$S = \frac{(u + at + u)}{2}t$$

$$S = \frac{2ut + at^2}{2}$$

$$\boxed{S = ut + \frac{1}{2}at^2} \text{.......................2}$$

3rd equation

S = average velocity x time

$S = \left(\frac{v+u}{2}\right)t$ But $t = \frac{v-u}{a}$

$$S = \left(\frac{v+u}{2}\right)\left(\frac{v-u}{a}\right)$$

$$S = \frac{v^2-u^2}{2a}$$

$$\boxed{v^2 = u^2 + 2as} \text{...................3}$$

Examples

1. A car is being driven along a road at a steady speed of 25 ms^{-1}, when suddenly the driver notices a fallen tree on the road 65m ahead. The driver immediately applies brakes giving the car a constant retardation of 5 ms^{-2}
   (a) How in front of the tree does the car come to rest
   (b) If the driver had not reacted immediately and the brakes were applied one second later, with what speed would the car have hit the tree.
   Solution

2. A particle travels in a straight line with a uniform acceleration. The particle passes through three points A, B and C lying in that order on the line, at time $t = 0$, $t = 2s$, and $t = 5s$ respectively. If BC= $30m$ and the speed of the particle when at B is $7ms^{-1}$, find the acceleration of the particle and its speed when at A
   Solution

3. An over loaded taxi travelling at a constant velocity of 90kmh^{-1}overtake a stationery traffic police car. $2s$ later, the police car sets off in pursuit , accelerating at a uniform rate of 6ms^{-2}. How far does the traffic car travel before catching up with the taxi. **Uneb 1999 no.4**
   Solution

$t_1 = time\ taken\ by\ the\ taxi$

$t_2 = time\ taken\ by\ the\ car$

$t_1 - t_2 = 2$

$t_1 = 2 + t_2 \ldots\ldots\ldots\ldots\ldots (i)$

$$s = \text{u}t + \frac{1}{2}at^2$$

Since it moves with a constant velocity a =0,

$u = \frac{90x1000}{3600} = 25m/s$

$S_T = 25t_1$---------------(1)

For car: $S_c = 0xt_2 + \frac{1}{2}x6{t_2}^2$

$S_c = 3{t_2}^2$

For car to catch taxi then : $S_T = S_C$

$25t_1 = 3{t_2}^2$

$25(2 + t_2) = 3{t_2}^2$

$50 + 25t_2 = 3{t_2}^2$

$3{t_2}^2 - 25t_2 - 50 = 0$

t= $\frac{25\pm\sqrt{25^2-4x3x(-50)}}{2x3}$

$t = 10s$ or $t = \frac{4}{3}s$

Since the car leaves 2s later then time 10s is correct since it gives a positive value

$S_c = 3{t_2}^2 = 3x10^2 = 300m$

4. A lorry starts from a point A and moves along a straight horizontal road with a constant acceleration of $2ms^{-2}$. At the same time a car moving with a speed of 20ms^{-1} and a constant acceleration of $3ms^{-2}$ is 400m behind the point A and moving in the same direction as the lorry. Find;
   (i) How far from A the car over takes the lorry
   (ii) The speed of the lorry when it is being over taken
   Solution

Exercise 13A

1. The speed of a boda- boda rider decreases from 90kh^{-1} to 18 kmh^{-1} in a distane of 120m. find the speed of the rider when it had covered a distance of 50m. **Uneb 2013 no3**
2. A train starts from station A with a uniform acceleration of $0.2ms^{-2}$ for 2 minutes and attains a maximum speed and moves uniformly for 15 minutes. It is then brought to rest at a constant retardation of $\frac{5}{3}ms^{-2}$ at station B. find the distance between A and B **Uneb 2003 no8 An(232112.8m)**
3. A motorcycle decelerated uniformly from 20kmh^{-1} to 8 kmh^{-1}in travelling 896m. find the rate of deceleration. **Uneb 1998 march no3 An(0.0145m/s²)**
4. Two stations A and B are a distance 6x meters apart along a straight line. A car starts from rest at A and accelerates uniformly to a speed of V m/s covering a distance x meters, then maintains this speed until it has travelled a further 3x meters, it then retards uniformly to rest at B. Prove that the time, T taken to travel from A to B is $T = \frac{9x}{V}$
5. Two stations P and Q are a distance x meters apart along a straight line. A train starts from rest at P and accelerates uniformly at $a\ ms^{-2}$ until it acquires a speed of V m/s, it then maintains this speed for T seconds, and then brought to rest at Q under a uniform retardation. Prove that
$$T = \frac{x}{V} - \frac{V}{a}$$
6. A particle moving in a straight line with a uniform acceleration passes $a\ ms^{-2}$ certain point with a velocity $u\ ms^{-1}$. 3 s later another particle, moving in the same straight line with a constant acceleration $^4/_3\, a\ ms^{-2}$, passes the same point with a velocity of $^1/_3\, u\ ms^{-1}$. The first particle is overtaken by the second particle when their velocities are $8.1ms^{-1}$ and $9.3ms^{-1}$ respectively. Find the;
   (i) value of u and a
   (ii) Distance travelled from the point **An($\mathbf{u = 0.9ms^{-1}, a = 0.15ms^{-2}, 216m}$)**

VERTICAL MOTION UNDER GRAVITY

When a body is projected **vertically downwards** , its subjected to an acceleration of $9.8ms^{-2}$ ie
$$a = g = 9.8ms^{-2}$$
Equations of motion become

$$v = u + g\,t \qquad h = ut + \frac{1}{2}gt^2 \qquad v^2 = u^2 + 2gh$$

When a body is projected **vertically upwards** , its subjected to a retardation of $9.8ms^{-2}$ ie
$a = -g = -9.8ms^{-2}$
Equations of motion become

$$v = u - g\,t \qquad h = ut - \frac{1}{2}gt^2 \qquad v^2 = u^2 - 2gh$$

Maximum/ greatest height

When a particle is projected vertically upwards, the final velocity is 0m/s at its maximum height
$$v^2 = u^2 - 2gh$$
$$0 = u^2 - 2gh_{max}$$
$$h_{max} = \frac{u^2}{2}$$

Time to reach maximum height
$$v = u - gt$$
$$0 = u - gt$$
$$t = \frac{u}{g}$$

Time of flight

$$T = \frac{2u}{g}$$

Examples

1. A stone is dropped from a point which is 40m above the ground. Find the time taken for the stone to reach the ground
Solution

2. A particle is projected vertically upwards with velocity of um/s. After $t\ seconds$ another particle is projected vertically upwards from the same point of projection and with the same initial velocity. Prove that the particles collides after $\left(\frac{t}{2} + \frac{u}{g}\right) s$. Hence show that they will meet at a height of $\frac{4u^2-(gt)^2}{8g}$
Solution

$$t_1 = time\ taken\ by\ the\ 1^{st}$$
$$t_2 = time\ taken\ by\ the\ 2^{st}$$
$$t_1 - t_2 = t \ldots\ldots\ldots\ldots\ldots (i)$$

$t_1\ an\ t_2$ are roots of the equation

$$h = \text{u}t - \frac{1}{2}gt^2$$
$$gt^2 - 2ut + 2h = 0$$
$$t_1 = \frac{2u + \sqrt{4u^2 - 8gh}}{2g}$$
$$t_2 = \frac{2u - \sqrt{4u^2 - 8gh}}{2g}$$
$$\frac{2u + \sqrt{4u^2 - 8gh}}{2g} - \frac{2u - \sqrt{4u^2 - 8gh}}{2g} = t$$
$$\sqrt{4u^2 - 8gh} = gt \ldots\ldots\ldots\ldots. (ii)$$
$$\boldsymbol{h} = \frac{4u^2 - (gt)^2}{8g}$$

$t_1 = \frac{2u+\sqrt{4u^2-8gh}}{2g}$ putting (ii)

$$t_1 = \frac{2u + gt}{2g}$$
$$t_1 = \frac{t}{2} + \frac{u}{g}$$

3. A particle is projected vertically upwards with velocity of 10m/s. After 2s another particle is projected vertically upwards from the same point of projection and with the same initial velocity. Find the height above the level of projection where the particles meet.
Solution

Exercise 13B

1. A stone is thrown vertically upwards with a velocity of $16ms^{-1}$ from a point H meters above the ground level. The stone hits the ground 4 seconds later. Calculate the: **UNEB 2019** No5
   (a) Value of H
   (b) Velocity of the stone as it hits the ground. **An(14.4m, 23.2m/s)**
2. A stone is thrown vertically upwards with a velocity of $21ms^{-1}$. Calculate the: **UNEB 2018** No.1
   (a) maximum height attained by the stone
   (b) time the stone takes to reach the maximum height. **An (a)=22.5m, (b)=2.143s**

CHAPTER 4: FORCE AND NEWTON'S LAWS OF MOTION

LAW I : Everybody continues in its state of rest or uniform motion in **a straight line** unless acted upon by an external force.

LAW II: The rate of change of momentum of a body is directly proportional to the applied force and takes place in the direction of the force.

$F = m\frac{(v-u)}{t}$ But $a = \frac{v-u}{t}$

$= ma$

Note: F must be the resultant force

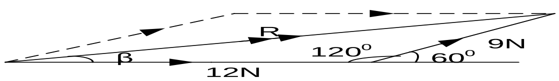
Examples

1. A body of mass 500g experiences a resultant force of 3N. find
   (a) Acceleration produced
   (b) Distance travelled by the body while increasing it speed from $1ms^{-1}$ to $7ms^{-1}$

   Solution

2. Two forces of magnitude 12N and 9N act on a particle producing an acceleration of 3.65ms^{-2}. The forces act an angle of 60^o to each other. Find the mass of the particle. **Uneb 2004 No.6**

   Solution



WHEN RESISTANCE/ FRICTION IS INVOLVED

1. Find the constant force necessary to accelerate a car of mass 600kg from rest to 25m/s in 12s, if the resistance to motion is
   (a) Zero (b) 350N **An(1250N, 1600N).**
2. A train of mass 60 tones is travelling at 40m/s when the brakes are applied. If the resultant braking force is 40kN, find the distance the train travels before coming to rest. **An(1200m).**
3. A train of mass 100 tones starts form rest at station A and accelerates uniformly at 1ms^{-2} until it attains a speed of 30m/s. it maintains this speed for a further 90s and then the brakes are applied, producing a resultant braking force of 52kN. If the train comes to rest at station B, find the distance between the two stations. **An(4050m).**

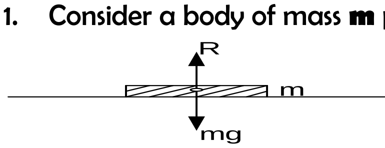
CALCULATIONS INVOLVING VECTOR FORM

1. Forces of $(a\tilde{i} + b\tilde{j} + c\tilde{k})N$ and $(2\tilde{i} - 3\tilde{j} + \tilde{k})N$ acting on a body of mass 2kg causing it to accelerate at$(4\tilde{i} + \tilde{k})ms^{-2}$. Find the constants a, b and c. **An**$(a = 6, b = 3, c = 1)$
2. A particle of mass 2.5kg is acted upon by a resultant force of magnitude 15N acting in the direction $(2\tilde{i} - \tilde{j} - 2\tilde{k})$. Find the magnitude of the acceleration

   Solution

LAW III: To every action there is an equal but opposite reactions.

1. Consider a body of mass **m** placed on a smooth horizontal surface

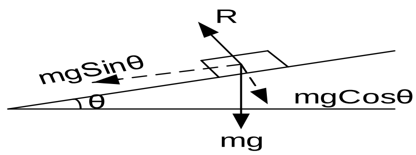


$\boxed{R = mg}$

R = normal reaction

Mg = gravitational pull [weight]

2. Mass m placed on a smooth inclined plane of angle of inclination θ



$$R = mgCos\theta$$

NB:

- All objects placed on, or moving on an inclined plane experience a force $mgsin\theta$ **down** the plane. [It doesn't matter what direction the body is moving]
- If the plane is **rough** the body experiences a frictional force whose direction is opposite to the direction of motion.

MOTION ON A HORIZONTAL PLANE

Examples:

1. A car of mass 1000kg is accelerating at 2ms^{-2}. If the resistance to the motion is 1000N.
   (i) Find the normal reaction of the car on the road surface
   (ii) What accelerating force acts on the car?

   Solution

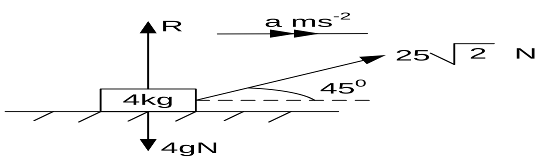
2. A car of mass 900kg tows a trailer of mass 600kg along a level road by means of a rigid bar. The car exerts experiences a resistance of 200N and the trailer a resistance of 300N, if the car engine exerts a forward force of 3kN, find the acceleration produced and the tension in the tow bar

   Solution

FORCE INCLINED AT ANGLE TO THE HORIZONTAL

1. A body of mass 4kg is acted upon by force of 25$\sqrt{2}N$ which is inclined at 45^o to a smooth horizontal surface. Find the acceleration of the body and the normal reaction between the body and the surface

   Solution



2. A body of mass 10kg is initially at rest on a rough horizontal surface. It is pulled along the surface by a constant force of 60N inclined at 60^o above the horizontal. If the resistance to motion totals 10N, find the acceleration of the body and the distance travelled in the first 3s

   Solution

Exercise 14A

MOTION ON AN INCLINED PLANE

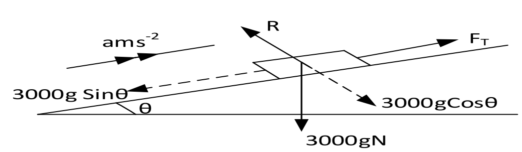
1. A lorry of mass 3 tones travelling at 90km/h starts to climb an incline of 1 in 5. Assuming the tractive pull between its tyres and the road remains constant and that its velocity reduces to 54km/h in a distance of 500m. Find the tractive pull

   Solution

$$u = 90km/h = \frac{90x1000}{36000} = 25ms^{-1}$$

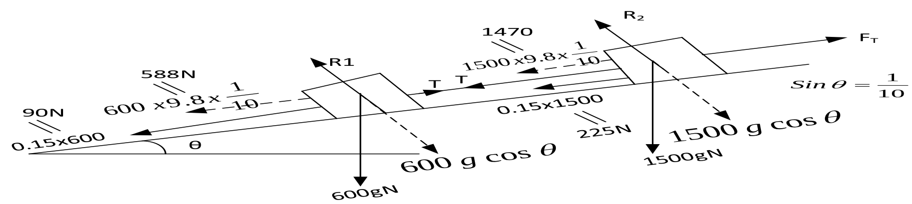
$$v = \frac{54km}{h} = \frac{54x1000}{36000} = 15ms^{-1}$$

$$a = \frac{v^2 - u^2}{2s} = \frac{25^2 - 15^2}{2x500} - 0.4ms^{-2}$$



2. A car of mass 1500kg is pulling a trailer of mass 600kg up a slope 1 in 10, the resistance to motion for the car and trailer is 0.15Nkg^{-1}. If the retardation is 0.5ms^{-2}, find the
    i. Tension in coupling between the car and the trailer
    ii. Tractive force exerted by the engine

Solution



Exercise 14B

1. A particle of mass 5kg resting on a smooth plane inclined at $tan^{-1}(\frac{1}{\sqrt{3}})$ to the horizontal. Find the magnitude of the horizontal force required to keep the particle in equilibrium and the normal reaction to the plane. **Uneb 2001 No.5 An (28.29N, 56.58N)**
2. The engine of a train exerts a force of 3,500N on a train of mass 240 tonnes and draws up a slope of 1 in 120 against resistances totaling to 160N per tonne. Find the acceleration of the train. **Uneb 2008, No6 An(0.00417ms^{-2})**
3. A car of mass 2.5 metric tonnes is drawn up a slope of 1 in 10 from rest with an acceleration of $1.2ms^{-2}$ against a constant frictional resistance of $\frac{1}{100}$ of the weight of the vehicle using a cable. Find the tension in the cable. **Uneb 2006, No6 An**(5695N)

CHAPTER 5: FRICTION

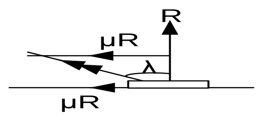
This is a force that opposes relative motion or attempted motion between two bodies in contact

Frictional force is given by $\boxed{F = \mu R}$

At limiting equilibrium, the body is on the point of moving (slip or slide) and frictional force is maximum

Angle of friction

This is the angle between the resultant force and the normal reaction force

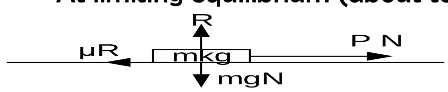


$$\tan\lambda = \frac{\mu R}{R}$$

$$\boxed{\tan\lambda = \mu}$$

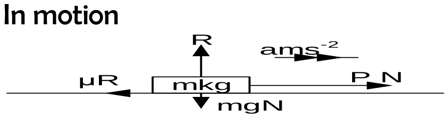
A horizontal plane

(i) At limiting equilibrium (about to slip or slid)



$$\mathrm{P} = \mu R$$

(ii) In motion



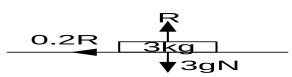
$$F = ma$$

$$\mathrm{P} - \mu R = \mathrm{ma}$$

Examples

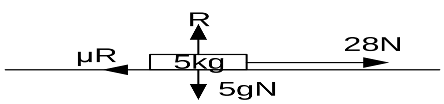
1. Calculate the maximum frictional force which can act when a block of mas 3kg rests on a rough horizontal surface, the coefficient of friction between the surface being 0.2

Solution



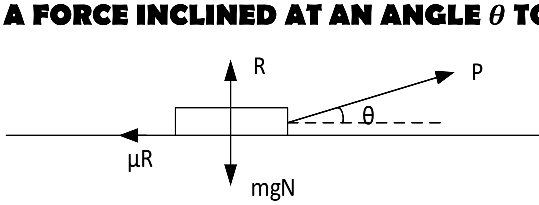
2. When a horizontal force of 28N is applied to a body of mass 5kg which is resting on a rough horizontal plane, the body is found to be in limiting equilibrium. Find the coefficient of friction between the body and the plane

Solution



Exercise 15A

A FORCE INCLINED AT AN ANGLE θ TO THE HORIZONTAL



At limiting equilibrium

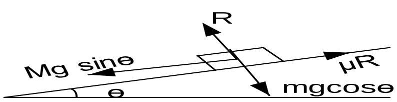
$(\rightarrow)$: $P\cos\theta = \mu R$

$(\uparrow)$: $\mathrm{R} + \mathrm{P}\sin\theta = \mathrm{mg}$

Examples

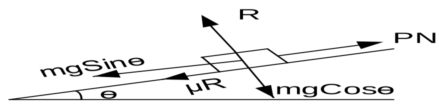
AN INCLINED PLANE

(i) At limiting equilibrium (about to slip down or slid down)



$$\text{mgsin}\theta = \mu R$$
$$\text{mgsin}\theta = \mu mgcos\theta$$
$$\mu = tan\theta$$

(ii) **A force P applied parallel to and up the plane to just move the particle upwards.**

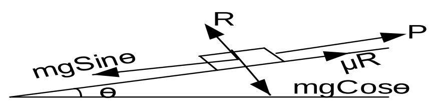


Normal to the plane: $\text{mgcos}\theta = R$

Parallel to the plane: $\text{mgsin}\theta + \mu R = P$

$$P = \text{mgsin}\theta + \mu\text{mgcos}\theta$$

(iii) **A force P applied parallel to and up the plane so that the particle is on the point of moving downwards (prevent it from moving downwards)**

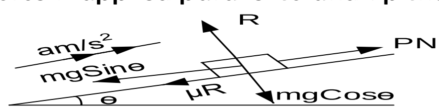


Normal to the plane: $\text{mgcos}\theta = R$

Parallel to the plane: $\text{mgsin}\theta = P + \mu R$

$$P = \text{mgsin}\theta - \mu\text{mgcos}\theta$$

(iv) **A force P applied parallel to and up the plane to move the particle upwards**



Normal to the plane: $\text{mgcos}\theta = R$

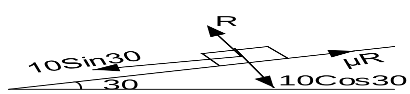
Parallel to the plane: $P - (\text{mgsin}\theta + \mu R) = ma$

$$P - \text{mgsin}\theta - \mu\text{mgcos}\theta = \text{ma}$$

Examples

1. A particle of weight 10N rests on a rough plane inclined at 30° to the horizontal and is just about to slip. Find the value of coefficient of friction between the plane and particle

Solution

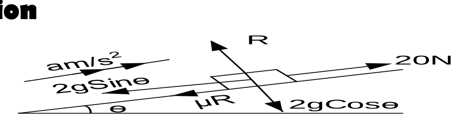


At limiting equilibrium

$$R = 10cos30$$
$$\mu R = 10\ sin30$$
$$\mu(10cos30) = 10\ sin30$$
$$\mu = 0.5774$$

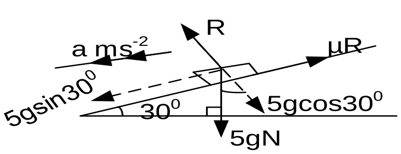
2. A body of mass 2kg lies on a rough plane which is incline at $sin^{-1}\left(\frac{5}{13}\right)$ to the horizontal. A force of 20N is applied to the body, parallel to and up the plane. If the body accelerates up the plane at $1.5ms^{-1}$, find the coefficient of friction between the body and the plane

Solution



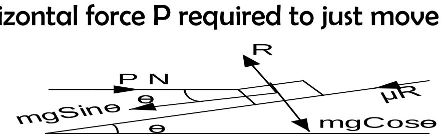
3. A body of mass 5kg is released from rest on a rough surface of a plane inclined at $30°$ to the horizontal. If the body takes $2\frac{1}{2}s$ to acquire a speed of 4ms^{-1} from rest, find the frictional force and the coefficient of friction

Solution



HORIZONTAL FORCE ON INCLINED PLANES

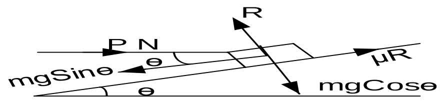
(i) A horizontal force P required to just move the particle upwards



Normal to the plane: $\text{mgcos}\theta + \text{Psin}\theta = R$

Parallel to the plane: $\text{mgsin}\theta + \mu R = Pcos\theta$

(ii) A horizontal force P required to prevent the particle from moving downwards



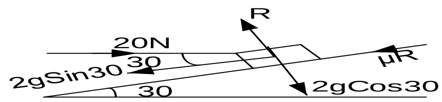
Normal to the plane: $mg\cos\theta + P\sin\theta = R$

Parallel to the plane: $mg\sin\theta - \mu R = P\cos\theta$

Examples

1. A body of mass 2kg lies on a rough plane which is inclined at 30° to the horizontal. When a horizontal force of 20N is applied to the body in an attempt to push it up the plane, the body is found to be on the point of moving up the plane. Find the coefficient of friction between the body and the plane

Solution



At limiting equilibrium

$$R = 2gcos30 + 20Sin30$$

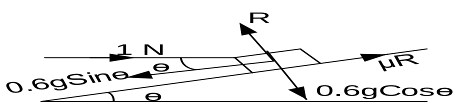
$$20\cos 30 = \mu R + 2g\sin 30$$

$$20\cos 30 = \mu(2gcos30 + 20Sin30) + 2g\sin 30$$

$$\mu = 0.279$$

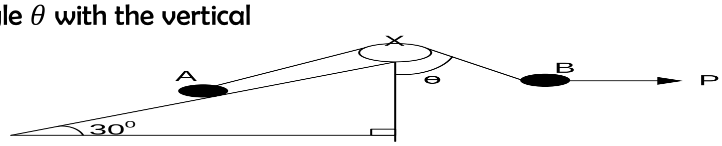
2. A horizontal force of 1N is just sufficient to prevent a brick of mass 600g sliding down a rough plane which is inclined at $Sin^{-1}\left(\frac{5}{13}\right)$ to the horizontal. Find the coefficient of friction between the brick and the plane

Solution



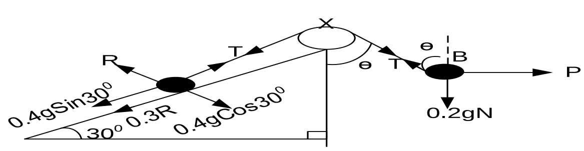
$Sin\theta = \frac{5}{13},\ cos\theta = \frac{12}{13}$

1. Particle A of mass 0.4kg and B of mass 0.2kg are attached to the end of a light inextensible sting. Particle A rests in equilibrium on a rough plane which is inclined at 30° to the horizontal. The string passes over a smooth pulley X at the top if the plane, and the part AX of the string is parallel to a line of greatest slope of the plane. Particle B is held in equilibrium by means of a horizontal force, P in such a way that, part XB of the sting makes angle θ with the vertical



The points A, X, B lie in the same vertical plane. The coefficient of friction between A and the sloping plane is 0.3. Given that A is about to slip up the plane, find the value of θ and the magnitude of the horizontal force P

Solution



At limiting equilibrium

Exercise: 15B

1. A box of mass 2kg rests on a rough inclined plane of angle 25°. The coefficient of friction between the box and the plane is 0.4.. Find the least force applied parallel to the plane which would move the box up the plane. **Uneb 1997 No.6**
An[15.39N]
2. A particle of mass 0.5kg is released from rest and slides down a rough plane inclined at 30° to the horizontal. It takes 6 seconds to go 3 meter. **Uneb 1997 No.14**
   i. Find the coefficient of friction between the particle and the plane
   ii. What minimum horizontal force is needed to prevent the particle from moving?
   An[0.557, 0.064N]
3. A particle of mass 2m rests on a rough plane inclined to the horizontal at an angle of rests on a

rough plane inclined to the horizontal at an angle if $tan^{-1}(3\mu)$ where μ is the coefficient of friction between the particle and the plane. The particle is acted upon by a force of P Newton's.

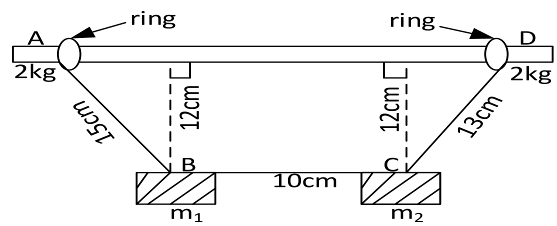
(a) Given that the force acts along the line of greatest slope and that the particle is on the point of slipping up, show that the maximum force possible to maintain the particle in equilibrium is $P_{max} = \frac{8\mu mg}{\sqrt{1+9\mu^2}}$

(b) Given that the force acts horizontally in a vertical plane through a line of greatest slope and that the particle is on the point of sliding down the plane, show thqt the minimum force required to maintain the particle in equilibrium is

$$P_{min} = \frac{4\mu mg}{1+3\mu^2}$$

4. A particle of weight 20N is placed on a rough plane inclined at an angle of 40° to the horizontal. The coefficient of friction between the plane and the particle is 0.25. When a horizontal force P is applied on the particle, it rests in equilibrium. Calculate the value of P
An(9.739N) **(Uneb 2020 Qn 7)**

5. The diagram below shows three strings $AB = 15cm$, $BC = 10cm$ and $CD = 13cm$, A and D are fixed to small rings each of mass 2 kg which can slide on a rough horizontal rail AD. Masses m_1 and m_2 are attached at B and C respectively. The system rests in equilibrium with BC at a distance of 12 cm below AD



(a) Show that $9m_1 = 5m_2$

(b) If the coefficient of friction between each ring and the rail is 0.25 and the ring A is on the point of slipping, determine the value of m_1
An($m_1 = 1kg$) **(Uneb 2020 Qn13)**

CHAPTER 6: CONNECTED PARTICLES

SIMPLE CONNECTIONS

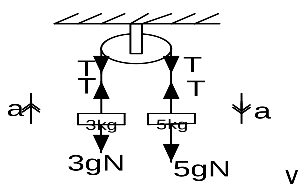
When two particles are connected by a light inextensible string passing over a smooth pulley and allowed to move freely, then as long as the string is taut, the following must be observed.

- Acceleration of particles is the same
- The tension in the uninterrupted string is constant
- The tensions in interrupted strings are different.

Examples

1. Two particles of masses 5kg and 3kg are connected by a light inelastic string passing over a smooth fixed pulley. Find;
   (i) Acceleration of the particles
   (ii) The tension in the string
   (iii) The force on the pulley

Solution



$$F = ma$$

For 5kg mass: $5g - T = 5a$............(i)

For 3kg mass: $T - 3g = 3a$ ....(ii)

Adding (i) and (ii)

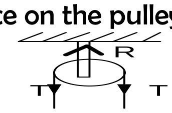
$2g = 8a$

$a = \frac{2x9.8}{8} = 2.45ms^{-2}$

ii) $T - 3g = 3a$

$T = 3x2.45 + 3x9.8 = 36.78N$

iii) Force on the pulley

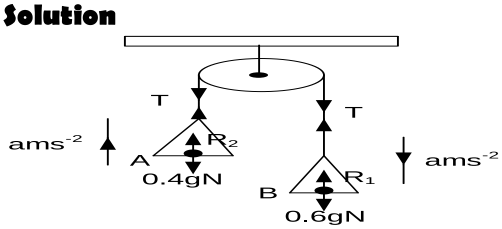


$R = 2T = 2x36.78 = 73.56N$

Force on the pulley is $73.56N$

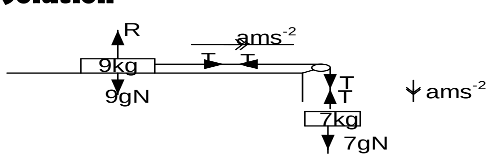
1. An inextensible string attached to two scale pans A and B each of weight 20g passes over a smooth fixed pulley. Particles of weight 3.8N and 5.8N are placed in pans A and B respectively, if the system is released from rest (take $g = 10ms^{-2}$). Find the;
   (i) Tension in the string
   (ii) Reaction of the scale pan holding the 3.8N weight

Solution

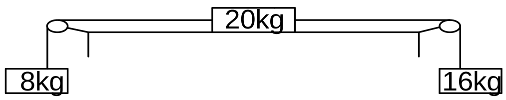


2. A mass of 9kg resting on a smooth horizontal table is connected by a light string passing over a smooth pulley at the edge of the table, to the pulley is a 7kg mass hanging freely 1.5m above the ground; find
   (i) Common acceleration
   (ii) The tension in the string
   (iii) The force on the pulley in the system if its allowed to move freely.
   (iv) Time taken for the 7kg mass to hit the ground

Solution



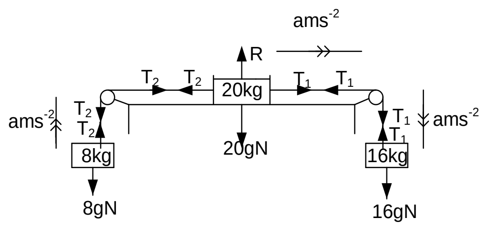
3.



The figure shows a block of mass 20 kg resting on a smooth horizontal table. Its connected by inelastic strings which pass over fixed pulleys at the edges of the table to two loads of masses 8kg and 16kg which hang vertically. When the system is released freely, Calculate;

(i) Acceleration of 16kg mass
(ii) Tension in each string
(iii) Reaction on each pulley

Solution



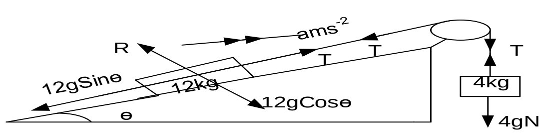
Exercise 16A

CONNECTED PARTICLES ON INCLINED PLANES

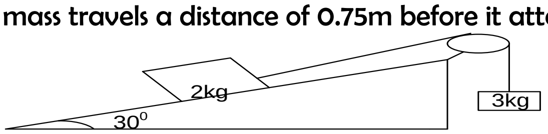
Examples

1. A mass of 12kg lies on a smooth incline plane 6m long and 1m high. One end of a light inextensible string is attached to this mass and the string passes up the line of greatest slope, over a smooth pulley fixed at the top of the plane is a freely suspended mass of 4kg at it's other end.
   If the system is released from rest, find the;
   (i) acceleration of the system (ii) Tension in the string
   (iii) Velocity with which the 4kg mass will hit the ground
   (iv) Time the 4kg mass takes to hit the ground **Uneb 2013, No 10**

Solution



2. A particle of mass 2kg on a rough plane inclined at 30^0 to the horizontal is attached by means of a light inextensible string passing over a smooth pulley at the top edge of the plane to a particle of mass 3kg which hangs freely. If the system is released from rest with above parts of the string taut, the 3kg mass travels a distance of 0.75m before it attains a speed of 2.25m/s

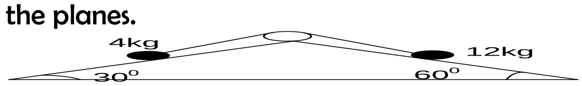


Find the;
(i) Acceleration of the system
(ii) Coefficient of friction between the plane and the 2kg mass
(iii) Tension in the string. **Uneb 2003, No11**

Solution

DOUBLE INCLINED PLANES

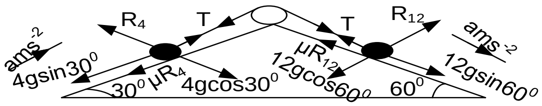
1. Two rough planes inclined at $30°$ and $60°$ to the horizontal and of thee same height are placed back to back. Masses of 4kg and 12kg are placed on the faces and connected by a light sting passing over a smooth pulley at the top of the planes.



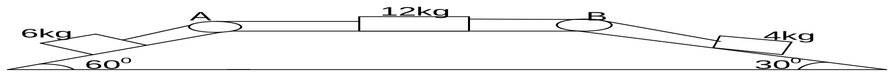
If the coefficient of friction is 0.5 on both faces , find the;

(i) Acceleration of the system (ii) Tension in the strings

Solution



2. The diagram below shows a 12kg mass on a horizontal rough plane. The 6kg and 4kg masses are on rough planes inclined at angles of 60° and 30° respectively. The masses are connected to each other by a light inextensible strings passing over a light smooth fixed pulleys A and B **Uneb 2015 No.11**



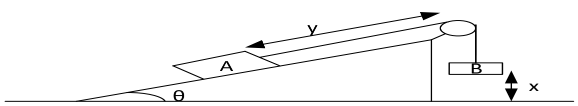
The planes are equally rough with coefficient of friction of $\frac{1}{12}$. If the system is released from rest, find the;

(i) Acceleration of the system (ii) Tension in the strings

Solution

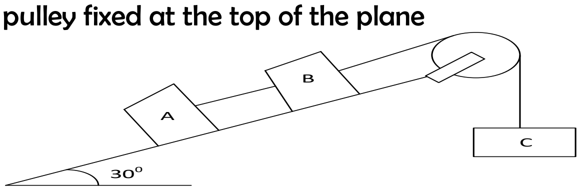
Exercise 16B

1. A mass A is of 4kg and the mass B is of 3kg are connected by a light in extensible string passing over a smooth pulley. The system is released from rest and mass A accelerates up along a smooth slope inclined at an angle θ to the horizontal where $\theta = 30°$



If $y = 3m$ and $x = 2.8m$, calculate the velocity with which A hits the pulley **An(2.42m/s)**

2. The diagram below shows particles A, B and C of masses 10kg, 8kg and 2kg respectively connected by a light in extensible strings. The string connecting B and C passes over a smooth light pulley fixed at the top of the plane



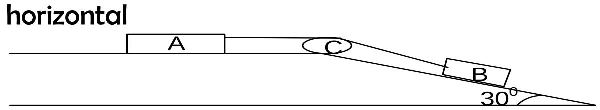
If the coefficient of friction between the plane and particles A and B are 0.22 and 0.25 respectively, calculate;

(i) Acceleration of the system

(ii) Tensions in the strings

An(1.6477ms^{-2}, 22.895N, 13.851N)

4. In the diagram particles A and B are of masses 2.4kg and 3.6kg respectively. A rest on a rough horizontal plane (coefficient of friction 0.5), it is connected by a light in extensible string passing over a smooth fixed pulley at C to a particle B resting on a smooth plane inclined at 30° to the horizontal



When the system is released from rest, find the

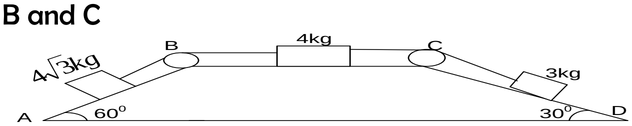
(i) acceleration in the system and the tension in the string

(ii) The force on the pulley at C

(iii) The velocity of the A mass after 2 seconds

An(0.98ms^{-2}, 14.112N, 7.3049N, 1.96ms^{-1})

5. The diagram below shows a 4kg mass on a horizontal rough plane with coefficient of friction 0.25. The $4\sqrt{3}$ kg mass rests on a smooth plane inclined at angles of 60° to the horizontal while the 3kg rests on a rough plane inclined at an angle 30° to the horizontal and coefficient of friction $\frac{1}{\sqrt{3}}$. The masses are connected to each other by a light inextensible strings passing over a light smooth fixed pulleys B and C



(a) Find the

(i) Acceleration of the system

(ii) Tension in the strings

(b) Work done against frictional forces when the particles each moved 0.5m

An(1.407ms^{-2}, 49.051N, 33.622N, 12.25J)

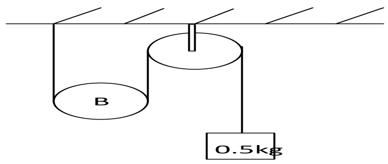
MULTIPLE CONNECTIONS

- Acceleration of a particles moving between two portions of the string is equal to half the net acceleration of the particle(s) attached to the end of the string
- The tension in the uninterrupted string is constant
- The tensions in interrupted strings are different.

Case 1: A pulley moving between two portions of a string

Examples

1. The diagram below shows particle A of mass 0.5kg attached to one end of a light inextensible string passing over a fixed light pulley and under a movable light pulley B. The other end of the string is fixed as shown below. **Uneb 1997**



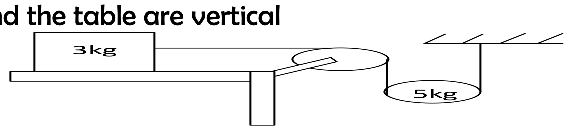
(i) What mass should be attached at B for the system to be equilibrium

(ii) If B is 0.8kg, what are the acceleration of particle A and pulley B

(iii) Find the tension in the string

Solution

2. A particle of mass $3kg$ on a smooth horizontal table is tied to one end of the string which passes over a fixed pulley at the edge and then under a movable pulley of mass 5kg, its other end being fixed so that the parts of the string beyond the table are vertical

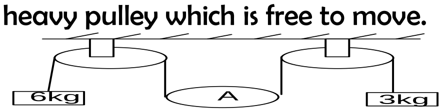


Find;

(i) Acceleration of 3kg and 5kg mass **(ii)** Tension in the string

Solution

3. In the pulley system below, A is a heavy pulley which is free to move.



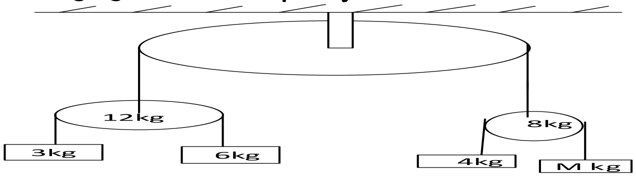
Find the mass of pulley A if it does not move upwards or downwards when the system is released from rest

Solution

Case 2: A pulley moving on one portion of a string

Examples

1. The diagram below shows two pulleys of masses 8kg and 12 kg connected by a light inextensible string hanging over a fixed pulley.

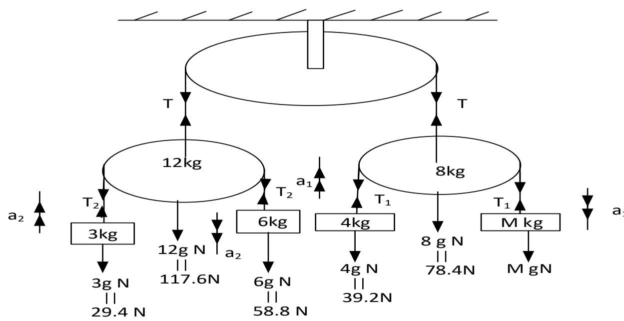


The hanging portions of the strings are vertical. Given that the string of the fixed pulley remains stationary, find the;

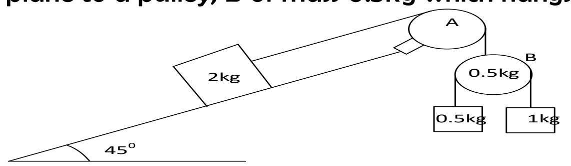
(i) Tensions in the strings

(ii) Value of m

Solution

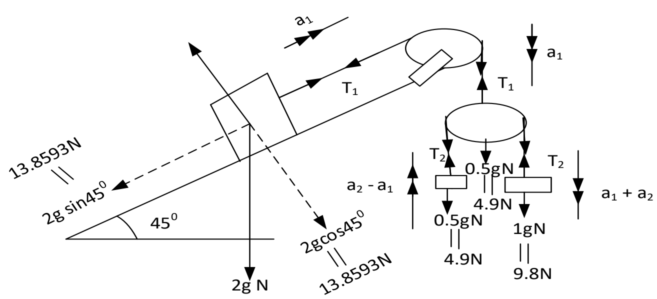


2. The diagram shows a particle of mass 2kg on a smooth plane inclined at 45^0 to the horizontal and attached by means of a light inextensible string passing over a smooth pulley, A at the top edge of the plane to a pulley, B of mass 0.5kg which hangs freely



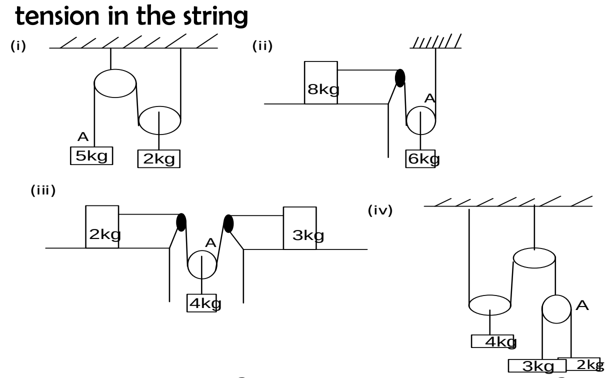
Pulley B carries two particles mass 0.5kg and 1kg on either side. Find
   (i) Acceleration of 2kg, 0.5kg and 1kg mass
   (ii) The tension in the string

Solution



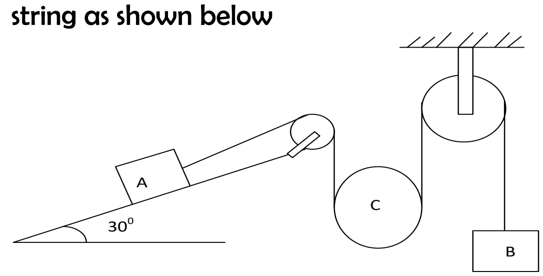
Exercise 16C

1. For each of the systems below; all the strings are light and inextensible, all pulleys are light and smooth and all surfaces are smooth.
In each case find the acceleration of A and the tension in the string



An((i) 7.127ms^{-2}, 13.364N, (ii) 1.547ms^{-2}, 24.758N, (iii) 3.564ms^{-2}, 10.691N, (iv) 4.731ms^{-2}, 12.166N, 24.331N)

2. Two particles **A** and **B** of masses 4kg and 2kg respectively and a movable pulley **C** of mass 6kg are connected by a light inextensible string as shown below



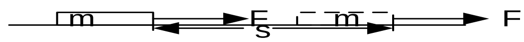
Given that the coefficient of friction between A and the plane is 0.2 and the system is released from rest, find the acceleration of A, B , C and the tension in the string. **An(A= -0.25ms^{-2}, B=2.9ms^{-2}, C= 1.325ms^{-2})**

CHAPTER 7: WORK, ENERGY AND POWER

WORK DONE BY A CONSTANT FORCE

Work is said to be done when energy is transferred from one system to another

When a block of mass m rests on a smooth horizontal



When a constant force F acts on the block and displaces it by s, then the work done by F is given by

$$W = Fs$$

Examples

1. A horizontal force pulls a body of mass 5kg a distance of 8m across a rough horizontal surface, coefficient of friction 0.25. The body moves with uniform velocity, find the work done against friction

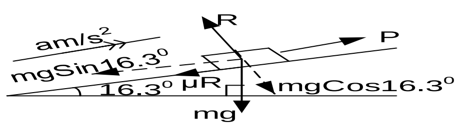
Solution

$W = \mu Rs$ | $W = 0.25x5x9.8x8$ | $w = 98J$

2. A rough surface is inclined at $\tan^{-1}\left(\frac{7}{24}\right)$ to the horizontal. A body of mass 5kg lies on the surface and is pulled at a uniform speed a distance of 75cm up the surface by a force acting along a line of greatest slope. The coefficient of friction between the body and the surface is $\frac{5}{12}$. Find;

a) Work done against gravity b) Work done against friction

Solution



$\theta = \tan^{-1}\left(\frac{7}{24}\right) = 16.3°$

a) Work done against friction

$W = \mu Rd$ But $R = mgcos\theta$

$$W = \mu mgcos\theta d$$

$$W = \frac{5}{12}x5x\frac{75}{100}x9.81cos16.3 = 14.71J$$

b) Work done against gravity

$$W = mgsin\,\theta xd$$

$$W = 5x9.81sin16.3x\frac{75}{100} = 10.35J$$

Exercise 17A

1. A rough surface is inclined at an angle θ to the horizontal. A body of mass mkg lies on the surface and is pulled at a uniform speed a distance of x meters up the surface by a force acting along a line of greatest slope. The coefficient of friction between the body and the surface is μ. Show that the total work done on the body is $mgx(sin\theta + \mu cos\theta)$
2. A particle of mass 15 kg is pulled up a smooth slope by a light inextensible string parallel to the slope. The slope is 10.5 m long and inclined at $sin^{-1}\left(\frac{4}{7}\right)$ to the horizontal. The acceleration of the particle is $0.98ms^{-2}$. Determine the:
   (a) tension in the string.
   (b) work done against gravity when the particle reaches the end of the slope.

UNEB 2018 No.4 An(98.7N, 882J)

WORK-ENERGY THEOREM

It states that the work done by the **net force** acting on a body is equal to the change in its kinetic energy. Consider a body of mass m accelerated from velocity, u by a constant force, F so that in a distance, s it gains velocity v

$a = \frac{v^2-u^2}{2s}$ ----------------------- [1]

Resultant $force = ma = \frac{m(v^2-u^2)}{2s}$

But work done $= Fxs$

$$W = \frac{m(v^2-u^2)}{2s}s = \frac{m(v^2-u^2)}{2}$$

$$\boxed{W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2}$$ work-energy theorem.

Examples

1. A car of mass 1000kg moving at 50ms^{-1} skids to rest in 4s under a constant retardation. Calculate the magnitude of the work done by the force of friction

Solution

a) Using $v = u + at$

$0 = 50 + 4a$

$a = -12.5m/s^2$

Frictional force $= ma$

$= 1000x - 12. = 12500N$

$S = ut + ½\, at^2$

$S = 50x4 + ½\, x - 12.5x4^2$

$S = 100m$

W= $FxS = 12500x100$

Work done = $1.25x10^6 J$

Alternatively

$W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

W= $\frac{1}{2}x1000x50^2 - \frac{1}{2}\, x1000x0^2$

Work done = $1.25x10^6 J$

2. A body of mass 4kg is moving with an initial velocity of 5m/s on a plane. The kinetic energy of the body is reduced by 16J in a distance of 40m. Find the deceleration of the body. **Uneb 2016**

Solution

$W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

$16 = \frac{1}{2}x4x(5^2 - v^2)$

$v^2 = 17$

$v^2 = u^2 + 2s$

$a = \frac{17 - 25}{2x40} = -0.1ms^{-2}$

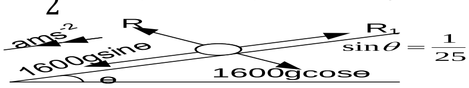
Incline planes

1. A car of mass 1600kg slides down a hill of slope 1 in 25. When the car descends 200m along at the hill, its speed increases from $3ms^{-1}$ to $10s^{-1}$. Calculate
   (i) The change in the total kinetic energy
   (ii) Average value of resistance to motion **UNEB 1992, No.6**

Solution

(i) $\Delta k.e = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

$= \frac{1}{2}1600(10^2 - 3^2) = 72,800J$



$v^2 = u^2 + 2as$

$a = \frac{10^2 - 3^2}{2x200} = 0.228ms^{-2}$

$using\ \ F = ma$

$1600g\ sin\theta - R_1 = 1600a$

$R_1 = 1600x9.8x\frac{1}{25} - 1600x0.228 = 262.4N$

OR $W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

$(1600gsin\theta - R)x200 = \frac{1}{2}x1600(10^2 - 3^2)$

$R = 263.2N$

Exercise 17B

1. A bullet of mass 50g travelling horizontally at 500ms^{-1} strikes as stationary block of wood and after travelling 10cm, it emerges from the block travelling at 100ms^{-1}. Calculate the average resistance of the block to the motion of the bullet. **An[60000N]**
2. A rough slope of length 10m is inclined at angle of $tan^{-1}\left(\frac{3}{4}\right)$ to the horizontal. A block of mass 50kg is released from rest at the top of the slope and travels down the slope, reaching the bottom of the slope with speed of 8m/s, find the
   (i) magnitude of the frictional force
   (ii) Work done by the frictional force
   (iii) Coefficient of friction
   An($134N, 1340J, 0.342$)
3. Point A is situated at the bottom of a rough slope inclined at angle of θ to the horizontal. A body of mass. mkg is projected form A along and up a line of greatest slope. The coefficient of friction between the body and the plane is μ. The body first comes to rest at point B a distance x m from A, before returning to A show;
   (i) Work done against friction when the body moves from A to B and back to A is given by $2\mu mgx\ cos\theta$
   (ii) The initial speed of the body $\sqrt{2gx\ (sin\theta + \mu cos\theta)}$
   (iii) Speed of the body on return to A **is** $\sqrt{2gx\ (sin\theta - \mu cos\theta)}$

POWER

It's the rate of doing work.

Motion of cars

Consider a car being driven along a road, the forward or tractive force F_T moves the car is supplied by the engine working a constant rate of P watts

Power = $\frac{\text{Work done}}{\text{time}}$

P= $\frac{F_T \text{ x d}}{\text{t}}$

P= $F_T x \ \frac{d}{t}$

$P = F_T x v$

$F_T = \frac{P}{v}$

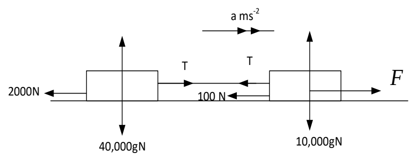
Numerical examples

1. A force on a particle of mass 15kg moves it along a straight line with a velocity of 10m/s. The rate at which work is done by the force is 50W. If the particle starts from rest, determine the time it takes to move a distance of 100m

Solution Uneb 2000 No7

2. A car of mass 10 tonnes tows a trailer of mass 40 tonnes along a level road. The car exerts experiences a resistance of 1000N and the trailer a resistance of 20,000N, if the car engine working at a constant rate of 4,000kW. Find the acceleration produced and the tension in the tow bar at the instant the speed is 72km/h

Solutions



$F = \frac{400,000}{20} = 20,000\ N$

$V = 72km/h = 20m/s$

10,000kg: $20{,}000 - (T + 100) = 10{,}000xa$

$19{,}900 - T = 10{,}000\text{a} \ldots\ldots (i)$

40,000kg: $T - 2000 = 40{,}000a \ldots\ldots (ii)$

$17{,}900 = 50{,}000xa$

$\text{a} = 0.358\text{ms}^{-2}$

Alternatively

50,000kg: $20{,}000 - 2000 - 100 = 50{,}000$

$17{,}900 = 50{,}000xa$

$\text{a} = 0.358\text{ms}^{-2}$

$T - 2000 = 40{,}000a$

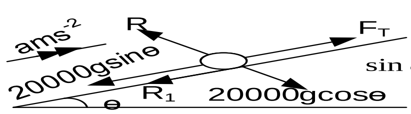
$T = 2000 + 40{,}000x0.358$

$T = 16{,}320\text{N}$

Inclined planes

1. A train of mass 20000kg moves at a constant speed of 72kmh^{-1} up a straight incline against a frictional force of 128N. The incline is such that the train rises vertically 1 meter for every 100m travelled along the incline. Calculates the necessary power developed by the train.

Solution



$\sin\theta = \frac{1}{100}$

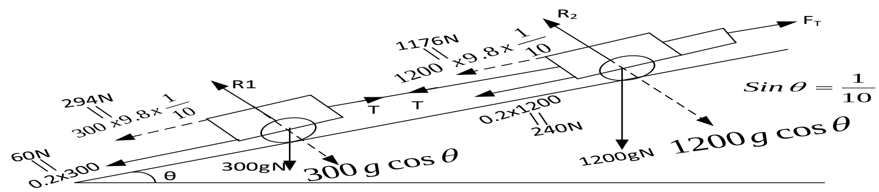
2. A car of mass 1000kg has a maximum speed of 150km/h on a level rough road and the engine is working at 60kW. **Uneb 2005 No13b**
   (i) Find the coefficient of friction between the car and the road if all resistance is due to friction
   (ii) Given that the tractive force remains unaltered and the non-gravitational resistance in both cases varies as square of the speed, find the greatest slope on which a speed of 120km/h could be maintained

Solution

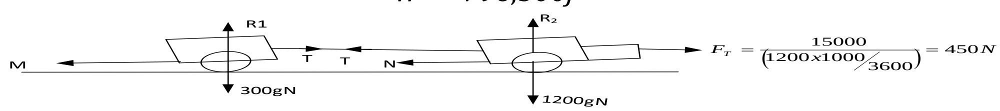
3. A car of mass 1200kg is pulling a trailer of mass 300kg up a slope 1 in 10, the resistance to motion for the car and trailer is $0.2Nkg^{-1}$. Given that the car moved at a constant speed of 1.5ms^{-1} for 5 minutes, find the
   i. Tension in coupling between the car and the trailer
   ii. Work done by the engine of the car during this time

iii. Total resistance, if the engine develops a power of 15kW at a maximum speed of 120km/h on a level road

Solution



$$W = 796{,}500J$$



1500kg: $450 - (M + N) = 1{,}500\text{x}0$

$$(M + N) = 450\text{N}$$

4. A car of mass mkg has an engine which works at a constant rate of 2H kW. The car has a constant speed of Vm/s along a horizontal road.

(a) Find in terms of m, H, V and θ the acceleration of the car when travelling

(i) Up a road of inclination θ with a speed of $^3/_4\, V\ ms^{-1}$

(ii) Down the same road with a speed of $^3/_5\, V\ ms^{-1}$, the resistance to the motion of the car apart from the gravitational force, being constant

(b) If the acceleration in (a)(ii) above is 3 times that of a(i) above, find the angle of inclination θ of the road

(c) If the car continues directly up the road, incase a(i) above, who that its maximum speed is $^{12}/_{13}\, V\ ms^{-1}$ **Uneb 2004 No13**

An $(i)\ a = \frac{2000H - 3mvg\ sin\theta}{3mv}$ $(ii)\ a = \frac{400H + 3mvg\ sin\theta}{3mv}$

Exercise 17C

1. The force opposing the motion of a car is $(a + bv^2)$N, where a and b are constants and v is the speed of the car in m/s. The power required to maintain a steady speed of 20m/s is 6.2kW and at 30m/s is 15.3kW.

(i) Find the value of a and b

(ii) Power developed for a steady speed of 40m/s **An(a=150,b=0.4, 31.6kW)**

2. With its engine working at a constant rate of 9.8kW, a car of mass 800kg can descend a slope of 1 in 56 at twice the steady speed that it can ascend the same slope, the resistances to motion remaining the same through it. Find the magnitude of the resistance and the speed if ascent **An[420N, 17.5m/s]**

3. The frictional resistance to motion of a car is (kv)N, where c is a constant and v is the speed of the car in m/s. the car ascends a hill of inclination 1 in 10 at a steady speed of 8m/s. The power exerted by the engine is 9.76kW

(i) Find the value of k

(ii) Find the steady speed at which the car ascends the hill if the power exerted by the engine is 12.8kW. When the car is traveling at this speed, the power exerted by the engine is increased by 2kW, find the immediate acceleration of the car **An(k=30, 10m/s, 0.2m/s²)**

4. A car is working at 5kW and is travelling at a constant speed of 72km/h. Find the resistance to motion. **Uneb 2007 No16 An (250N)**

5. The engine of a lorry of mass 5,000kg is working at a steady rate of 350kW against a constant resistance force of 1,000N. The lorry ascends a slope of inclination $\theta°$ to the horizontal. If the maximum speed of the lorry is $20ms^{-1}$. Find the value of θ **Uneb 2017 No8. An** $19.68°$

PUMP RAISING AND EJECTING WATER.

Consider a pump which is used to raise water from a source and then eject it at a given speed.
The work done per second gives the rate (power) at which the pump is working.

$$\boxed{\text{work done per second} = P.E \text{ given to water per second} + K.E \text{ given to water per second}}$$

Examples

1. A pump raises water through a height of 3.0m at a rate of 300kg per minute and delivers it with a velocity of 8.0ms^{-1}. Calculate the power output of the pump

Solution

2. A pump raises 2$m^3$of water through a vertical distance of 10m in one and half minutes, and discharges it at a speed of 2.5m/s. Show that the power developed is approximately 2.25kW. **Uneb 2005 No13a**

Solution

$$\textbf{Power} = P.E \text{ given to water per second} + K.E \text{ given to water per second}$$

$$\textbf{Power} = (mass\ per\ second x g x h) + \left(\frac{1}{2} x mass\ per\ second x v^2\right)$$

$$\textbf{Powerwork done per second} = \left(\frac{2x1000}{90}x9.8x10\right) + \left(\frac{1}{2}x\frac{2x1000}{90}x2.5^2\right)$$

$$\textbf{Power} = 2247.22W = 2.25kW$$

EXERCISE: 17D

CHAPTER 8: VARIABLE ACCELERATION

This is when the rate of change of velocity is not constant

Differential calculus

Let = displacement, v = velocity and a = acceleration and are functions of t:

$velocity, \underbrace{v} = \frac{dr}{dt}$ where r is displacement

$differentiation: displacement \Rightarrow velocity$
$\Rightarrow acceleration$

$acceleration, \underbrace{a} = \frac{dv}{dt} = \frac{d^2r}{dt^2}$ where v is velocity

Examples

1. A particle moves a long a straight line such that after t seconds its displacement from a fixed point is s meters where $s = 2sint\,\hat{\imath} + 3cost\hat{\jmath}$. Find
   (a) acceleration after t second
   (b) acceleration after $\frac{\pi}{2}s$
   (c) magnitude of the acceleration after $\frac{\pi}{2}s$

   Solution

   (a) $v_{(t=t)} = \frac{ds}{dt} = \frac{d}{dt}(2sint\,\hat{\imath} + 3cost\hat{\jmath})$
   $v_{(t=t)} = 2cost\,\hat{\imath} - 3sint\hat{\jmath}$
   $a_{(t=t)} = \frac{dv}{dt} = \frac{d}{dt}(2cost\,\hat{\imath} - 3sint\hat{\jmath})$
   $a_{(t=t)} = (-2sint\,\hat{\imath} - 3cost\hat{\jmath})\,ms^{-2}$

   (b) **When** $t = \frac{\pi}{2}s$
   $a_{\left(t=\frac{\pi}{2}\right)} = -2sin\left(\frac{\pi}{2}\right)\hat{\imath} - 3cos\left(\frac{\pi}{2}\right)\hat{\jmath} = -2\,\hat{\imath}\,ms^{-2}$

   (c) $\left|a_{\left(t=\frac{\pi}{2}\right)}\right| = \sqrt{(-2)^2 + (0)^2} = 2\,ms^{-2}$

2. A particle moves in the x-y plane such that its position vector at any time t is given by $r = (3t^2 - 1)\hat{\imath} + (4t^3 + t - 1)\hat{\jmath}$. Find **Uneb 2002 No.8**
   (a) Speed after $t = 2$
   (b) Magnitude of the acceleration after $t = 2s$

   Solution

Exercise 18A

1. The position vector of a particle at any time (t) is given by
   $\boldsymbol{r(t)} = \left(\left(\boldsymbol{t^2 + 4t}\right)\boldsymbol{i} + \left(\boldsymbol{3t - t^3}\right)\boldsymbol{j}\right)\boldsymbol{m}$. Find the speed of the particle at $t = 3$ seconds
   UNEB 2020 NO.1. An$= 26\,ms^{-1}$

2. The displacement of a particle after t seconds is given by $s = 2\sqrt{3}sint\hat{\imath} + 8cost\hat{\jmath}$. Find the speed when $t = \frac{\pi}{6}s$ **An** $5\,ms^{-1}$

Integral calculus

If r, v or a are functions of time t:

$\boxed{velocity, \underbrace{v} = \int adt + \underbrace{c}}$ and $\boxed{displacement, \underbrace{r} = \int vdt + \underbrace{c}}$

$integration: acceleration \Rightarrow velocity \Rightarrow displacement$

Examples

1. A particle starts from rest origin $(0,0)$. Its acceleration in ms^{-2} at time t second is given by $a = 6t\hat{\imath} - 4\hat{\jmath}$. Find its speed when $t = 2$ second **Uneb 2014 No.2**

   Solution

2. A particle starts from rest at a point $(2,0,0)$ and moves such that Its acceleration in ms^{-2} at time t second is given by $a = \left[16cos4t\hat{\imath} + 8sin2t\hat{\jmath} + (sint - 2sin2t)\hat{k}\right]ms^{-2}$. Find the; **Uneb 2016 No.13**

(a) speed when $t = \frac{\pi}{4}$

Solution

$$v_{(t=t)} = \int a dt + C = \int \begin{pmatrix} 16cos4t \\ 8sin2t \\ sint - 2sin2t \end{pmatrix} dt + c$$

$$v_{(t=t)} = \begin{pmatrix} 4sin4t \\ -4cos2t \\ -cost + cos2t \end{pmatrix} + c$$

At $t = 0, v = 0$; $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4sin4x0 \\ -4cos2x0 \\ -cos0 + cos2x0 \end{pmatrix} + c$

$$c = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$$

$$v_{(t=t)} = \begin{pmatrix} 4sin4t \\ 4 - 4cos2t \\ -cost + cos2t \end{pmatrix}$$

When $t = \frac{\pi}{4}$: $v_{\left(t=\frac{\pi}{4}\right)} = \begin{pmatrix} 4sin4x\frac{\pi}{4} \\ 4 - 4cos2x\frac{\pi}{4} \\ -cosx\frac{\pi}{4} + cos2x\frac{\pi}{4} \end{pmatrix}$

$$v_{\left(t=\frac{\pi}{4}\right)} = \begin{pmatrix} 0 \\ 4 \\ -\sqrt{2}/_2 \end{pmatrix}$$

$$\left|v_{\left(t=\frac{\pi}{4}\right)}\right| = \sqrt{(0)^2 + (4)^2 + \left(-\sqrt{2}/_2\right)^2} = 4.062ms^{-1}$$

$$r_{(t=t)} = \int vdt + c = \int \begin{pmatrix} 4sin4t \\ 4 - 4cos2t \\ -cost + cos2t \end{pmatrix} dt + c$$

(b) Distance from origin when $t = \frac{\pi}{4}$

$$r_{(t=t)} = \begin{pmatrix} -cos4t \\ 4t - 2sin2t \\ -sint + 0.5sin2t \end{pmatrix} + c$$

At $t = 0, r = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -cos4x0 \\ 4x0 - 2sin2x0 \\ -sin0 + 0.5sin2x0 \end{pmatrix} + c$$

$$c = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

$$r_{(t=t)} = \begin{pmatrix} 3 - cos4t \\ 4t - 2sin2t \\ -sint + 0.5sin2t \end{pmatrix}$$

$$v_{\left(t=\frac{\pi}{4}\right)} = \begin{pmatrix} 3 - cos4x\frac{\pi}{4} \\ 4x\frac{\pi}{4} - 2sin2x\frac{\pi}{4} \\ -sin\frac{\pi}{4} + 0.5sin2x\frac{\pi}{4} \end{pmatrix}$$

$$r_{\left(t=\frac{\pi}{4}\right)} = \begin{pmatrix} 4 \\ 1.1416 \\ -0.2071 \end{pmatrix}$$

$$\left|r_{\left(t=\frac{\pi}{4}\right)}\right| = \sqrt{(4)^2 + (1.1416)^2 + (-0.2071)^2}$$

$$= 4.1649m$$

Vector approach of finding work and power

Work done by a variable force is obtained using a dot product

$W = F.d$ or

$$W_{(t=t)} = \int F.V\, dt$$

Where Power, P is given by $P = F.V$

Examples

1. A particle of mass 4 kg starts from rest at the origin. It acted upon by a force $F = \left(2t\check{\imath} + 3t^2\tilde{\jmath} + 5\tilde{k}\right)N$. Find the work done by the force F after 3 seconds.

   Solution

2. A particle of mass 10 kg starts from rest at a point A with position vector $\left(4\hat{\imath} + 3\hat{\jmath} + 2\hat{k}\right)m$. It is acted upon by a constant force, $F = \left(8\hat{\imath} + 4\hat{\jmath} + 6\hat{k}\right)N$ causing it to accelerate to B after 4 seconds. Find the:

   (i) Magnitude of the acceleration of the particle
   (ii) velocity at any time *t*.
   (iii) Position vector of point B
   (iv) the displacement vector AB
   (v) work done by the force F after 4 seconds.

   Solution

   $(i)\quad F = ma$

$$a_{(t=t)} = \frac{1}{10}(8\hat{\imath} + 4\hat{\jmath} + 6\hat{k})ms^{-2}$$

$$a_{(t=t)} = \frac{1}{10}\sqrt{8^2 + 4^2 + 6^2} = 1.077s^{-2}$$

(ii) $v_{(t=t)} = \int a dt + c = \frac{1}{10}\int(8\check{\imath} + 4\tilde{\jmath} + 6\tilde{k})dt + c$

$$v_{(t=t)} = \frac{1}{10}(8t\check{\imath} + 4t\tilde{\jmath} + 6t\tilde{k}) + c$$

At $t = 0, v = 0$: $v = \frac{1}{10}(8x0\check{\imath} + 4x0\tilde{\jmath} + 6x0\tilde{k}) + c$

$$c = 0$$

$$v_{(t=t)} = \frac{1}{10}(8t\check{\imath} + 4t\tilde{\jmath} + 6t\tilde{k})$$

(iii) $r_{(t=t)} = \int \frac{1}{10}(8t\check{\imath} + 4t\tilde{\jmath} + 6t\tilde{k})dt + c$

$$r_{(t=t)} = \frac{1}{10}(4t^2\check{\imath} + 2t^2\tilde{\jmath} + 3t^2\tilde{k}) + c$$

At $t = 0$, OA= $(4\hat{\imath} + 3\hat{\jmath} + 2\hat{k})$

$$(4\hat{\imath} + 3\hat{\jmath} + 2\hat{k}) = \frac{1}{10}(4x0^2\check{\imath} + 2x0^2\tilde{\jmath} + 3x0^2\tilde{k}) + c$$

$$c = (4\hat{\imath} + 3\hat{\jmath} + 2\hat{k})$$

$$r_{(t=t)} = \frac{1}{10}([4t^2 + 40]\check{\imath} + [2t^2 + 30]\tilde{\jmath} + [3t^2 + 20]\tilde{k})$$

(iv) $OB_{(t=4)} = {}^1/_{10}\begin{pmatrix} 4x4^2 + 40 \\ 2x4^2 + 30 \\ 3x4^2 + 20 \end{pmatrix} = \begin{pmatrix} 10.4 \\ 6.2 \\ 6.8 \end{pmatrix}m$

$$\overrightarrow{AB} = OB - OA = \begin{pmatrix} 10.4 \\ 6.2 \\ 6.8 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 6.4 \\ 3.2 \\ 4.8 \end{pmatrix}$$

(v) $W_{(t=4)} = F.r = \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix}.\begin{pmatrix} 6.4 \\ 3.2 \\ 4.8 \end{pmatrix} = 92.8J$

Alternative 1

$$v_{(t=4)} = {}^1/_{10}\begin{pmatrix} 8x4 \\ 4x4 \\ 6x4 \end{pmatrix} = \begin{pmatrix} 3.2 \\ 1.6 \\ 2.4 \end{pmatrix}m/s$$

$$|v_{(t=4)}| = \sqrt{3.2^2 + 1.6^2 + 2.4^2} = \sqrt{18.56}\,m/s$$

$$W_{(t=0\ and\ t=4)} = {}^1/_2 m(v_{(t=4)}{}^2 - v_{(t=0)}{}^2)$$

$$= {}^1/_2 x10(18.56 - 0) = 92.8J$$

Alternative 2

$$P_{(t=t)} = F.V$$

$$P_{(t=t)} = \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix}.{}^1/_{10}\begin{pmatrix} 8t \\ 4t \\ 6t \end{pmatrix} = 11.6t$$

$$W_{(t=0\ and\ t=4)} = \int_0^4 F.V\,dt = \int_0^4 (11.6t)\,dt$$

$$= [5.8t^2]_0^4$$

$$= (5.8x4^2) - 0 = 92.8J$$

Exercise 18B

1. A force $F = (2t\hat{\imath} + \hat{\jmath} - 3t\hat{k})N$ acts on a particle of mass 2kg. The particle is initially at a point $(0,0,0)$ and moving with a velocity $(\hat{\imath} + 2\hat{\jmath} - \hat{k})m/s$. Determine the, **Uneb 2019 No.12**
   i. Magnitude of the acceleration of the particle after 2s
   ii. Velocity of the particle after 2s
   iii. Displacement of the particle after 2s.

   An= $3.6401ms^{-2}$, $\begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}ms^{-1}$, $\begin{pmatrix} {}^{10}/_3 \\ 5 \\ -4 \end{pmatrix}m$

2. A particle of mass 4 kg starts from rest at a point $(2\hat{\imath} - 3\hat{\jmath} + \hat{k})m$. It moves with acceleration $a = (4\hat{\imath} + 2\hat{\jmath} - 3\hat{k})ms^{-2}$ when a constant force F acts on it. Find the: **Uneb 2018 No.10**
   (i) force F.
   (ii) velocity at any time t.
   (iii) work done by the force F after 6 seconds.

   An $\begin{pmatrix} 16 \\ 8 \\ -12 \end{pmatrix}N$, $\begin{pmatrix} 4t \\ 2t \\ -3t \end{pmatrix}m/s, 2088J$

CHAPTER 9: LINEAR MOMENTUM

This is the product of mass of a body and its velocity

$$Momentum = mas\ x\ velocity$$

Collisions

Case 1: Bodies separate after collision

Consider two bodies A and B with body A having a mass of M_A, initial velocity U_A , and body B having a mass of M_B, initial velocity U_B, after collision body A has a final velocity V_A and body B has a final velocity V_B

$$M_AU_A + M_BU_B = M_AV_A + M_BV_B$$

$$loss\ in\ k.e = \left(\frac{1}{2}M_AU_A^2 + \frac{1}{2}M_BU_B^2\right) - \left(\frac{1}{2}M_AV_A^2 + \frac{1}{2}M_BV_B^2\right)$$

$$\%loss\ in\ k.e = \frac{\left(\frac{1}{2}M_AU_A^2 + \frac{1}{2}M_BU_B^2\right) - \left(\frac{1}{2}M_AV_A^2 + \frac{1}{2}M_BV_B^2\right)}{\left(\frac{1}{2}M_AU_A^2 + \frac{1}{2}M_BU_B^2\right)}$$

Case 2: Bodies stick together and move with a common velocity after collision

Consider two bodies A and B with body A having a mass of M_A, initial velocity U_A , and body B having a mass of M_B, initial velocity U_B, after collision body A and body B stick together and move with a common velocity V.

$$M_AU_A + M_BU_B = (M_A + M_B)V$$

$$loss\ in\ k.e = \left(\frac{1}{2}M_AU_A^2 + \frac{1}{2}M_BU_B^2\right) - \frac{1}{2}(M_A + M_B)V^2$$

$$\%loss\ in\ k.e = \frac{\left(\frac{1}{2}M_AU_A^2 + \frac{1}{2}M_BU_B^2\right) - \frac{1}{2}(M_A + M_B)V^2}{\left(\frac{1}{2}M_AU_A^2 + \frac{1}{2}M_BU_B^2\right)}$$

Examples:

1. A particle of mass 2kg moving with a speed 10m/s collides with a stationary particle of mass 7kg. Immediately after impact, the particle move with the same speed but in opposite directions. Find the loss in kinetic energy **Uneb 2007 No 4**

   Solution

2. Two particles are moving towards each other along a straight line. The first particle has a mass of 0.2kg and moving with a velocity 4m/s and then the second has a mass of 0.4kg moving with a velocity of 3m/s. On collision, the first particle reverses its direction and moves with a velocity of 2.5m/s. Find the percentage loss in kinetic energy **Uneb 1997 N013**

   Solution

$$M_AU_A + M_BU_B = M_AV_A + M_BV_B$$
$$0.2x4 + 0.4x - 3 = 0.2x - 2.5 + 0.4xv$$
$$v = 0.25m/s$$
$$k.e\ beore = \frac{1}{2}x0.2x4^2 + \frac{1}{2}x0.4x(-3)^2 = 3.4J$$

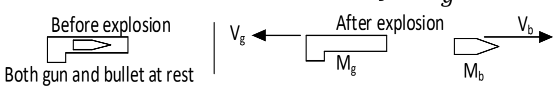
$$k.e\ after = \frac{1}{2}x0.2x(-2.5)^2 + \frac{1}{2}x0.4x0.25^2$$
$$= 0.6375J$$
$$\%loss\ in\ k.e = \frac{3.4 - 0.6375}{3.4}x100\% = 81.25\%$$

3. Two bodies A and B of mass 7.5kg and 5.0kg moving with velocities of $(-\hat{\imath} - 2\hat{\jmath})$m/s and $(9\hat{\imath} + 8\hat{\jmath})$m/s respectively collide. After collision bodies stick together and move with a common velocity find the;
   i. Common velocity   ii. Percentage loss in kinetic energy.

Solution

Recoil velocity of a gun and muzzle velocity of a bullet

When a bullet of mass m_b is fired with a muzzle velocity of V_b from a gun of mass M_g, the gun jerks back ward with a recoil velocity of V_g



$$M_g x0 + M_b x0 = M_g x - V_g + \boldsymbol{M_b} V_b$$
$$M_g V_g = M_b V_b$$

Examples

1. A bullet of mass $60g$ is fired from a gun of mass $3kg$. The bullet leaves the gun with a velocity of 400m/s. Find the initial speed of recoil of the gun and gain in kinetic energy of the system.

Solution

2. A gun of mass 3000kg fires horizontally a shell at an initial velocity of 300m/s. If the recoil of the gun is brought to rest by a constant opposing force of 9000N in 2 second, find the;
   (a) (i) initial velocity of the recoil gun   (ii) mass of the shell
   (iii) Gain in kinetic energy of the shell just after firing
   (b) (i) displacement of the gun
   (ii) work done against the opposing force

Solution

i. **(i)** $F = ma$

$$-9000 = 3000a$$
$$a = -3ms^{-2}$$
$$v = u + at$$
$$0 = u - 3x2$$
$$u = 6m/s$$

(ii) $M_g V_g = M_b V_b$

$$3000x6 = M_b x300$$
$$M_b = 60kg$$

(iii) Gain in k.e of bullet$= \frac{1}{2} M_b {V_b}^2 = \frac{1}{2} x60x300^2$

$$= 2.7x10^6 J$$

(b) (i) $s = \frac{v^2 - u^2}{2a} = \frac{0^2 - 6^2}{2x-3} = 6m$

$(ii)\ w = Fs = 9000x6 = 54{,}000J$

Exercise: 19

1. Two bodies A and B of masses 3 kg and 2 kg respectively are 7 m apart on a smooth horizontal surface. A moving directly towards B with a speed of 2 ms^{-1} and acceleration of 0.3 ms^{-2}. B is moving in the same direction as A with a speed of 5 ms^{-1} and retardation of 0.2ms^{-2}. If the bodies collide and coalesce, calculate the; **Uneb 2020 No 10**
   (a) Time taken before collision occurs
   (b) Common velocity immediately after the collision
   An((a)=14s, (b)=4.6m/s)
2. A bullet of mass 50g travelling horizontally at 80m/s hits a block of wood of mass 10kg resting on a smooth horizontal plane. If the bullet emerges with a speed of 50m/s, find the speed with which the block moves **Uneb 2016 No 8**
   An(0.15m/s)
3. A bullet of mass 0.1kg travelling horizontally at 420m/s hits a block of wood of mass 2kg resting on a smooth horizontal plane. If the bullet becomes embedded on the block, find the speed with which the block moves after impact.
   An(20m/s)

CHAPTER 10: PROJECTILE MOTION

This is the motion of a body which after being given an initial velocity moves freely under the influence of gravity.

TERMS USED IN PROJECTILES

1. **Angle of projection** θ**.** Angle the initial velocity makes with the horizontal.
2. **MAXIMUM HEIGHT [GREATEST HEIGHT].** The greatest height reached by projectile.
3. **TIME OF FLIGHT [T].** Time taken for a projectile to complete motion.
   Note: The time of flight is twice the time to maximum height
4. **RANGE [R].** Horizontal distance covered by projectile.
5. **MAXIMUM RANGE [R_{max}].** The greatest horizontal distance covered.
6. **A TRAJECTORY.** A trajectory is a path described by a projectile.

a. An object projected horizontally from a height above the ground

Horizontal motion: $u_x = u$, $a = 0$

$$s = ut + ½at^2$$
$$x = ut \ldots\ldots(1)$$
$$v = u + at$$
$$v_x = u \ldots\ldots\ldots.(2)$$

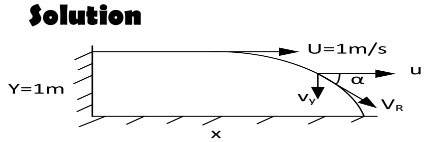
vertical motion: $u_y = 0$, $a == -9.8ms^{-2}$

$$s = ut + ½at^2$$
$$y = {}^1\!/_2\, gt^2 \ldots\ldots(3)$$
$$v = u + at$$
$$v_y = gt \ldots\ldots\ldots.(4)$$

Examples

1. A ball rolls off the edge of a table top 1m high above the floor with a horizontal velocity 1ms^{-1}. Find;
   i) The time it takes to hit the floor
   ii) The horizontal distance it covered
   iii) The velocity when it hits the floor

Solution



u=1ms^{-1} θ=0° y=-1m below the point of projection

(i) vertical motion: $y = {}^1\!/_2\, gt^2$

$$-1 = {}^1\!/_2\, x - 9.8t^2$$
$$t = 0.4518s$$

$$(ii)\ \ x = ut = 1x0.4518m = 0.4518m$$

velocity when it hits the ground

$$v_x = u = 1m/s$$
$$v_y = gt = -9.8x0.4518 = -4.428m/s$$
$$v = \sqrt{V_x^2 + V_y^2} = \sqrt{(1)^2 + (-4.428)^2} = 4.54ms^{-1}$$
$$\alpha = tan^{-1}\left(\frac{V_y}{V_x}\right) = tan^{-1}\left(\frac{4.428}{1}\right) = 77.3°$$

The velocity is $4.5ms^{-1}$ at 77.3°to the horizontal

2. At time $t = 0$, a particle is projected with a velocity of $3\hat{\imath}ms^{-1}$ from a point with position vector $(5\hat{\imath} + 25\hat{\jmath})m$. Find the;
   (i) Speed and direction of the particle when $t = 2s$
   (ii) Position vector of the particle when $t = 2s$

Solution

$$v_x = u = 3m/s$$
$$v_y = gt = -9.8x2 = -19.6m/s$$
$$v_R = \sqrt{V_x^2 + V_y^2} = \sqrt{(3)^2 + (-19.6)^2} = 19.83ms^{-1}$$
$$\alpha = tan^{-1}\left(\frac{V_y}{V_x}\right) = tan^{-1}\left(\frac{19.6}{3}\right) = 81.3°$$

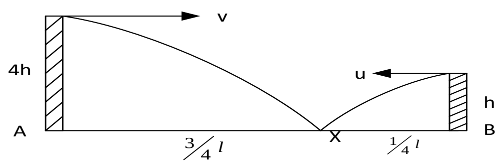
The speed is $19.83ms^{-1}$ at $81.3°$ to the horizontal

$$P_{(t=t)} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}$$
$$P_{(t=2)} = \begin{pmatrix} 5 \\ 25 \end{pmatrix} + \begin{pmatrix} ut \\ -{}^1\!/_2\, gt^2 \end{pmatrix}$$
$$P_{(t=2)} = \begin{pmatrix} 5 \\ 25 \end{pmatrix} + \begin{pmatrix} 3x2 \\ -{}^1\!/_2\, x9.8x2^2 \end{pmatrix} = \begin{pmatrix} 11 \\ 5.4 \end{pmatrix} m$$

3. A and B are two points on level ground. A vertical tower of height $4h$ has its base at A and a vertical tower of height h has its base at B. When a stone is thrown horizontally with speed v from the top of the taller tower

towards the smaller tower, it lands a point X where $AX = {}^{3}/_{4}AB$. When a stone is thrown horizontally with speed u and from the top of the smaller tower towards the taller tower, it also lands at the point X . Show that $3u = 2v$

Solution:



For A: Vertical motion, $y = {}^{1}/_{2}gt^2$

$$4h = {}^{1}/_{2}gt^2$$

$$t = \sqrt{\frac{8h}{g}}$$

Horizontal motion, $x = vt$

$${}^{3}/_{4}l = vt$$

$$l = {}^{4}/_{3}vt = {}^{4}/_{3}v\left(\sqrt{\frac{8h}{g}}\right) \ldots..(i)$$

For B: Vertical motion, $y = {}^{1}/_{2}gt^2$

$$h = {}^{1}/_{2}gt^2$$

$$t = \sqrt{\frac{2h}{g}}$$

Horizontal motion, $x = ut$

$${}^{1}/_{4}l = ut$$

$$l = 4ut = 4u\left(\sqrt{\frac{2h}{g}}\right) \ldots \ldots.(ii)$$

$$(ii) = (i): \quad 4u\left(\sqrt{\frac{2h}{g}}\right) = {}^{4}/_{3}v\left(\sqrt{\frac{8h}{g}}\right)$$

$$u = {}^{2}/_{3}v$$

$$3u = 2v$$

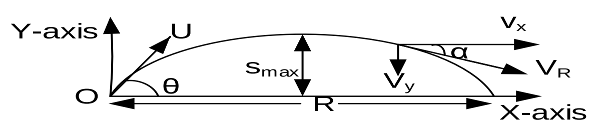
Exercise: 20A

1. At time $t = 0$, a particle is projected with a velocity of $2\hat{\imath}ms^{-1}$ from a point with position vector $(10\hat{\imath} + 150\hat{\jmath})m$. Find the;
   (i) Speed and direction of the particle when $t = 5s$
   (ii) Position vector of the particle when $t = 5s$

   An [49.04m/s at 87.6°, $\begin{pmatrix}20\\27.5\end{pmatrix}m$**]**
2. A batsman strikes a ball horizontally when it is 1m above the ground. The ball is caught 10cm above the ground by a fielder standing 12m from the batsman. Find the speed with which the batsman hits the ball. **An[28ms^{-1}]**
3. An aero plane moving horizontally at 150ms^{-1} releases a bomb at a height of 500m. The hits the intended target. What was the horizontal distance of aero plane from the target when the bomb was released? **An(1500m)**

b. Objects projected upwards from the ground at an angle to the horizontal

Suppose an object is projected with velocity u at an angle θ from a horizontal ground.



Horizontally: $u = u_x = u\cos\theta, \quad a = 0$

$$v = u + at$$

$$\boxed{v_x = u\cos\theta}$$

$$s = ut + {}^{1}/_{2}at^2$$

$$\boxed{x = u\cos\theta\, t}$$

Vertically: $u = u_y = u\sin\theta, \quad a = -g = -9.8ms^{-2}$

$$v = u + at$$

$$\boxed{v_y = u\sin\theta - gt}$$

$$s = ut + {}^{1}/_{2}at^2$$

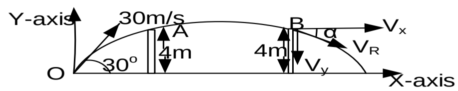
$$\boxed{y = u\sin\theta t - \frac{1}{2}gt^2}$$

Examples

1. A Particle is projected with a velocity of 30ms^{-1} at an angle of elevation of 30°. Find
   i) The greatest height reached
   ii) The time of flight
   iii) Horizontal range

iv) The velocity and direction of motion at a height of 4m on its way upwards

Solution



2. A particle is projected from the origin at a velocity of $(10\hat{\imath} + 20\hat{\jmath})ms^{-1}$. Find the position and velocity vectors of the particle 3 seconds after projection. (Take $g = 10ms^{-2}$)

Solution

$$P_{(t=t)} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}$$

$$P_{(t=3)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} u\cos\theta\, t \\ \text{usin } \theta t - \frac{1}{2}\text{gt}^2 \end{pmatrix}$$

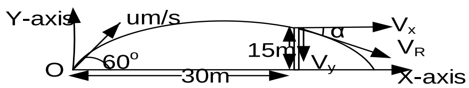
$$P_{(t=3)} = \begin{pmatrix} 10x3 \\ 20\text{x}3 - \frac{1}{2}\text{x}10\text{x}3^2 \end{pmatrix} = \begin{pmatrix} 30 \\ 15 \end{pmatrix} m$$

$$V_{(t=t)} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} u\cos\theta \\ u\sin\theta - gt \end{pmatrix}$$

$$V_{(t=3)} = \begin{pmatrix} 10 \\ 20 - 10x3 \end{pmatrix} = \begin{pmatrix} 10 \\ -10 \end{pmatrix} ms^{-1}$$

3. A projectile fired at an angle of 60° above the horizontal strikes a building 30m away at a point 15m above the point of projection. Find
   (i) Speed of projection (ii) Velocity when it strikes a building

Solution



4. A football player projects a ball at a speed of 8m/s at an angle of 30° with the ground. The ball strikes the ground at a point which is level with the point of projection. After impact with the ground, the ball bounces and the horizontal component of the velocity of the ball remains the same but the vertical component is reversed in direction and halved in magnitude. The player running after the ball, kicks it again at a point which is at a horizontal distance of 1.0m form the point where it bounced, so that the ball continues in the same direction. Find the ; **Uneb 2011 No.13**
   (a) Horizontal distance between the point of projection and the point at which the ball first strikes the ground (take $g = 10ms^{-2}$)
   (b) (i) the time interval between the ball striking the ground and the player kicking it again
   (ii)The height of the ball above the ground when it is kicked again(take $g = 10ms^{-2}$)

Solution

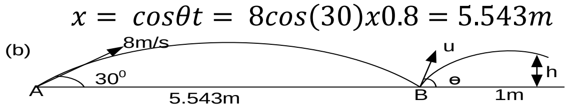
$(a)\ y = usin\theta t - ½\, gt^2$

At AB: $0 = 8sin30xt - ½\, x10xt^2$

$0 = (8sin30 - ½\, x10)t$

$\boldsymbol{t = 0.8s}$

$x = cos\theta t = 8cos(30)x0.8 = 5.543m$



$(i)\ x = ucos\theta t$

$1 = 8cos(30)xt$

$t = 0.1443s$

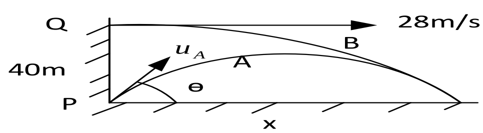
$(ii)\ y = usin\theta t - ½\, gt^2$

$h = 4sin30x0.1443 - ½\, x10x(0.1443)^2$

$h = 0.185m$

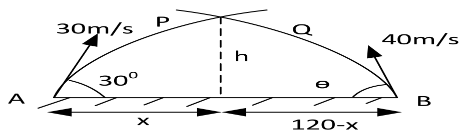
6. Two objects A and B are projected simultaneously from different points. B is projected from the top of a vertical cliff and A from the base. Particle B is projected horizontally with a speed 28 ms^{-1} and A is projected at an angel θ above the horizontal. The height of the cliff is 40m and the particles hit the same point on the ground, find:
   (a) time taken and distance from P to where they hit
   (b) speed and angle of projection of A.

Solution:



7. A particle P is projected from a point A with initial velocity 30m/s at an angle of elevation $30°$ to the horizontal. At the same instant a particle Q is projected in opposite direction with initial speed of 40m/s from a point at the same level with A and 120m from A. Given that the particles collide. Find;
   (i) Angle of projection of Q
   (ii) Time when collision occurs

Solution

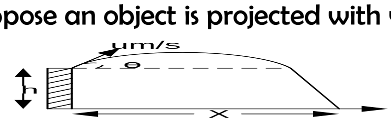


Exercise: 20B

1. A particle is projected from a point with speed $20m/s$ at angle of $60°$ to the horizontal. Express in vector form its velocity v and its displacement r from O at any time t seconds. **Uneb 2017 No.1**
   An $\begin{pmatrix} 10 \\ 10\sqrt{3} - gt \end{pmatrix} ms^{-}$, $\begin{pmatrix} 10t \\ 10\sqrt{3} - \frac{g}{2}t^2 \end{pmatrix} m$
2. A particle is projected at an angle of 60° to the horizontal with a velocity of 20m/s. Calculate the greatest height the particle attains **Uneb 2004 No.4 An[15.29m]**
3. A stone is thrown upwards from level ground towards a vertical pole, 4m high and 30m away from the point of projection. It just passes the pole in one second. Find **Uneb 2002 N**o.16
   i) Its initial speed and angle of projection **An [31.29m/s, 16.5°]**
   ii) The distance beyond the pole where the particle will fall **An [24.47m]**

b. Objects projected upwards from a point above the ground at an angle to the horizontal

Suppose an object is projected with velocity u at an angle θ from a height h



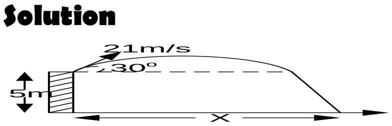
Horizontal: $u_x = ucos\theta,\ a = 0$

Vertical: $u_y = usin\theta,\ a = -g = -9.8$

Examples

1. A particle is projected at an angle of elevation of 30° with a speed of 21m/s. If the point of projection is 5m above the horizontal ground, find the horizontal distance that the particle travels before striking the ground . **Uneb 2006 No.16a**

Solution



$y = -5m$ since it's below the point of projection

For vertical motion: $y = usin\theta t - ½ gt^2$

$-5 = 21sin30T - \frac{9.8T^2}{2}$

$4.9T^2 - 10.5T - 5 = 0$

$$T = \frac{10.5 \pm \sqrt{(-10.5)^2 - 4x4.9x - 5}}{2x4.9}$$

T= $2.54s$ or T = $-0.40s$

Time of flight $T = 2.54s$

For horizontal motion

$x = ucos\theta xT = 21(cos30)\,x2.54 = 46.19m$

The horizontal distance = $46.19m$

Alternatively $y = xtan\theta - \frac{g\,x^2(\,1+tan^2\theta)}{2\,u^2}$

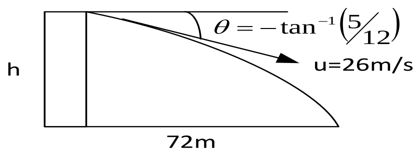
$$-5 = xtan30 - \frac{9.8\,x^2(\,1 + tan^2 30)}{2\,x21^2}$$

$$0.0148x^2 - 0.5774x - 5 = 0$$

$$\boldsymbol{x = 45.15m}$$

2. A stone is thrown from the edge of a vertical cliff and has an initial velocity of 26m/s at an angle of $tan^{-1}\left(\frac{5}{12}\right)$ below the horizontal. The stone hits the sea at a point level with the base of the cliff and 72m from it. Find the height of the cliff and the time for which the stone is in the air. Take $g = 10m/s^2$

Solution



$$y = xtan\theta - \frac{g\,x^2(\,1 + tan^2\theta)}{2\,u^2}$$

$$\mathrm{h} = 72x\left(-\frac{5}{12}\right) - \frac{10\mathrm{x}\,72^2\left(\,1 + \left[-{}^{5}/_{12}\right]^2\right)}{2\,x26^2}$$

$$\mathrm{h} = -75\mathrm{m}$$

$h = 75m$ below point of projection

$$x = ucos\theta xt$$

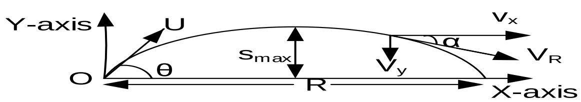
$$72 = 26(cs\left[-tan^{-1}\,{}^{5}/_{12}\right])t$$

$$t = 3s$$

EXERCISE 20C

Standard equations of projectiles

Suppose an object is projected with velocity u at an angle θ from a horizontal ground.



1. **MAXIMUM HEIGHT [GREATEST HEIGHT] [H]**

For vertical motion : at max height v=0,

u_y = $usin\,\theta$, $a = -g$, s= H

$$v_y{}^2 = u_y{}^2 + 2gs$$

$$0 = (u\,sin\,\theta)^2 - 2g\mathrm{H}$$

$$2g\mathrm{H} = u^2\,sin^2\theta$$

$$\boxed{H = \frac{u^2\,sin^2\theta}{2g}}$$

Note : $sin^2\theta = (sin\theta)^2$ but $sin^2\theta \neq sin\theta^2$

2. **TIME TO REACH MAX HEIGHT [t]**

Vertically $v = u_y + at$ at max height v=0

$u_y = u\,sin\,\theta$, $a = g$

$$0 = usin\,\theta - gt$$

$$\boxed{t = \frac{u\,sin\,\theta}{g}}$$

3. **TIME OF FLIGHT [T]**

Vertically: S_y=u_y t + ½ at²

at point A when the projectile return to the plane S_y=0,

t=T(time of flight), $a = -g$ $u_y = u\,sin\,\theta$

$$0 = usin\theta T - \frac{gT^2}{2}$$

$$\mathrm{T}\left(usin\theta - \frac{gT}{2}\right) = 0$$

Either $T = 0$ or $\left(usin\theta - \frac{gT}{2}\right) = 0$

$$\left(usin\theta - \frac{gT}{2}\right) = 0$$

$$usin\theta = \frac{gT}{2}$$

$$\boxed{T = \frac{2\,u\,sin\theta}{g}}$$

Note: The time of flight is twice the time to maximum height

4. **RANGE [R]**

Horizontally : $S_x = u_x\,t + ½\,at^2$

$u_x = ucos\theta$, a=0 (constant velocity), t=T

$$R = ucos\theta\,T + ½\,x0xT^2$$

$$R = ucos\theta T$$

But $T = \frac{2\,u\,sin\theta}{g}$

$$R = \frac{\boldsymbol{u^2\,2sin\theta\,cos\theta}}{\boldsymbol{g}}$$

But from trigonometry $2sin\theta cos\theta = sin2\theta$

$$\boxed{R = \frac{u^2 \sin 2\theta}{g}}$$

5. MAXIMUM RANGE [R_{max}]

For maximum range sin2θ =1, R=R_{max}

2 θ= $sin^{-1}(1)$

$$2\theta = 90°$$

$$R_{max} = \frac{u^2 sin90}{g}$$

$$\boxed{R_{max} = \frac{u^2}{g}}$$

6. EQUATION OF A TRAJECTORY

A trajectory is expressed in terms of horizontal distance x and vertical distance y.

For horizontal motion at any time t

$= ucos\,\theta t$

$t = \frac{x}{ucos\theta}$ ----------------[1]

For vertical motion at any time t

$y = usin\theta t - ½\, gt^2$ -------------[2]

Putting t into equation [2]

$$y = usin\theta\, \frac{x}{ucos\theta} - \frac{1}{2}g\left(\frac{x}{ucos\theta}\right)^2$$

$$\boxed{\boldsymbol{y} = \boldsymbol{x}tan\theta - \frac{gx^2}{2u^2\cos^2\theta}}$$

Either $\boxed{y = xtan\theta - \frac{g\,x^2\,Sec^2\theta}{2\,u^2}}$

Or $\boxed{y = xtan\theta - \frac{g\,x^2(\,1+tan^2\theta)}{2\,u^2}}$

Examples

4. If the horizontal range of a particle with a velocity u is R, show that the greatest height H is satisfied by the equation $16gH^2 - 8Hu^2 + gR^2 = 0$

Solution

$$H = \frac{u^2\,Sin^2\theta}{2g}$$

$$Sin^2\theta = \frac{2gH}{u^2} \ldots..(i)$$

$$R = \frac{u^2\,Sin\,2\theta}{g}$$

$$gR = u^2 2\,cosSin\,\theta$$

$$cos\theta = \frac{gR}{2u^2 Sin\,\theta}$$

$$cos^2\theta = \frac{(gR)^2}{4u^4 Sin^2\,\theta}$$

$$cos^2\theta = \frac{(gR)^2}{4u^4\left(\frac{2gH}{u^2}\right)}$$

$$cos^2\theta = \frac{gR^2}{8u^2H} \ldots\ldots(ii)$$

$$cos^2\theta + Sin^2\theta = 1$$

$$\frac{gR^2}{8u^2H} + \frac{2gH}{u^2} = 1$$

$$gR^2 + 16gH^2 = 8u^2H$$

$$16gH^2 - 8Hu^2 + gR^2 = 0$$

7. A ball is projected from point A and falls at a point which is in level with A and at a distance of 160m from A. The greatest height of the ball attained is 40m. Find the ; **Uneb 2015 No.13**
   a. Angle and velocity at which the ball is projected
   b. Time taken for the ball to attain its greatest height.

Solution

$$H = \frac{u^2\,Sin^2\theta}{2g}$$

$$40 = \frac{u^2\,Sin^2\theta}{2g}$$

$$Sin^2\theta = \frac{784}{u^2} \ldots..(i)$$

$$R = \frac{u^2\,Sin\,2\theta}{g}$$

$$9.8x160 = u^2 2\,cos\theta Sin\,\theta$$

$$cos\theta = \frac{784}{u^2 Sin\,\theta}$$

$$cos^2\theta = \frac{614656}{u^4 Sin^2\,\theta}$$

$$cos^2\theta = \frac{614656}{u^4\left(\frac{784}{u^2}\right)}$$

$$cos^2\theta = \frac{784}{u^2} \ldots\ldots(ii)$$

$$cos^2\theta + Sin^2\theta = 1$$

$$\frac{784}{u^2} + \frac{784}{u^2} = 1$$

$$u^2 = 1568$$

$$u = 39.6m/s$$

$$Sin^2\theta = \frac{784}{u^2}$$

$$Sin^2\theta = \frac{784}{(39.6)^2}$$

$$\theta = Sin^{-1}\left(\sqrt{\frac{784}{(39.6)^2}}\right) = 45°$$

$$t = \frac{u\,sin\theta}{g}$$

$$t = \frac{39.6x\,sin45}{9.8} = 2.85s$$

8. A boy throws a ball at an initial speed of 40m/s at an angle of elevation, θ. Show taking g to be $10ms^{-2}$, that the times of flight corresponding to a horizontal range of 80m are positive roots of equation $T^4 - 64T^2 + 256 = 0$ **Uneb 2006 No.16 b**

Solution

$$R = \frac{u^2\, 2sin\theta\, cos\theta}{g}$$

$$80 = \frac{40^2 x 2 sin\theta\, cos\theta}{10}$$

$$sin\theta\, cos\theta = 0.25$$

$$sin\theta = \frac{1}{4\, cos\theta} \ldots .(i)$$

$$x = ucos\theta t$$

$$80 = 40xcos\theta xT$$

$$cos\theta = \frac{2}{T} \ldots \ldots (ii)$$

but $sin^2\theta + cos^2\theta = 1$

$$\left(\frac{1}{4\, cos\theta}\right)^2 + \left(\frac{2}{T}\right)^2 = 1$$

$$\left(\frac{1}{4\, x\, ^2/_T}\right)^2 + \left(\frac{2}{T}\right)^2 = 1$$

$$\left(\frac{T}{8}\right)^2 + \left(\frac{2}{T}\right)^2 = 1$$

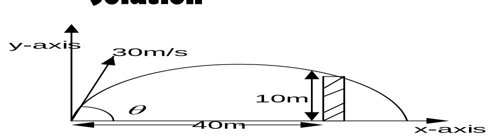
$$\frac{T^2}{64} + \frac{4}{T^2} = 1$$

$$\frac{T^4 + 256}{64T^2} = 1$$

$$T^4 - 64T^2 + 256 = 0$$

9. A particle is projected from a point O on the level ground, with initial speed 30m/s to pass through a point which is a horizontal distance **40m** from O and a distance **10m** vertically above the level O
   (i) show that there are two possible angles of projection
   (ii) If these angles are α and β, prove that $tan(\alpha + \beta) = -4$. take $g = 10ms^{-2}$

Solution



$$y = xtan\theta - \frac{g\, x^2(1 + tan^2\theta)}{2\, u^2}$$

$$10 = 40tan\theta - \frac{10x40^2(1 + tan^2\theta)}{2x(30)^2}$$

$$10 = 40tan\theta - \frac{80}{9}(1 + tan^2\theta)$$

$$8tan^2\theta - 36tan\theta + 17 = 0$$

since its a quadratic equation in $tan\theta$, its has two roots and two values of $\theta < 90$

$$tan\alpha + tan\beta = \frac{36}{8} \ldots \ldots \ldots ..(i)$$

$$tan\alpha x tan\beta = \frac{17}{8} \ldots \ldots \ldots (ii)$$

$$tan(\alpha + \beta) = \frac{tan\alpha + tan\beta}{1 - tan\alpha x tan\beta} \ldots \ldots (iii)$$

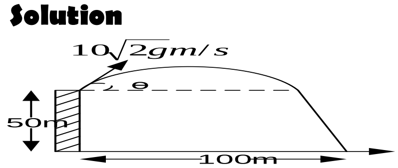
$$tan(\alpha + \beta) = \frac{\frac{36}{8}}{1 - \left(\frac{17}{8}\right)} = \frac{\frac{36}{8}}{\left(\frac{8 - 17}{8}\right)}$$

$$= \left(\frac{36}{8}\right) x \frac{8}{-9}$$

$$tan(\alpha + \beta) = -4$$

10. A particle is projected with a speed of $10\sqrt{2g}\ m/s$ from the top of a cliff 50m high. The particle hits the see at a distance 100m from the vertical through the point of projection. **Uneb 1998 No.10**
   (i) Show that there are two possible directions of projection which are perpendicular
   (ii) Determine the time taken from the point of projection in each case.

Solution



OR $y = xtan\theta - \frac{g\, x^2(1+tan^2\theta)}{2\, u^2}$

$$-50 = 100tan\theta - \frac{gx100^2(1 + tan^2\theta)}{2x100x2g}$$

$$-50 = 100tan\theta - 25(1 + tan^2\theta)$$

$$tan^2\theta - 4tan\theta - 1 = 0$$

$$tan\theta = \frac{4 \pm \sqrt{16 - 4x1x - 1}}{2}$$

$tan\theta_1 = 2 + \sqrt{5}$ and $tan\theta_2 = 2 - \sqrt{5}$

$$tan\theta_1 . tan\theta_1 = (2 + \sqrt{5})(2 - \sqrt{5}) = -1$$

Hence they perpendicular

For horizontal motion

$$x = ucos\theta xt$$

$$tan\theta_1 = 2 + \sqrt{5}$$

$$\theta_1 = 76.72°$$

$$t = \frac{100}{10\sqrt{2g}\ (cos76.72)} = 9.83s$$

$$tan\theta_2 = 2 - \sqrt{5}$$

$$\theta_1 = -13.28°$$

$$t = \frac{100}{10\sqrt{2g}\ (cos - 13.28)} = 2.32s$$

Exercise 20D

1. A girl throws a stone from a height of 1.5m above level ground with a speed of 10m/s and hits a bottle standing on a wall 4m high and 5m from her. (Taking $g = 10ms^{-2}$)
   (i) Show that if α is the angle of projection of the stone as it leaves her hand then $1.25tan^2\alpha - 5tan\alpha + 3.75 = 0$
   (ii) The horizontal component of the stones velocity has to be at least 6m/s for the bottle to be knocked off. By solving the above equation or otherwise, show that α has to be $45°$ for the bottle to be knocked off.
   (iii) If α is $45°$, find the direction in which the stone is moving when it hits the bottle
   (iv) If the bottle has a velocity of 3m/s after being struck, find where it hits the ground **An[horizontally,** $2.68m$ from wall]
2. A basket ball is released from a players hands with a speed of 8m/s at an inclination of $\beta°$ above the horizontal so as to land in the center of the basket, which is 4m horizontally from the point of release and a vertical height of 0.5m above it. (Taking $g = 10ms^{-2}$)
   (i) Show that β satisfies the quadratic equation $5tan^2\beta - 16tan\beta + 7 = 0$
   (ii) Given that the player throws the ball at a large angle of projection find β for the ball to land in the basket **An[**$70°, 1.43s$**]**
3. A basket ball is released from a players hands with a speed of vm/s at an inclination of $\beta°$ above the horizontal so as to land in the center of the basket, which is 4m horizontally from the point of release and a vertical height of 1m above it. (Taking $g = 9.8ms^{-2}$)
   (i) Show that β satisfies the equation $78.4(1 + tan^2\beta) + v^2(1 - tan\beta) = 0$
   (ii) If β is $45°$, find the speed of projection
   (iii) Find the two possible trajectories if $v = 8m/s$ **An[7.23m/s,** $35.4°, 68.6°$**]**

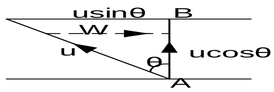
CHAPTER 11: VECTOR MECHANICS

CROSSING A RIVER

There are three cases to consider when crossing a river

a. Case I (shortest route)

If the water is not still and the boat man wishes to cross **directly opposite** to the starting point.
In order to cross point A to another point B directly opposite A (perpendicularly), then the course set by the boat must be upstream of the river.



u is the speed of the boat in still water,
w is the speed of the running water
At point B: $usin\theta = w$

$\theta = Sin^{-1}\left(\frac{W}{u}\right)$

θ is the direction to the vertical but the direction to the bank is (90-θ)°

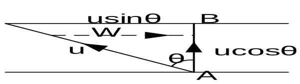
$\textbf{Time taken} = \frac{AB}{uCos\theta}$

Examples

1. A man who can swim at 6km/h in still water would like to swim between two directly opposite points on the banks of the river 300m wide flowing at 3km/hr. Find the time he would take to do this.

Solution

$AB = 300m \qquad AB = 0.3km$



$\theta = Sin^{-1}\left(\frac{W}{u}\right) = Sin^{-1}\left(\frac{3}{6}\right) = 30°$

Time taken $= \frac{AB}{uCos\theta} = \frac{0.3}{6Cos30}$

Time $= 0.058hrs = 3.46minute$

He must swim at 30° to AB in order to cross directly and it will take 3.46minutes

Alternatively

Using Pythagoras theorem:

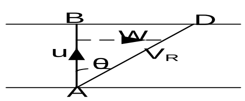
$6^2 = 3^2 + V_{AB}^2$

$V_{AB} = \sqrt{36-9} = 5.1962km/h$

Time taken $= \frac{AB}{V_{AB}} = \frac{0.3}{5.1962} = 0.058h$

Case II. The shortest time/as quickly as possible

If the boat man wishes to cross the river as quickly as possible, then he should steer his boat directly from A to B as shown. The river pushes him down stream.



Time to cross the river **t** $= \frac{AB}{u}$

Distance covered downstream is = wxt

Or distance downstream = $w\frac{AB}{u}$

tan$\theta = \frac{w}{u} \qquad \theta = tan^{-1}\frac{w}{u}$

The resultant velocity downstream V_R

$V_R^2 = W^2 + u^2$

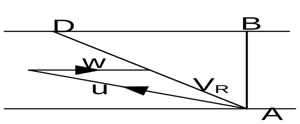
$V_R = \sqrt{w^2 + u^2}$

Examples

1. A man who can swim at 2m/s at in still water wishes to swim across a river 120m wide as quickly as possible. If the river flows at 0.5m/s, find the time the man takes to cross and how far down streams he travels.

Solution

C. Case III



Resultant velocity $\overrightarrow{V_R} = \overrightarrow{W} + \overrightarrow{U}$

RELATIVE MOTION

It comprises of;
1-Relative velocity 2-Relative path

(a) **Relative velocity**

This is the velocity a body would have as seen by an observer on another body. Suppose A and B are two moving bodies, the velocity of A relative to B is the velocity of A as it appears to an observer on B.

It's denoted by $V_{AB} = V_A - V_B$

Note that $V_{AB} \neq V_{BA}$ since $[V_{BA} = V_B - V_A]$

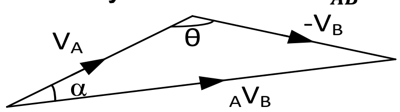
Numerical calculations

There are two methods used in calculations

- Geometric method
- Vector method

1. Geometrical method

If V_A and V_B are not given in vector form and the velocity of A relative to B is required, then we can reverse the velocity of B such that $V_{AB} = V_A + (-V_B)$ and the vector triangle is drawn as below



$${V_{AB}}^2 = {V_A}^2 + {V_B}^2 - 2\, V_A x\, V_B\, Cos\theta$$
and
$$\frac{V_{AB}}{Sin\theta} = \frac{V_B}{Sin\alpha}$$

2. Vector method

Find component of velocity for each object separately

Therefore $V_{AB} = V_A - V_B$

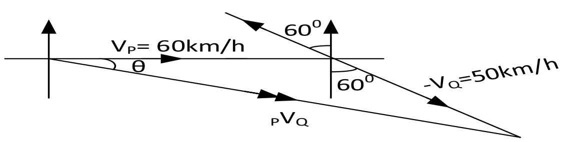
Examples

1. A girl walks at $5kmh^{-1}$ due west and a boy runs at $12kmh^{-1}$ at a bearing of $150°$. Find the velocity of the boy relative to the girl.

Method I (geometrical)

2. Ship P is steaming at 60km/h due east while ship Q is steaming in the direction N60°W at 50km/h. Find the velocity of P relative to Q.

Method I (geometrical)

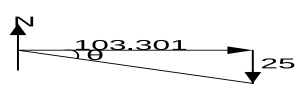


$$V_{PQ} = \sqrt{60^2 + 50^2 - 2x60x50cos150°} = 106.28kmh^{-1}$$
$$\frac{50}{Sin\theta} = \frac{106.28}{Sin150}$$
$$\theta = sin^{-1}\left(\left(\frac{50xsin150}{106.28}\right)\right) = 13.6°$$

The relative velocity is $106.28kmh^{-1}$ at $S76.4°E$

Method II (Vector)

$$V_{PQ} = V_P - V_Q$$
$$V_{PQ} = \begin{pmatrix}60\\0\end{pmatrix} - \begin{pmatrix}-50Sin60\\50Cos60\end{pmatrix} = \begin{pmatrix}103.301\\-25\end{pmatrix}$$
$$|V_{PQ}| = \sqrt{(103.301)^2 + (-25)^2} = 106.3kmh^{-1}$$



$$\theta = tan^{-1}\left(\frac{25}{103.301}\right) = 13.60°$$

Direction $S(90 - 13.6)°E$

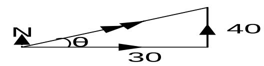
Relative velocity is $106.3\ kmh^{-1}$at $S76.4°E$

Finding true velocity

1. To a cyclist riding due north at $40\ kmh^{-1}$, a steady wind appears to blow from west at $30\ kmh^{-1}$. find the true velocity of the wind.

Solution

$$V_{WC} = V_w - V_C$$
$$\begin{pmatrix}30\\0\end{pmatrix} = V_w - \begin{pmatrix}0\\40\end{pmatrix}$$
$$V_w = \begin{pmatrix}30\\40\end{pmatrix}$$



$|V_w| = \sqrt{30^2 + 40^2} = 50kmh^{-1}$

$\theta = tan^{-1}\left(\frac{40}{30}\right)$ =**53.13°**

Direction $N(90 - 53.13)°E = N$**36.87°E**

2. To a man travelling due north at 10 kmh^{-1}, a steady wind appears to blow from East. When he travels in direction N60°W at 8 kmh^{-1}, it appears to come from south. Find the true velocity of the wind

Solution

Exercise 21B

1. A, B, and C are three aircrafts. A has velocity $(200i + 170j)$m/s. To the pilot of A it appears that B has a velocity $(50i - 270j)$m/s. To the pilot of B it appears that C has a velocity $(50i + 170j)$m/s. Find the velocities of B and C. **An** $(250i - 100j)$m/s, $(300i + 70j)$m/s,

2. To a bird fling due east at 10m/s, the wind seems to come from the south. When the bird alters its direction of flight to N$30°E$ with out altering its speed, the wind seems to come from the north-west. Find the true velocity of the wind. **An**(10.6m/s, from $S69.9°W$)

(b) RELATIVE POSITION

Consider two bodies A and B moving with V_A and V_B from points with position vectors 0A and OB respectively.

Position of A at time t is $R_{A(t=t)} = R_{A(t=0)} + t\ xV_A$

Position of B at time t is $R_{B(t=t)} = R_{B(t=0)} + t\ x\ V_B$

Position of A relative to B at time t is $R_{AB(t=t)} = R_{A(t=t)} - R_{B(t=t)}$

$R_{AB(t=t)} = (R_{A(t=0)} + t\ xV_A) - (R_{B(t=0)} + t\ x\ V_B)$

$R_{AB(t=t)} = (R_{A(t=0)} - R_{B(t=0)}) + (V_{AB})t$

Examples

1. The velocities of two ships P and Q are $(i + 6j)km/h$and $(-i + 3j)km/h$. At a certain instant, the displacement between the two ships is $(7i + 4j)km$. Find the: **Uneb 2002 No.15a**
   (i) Relative velocity of ship P to Q
   (ii) Magnitude of displacement between ships P and Q after 2 hours

Solution

DISTANCE AND TIME OF CLOSEST APPROACH
(SHORTEST DISTANCE AND TIME TO SHORTEST DISTANCE)

When two particles are moving simultaneously with specific velocities, time will come when they are closest to each other **without** colliding

There three methods used for time the distance and time of closest approach

- Geometrical
- Vector
- Differential

1. Vector

Consider particles A and B moving with velocities V_A and V_B from point with positions vectors OA and OB respectively.

For minimum distance to be attained then $V_{AB} . R_{AB(t=t)} = 0$ This gives the time

Then **shortest distance** **d**= $\left|R_{AB(t=t)}\right|$

2. Differential

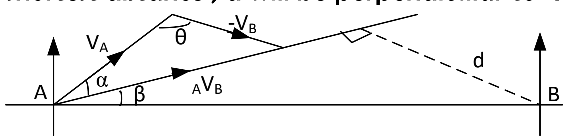
The minimum distance is reached when $\frac{d}{dt}\left|R_{AB(t=t)}\right|^2 = 0$ This gives the time

Minimum distance **d** $= \left|R_{AB(t=t)}\right|$

3. Geometrical

If V_A and V_B are not given in vector form, then we can reverse the velocity of B such that $V_{AB} = V_A + (-V_B)$ and the vector triangle is drawn as below.

The shortest distance , d will be perpendicular to V_{AB}



$V_{AB}{}^2 = V_A{}^2 + V_B{}^2 - 2\,V_A x\,V_B\,Cos\theta$ and $\frac{V_{AB}}{Sin\theta} = \frac{V_B}{Sin\alpha}$

Shortest distance

$$d = ABsin\beta$$

Time to shortest distance

$$t = \frac{A\,B\,Cos\beta}{V_{AB}}$$

Examples

1. A particle P starts from rest from a point with position vector $2j + 2k$ with a velocity $(j + k)$m/s. A second particle Q starts at the same time from a point whose position vector is $-11i - 2j - 7k$ with a velocity of $(2i + j + 2k)$m/s . Find;
   i) The shortest distance between the particles
   ii) The time when the particles are closest together
   iii) How far each has travelled by this time

Method I vector

(ii) $V_{PQ} = V_P - V_Q = \begin{pmatrix}0\\1\\1\end{pmatrix} - \begin{pmatrix}2\\1\\2\end{pmatrix} = \begin{pmatrix}-2\\0\\-1\end{pmatrix}$

$R_{PQ(t=t)} = (R_{P(t=0)} - R_{Q(t=0)}) + (V_{PQ})t$

$R_{PQ(t=t)} = \left[\begin{pmatrix}0\\2\\2\end{pmatrix} - \begin{pmatrix}-11\\-2\\-7\end{pmatrix}\right] + \begin{pmatrix}-2\\0\\-1\end{pmatrix}t$

$R_{PQ(t=t)} = \begin{pmatrix}11\\4\\9\end{pmatrix} + \begin{pmatrix}-2\\0\\-1\end{pmatrix}t$

For minimum distance: $V_{PQ}\,.R_{PQ(t=t)} = 0$

$\begin{pmatrix}-2\\0\\-1\end{pmatrix} . \begin{pmatrix}11-2t\\4\\9-t\end{pmatrix} = 0$

$-22 + 4t + 0 - 9 + t = 0$

$t = \frac{31}{5}$ $\therefore\ t = 6.2s$

i) Shortest distance $d = \left|R_{PQ(t=t)}\right|$

$R_{PQ(t=6.2)}\begin{pmatrix}11\\4\\9\end{pmatrix} + \begin{pmatrix}-2\\0\\-1\end{pmatrix}x6.2 = \begin{pmatrix}-1.4\\4\\2.8\end{pmatrix}$

$\left|R_{PQ(t=t)}\right| = \sqrt{(-1.4)^2 + 4^2 + 2.8^2} = 5.08m$

ii) How far each has travelled

$R_{P(t=t)} = R_{P(t=0)} + (V_P)t$

$R_{P(t=6.2)} = \begin{pmatrix}0\\2\\2\end{pmatrix} + \begin{pmatrix}0\\1\\1\end{pmatrix}6.2 = \begin{pmatrix}0\\8.2\\8.2\end{pmatrix}$

$\left|R_{P(t=6.2)}\right| = \sqrt{0^2 + 8.2^2 + 8.2^2} = 11.6m$

$R_{Q(t=t)} = R_{Q(t=0)} + (V_Q)t$

$R_{Q(t=t)} = \begin{pmatrix}-11\\-2\\-7\end{pmatrix} + \begin{pmatrix}2\\0\\1\end{pmatrix}6.2 = \begin{pmatrix}1.4\\4.2\\5.4\end{pmatrix}$

$\left|R_{Q(t=6.2)}\right| = \sqrt{1.4^2 + 4.2^2 + 5.2^2} = 6.8m$

Method II (differential)

$\frac{d}{d}\left|R_{PQ(t=t)}\right|^2 = 0$

$R_{PQ(t=t)} = \begin{pmatrix}11\\4\\9\end{pmatrix} + \begin{pmatrix}-2\\0\\-1\end{pmatrix}t = \begin{pmatrix}11-2t\\4\\9-t\end{pmatrix}$

$\left|R_{PQ(t=t)}\right|^2 = (11-2t)^2 + (4)^2 + (9-t)^2$

$\left|R_{PQ(t=t)}\right|^2 = 121 - 44t + 4t^2 + 16 + 81 - 18t + t^2$

$\left|R_{PQ(t=t)}\right|^2 = 218 - 62t + 5t^2$

$\frac{d}{dt}\left|R_{PQ(t=t)}\right|^2 = -62 + 10t = 0$

$t = 6.2s$

$R_{PQ(t=6.2)} = \begin{pmatrix}11-2x6.2\\4\\9-6.2\end{pmatrix} = \begin{pmatrix}-1.4\\4\\2.8\end{pmatrix}$

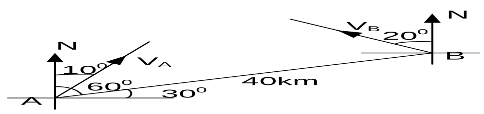
$\left|R_{PQ(t=6.2)}\right| = \sqrt{(-1.4)^2 + (4)^2 + (2.8)^2}$ = **5.08m**

2. At noon a boat A is 30km from boat B and its direction from B is 286°. Boat A is moving in the North east direction at 16km/h and boat B is moving in the northern direction at 10km/h. Determine when they are closest to each other. What is the distance between them. **Uneb 1997 No.16**

Solution

3. Two planes A and B are both flying above the pacific ocean. Plane A is flying on a course of 010° at a speed of 300km/h and plane B is flying on a course of 340° at 200km/h. At a certain instant, plane B is 40 km from plane A. Plane A is then on a bearing of 060°. After what time will they come closest together and what will be their minimum distance apart? **Uneb 2004 No.16**

Solution



$$V_{AB} = V_A - V_B$$

$$V_{AB} = \begin{pmatrix} 300sin10 \\ 300cos10 \end{pmatrix} - \begin{pmatrix} -200sin20 \\ 200cos20 \end{pmatrix} = \begin{pmatrix} 120.4985 \\ 107.5038 \end{pmatrix}$$

$$R_{A(t=0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} km, \ R_{B(t=0)} = \begin{pmatrix} 40cos30 \\ 40sin30 \end{pmatrix} km$$

$$R_{AB(t=t)} = (R_{A(t=0)} - R_{B(t=0)}) + (V_{AB})t$$

$$R_{AB(t=t)} = \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 40cos30 \\ 40sin30 \end{pmatrix}\right] + t\begin{pmatrix} 120.4985 \\ 107.5038 \end{pmatrix}$$

$$R_{AB(t=t)} = \begin{pmatrix} -34.641 + 120.4985t \\ -20 + 107.5038t \end{pmatrix}$$

For least distance: $V_{AB} .R_{AB(t=t)} = 0$

$$\begin{pmatrix} 120.4985 \\ 107.5038 \end{pmatrix} . \begin{pmatrix} -34.641 + 120.4985t \\ -20 + 107.5038t \end{pmatrix} 0$$

$$-6324.2645 + 26076.9555t = 0$$

$$t = 0.2425hrs$$

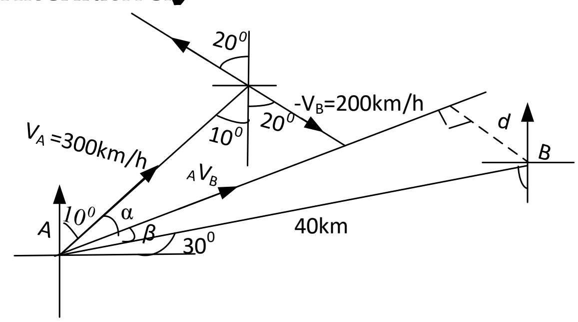
least distance d = $|R_{AB(t=0.2425)}|$

$$R_{AB(t=0.2425)} = \begin{pmatrix} -34.641 + 120.4985x0.2425 \\ -20 + 107.5038x0.2425 \end{pmatrix}$$

$$= \begin{pmatrix} 5.4202 \\ 6.0692 \end{pmatrix}$$

$$|R_{AB(t=0.2425)}| = \sqrt{(5.4202)^2 + 6.0692^2} = 8.14\text{km}$$

Alternatively



$$V_{AB}{}^2 = V_A{}^2 + V_B{}^2 - 2 V_A x V_B cos30°$$

$$V_{AB} = \sqrt{300^2 + 200^2 - 2x300x200cos30°}$$

$$V_{AB} = 161.484kmh^{-1}$$

$$\frac{200}{Sin\alpha} = \frac{161.484}{Sin30}$$

$$\alpha = sin^{-1}\left(\left(\frac{200xsin30}{161.484}\right)\right) = 38.26°$$

$$\alpha + \beta = 50°$$

$$\beta = 11.74°$$

$$d = ABsin\beta$$

$$d = 40sin\ 11.74° = 8.14km$$

Time $t = \frac{A\ B\ Cos\theta}{V_{AB}} = \frac{40\ Cos11.74°}{161.484} = 0.2425hrs$

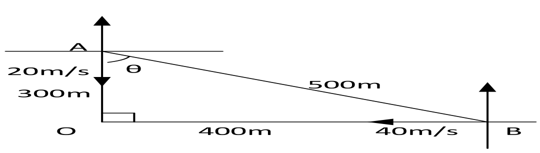
4. At a given instant two cars are at distances 300m are at distances 300m and 400m from the point of intersection O of two roads crossing at right angles and are approaching O at uniform speeds of 20m/s and 40m/s respectively. Find;

(i) Initial distance between the two cars

(ii) Time taken to reach this point

(iii) Shortest distance the between the cars

Solution



$$AB = \sqrt{300^2 + 400^2} = 500m$$

$$\theta = tan^{-1}(^{400}/_{300}) = 53.1°$$

$$R_{A(t=0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} km, \ R_{B(t=0)} = \begin{pmatrix} 50cos53.1 \\ -500sin53.1 \end{pmatrix} km$$

$$V_{AB} = V_A - V_B = \begin{pmatrix} 0 \\ -20 \end{pmatrix} - \begin{pmatrix} -40 \\ 0 \end{pmatrix} = \begin{pmatrix} 40 \\ -20 \end{pmatrix} ms^{-1}$$

$$R_{AB(t=t)} = (R_{A(t=0)} - R_{B(t=0)}) + (V_{AB})t$$

$$R_{AB(t=t)} = \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 500sin53.1 \\ -500cos53.1 \end{pmatrix}\right] + \begin{pmatrix} 40 \\ -20 \end{pmatrix} t$$

$$R_{AB(t=t)} = \begin{pmatrix} -399.842 + 40t \\ 300.21 - 20t \end{pmatrix}$$

For least distance: $V_{AB} .R_{AB(t=t)} = 0$

$$\begin{pmatrix} 40 \\ -20 \end{pmatrix} . \begin{pmatrix} -399.842 + 40t \\ 300.21 - 20t \end{pmatrix} 0$$

$$-21,997.88 + 2000t = 0$$

$$t = 11s$$

least distance d $= |R_{AB(t=11)}|$

$$R_{AB(t=11)} = \begin{pmatrix} -399.842 + 40x11 \\ 300.21 - 20x11 \end{pmatrix} = \begin{pmatrix} 40.158 \\ 80.21 \end{pmatrix} \text{m}$$

$\left|R_{AB(t=11)}\right| = \sqrt{40.158^2 + 80.21^2} = 89.7012\text{m}$

5. A road running north-south crosses a road running east-west at a junction O. Initially Paul is on the east-west road, 1.7km west of O and is cycling towards O at 15km/h. At the same time John is at O cycling due north at 8km/h. If Paul and John do not alter their velocities, Find the;.

- **(i)** Relative velocity of Paul to John
- (ii) Shortest distance the between Paul and John

Solution

Exercise 21C

1. A particle P moves with a constant velocity $(2i + 3j + 8k)$ passes a point with position vector $6i - 11j + 4k$. At the same instant particle Q passes through a point whose position vector is $i - 2j + 5k$ moving with a constant velocity of $(3i + 4j - 7k)$. Find; **Uneb 2005 No.14**
   - i) Position and velocity of Q relative to P at that instant
   - ii) The shortest distance between the particles
   - iii) Time that elapses before the particles are nearest each other

   An($= 0.0485\ units$ **d=10.32uints,)**

2. Car A is 80m North west of point O. Car B is 50m $N30°E$ of O. Car A is moving at 20m/s on a straight road towards O. Car B is also moving at 10m/s on another straight road towards O. Determine the:
   - (i) Initial distance between the two cars
   - (ii) Velocity of A relative to B
   - (iii) Shortest distance between the two cars as they approach O **UNEB 2019 No. 9**

   An(82.6404m, 19.9116m/s at E35.98$°S$**, 9.7022m)**

3. Two airstrips P and Q are 100km apart, P being west of Q. Two Helicopters A and B fly simultaneously from P and Q respectively, at 11:00a.m. Helicopter A is flying with a constant speed of 400km/h in the direction N50°E. Helicopter B flying at a constant speed of 500km/h in direction N70°W. Find the;
   - (i) Time when the helicopters are closest together
   - (ii) Closest distance between the helicopters

   UNEB 2020 No.16

   An(11.08am, 11.0247km)

COURSE OF CLOSEST APPROACH

If A is to pass as close as possible to B, then velocity of A must be perpendicular to the relative velocity

$$V_{AB}.V_A = 0$$

Examples

1. Two particles P and Q initially at positions $(3i + 2j)m$ and $(13i + 2j)m$ respectively begin moving. Particle P moves with a constant velocity $(2i + 6j)$m/s. A second particle Q moves with a constant velocity of $(5j)$m/s . **UNEB 2003 No.16**
   - (a) Find;
     - i) The time when the particles are closest together
     - ii) Bearing of particle P from Q when they are closest to each other
   - (b) Given that half the time, the two particles are moving closest to each other, particle P reduces its speed to half its original speed, in the direction to approach particle Q and the velocity of Q remains unchanged, find the direction of particle P

Solution:

(a) $R_{P(t=0)} = \begin{pmatrix}3\\2\end{pmatrix}m,\ R_{Q(t=0)} = \begin{pmatrix}13\\2\end{pmatrix}m$

$$V_{PQ} = V_P - V_Q = \begin{pmatrix}2\\6\end{pmatrix} - \begin{pmatrix}0\\5\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}ms^{-1}$$

$$R_{PQ(t=t)} = (R_{P(t=0)} - R_{Q(t=0)}) + (V_{PQ})t$$

$$R_{PQ(t=t)} = \left[\begin{pmatrix}3\\2\end{pmatrix} - \begin{pmatrix}13\\2\end{pmatrix}\right] + \begin{pmatrix}2\\1\end{pmatrix} t = \begin{pmatrix}-10+2t\\t\end{pmatrix}m$$

For minimum distance: $V_{PQ}.R_{PQ(t=t)} = 0$

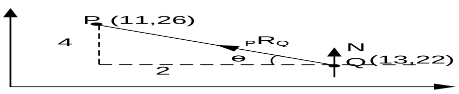
$$\begin{pmatrix}2\\1\end{pmatrix} \cdot \begin{pmatrix}-10+2t\\t\end{pmatrix} = 0$$

$$-20 + 4t + t = 0$$

$t = 4\,s$

$R_{P(t=2)} = R_{P(t=0)} + V_P xt = \begin{pmatrix}3\\2\end{pmatrix} + \begin{pmatrix}2\\6\end{pmatrix} 4 = \begin{pmatrix}11\\26\end{pmatrix}$

$R_{Q(t=2)} = R_{Q(t=0)} + V_Q xt = \begin{pmatrix}13\\2\end{pmatrix} + \begin{pmatrix}0\\5\end{pmatrix} x\, 4 = \begin{pmatrix}13\\22\end{pmatrix}$



$tan\theta = \frac{4}{2}$ $\quad \theta = 63.4°$

$N26.6°W$

(c) At $t = 2s$ $V_P = \begin{pmatrix}1\\3\end{pmatrix} ms^{-1}$

$/V_P/ = \sqrt{1^2 + 3^2} = \sqrt{10}m/s$

Let P move at angle θ to x-axis

$V_{PQ} = \sqrt{10}\begin{pmatrix}cos\theta\\sin\theta\end{pmatrix} - \begin{pmatrix}0\\5\end{pmatrix} = \begin{pmatrix}\sqrt{10}cos\theta\\\sqrt{10}sin\theta - 5\end{pmatrix}$

If P is to approach Q: $V_{PQ}\,.V_P = 0$

$$\begin{pmatrix}\sqrt{10}cos\theta\\\sqrt{10}sin\theta - 5\end{pmatrix}.\begin{pmatrix}\sqrt{10}cos\theta\\\sqrt{10}sin\theta\end{pmatrix} = 0$$

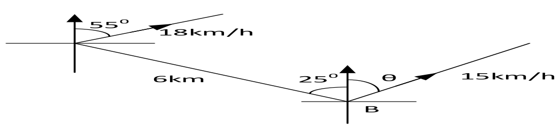
$$10cos^2\theta + 10sin^2\theta - 5\sqrt{10}sin\theta = 0$$

$$sin\theta = \frac{10}{5\sqrt{10}} \quad \theta = 39.2°$$

$N50.8°E$

2. A ship A is moving with a constant speed of 18km/h in a direction $N55°E$ and is initially 6km from a second ship B, the bearing of A from B being $N25°W$. If B moves with a constant speed of 15km/h, find;
   - **(iii)** Course that B should set in order to get as close as possible to A
   - **(iv)** Closest distance and time taken for the situation to occur

Solution



(i) $V_{BA} = \begin{pmatrix}15\,\sin\theta\\15\,cos\theta\end{pmatrix} - \begin{pmatrix}18sin55\\18cos55\end{pmatrix}$

If B is to approach A, then $V_{BA}.V_B = 0$

(ii) $\left[\begin{pmatrix}15\,\sin\theta\\15\,cos\theta\end{pmatrix} - \begin{pmatrix}18sin55\\18cos55\end{pmatrix}\right].\begin{pmatrix}15\,\sin\theta\\15\,cos\theta\end{pmatrix} = 0$

$$225cos^2\theta + 225sin^2\theta - 154.866cos\theta - 221.171\ sin\theta = 0$$

$$221.171sin\theta + 154.866\,cos\theta = 225$$

But $sin\theta = \frac{2T}{1+T^2}$ and $cos\theta = \frac{1-T^2}{1+T^2}$

$$221.171\left(\frac{2T}{1+T^2}\right) + 154.866\left(\frac{1-T^2}{1+T^2}\right) = 225$$

$$379.866T^2 - 442.34T + 70.134 = 0$$

$$T = \frac{442.34 \pm \sqrt{(-442.34)^2 - 4x379.866x70.134}}{2x379.866}$$

$T = 0.189$ or $T = 0.975$

$$sin\theta = \frac{2T}{1+T^2}$$

$T = 0.189$: $\theta = sin^{-1}\left(\frac{2x0.189}{1+0.189^2}\right) = 21.4°$

$T = 0.975$: $\theta = sin^{-1}\left(\frac{2x0.975}{1+0.975^2}\right) = 88.5°$

Bearing $N21.4°E$

$R_{A(t=0)} = \begin{pmatrix}-6\sin25\\6\cos25\end{pmatrix}$ $\quad R_{B(t=0)} = \begin{pmatrix}0\\0\end{pmatrix}$

$R_{BA(t=t)} = (R_{B(t=0)} - R_{A(t=0)}) + (V_{BA})t$

$= \left[\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}-6\sin25\\6\cos25\end{pmatrix}\right] + t\left[\begin{pmatrix}15\,\sin21.4\\15\,cos21.4\end{pmatrix} - \begin{pmatrix}18sin55\\18cos55\end{pmatrix}\right]$

$R_{BA(t=t)} = \begin{pmatrix}2.536 - 9.272t\\-5.438 + 3.641t\end{pmatrix}$

For minimum distance: $V_{BA}\,.R_{BA(t=t)} = 0$

$\begin{pmatrix}-9.272\\3.641\end{pmatrix}.\begin{pmatrix}2.536 - 9.272t\\-5.438 + 3.641t\end{pmatrix} = 0$

$99.227t - 43.314 = 0$

$t = 0.437\text{h}$

least distance d = $|R_{BA(t=0.437)}|$

$R_{BA(t=0.437)} = \begin{pmatrix}2.536 - 9.272x0.437\\-5.438 + 3.641x0.437\end{pmatrix} = \begin{pmatrix}-1.516\\-3.847\end{pmatrix}$

$|R_{BA(t=0.437)}| = \sqrt{(-1.516)^2 + (-3.847)^2} = 4.135km$

3. Two air craft A and B are flying at the same altitude with A initially 10km due north of B and flying at a constant speed of 300m/s on a bearing of $060°$, If B flies at a constant speed of 200m/s. find ;
   - **(i)** Course that B should set in order to get as close as possible to A
   - **(ii)** Closest distance and time taken for the situation to occur

Solution

Exercise 21D

INTERCEPTION AND COLLISION

Consider two bodies A and B moving with V_A and V_B from points with position vectors OA and OB respectively.

Position of A after time t is

$R_{A(t=t)} = R_{A(t=0)} + t\ xV_A$

Position of B after time t is

$R_{B(t=t)} = R_{B(t=0)} + t\ x\ V_B$

For collision to occur $R_{A(t=t)} = R_{B(t=t)}$

$R_{A(t=0)} + t\ xV_A = R_{B(t=0)} + t\ x\ V_B$

$(R_{A(t=0)} - R_{B(t=0)}) + t(V_{AB}) = 0$

Hence $R_{AB(t=t)} = 0$

SHOWING THAT PARTICLES COLLIDE

Equate the corresponding unit vector form both side to show that y is constant in both directions
For vectors in three dimensions at least any two unit vectors be constant

Examples

1. The position vectors $r_A = (5\tilde{i} - 3\tilde{j} + 4\tilde{k})m$ and $r_B = (7\tilde{i} + 5\tilde{j} - 2\tilde{k})m$ are for two particles with velocities $v_A = (2\tilde{i} + 5\tilde{j} + 3\tilde{k})m/s$ and $v_B = (-3\tilde{i} - 15\tilde{j} + 18\tilde{k})m/s$ respectively. Show that if the velocities remain constant, a collision will occur

Solution:

$R_{A(t=0)} + t\ xV_A = R_{B(t=0)} + t\ x\ V_B$

$$\begin{pmatrix}5\\-3\\4\end{pmatrix} + \begin{pmatrix}2\\5\\3\end{pmatrix}t = \begin{pmatrix}7\\5\\-2\end{pmatrix} + \begin{pmatrix}-3\\-15\\18\end{pmatrix}t$$

$$\begin{pmatrix}-2\\-8\\6\end{pmatrix} = \begin{pmatrix}-5\\-20\\15\end{pmatrix}t$$

Along the i direction; $-2 = -5t$

$t = 0.4s$

Along the j direction; $-8 = -20t$

$t = 0.4h$

Along the k direction; $6 = 15t$

$t = 0.4h$

Since t is constant in all directions collision occurred

2. At 11:30am a battle ship is at a place with position vectors $(-6\tilde{i} + 12\tilde{j})km$ and is moving with velocity vector$(16\tilde{i} - 4\tilde{j})km/h$. At 12:00 noon a cruiser is at a place with position vectors $(12\tilde{i} - 15\tilde{j})km$ and is moving with velocity vector $(8\tilde{i} + 16\tilde{j})km/h$ Assuming velocities do not change,
   (i) Show that collision will occur
   (ii) Find the time at which collision occur
   (iii) Find the position vector of the location during collision

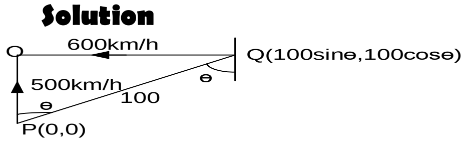
   Solution:

3. At 11pm two ships A and B are 10km apart with B due north of A. A is travelling north east at a speed of 18km/h and ship B is travelling due east at 9$\sqrt{2}\ km/h$. Show that, if the two ships do not change their velocities, they collide and find to the nearest minute when the collision occurs

Solution

4. Two aircrafts P and Q are flying at the same height. P is flying due north at 500km/h while Q is flying due west at 600km/h. When the aircrafts are 100km apart, the pilots realize that they are about to collide. The pilot of P then changes course to 345° and maintains the speed of 500km/h. The pilot of Q maintains his course but increases the speed. Determine the. **Uneb 2010 No.15**
   (i) Distance each aircraft would have travelled if the pilots had not realized that they were about to collide
   (ii) New speed beyond which the aircraft Q must fly in order to avoid collision

   Solution



$$\theta = \tan^{-1}\left(\frac{600}{500}\right) = 50.2°$$

For collision

$OP + t\ x\ V_P = OQ + t\ x\ V_Q$

$\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}0\\500\end{pmatrix}t = \begin{pmatrix}100sin\theta\\100cos\ \theta\end{pmatrix} + \begin{pmatrix}-600\\0\end{pmatrix}t$

$\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}0\\500\end{pmatrix}t = \begin{pmatrix}100xsin50.2\\100xcos50.2\end{pmatrix} + \begin{pmatrix}-600\\0\end{pmatrix}t$

Along the i direction; $0 = 100xsin50.2 - 600t$

$t = 0.128\text{h}$

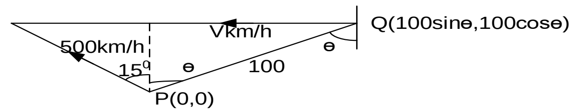
Distance moved by P

$d_p = \begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}0\\500\end{pmatrix}x0.128 = \begin{pmatrix}0\\64\end{pmatrix} = 64km$

Distance moved by Q

$d_Q = \begin{pmatrix}100xsin50.2\\100xcos50.2\end{pmatrix} + \begin{pmatrix}-600\\0\end{pmatrix}0.128 = \begin{pmatrix}0.0284\\64.011\end{pmatrix}$

$= 64.011km$



For collision: $OP + \mathbf{t}\ x\ V_P = OQ + \mathbf{t}\ x\ V_Q$

$\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}-500sin15\\500cos15\end{pmatrix}t = \begin{pmatrix}100sin\theta\\100cos\ \theta\end{pmatrix} + \begin{pmatrix}-v\\0\end{pmatrix}t$

$\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}-500sin15\\500cos15\end{pmatrix}t = \begin{pmatrix}100xsin50.2\\100xcos50.2\end{pmatrix} + \begin{pmatrix}-v\\0\end{pmatrix}t$

Along the i direction; $-500sin15\ t = 76.8284 - vt..(i)$

Along the j direction; $500cos15\ t = 64.011 \ldots\ldots(ii)$

$t = 0.1325h$

$vt = 76.8284 + 500sin15\ t$

$$v = \frac{76.8284 + 500x0.1325sin15}{0.1325} = 709.2837km/h$$

COURSE OF INTERCEPTION

Suppose particle A moving with a speed V_A is to intercept particle B moving with a speed V_B, then;

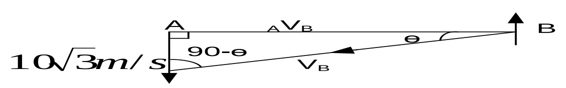
- Draw a sketch a diagram showing the initial position and velocities of the two particles
- For interception to occur, the relative velocity must be in the direction if the initial displacements of the particles

$$t = \frac{AB}{AV_B}$$

Examples

1. At any instant a body A travelling south at $10\sqrt{3}m/s$ is 150m west of B. Show that B will intercept A if B is travelling S30°W at 20m/s and find the time that elapses before the collision occurs

Solution



A $= \begin{pmatrix}0\\0\end{pmatrix}m$ B$= \begin{pmatrix}150\\0\end{pmatrix}m$

$V_A = \begin{pmatrix}0\\-10\sqrt{3}\end{pmatrix}ms^{-1}$, $V_B = \begin{pmatrix}-20cos\theta\\-20sin\theta\end{pmatrix}ms^{-1}$

$OA + \mathbf{t}\ x\ V_A = OB + \mathbf{t}\ x\ V_B$

$\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}0\\-10\sqrt{3}\end{pmatrix}t = \begin{pmatrix}150\\0\end{pmatrix} + \begin{pmatrix}-20cos\theta\\-20sin\theta\end{pmatrix}t$

$\hat{\imath}$: $0 = 150 - 20tcos\theta$

$t = {7.5}/{cos\theta} \ldots.(i)$

$\hat{\jmath}$: $10\sqrt{3}t = 20tsin\theta$

$sin\theta = {\sqrt{3}}/{2}$

$\theta = 60°$

Bearing $S30°W$

$t = {7.5}/{cos\theta}$

$t = {7.5}/{cos60°} = 15s$

Alternatively

$$\frac{sin90}{20} = \frac{sin\theta}{10\sqrt{3}}$$

$\theta = 60°$

Bearing $S30°W$

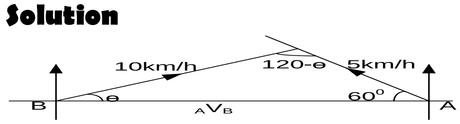
Also $\frac{sin90}{20} = \frac{sin(90-\theta)}{AV_B}$

$AV_B = 20xsin(90 - 60) = 10m/s$

$$t = \frac{AB}{AV_B} = \frac{150}{10} = 15s$$

2. At 9:00 am two ships A and B are 15km apart with B on a bearing of 270° from A. Ship A moves at 5km/h on a bearing of 330°. If the maximum speed of B is 10km/h, Find the;
   (i) Direction B should set in order to intercept A as soon as possible
   (ii) Time taken for the interception to occur

   Solution



B $= \begin{pmatrix}0\\0\end{pmatrix}km$ A$= \begin{pmatrix}15\\0\end{pmatrix}km$

$V_B = \begin{pmatrix}10cos\theta\\10sin\theta\end{pmatrix}kmh^{-1}$, $V_A = \begin{pmatrix}-5cos60°\\5sin60°\end{pmatrix}kmh^{-1}$

$$OA + t \times V_A = OB + t \times V_B$$

$$\begin{pmatrix}15\\0\end{pmatrix} + \begin{pmatrix}-5cos60°\\5sin60°\end{pmatrix}t = \begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}10cos\theta\\10sin\theta\end{pmatrix}t$$

$\hat{\imath}$: $15 - 5tcos60° = 10tcos\theta$

$$t = {}^{15}/_{10cos\theta + 2.5} \quad ....(i)$$

$\hat{\jmath}$: $5tsin60° = 10tsin\theta$

$$\theta = sin^{-1}\left(\frac{5sin60}{10}\right) = 25.7°$$

Bearing $E25.7°N$

$$time, t = {}^{15}/_{10cos\theta + 2.5}$$

$$t = {}^{15}/_{10cos\theta 25.7 + 2.5} = 1.3h$$

$$= 1.3h = 1.3x60 = 78mins$$

Alternatively

$$\frac{sin60}{10} = \frac{sin(\theta)}{5}$$

$$= 25.7°$$

Bearing $N\ (90 - 25.7)E = N64.3°E$

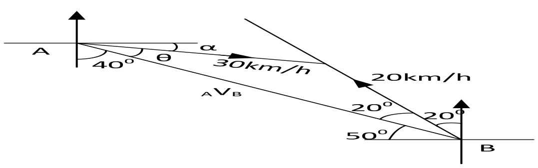
Also; $\frac{sin60}{10} = \frac{sin(120-\theta)}{AV_B}$

$$AV_B = 11.515km/h$$

$$t = \frac{AB}{AV_B} = \frac{15}{11.515} = 1.303h = 78minutes$$

3. At 12:00 noon two ships A and B are 12km apart with B on a bearing of 140° from A. Ship A moves at 30km/h to intercept B which is travelling at 20km/h on a bearing of 340°. Find the;
   (i) Direction A should set order to intercept B
   (ii) Time taken for the interception to occur

Solution



A $= \begin{pmatrix}0\\0\end{pmatrix}km$ B$= \begin{pmatrix}12cos50°\\-12sin50°\end{pmatrix}km$

$$V_A = \begin{pmatrix}30cos\alpha\\-30sin\alpha\end{pmatrix}kmh^{-1},\ V_B = \begin{pmatrix}-20sin20°\\20cos20°\end{pmatrix}kmh^{-1}$$

$$OA + t \times V_A = OB + t \times V_B$$

$$\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}30cos\alpha\\-30sin\alpha\end{pmatrix}t = \begin{pmatrix}12cos50°\\-12sin50°\end{pmatrix} + \begin{pmatrix}-20sin20°\\20cos20°\end{pmatrix}t$$

$\hat{\imath}$: $30tcos\alpha = 12cos50° - 20tsin20°$

$$t = {}^{7.713}/_{30cos\alpha + 6.84} \quad ....(i)$$

$\hat{\jmath}$: $-30tsin\alpha = -12sin50° + 20tcos20°$

$$t = {}^{9.193}/_{18.794 + 30sin\alpha} \quad ....(ii)$$

$$(i) = (ii): \frac{7.713}{30cos\alpha + 6.84} = \frac{9.193}{18.794 + 30sin\alpha}$$

$$144.958 + 231.39sin\alpha = 62.88 + 275.79cos\alpha$$

$$231.39sin\alpha - 275.79cos\alpha = -82.078$$

But $sin\alpha = \frac{2T}{1+T^2}$ and $cos\alpha = \frac{1-T^2}{1+T^2}$

$$231.39\left(\frac{2T}{1+T^2}\right) - 275.79\left(\frac{1-T^2}{1+T^2}\right) = -82.078$$

$$357.868T^2 + 462.78T - 193.712 = 0$$

$$T = \frac{-462.78 \pm \sqrt{462.78^2 + 4x357.868x193.712}}{2x357.868}$$

$T = -1.626$ or $T = 0.333$

$$\therefore T = 0.333$$

$$sin\alpha = \frac{2T}{1+T^2}$$

$$T = 0.333: \alpha = sin^{-1}\left(\frac{2x0.333}{1+0.333^2}\right) = 36.8°$$

Bearing $E36.8°S$

$$time, t = {}^{7.713}/_{30cos\alpha + 6.84}$$

$$t = {}^{7.713}/_{30cos36.8 + 6.84} = 0.25h$$

$$= 0.25h = 0.25x60 = 15mins$$

Alternatively

$$\frac{sin20}{30} = \frac{sin(\theta)}{20}$$

$$\theta = 13.2°$$

Bearing $E\ (50 - 13.2)S$

$E36.8°S$ or $S53.2°E$

Also; $\frac{sin20}{30} = \frac{sin(180-[20+13.2])}{AV_B}$

$$AV_B = 48.03km/h$$

$$t = \frac{AB}{AV_B} = \frac{12}{48.03} = 0.250h = 15minutes$$

Exercise 21E

1. At 11:30am a jumbo jet has a position vectors $(-100\tilde{\imath} + 220\tilde{\jmath})km$ and is moving with velocity vector$(300\tilde{\imath} + 400\tilde{\jmath})km/h$. At 11:45 a cargo plane has a position vectors $(-60\tilde{\imath} + 355\tilde{\jmath})km$ and is moving with velocity vector $(400\tilde{\imath} + 300\tilde{\jmath})km/h$ Assuming velocities do not change,
   (i) Show that the planes will crash into each other
   (ii) Find the time of the crash

(iii) Find the position vector of the crash
An$(12:06pm, (80\hat{\imath} + 460\hat{\jmath})km,)$

2. At 2pm the position vectors r and velocity vectors v of three ships A, B and C are as follows
$r_A = (5\tilde{\imath} + \tilde{\jmath})km,\quad V_A = (9\tilde{\imath} + 18\tilde{\jmath})kmh^{-1}$
$r_B = (12\tilde{\imath} + 5\tilde{\jmath})km,\quad V_B = (-12\tilde{\imath} + 6\tilde{\jmath})kmh^{-1}$
$r_C = (13\tilde{\imath} - 3\tilde{\jmath})km,\quad V_C = (9\tilde{\imath} + 12\tilde{\jmath})kmh^{-1}$
Assuming velocities do not change,
(i) Show that ship A and B will collide and find the when and where the collision occurs
(ii) Find the position vector of C when A and B collide and find how far C is from the collision
(iii) When the collision occurs, C immediately changes its course but not its speed and streams direct to the scene. When does C arrive **An**$((a)\ 2:20pm, (8\hat{\imath} + 7\hat{\jmath})km,$
$(b)\ (16\hat{\imath} + \hat{\jmath})km, 10km,\ (c)\ 3:00pm)$

3. A jet fighter travelling at 30km/h wishes to intercept a plane travelling at 20km/h in a course of 200°. Initially the plane is 40km away on a bearing of $11°$ from the jet fighter. Find;
(i) Course the jet fighter should set so as to reach the plane as quickly as possible
(ii) Time taken for interception to occur
An$(N5°E, 48minutes\ and\ 24seconds)$

4. A batsman hits a ball at 15m/s in a direction $S80°W$ A fielder , 45m and $S65°W$ form the batmans, runs at 6m/s to intercept the ball. Assuming the velocities remain unchanged,
(i) find in what direction the fielder must run to intercept the ball as quickly as possible
(ii) How long did it take him,
An$(N24.7°E, 2.4s)$

CHAPTER 12: CENTRE OF GRAVITY

This is the point where the resultant force due to attraction acts

General formula for COG

Consider a system of particle of weight $W_1, W_2, \dots\dots\dots\dots\dots\dots\dots\dots W_n$ located at the points with coordinates $x_1y_1,\ x_2y_2,, \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots x_ny_n$ in the $x-y$ plane.

The resultant of weight $W_1 + W_2 + \dots\dots\dots\dots\dots\dots\dots\dots + W_n$ have a C.O.G at a point G($\bar{x}, \bar{y}$)

Taking moments along the y-axis

$$(W_1 + W_2 + \dots\dots\dots\dots\dots\dots\dots\dots + W_n)\bar{x} = W_1\,x_1 + W_2\,x_2 + \dots\dots\dots\dots\dots\dots\dots\dots + W_n\,x_n$$

$$\bar{x} = \frac{\sum W_i\,x_i}{\sum W_i}$$

Similarly: Taking moments along the X-axis : $\bar{y} = \frac{\sum W_i\,y_i}{\sum W_i}$

Alternatively : $$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{\left(W_1\begin{pmatrix}x_1\\ y_1\end{pmatrix} + W_2\begin{pmatrix}x_2\\ y_2\end{pmatrix} + \dots\dots\dots\dots + W_n\begin{pmatrix}x_n\\ y_n\end{pmatrix}\right)}{(W_1 + W_2 + \dots\dots\dots\dots + W_n)}$$

OR $$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{\left(m_1g\begin{pmatrix}x_1\\ y_1\end{pmatrix} + m_2g\begin{pmatrix}x_2\\ y_2\end{pmatrix} + \cdots\dots\dots\dots m_ng\begin{pmatrix}x_n\\ y_n\end{pmatrix}\right)}{(m_1g + m_2g + \cdots\dots\dots\dots m_ng)}$$

FOR CENTRE OF MASS

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{\left(m_1\begin{pmatrix}x_1\\ y_1\end{pmatrix} + m_2\begin{pmatrix}x_2\\ y_2\end{pmatrix} + \cdots\dots\dots\dots\dots\dots m_n\begin{pmatrix}x_n\\ y_n\end{pmatrix}\right)}{(m_1 + m_2 + \cdots\dots\dots\dots\dots\dots m_n)}$$

Examples

1. Find the position of the center of gravity of three particles of masses 1kg, 5kg, 2kg which lie on the y-axis at the points (0,2), (0,4) and (0,5) respectively . **UNEB 1996 NO.9(a)**

Solution

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{1g\begin{pmatrix}0\\ 2\end{pmatrix} + 5g\begin{pmatrix}0\\ 4\end{pmatrix} + 2g\begin{pmatrix}0\\ 5\end{pmatrix}}{(1g + 5g + 2g)}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{\begin{pmatrix}0\\ 32\end{pmatrix}}{8} = \begin{pmatrix}0\\ 4\end{pmatrix}$$

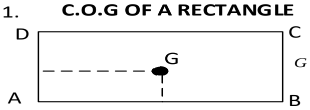
2. Find the co-ordinates of the center of mass of four particles of masses 5kg, 2kg, 2kg and 3kg which are situated at (3,1), (4,3), (5,2) and (-3,1) respectively

Solution

Exercise 22A

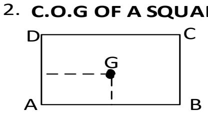
CENTRE OF GRAVITY OF LAMINAE

1. C.O.G OF A RECTANGLE



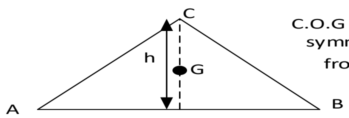
$G\left(\frac{AB}{2}, \frac{AD}{2}\right)$

2. C.O.G OF A SQUARE



$G\left(\frac{AB}{2}, \frac{AD}{2}\right)$

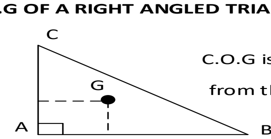
3. C.O.G OF AN ISOSCELES TRIANGLE



C.O.G lies along the line of symmetry at a distance $\frac{h}{3}$ from the straight face AB

$G\left(\frac{AB}{2}, \frac{h}{3}\right)$

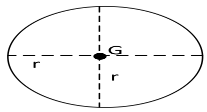
4. C.O.G OF A RIGHT ANGLED TRIANGLE



C.O.G is at a distance $\frac{AB}{3}$ and $\frac{AC}{3}$ from the right angled faces AB and AC

$G\left(\frac{AB}{3}, \frac{AC}{3}\right)$

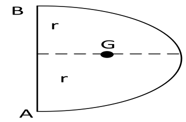
5. **C.O.G OF A CIRCLE**



C.O.G lies along the centre

$G(r, r)$

6. **C.O.G OF A SEMI CIRCLE**



C.O.G lies along the line of symmetry at a distance $\frac{4r}{3\pi}$ from straight face AB

$G\left(\frac{4r}{3\pi}, r\right)$

7. C.O.G of **sector of a circle** subtending an angle 2α at the centre lies along the line of symmetry at a distance $\frac{2rsin\alpha}{3\alpha}$ from the centre
8. C.O.G of **a circular arc** subtending an angle 2α at the centre lies along the line of symmetry at a distance $\frac{rsin\alpha}{\alpha}$ from the centre
9. C.O.G of <u>a semi circular arc</u> of radius r is at a distance $\frac{2r}{\pi}$ from the centre.

Examples

1. Find the position of the center of gravity of a uniform lamina in form of a triangle whose co-ordinates are

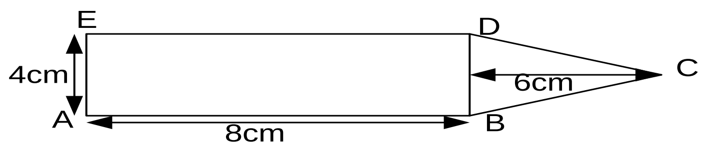
(i) $(0,0), (2,6)$, and $(4,0)$. | **(iii)** $(0,0), (0,6)$, and $(6,0)$. | **(v)** $(1,0), (5,0)$, and $(0,6)$.

(ii) $(0,3), (3,0)$, and $(6,3)$. | **(iv)** $(0,0), (0,6)$, and $(3,0)$. | **(vi)** $(3,0), (6,0)$, and $(0,6)$.

Solution

$$\binom{\bar{x}}{\bar{y}} = \frac{1}{3}(\mathbf{0+2+4, 0+6+0})$$
$$= (\mathbf{2, 2})$$
$$(ii) = (\mathbf{3, 2})\ (\boldsymbol{iii}) = (\mathbf{2, 2})\ (\boldsymbol{iv}) = (\mathbf{1, 2})\ (\boldsymbol{v}) = (\mathbf{2, 2})\ (\boldsymbol{vi}) = (\mathbf{3, 2})$$

2. The figure below shows a lamina formed by joining together a rectangular solid and triangular solid. Find the C.O.G of the composite lamina from side AE and AB



Solution

Let W be weight per unit area

lamina	Area	Weight	Distance of C.O.G from AE	AB
ABDE	32cm^2	32 W	4	2
BDC	12cm^2	12 W	10	2
composite	44cm^2	44 W	$\bar{x}$	$\bar{y}$

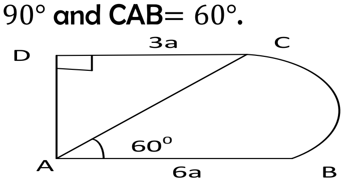
c.o.g from AE; $44W\bar{x} = 12Wx10 + 32Wx4$

$\bar{x} = 5.64cm$

c.o.g from AB; $44W\bar{y} = 12Wx2 + 32W2$

$\bar{y} = 2cm$

3. The figure below shows a uniform lamina consisting of a sector ABC of a circle centre A and of radius 6a and triangle ADC, where angle ADC= $90°$ and CAB= $60°$.



Show that the distance of the centre of gravity of the composite body from AD is $\frac{27a\sqrt{3}}{4\pi+3\sqrt{3}}$

Solution

$$AD = \sqrt{(6a)^2 - (3a)^2} = 3a\sqrt{3}$$

Let W be weight per unit area

Portion	Area	Weight	C.O.G from AD
	$\frac{1}{2}x3ax3a\sqrt{3}$	$4.5\sqrt{3}a^2W$	$\frac{3a}{3}=a$
	$\frac{60^0}{360^0}x\Pi(6a)^2$	$6\Pi a^2W$	$\frac{2x6ax\sin(30^0)}{3x\left(30^0\Pi/180^0\right)}\cos 30^0=\frac{6a}{\Pi}\sqrt{3}$
Composite		$(4.5\sqrt{3}+6\Pi)a^2W$	$\bar{x}$

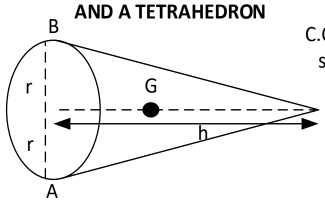
$$\bar{x}\left(\frac{9}{2}\sqrt{3}+6\pi\right)a^2W=6\pi a^2Wx\frac{6a}{\pi}\sqrt{3}+\frac{9}{2}\sqrt{3}a^2Wxa$$

$$\bar{x}(9\sqrt{3}+12\pi)=81a\sqrt{3}$$

$$\bar{x}=\frac{81a\sqrt{3}}{12\pi+9\sqrt{3}}=\frac{27a\sqrt{3}}{4\pi+3\sqrt{3}}$$

CENTRE OF GRAVITY OF SOLIDS

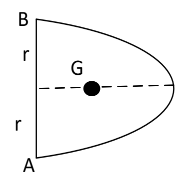
1. **C.O.G OF A SOLID CONE, PRAMID AND A TETRAHEDRON**



C.O.G lies along the line of symmetry at a distance $\frac{h}{4}$ from the base

$G\left(\frac{h}{4}, r\right)$

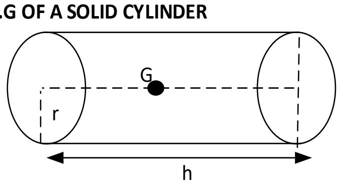
2. **C.O.G OF A SOLID HEMISPHERE**



C.O.G lies along the line of symmetry at a distance $\frac{3r}{8}$ from the straight face AB

$G\left(\frac{3r}{8}, r\right)$

3. **C.O.G OF A SOLID CYLINDER**



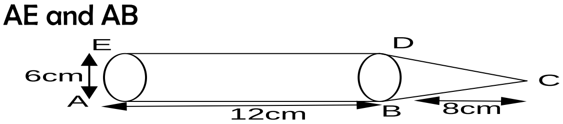
C.O.G lies along the line of symmetry at a distance $\frac{h}{2}$ from the base

$G\left(\frac{h}{2}, r\right)$

4. C.O.G of <u>a hollow (thin) hemispehere</u> of radius r is at a distance $\frac{r}{2}$ from the base.
5. C.O.G of <u>a hollow (thin) cone, pyramid or tetrahedron</u> of height h is at a distance $\frac{h}{3}$ from the base.

Examples

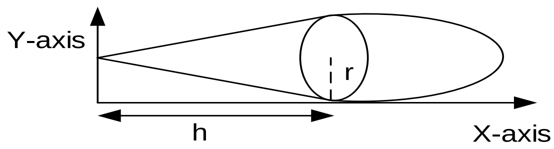
1. The figure below shows a lamina formed by joining together a cylindrical solid to a conical solid. Find the C.O.G of the composite lamina from side AE and AB



Solution

2. A body of consists of a solid hemisphere of radius r joined to a solid right circular cone of base radius r and perpendicular h. The pane surface of the cone and hemisphere coincide and both solids are made of the same uniform material. Show that the C.O.G of the body lies on the axis of symmetry at a distance $\frac{3r^2-h^2}{4(h+2r)}$ from the base of the cone

Solution



W is weight per unit volume

lamina	Weight	C.O.G from y axis
Cone	$\frac{1}{3}\pi r^2 hW$	$\frac{3}{4}h$
hemisphere	$\frac{2}{3}\pi r^3 W$	$h+\frac{3r}{8}$
Composite	$\frac{1}{3}\pi r^2(h+2r)W$	$\bar{x}$

$$\frac{1}{3}\pi r^2(h+2r)W\bar{x} = \frac{2}{3}\pi r^3\left(h+\frac{3r}{8}\right)W + \frac{1}{3}\pi r^2 h\left(\frac{3h}{4}\right)W$$

$$(h+2r)\bar{x} = 2r\left(h+\frac{3r}{8}\right)+\frac{3h^2}{4}$$

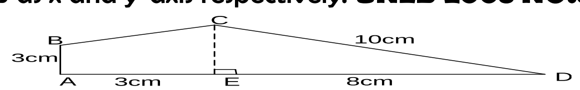
$$\bar{x} = \frac{\mathbf{3h^2+8hr+3r^2}}{\mathbf{4}(h+2r)}$$

Centre of gravity from the base= $\frac{3h^2+8hr+3r^2}{4(h+2r)} - h$

$$= \frac{3r^2-h^2}{4(h+2r)}$$

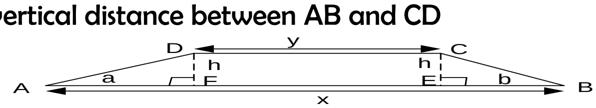
Exercise 22B

1. Find the position of the center of gravity of a uniform lamina in form of a triangle whose $(2,2)$, $(4,6)$, and $(0,3)$ respectively. **UNEB 2007 NO.2** **An**$(2,\frac{11}{3})$
2. Find the co-ordinate if the centre of mass of the lamina shown below. Take A as the origin and AD, AB as x and y-axis respectively. **UNEB 2008 NO.8**



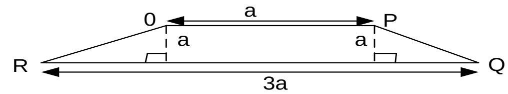
An(4.227cm, 2.12cm)

3. The figure ABCD below shows a metal sheet of uniform material cut in the shape of a trapezium $\overline{AB} = x$, $\overline{CD} = y$, $\overline{AF} = a$, $\overline{EB} = b$ and h is the vertical distance between AB and CD



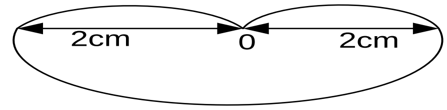
Prove that the distance of the centre of gravity of the sheet is at a distance $\frac{1}{3}h\left(\frac{3y+a+b}{x+y}\right)$ from side AB

4. The figure OPQR below shows a metal sheet of uniform material cut in the shape of a trapezium $\overline{OP} = a$, $\overline{RQ} = 3a$, and the vertical height of P from $RQ = a$ **UNEB 2006 NO.5**



Calculate the centre of mass of OPQR **An**$(\frac{3}{2}a, \frac{5}{12}a)$.

5. A badge is cut from a uniform thin sheet of metal. The badge is formed by joining the diameter of two semicircle, each of radius 1cm to the diameter of semicircle of radius 2cm as shown below

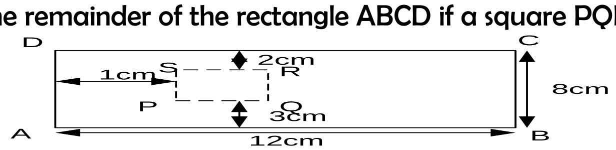


The point of contact of two smaller semicircle is O. Determine in terms of π, the distance from O of the centre of mass of the badge. **An**$(,\frac{4}{3\pi}cm)$.

CENTRE OF GRAVITY OF A REMAINDER

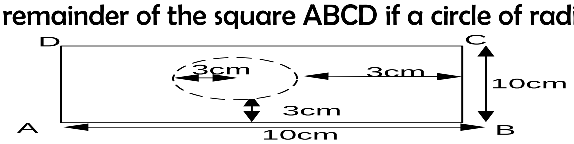
Examples

1. Find the centre of gravity of the remainder of the rectangle ABCD if a square PQRS is removed as shown below



Solution

2. Find the centre of gravity of the remainder of the square ABCD if a circle of radius r is removed as shown below



Solution

W is the weight per unit area

lamina	area	Weight	Distance of C.O.G from AD	AB
ABCD	100cm^2	100 W	5	5
circle	28.27cm^2	28.27 W	4	6
Remainder	71.73cm^2	71.73 W	$\bar{x}$	$\bar{y}$

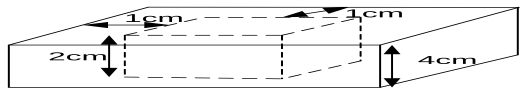
c.o.g from AD: $71.73W\bar{x} = 100Wx\,5 - 28.27Wx4$

$\bar{x} = 5.39cm$

c.o.g from AB; $71.73W\bar{y} = 100Wx\,5 - 28.27Wx6$

$\bar{y} = 4.61cm$

3. A solid cube of side 4cm is made from a uniform material. From this a smaller cube of side is removed as shown below

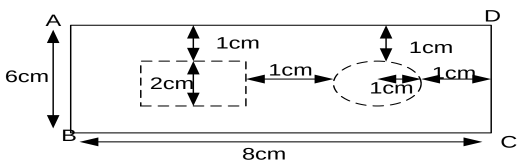


Find the position of the centre of gravity of the remaining body

Solution

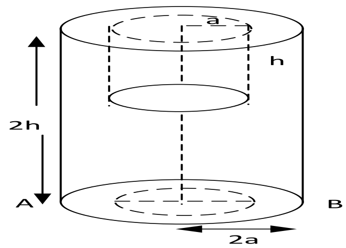
Exercise 22C

1. ABCD is a uniform rectangular sheet of card board of length 8cm and width 6cm. A square and a circular hole are cut off from the card board as shown above. **UNEB 1999 NO.16**



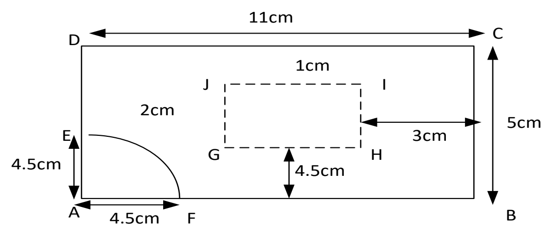
Calculate the position of the C.O.G of the remaining sheet **An**$(3.944cm, 2.825cm)$

2. The diagram below shows a uniform cylinder of radius 2a and height 2h with a cylindrical hole of radius a and height h drilled centrally at one plane end



Show that the centre of gravity of the remaining solid is $^{13}/_{14}\,h$ from the base AB

3. The figure below shows a uniform rectangular lamina ABCD with a square GHIJ of side 3cm and a quarter circular section AFE of radius 4.5 cm is cut off.

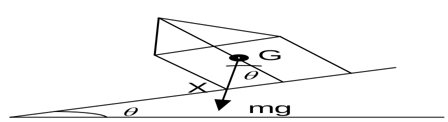


Find the coordinates of centre of gravity from sides AB and AD taken as the x and y axes respectively **An**$(6.149\text{cm}, 4.874\text{cm})$

TOPPLING

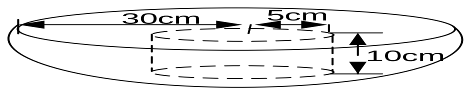
Consider a body which is resting on a slop which is rough enough to prevent slipping.

When the angle of the slope is such that the weight acts through X the body will be on the point of toppling



Examples

1. The figure below shows a uniform hemispherical solid of radius 30cm with a cylindrical hole of radius 5cm and height 10cm centrally drilled in it



(a) Find the distance of centre of gravity of the figure from a flat surface
(b) If the figure is placed on an inclined plane with the flat surface in contact with the plane, calculate the angle that should be inclined, before toppling occurs assuming that sliding does not occur

Solution

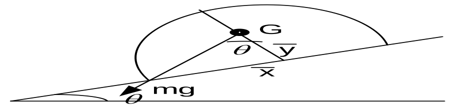
lamina	Volume	Weight	C.O.G from Flat face	C.O.G from Y-axis
hemisphere	$\frac{2}{3}\pi(30)^3 = 56548.668$	$56548.668W$	$\frac{3x30}{8} = 11.25$	30cm
Cylinder	$\pi(5)^2 x10 = 785.398$	$785.398W$	5cm	30cm
Remainder	55763.270	55763.270W	$\bar{y}$	$\bar{X}$

c.o.g from flat face

$$55763.27\bar{y} = 56548.668x11.25 - 785.398x5$$
$$\bar{y} = 11.34$$

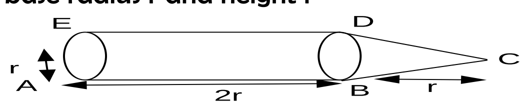
c.o.g from y-axis

$$55763.27\bar{x} = 56548.668x30 - 785.398x30$$
$$\bar{x} = 30$$



$$\theta = tan^{-1}\left(\frac{30}{11.34}\right) = 69.3°$$

2. A body consists of a uniform solid cylinder of mass 6m, base radius r and height 2r, attached to a plane face of a uniform solid cone of mass 4m base radius r and height r

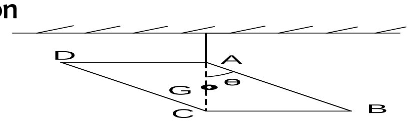


(i) Find the position of the centre of gravity of the body
(ii) The body is now placed with its plane face AE in contact with a horizontal table. The surface of the table is rough enough to prevent the body slipping as the table is slowly tilted. Find the angle through which the table has been tilted when the body is on the point of toppling

Solution

EQUILIBRIUM OF SUSPENDED LAMINA

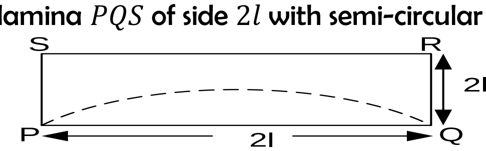
Consider a lamina freely suspended from point A. The centre of gravity passes through the point of suspension



$$tan\,\theta = \frac{BC}{AB}$$

Examples

1. The figure below shows a uniform lamina PQS of side $2l$ with semi-circular lamina cut off as show below



(a) Show that the distance of the centre of gravity of the figure from PQ is $\frac{20l}{3(8-\pi)}$

(b) The figure is freely suspended from the point R. find the angle that RS makes with the vertical

Solution

W is the weight per unit area

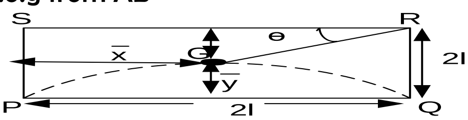
lamina	Area	Weight	Distance of C.O.G from PQ
PQRS	$4l^2$	$4l^2$ W	l
Semi-cirlce	$\frac{1}{2}\pi l^2$	$\frac{1}{2}\pi l^2 W$	$\frac{4\ l}{3\ \pi}$
remainder	$\left(4-\frac{1}{2}\pi l^2\right)$	$\left(4-\frac{1}{2}\pi\right)l^2 W$	$\bar{y}$

c.o.g from PQ: $\left(4-\frac{\pi}{2}\right)l^2\bar{y} = 4l^2 x\ l - \frac{\pi}{2}l^2 x\frac{4\ l}{3\pi}$

$$\left(\frac{8-\pi}{2}\right)\bar{y} = 4l - \frac{2\ l}{3}$$
$$\left(\frac{8-\pi}{2}\right)\bar{y} = \frac{10\ l}{3}$$

$$\bar{y} = \frac{20\,l}{3(8-\pi)}$$

c.o.g from AB



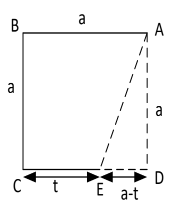
C.O.G from RS= $2l - \frac{20\,l}{3(8-\pi)} = \frac{48\,l - 6l\pi - 20l}{3(8-\pi)}$

$$= \frac{28\,l - 6l\pi}{3(8-\pi)}$$

$$\theta = tan^{-1}\left[\frac{28\,l - 6l\pi}{3(8-\pi)}\Big/ l\right] = \frac{28\ - 6\pi}{3(8-\pi)} = 32.12°$$

2. ABCD is a uniform square lamina of side a from which a triangle ADE is removed. E being a point on CD of distance t from C
   i. Show that the centre of masss of the remaining lamina is at a distance $\frac{a^2+at+t^2}{3(a+t)}$ from BC
   ii. If the lamina is placed in a vertical plane with CE resting on a horizontal table, show that equilibrium will not be possible if $t < \frac{a(\sqrt{3}-1)}{2}$

Solution



Let W be mass per unit area

portion	Area	Mass	C.O.G from BC
□	a^2	a^2W	$\frac{a}{2}$
◺	$\frac{1}{2}a(a-t)$	$\frac{1}{2}a(a-t)W$	$a - \frac{1}{3}(a-t) = \frac{2a+t}{3}$
Remainder	$\frac{1}{2}a(a+t)$	$\frac{1}{2}a(a+t)W$	x̄

$$\bar{x}\frac{(a+t)a}{2}W = a^2Wx\frac{a}{2} - \frac{a(a-t)W}{2}x\frac{(2a+t)}{3}$$

$$\bar{x}(a+t) = \frac{3a^2 - (a-t)(2a+t)}{3}$$

$$\bar{x} = \frac{a^2+at+t^2}{3(a+t)}$$

(ii) $\quad t < \bar{x}$

$$t < \frac{a^2+at+t^2}{3(a+t)}$$

$$3at + 3t^2 < a^2 + at + t^2$$

$$2t^2 - 2at - a^2 < 0$$

$$t < \frac{-2a \pm \sqrt{(2a)^2 - 4x2x{-a^2}}}{2x2}$$

$$t < \frac{-a \pm a\sqrt{3}}{2}$$

$$t < \frac{a(-1-\sqrt{3})}{2}$$

Or $t < \frac{a(\sqrt{3}-1)}{2}$

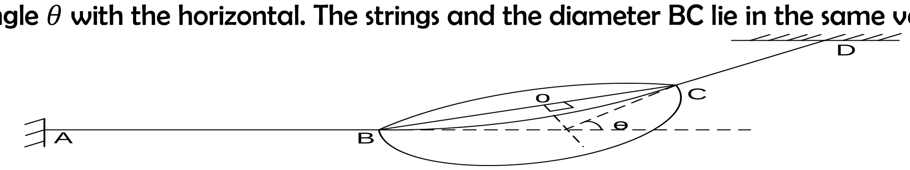
Since $\frac{a(-1-\sqrt{3})}{2} < a$ then

$$t < \frac{(\sqrt{3}-1)}{2}$$

3. ABCD is a uniform square lamina of side $2cm$ from which a triangle EDC is removed. E being a point on AD such that $ED = x\ cm$
   i. Show that the centre of gravity of the remaining lamina is at a distance $\frac{12-6x+x^2}{3(4-x)}$ from AB
   ii. If the lamina is placed in a vertical plane with AE resting on a rough horizontal table, show that it will topple if $x > 3 - \sqrt{3}$

Solution

4. A uniform solid hemisphere is kept in equilibrium in space by two in elastic strings, AB which is horizontal and CD which makes an angle θ with the horizontal. The strings and the diameter BC lie in the same vertical plane.

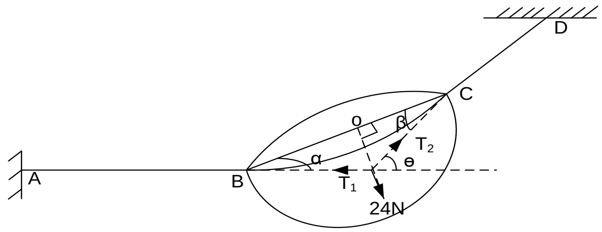


Given that the weight of the hemisphere is 24N, show that

(i) $tan\theta = \frac{48}{55}$

(ii) Tension in strings CD and AB are $36.5N$ and $27.5N$ respectively

Solution



$tan\alpha = {}^{\frac{3r}{8}}\!/_{r} = \frac{3}{8}$ and $tan\beta = {}^{\frac{3r}{8}}\!/_{r} = \frac{3}{8}$

$$\theta = \alpha + \beta$$

$$tan\theta = tan(\alpha+\beta) = \frac{tan\alpha + tan\beta}{1 - tan\alpha\ tan\beta}$$

$$tan\theta = \frac{\frac{3}{8}+\frac{3}{8}}{1-\frac{3}{8}x\frac{3}{8}} = {}^{\frac{3}{4}}\!/_{\frac{55}{64}} = \frac{48}{55}$$

$(\rightarrow)\ T_2cos\theta = T_1 \ldots\ldots..(i)$

$(\uparrow)\ T_2sin\theta = 24 \ldots\ldots\ldots\ldots(ii)$

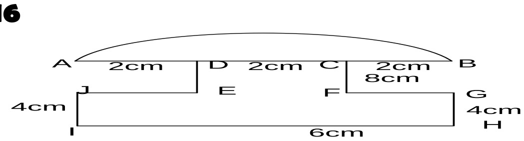
$\frac{T_2sin\theta}{T_2cos\theta} = \frac{24}{T_1}$ $\quad\therefore tan\theta = \frac{24}{T_1}$

$$T_1 = \frac{24}{tan\theta} = {}^{24}\!/_{\frac{48}{55}} = 27.5N$$

$$T_2 = \frac{24}{sin\theta} = \frac{24}{sin\left(tan^{-1}\frac{48}{55}\right)} = 36.5N$$

Exercise 22D

1. The figure ABCDEFGHIJ shows a symmetrical composite lamina made up of a semicircle radius 3cm, a rectangle CDEF 2cm by 8cm and a nother rectangle GHIJ 6cm by 4cm **UNEB Mar 1998 NO.16**



Find the distance of the C.O.G of this lamina from IH. If the lamina is suspended from H by means of a peg through a hole, calculate the angle of inclination of HG to the vertical **An** $\mathbf{6.72cm, 24.1°,}$**)**

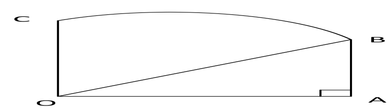
3. (a) A, B, C and D are the points (0,0), (10,0), (7,4) and (3,4) respectively. If AB, BC, CD and DA are made of a thin wire of uniform mass, find the co-ordinates of centre of gravity. **UNEB 1994 NO.6**

$(b)(i)$ If instead AB is uniform lamina, find its centre of gravity

(ii) If lamina is hung from B, find the angle AB makes with vertical **An** $\mathbf{(a)(5, 1.5), (b)(i)(5, 1.7), 18.°,}$**)**

4. (a) Prove that the centre of mass of a solid cone is $\frac{1}{4}$ of the vertical height from the base. **UNEB 2001 NO.15**

5. The figure below shows a uniform lamina consisting of a sector OBC of a circle centre O and of radius 6cm and triangle OAB, where angle AOC= $90°$ and COB= $60°$.

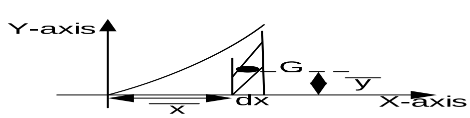


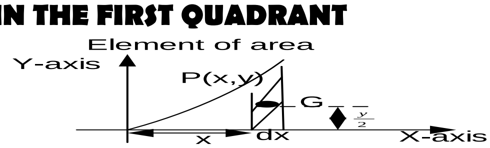
(a) Find the coordinates of centre of gravity from sides OC and OA

(b) If the lamina is suspended from point C, find the angle that OC makes with the vertical at equilibrium **An**($2.3645cm, 2.6328m,\ 35.1°$).

CENTRE OF GRAVITY OF THE LAMINA WHOSE AREA IS BOUNDED

(i) C.O.G OF THE AREA BOUNDED IN THE FIRST QUADRANT





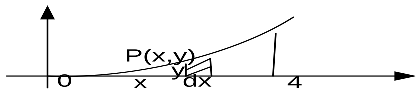
C.O.G from y-axis: $W\bar{x}\int ydx = W\int xydx$ | C.O.G from the x-axis: $W\bar{y}\int ydx = W\int \frac{y}{2}ydx$

Where $W -$ weight per unit area

Examples

1. Find the co-ordinates of the centre of gravity of the uniform lamina enclosed by the curve $y = x^2$, the x-axis and the line $x = 4$

Solution



$$W\bar{x}\int_0^4 ydx = W\int_0^4 xydx$$

$$\bar{x}\int_0^4 x^2dx = \int_0^4 x(x^2)dx$$

$$\bar{x}\left[\frac{x^3}{3}\right]_0^4 = \left[\frac{x^4}{4}\right]_0^4$$

$$\bar{x} = 3$$

$$W^{-}\int_0^4 ydx = W\int_0^4 \frac{y^2}{2}dx$$

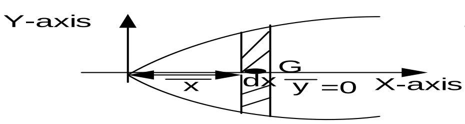
$$\bar{y}\int_0^4 x^2dx = \frac{1}{2}\int_0^4 x^4dx$$

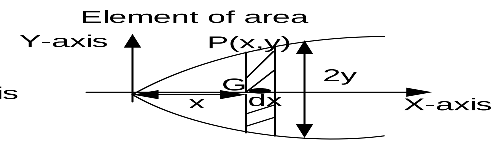
$$\bar{y}\left[\frac{x^3}{3}\right]_0^4 = \frac{1}{2}\left[\frac{x^5}{5}\right]_0^4$$

$$\bar{y} = 4.8$$

2. Find the co-ordinates of the centre of gravity of the uniform lamina enclosed by the curve $y^2 = 9x$, the x-axis and the line $x = 1$ and $x = 4$ and lying in the first quadrant

(ii) C.O.G OF THE AREA BOUNDED IN THE FIRST AND FOURTH QUADRANT

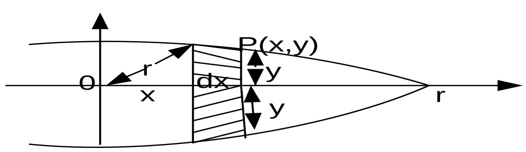




Taking moments about the y-axis: $W\bar{x}\int 2ydx = W\int x(2y)dx$
Where $W -$ weight per unit area

Examples

1. Find the co-ordinates of the centre of gravity of the uniform lamina enclosed by the curve $y^2 = 4x$, and the line $x = 9$
Solution
2. Show that the position of C.O.G of a uniform semi-circular lamina of radius r is $\frac{4r}{3\pi}$ from the straight edge
Solution



Area of semi circle= area of element of semicircle

$$W\frac{1}{2}\pi r^2\bar{x} = W\int_0^r x(2y)dx$$

But a semi cycle is part of a circle of radius r whose equation is $x^2 + y^2 = r^2$

$$y = (r^2 - x^2)^{1/2}$$

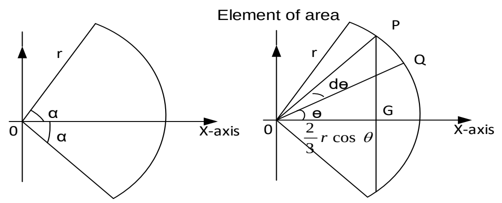
$$\frac{1}{2}\pi r^2\bar{x} = 2\int_0^r x\left((r^2 - x^2)^{1/2}\right)dx$$

$$\frac{1}{2}\pi r^2\bar{x} = 2\left[-\frac{(r^2 - x^2)^{3/2}}{3}\right]_0^r$$

$$\frac{1}{2}\pi r^2\bar{x} = \frac{2\,r^3}{3}$$

$$\bar{x} = \frac{4r}{3\pi}$$

3. Show that the centre of gravity of a uniform lamina in the shape of <u>a sector of a circle</u> of radius r and subtending an angle 2α at the centre O is given by $\frac{2}{3}\frac{rsin\alpha}{\alpha}$ from O.
Solution



The strip OPQ approximates a triangle and its C.O.G is at a distance $\frac{2}{3}r$ from O. The distance of C.O.G from O is therefore, $\frac{2}{3}rcos\theta$

Area of a sector= element of the area of sector

$$W\frac{1}{2}r^2(2\alpha)\bar{x} = W\int_{-\alpha}^{\alpha} xydx$$

$$\frac{1}{2}r^2(2\alpha)\bar{x} = W\int_{-\alpha}^{\alpha} \frac{2}{3}rcos\theta\left(\frac{1}{2}r^2\right)d\theta$$

$$\alpha\bar{x} = \frac{1}{3}\int_{-\alpha}^{\alpha} rcos\theta d\theta$$

$$\alpha\bar{x} = \frac{r}{3}[\sin\theta]_{-\alpha}^{+\alpha}$$

$$\bar{x} = \frac{2}{3}\frac{rsin\alpha}{\alpha}$$

For a complete semi –circle, $\alpha = \frac{\pi}{2}$ there fore

$$\bar{x} = \frac{4r}{3\pi}$$

CENTRE OF GRAVITY OF SOLIDS OF REVOLUTION

Taking moments about the y-axis: $\bar{x}\, W\pi\int y^2dx = W\pi\int xy^2dx$

Where W – weight per unit volume

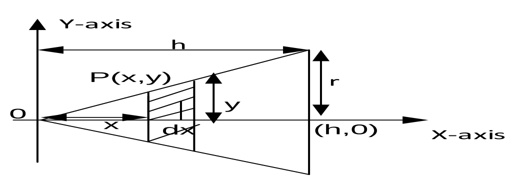
Examples

1. Find the centre of gravity of the solid generated by rotating about the x-axis, the area under $y = x$ from $x = 0$ and $x = 3$

Solution

4. Show that the position of C.O.G of a uniform <u>solid right circular cone</u> of base radius r and height h is given by $\frac{h}{4}$ from the straight edge

Solution



Volume of a cone= volume of element of a cone

$$\bar{x}W\frac{1}{3}\pi r^2h = W\pi\int xy^2dx$$

$$\bar{x}\frac{1}{3}r^2h = \int_0^h xy^2dx$$

From similarity $\frac{y}{x} = \frac{r}{h}$

$$y = \frac{r}{h}x$$

$$\bar{x}\frac{1}{3}r^2h = \int_0^h x\left(\frac{r}{h}x\right)^2dx$$

$$\bar{x}\frac{1}{3}r^2h = \left(\frac{r^2}{h^2}\right)\left[\frac{x^4}{4}\right]_0^h$$

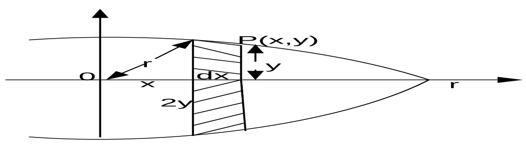
$$\bar{x} = \frac{3h}{4}$$

From the straight edge

$$\bar{x} = h - \frac{3h}{4} = \frac{h}{4}$$

5. Show that the position of C.O.G of a uniform <u>solid hemisphere</u> of radius r is $\frac{3r}{8}$ from the straight edge

Solution



$$W\frac{2}{3}\pi r^3\bar{x} = W\pi\int_0^r xy^2dx$$

But $x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2$

$$\frac{2}{3}\pi r^3\bar{x} = \pi\int_0^r x(r^2 - x^2)dx$$

$$\frac{2}{3}r^3\bar{x} = \left[r^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^r$$

$$\frac{2}{3}r^3\bar{x} = \frac{r^4}{2} - \frac{r^4}{4}$$

$$\bar{x} = \frac{3r}{8}$$

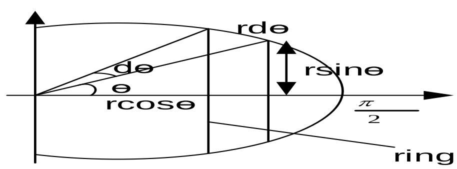
Exercise 22E

1. Find the co-ordinates of the centre of gravity of the uniform lamina enclosed by the curve $y = x^2$, the x-axis and the line $x = 2$ **An**$(1.5,1.2)$ Find the co-ordinates of the centre of gravity of the uniform lamina enclosed by the curve $y = 2x - x^2$, and the x-axis **An**$(1.0,0.4)$
2. Find the co-ordinates of the centre of gravity of the uniform lamina enclosed by the curve $y = x^2 + 2$, the x-axis and the line $x = 1$ and $x = 2$ **An**$(1.56,2.25)$
3. The area enclosed by the curve $y^2 = x$, the x-axis, the line $x = 2, x = 4$ and lying in the first quadrant is rotated about the x-axis through one revolution. Find the co-ordinates of the centre of gravity of the uniform solid formed **An**$(3.39,0)$ **UNEB 1996 NO.9(b)**
4. The area enclosed by the curve $y^2 = x$, the x-axis, the line $= 0, x = 2$, and $x = 4$ is rotated about the x-asis through one revolution. Find the co-ordinates of the centre of gravity of the uniform solid formed **An**$(3.39,0)$
5. The area enclosed by the curve $y = x^3$, the x-axis and the line $x = 3$ is rotated about the x-asis through one revolution. Find the co-ordinates of the centre of gravity of the uniform solid formed **An**$(2.625,0)$

SURFACE OF REVOLUTION

1. Show that the centre of gravity of a uniform <u>thin hemispherical</u> cup of radius r is at a distance $\frac{r}{2}$ from the base. **Uneb 2012 No.13**

Solution



Surface area of a hemisphere= element of the surface

$$W2\pi r^2\bar{x} = W2\pi\int_0^{\frac{\pi}{2}} xdx$$

$$W2\pi r^2\bar{x} = W2\int_0^{\frac{\pi}{2}}(rsin\theta\ xrcos\theta)rd\theta$$

$$W2\pi r^2\bar{x} = W\pi r^3\int_0^{\frac{\pi}{2}} 2sin\theta\ cos\theta d\theta$$

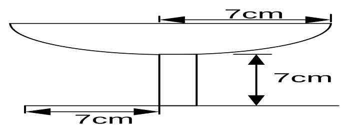
$$\bar{x} = \frac{1}{2}r\int_0^{\frac{\pi}{2}} sin2\theta d\theta$$

$$\bar{x} = \frac{1}{2}r\left[-\frac{1}{2}cos2\theta\right]_0^{\frac{\pi}{2}}$$

$$\bar{x} = -\frac{1}{4}r\left[cos2\left(\frac{\pi}{2}\right) - cos2(0)\right]_0^{\frac{\pi}{2}}$$

$$\bar{x} = -\frac{1}{4}r(-1-1) = \frac{r}{2}$$

4. The figure below is made up of a thin hemispherical cup of radius 7cm. it is welded to a stem of length 7cm and then to a circular base of the same material and of radius 7cm. The base of the stem is one-quarter that of the cup



Find the distance from the base of the centre of gravity of the figure

Solution

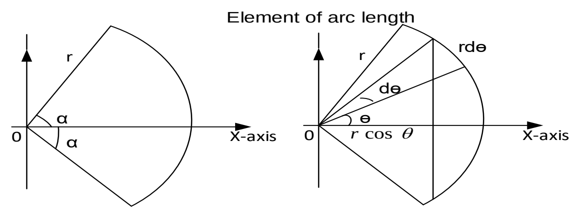
lamina	Area	Weight	C.o.g from base
Circular base	$\pi r^2 = \pi[7]^2 = 49\pi$	$49\pi w$	0
stem	$98\pi/4$	$24.5\pi w$	3.5
Hemispherical cup	$2\pi r^2 = 2\pi[7]^2 = 98\pi$	$98\pi w$	10.5
composite	171.5π	$171.5\pi w$	$\bar{x}$

$$171.5\pi W\bar{x} = 98\pi Wx\ 10.5 + 24.5\pi Wx3.5$$

$$\bar{x} = 6.5cm$$

5. Show that the centre of gravity of a uniform lamina in the shape of <u>an arc of a circle</u> of radius r and subtending an angle 2α at the centre O is given by $\frac{r\sin\alpha}{\alpha}$ from O.

Solution



Length of the arc= element of the arc length

$$Wr(2\alpha)\bar{x} = W\int_{-\alpha}^{\alpha} xdx$$

$$Wr(2\alpha)\bar{x} = W\int_{-\alpha}^{\alpha} (r\cos\theta)rd\theta$$

$$2\alpha\bar{x} = \int_{-\alpha}^{\alpha} r\cos\theta d\theta$$

$$2\alpha\bar{x} = r[\sin\theta]_{-\alpha}^{+\alpha}$$

$$\bar{x} = \frac{r\sin\alpha}{\alpha}$$

For a semi –circle arc, $\alpha = \frac{\pi}{2}$ there fore

$$\bar{x} = \frac{2r}{\pi}$$

CHAPTER 13: COPLANAR FORCES (RIGID BODIES)

A rigid body is one in which distances between its various parts remain fixed

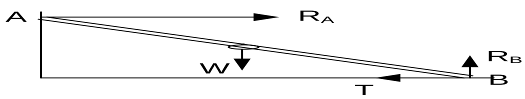
Equilibrium of a rigid body

(i) Sum of Force acting in one direction is equal to sum of forces acting in opposite direction

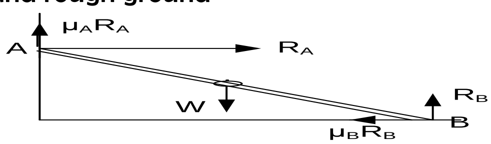
(ii) Sum of Clockwise moments about a point is equal to sum of anticlockwise moments about the same point

Points to note

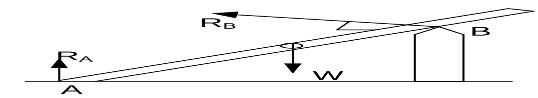
- When a rigid body rests in contact with a smooth wall and a string tied at the base, the reaction on the body is perpendicular to the surface



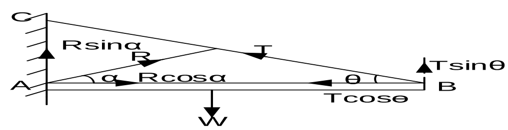
- When a rigid body rest in contact with a rough wall and rough ground



- A rigid body resting against a smooth peg or bar, the reaction on the rigid body is perpendicular to the bar



- When a rod is hinged, the reaction at the hinge acts at an angle to either the horizontal or vertical



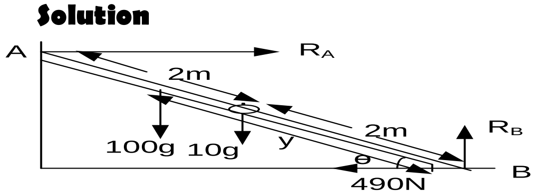
SMOOTH CONTACTS AT THE LADDER

1. A uniform ladder AB of mass 30kg rests with its upper end A against a smooth vertical wall and lower end B on a smooth horizontal ground. A light horizontal inextensible string, which has one end attached to B and the other end attached to the wall, keeps the ladder in equilibrium inclined at $60°$ to the horizontal. The vertical plane containing the ladder and the string is at right angles to the wall. Find
   (i) Tension in the string
   (ii) Normal reactions at points A and B

   Solution

2. A uniform ladder AB of mass 10kg and length 4m rests with its upper end A against a smooth vertical wall and lower end B on a smooth horizontal ground. A light horizontal string, which has one end attached to B and the other end attached to the wall, keeps the ladder in equilibrium at an angle $tan^{-1}(2)$ to the horizontal. The vertical plane containing the ladder and the string is at right angles to the wall. A man of mass 100kg ascends the ladder.
   (i) If the string will break when the tension exceeds 490N, find how far up the ladder the man can go before this occurs
   (ii) What tension must the string be capable of withstanding if the man is to reach the top of the ladder

   Solution



(i) $(\rightarrow)$ $R_A = 490N \ldots\ldots(i)$

B↶ $R_A\ x\ 4sin\theta = 10gx2cos\theta + 100gycos\theta$

$R_A\ \ x\ 4tan\theta = 20g + 100gy$

$490\ \ x\ 4tan\theta = 20g + 100gy$

$y = \frac{490x4x2-20g}{100g} = 3.8m$

(ii) $(\rightarrow)$ $R_A = T \ldots\ldots(i)$

B↶ $R_A\ x\ 4sin\theta = 10gx2cos\theta + 100gx4cos\theta$

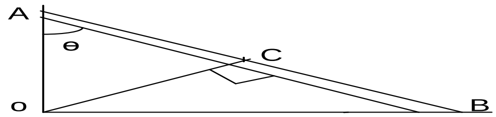
$R_A\ \ x\ 4tan\theta = 20g + 400g$

$R_A = 514.5N$

T = 514.5N

3. A uniform rod AB of weight W and length l rests with its upper end A against a smooth vertical wall and lower end B on a smooth horizontal table. The ladder is kept in equilibrium inclined at θ to the wall by a

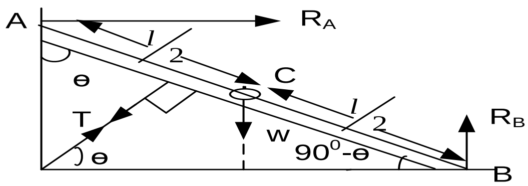
light inextensible string OC. C being a point on AB such that OC is perpendicular to AB and O on the point of intersection of the wall and the table. Angle AOB is $90°$ **UNEB 2000 No.13**



Find;

(i) Tension in the string

Solution



(↑) $R_B = W + Tsin\theta \ldots\ldots(i)$

(→) $R_A = T\cos\theta \ldots\ldots(ii)$

Taking moments about O

$R_B \ x\ lsin\theta = Wx\ ^l/_2\ sin\theta + R_A\ x\ lcos\theta$

$(W + Tsin\theta)lsin\theta = W\ ^l/_2\ sin\theta + T\cos\theta\ lcos\theta$

(j) Reactions at A and B in terms of θ and W

$T(cos^2\theta - sin^2\theta) = \frac{W(2sin\theta - sin\theta)}{2}$

$T = \frac{Wsin\theta}{2(cos^2\theta - sin^2\theta)}$

$R_A = T\cos\theta = \left[\frac{Wsin\theta}{2(cos^2\theta - sin^2\theta)}\right]\cos\theta$

$R_A = \frac{Wsin\theta\cos\theta}{2(cos^2\theta - sin^2\theta)}$

$R_B = W + Tsin\theta = W + \left[\frac{Wsin\theta}{2(cos^2\theta - sin^2\theta)}\right]sin\theta$

$R_B = \left[\frac{2cos^2\theta - 2sin^2\theta + sin^2\theta}{2(cos^2\theta - sin^2\theta)}\right]W = \frac{W(2cos^2\theta - sin^2\theta)}{2(cos^2\theta - sin^2\theta)}$

Exercise 23A

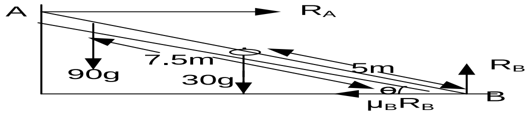
ROUGH CONTACT AT THE FOOT AND SMOOTH CONTACT AT THE TOP OF THE LADDER

1. A non uniform ladder AB 10m long and mass 8kg lies in limiting equilibrium with its lower end resting on a rough horizontal ground and the upper end resting against a smooth vertical wall. If the centre of gravity of the ladder is 3m from the foot of the ladder and the ladder makes an angle of $30°$ with horizontal, find the
   (i) Coefficient of friction between the ladder and the ground
   (ii) Reaction at the wall **UNEB 2004 No.15**

Solution

3. A uniform ladder AB 10m long and mass 30kg lies in limiting equilibrium with its lower end resting on a rough horizontal ground and the upper end resting against a smooth vertical wall. If the ladder makes an angle of $60°$ with horizontal, with a man of mass 90kg standing on the ladder at a point 7.5m from its base. find the;
   (i) Magnitude of the normal reaction and of the frictional force at the ground
   (ii) The minimum value for the coefficient of friction between the ladder and the ground that would enable the man to climb to the top of the ladder

Solution



(i) (↑) $R_B = 30g + 90g = 120x9.8 = 1176N$

(→) $R_A = \mu_B R_B \ldots\ldots(ii)$

$1176\mu_B = R_A$

Taking moments about B

$R_A\ x\ 10sin60 = 30gx5cos60 + 90gx7.5cos60$

$R_A\ 10sin60 = 825gcos60$

$R_A = 466.788N$

$1176\mu_B = R_A$

$1176\mu_B = 466.788$

$\mu_B = 0.397$

(ii) $R_A\ x\ 10sin60 = 30gx5cos60 + 90gx10cos60$

$R_A\ 10sin60 = 1050gcos60$

$R_A = 594.093N$

$1176\mu_B = R_A$

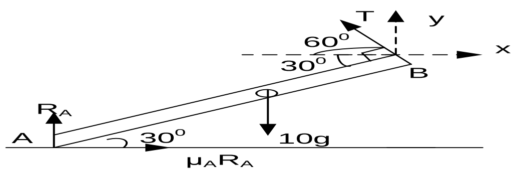
$1176\mu_B = 594.093$

$\mu_B = 0.5052$

ROUGH CONTACT AT THE FOOT OF THE LADDER

1. A uniform pole AB of mass 10kg has its lower end A on a rough horizontal ground and being raised to vertical position by a rope attached to B. The rope and the pole lie in the same vertical plane and A does not slip across the ground. If the rope is at right angles to the pole and the pole is $30°$ to the horizontal, find;
   (i) Tension in the rope | (ii) Coefficient of friction on the ground

Solution



(i) Taking moments about A

$T\ x\ 2l\ = 10gxlcos30$

$T\ = 42.44N$

(ii) $(\uparrow)\ R_A + Tsin60 = 10g$

$R_A + 42.44xsin60 = 10x9.8$

$R_A = 61.25N$

$(\rightarrow)\ Tcos60 = \mu_A R_A \ldots\ldots(iii)$

$42.44cos60 = \mu_A x61.25$

$\mu_A = 0.346$

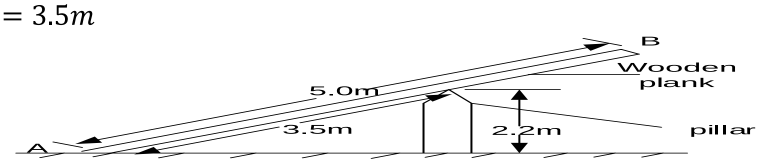
2. A uniform ladder AB of length 2l rests in limiting equilibrium with its lower end A resting on a rough horizontal ground . A point C on the beam rests against a smooth support. AC is of length $\frac{3l}{2}$ with C higher than A and AC makes an angle of $60°$ with the horizontal. Find the coefficient of friction between the ladder and the ground

Solution

3. A uniform rod of length 2l inclined at an angle θ to the horizontal rests in a vertical plane against a smooth horizontal bar at a height h above the ground. Given that the lower end of the rod is on a rough ground and the rod is about to slip. Show that the coefficient of friction between the rod and the ground is $\frac{l\ sin^2\theta\ cos\theta}{h-l\ cos^2\theta\ sin\theta}$

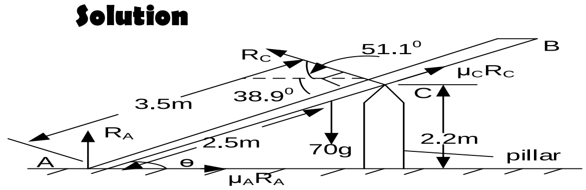
Solution

4. The diagram below shows a uniform wooden plank AB of mass 70kg and length 5m. The end A rests on a rough horizontal ground. The plank is in contact with the top of a rough pillar at C. The height of the pillar is $2.2m$ and $\overline{AC} = 3.5m$



Given that the coefficient of friction at the ground is 0.6 and the plank is just about to slip, find the
   (i) Angle the plank makes with the ground at A
   (ii) Normal reaction at A and normal reaction at C
   (iii) Coefficient of friction at C

Solution



(i)

$sin\theta = \frac{2.2}{3.5}\quad \theta = 38.9°$

(ii) Taking moments about A

$R_C\ x\ 3.5\ = 70gx2.5cos38.9$

$R_C\ = 381.34N$

$(\uparrow)\ R_A + \mu_C R_C sin38.9 + R_C sin51.1 = 70g \ldots(i)$

$R_A = 70g - \mu_C 381.34sin38.9 - 381.34sin51.1$

$R_A = 389.2248 - 239.4674\mu_C$

$(\rightarrow)\ R_C cos51.1 = 0.6R_A + \mu_C R_C cos38.9..(ii)$

$381.34xcos51.1 = 0.6R_A + \mu_C x381.34cos38.9$

$239.4674 = 0.6(389.2248 - 239.4674\mu_C) + 296.7752\mu_C$

$\mu_c = \frac{5.93252}{153.0948} = 0.0388$

$R_A = 389.2248 - 239.4674x0.0388$

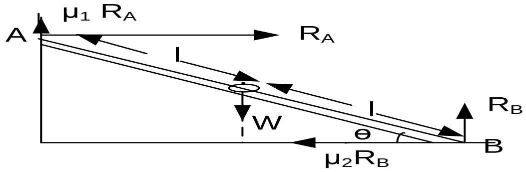
$R_A = 379.933N$

Exercise 23B

ROUGH CONTACTS BOTH AT THE TOP AND FOOT OF A LADDER

1. A uniform ladder rests in limiting equilibrium with the top end against a rough vertical wall with coefficient of friction μ_1 and its base on a rough horizontal floor with coefficient of friction μ_2. If the ladder makes an angle of θ with the floor, prove that $tan\theta = \frac{1-\mu_1\mu_2}{2\mu_2}$

Solution



$(\uparrow)\ R_B + \mu_1 R_A = W \ldots\ldots (i)$

$(\rightarrow)\ R_A = \mu_2 R_B \ldots\ldots (i)$

$R_B = \frac{1}{\mu_2} R_A$ put into (i)

$R_B + \mu_1 R_A = W$

$\frac{1}{\mu_2} R_A + \mu_1 R_A = W$

$\frac{(1+\mu_1\mu_2)}{\mu_2} R_A = W$

$R_A = \frac{\mu_2 W}{1+\mu_1\mu_2}$

Taking moments about B

$R_A\ x\ 2lsin\theta + \mu_1 R_A x 2lcos\theta = Wxlcos\theta \ldots\ldots (iii)$

$$\frac{\mu_2 W}{1+\mu_1\mu_2} 2lsin\theta + \mu_1 x \frac{\mu_2 W}{1+\mu_1\mu_2} 2lcos\theta = Wxlcos\theta$$

$$\frac{2\mu_2}{1+\mu_1\mu_2} sin\theta = cos\theta - \frac{2\mu_1\mu_2}{1+\mu_1\mu_2} cos\theta$$

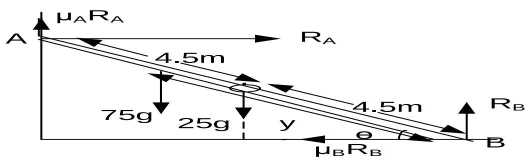
$$\frac{2\mu_2}{1+\mu_1\mu_2} sin\theta = \frac{1+\mu_1\mu_2 - 2\mu_1\mu_2}{1+\mu_1\mu_2} cos\theta$$

$$2\mu_2 sin\theta = (1-\mu_1\mu_2)cos\theta$$

$$tan\theta = \frac{1-\mu_1\mu_2}{2\mu_2}$$

2. The foot of a ladder length of 9m and mass 25kg rests on a rough horizontal surface while the upper end rests in contact with a rough vertical wall. The ladder being in vertical plane perpendicular to the wall. If the first rung is 30cm from the foot and the rest at the interval of 30cm. find the highest rung to which a man of mass 75kg can climb without causing the ladder to slip. When the ladder is inclined at 60° to the horizontal and the coefficient of friction at each end is 0.25

Solution



$(\uparrow)\ R_B + {}^1/_4 R_A = 25g + 75g \ldots\ldots (i)$

$(\rightarrow)\ R_A = {}^1/_4 R_B \ldots\ldots (ii)$

$R_B = 4R_A$ put into (i)

$4R_A + {}^1/_3 R_A = 980$

$R_A = \frac{3}{13}(980) = 226.154N$

Taking moments about B

$R_A\ x\ 9sin\theta + {}^1/_4 R_A x 9cos\theta = 25gx4.5cos\theta + 75gycos\theta$

$226.154\ x\ 9sin60 + {}^1/_4 x226.154x9cos60 = 25gx4.5cos60 + 75gycos60$

$y = 4m$

Number of rungs$= \frac{4}{0.3} = 13$

3. A uniform ladder of length 2l and weight W rests in a vertical plane with one end on a rough horizontal ground and the other against a rough vertical wall, the angle of friction being respectively $tan^{-1}({}^1/_3)$ and $tan^{-1}({}^1/_2)$.

UNEB 2005 No.16

(a) Find the inclination of the ladder to the horizontal when it is in limiting equilibrium at either end
(b) A man of weight 10 times that of the ladder begins to ascent it . How far will he climb before the ladder slips

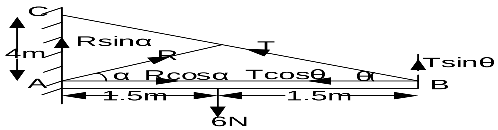
Solution

Exercise 23C

BEAMS HINGED AND MAINTAINED IN A HORIZONTAL POSITION

1. A Uniform beam AB, 3.0m long and of weight 6N is hinged at a wall at A and is held stationary in a horizontal position by a rope attached to B and joined to a point C on the wall, 4.0m vertically above A. Find
   (i) the tension T in the rope
   (ii) the magnitude and direction of the Reaction R at the hinge.

Solution



tan$\theta = \frac{4}{3}$ θ= 53.13°

Taking moments about A at equilibrium

$(Tsin53.13)\ x\ 3\ =\ 6\ x\ 1.5$

$T\ =\ 3.75N$

$(\uparrow)\ Rsin\alpha\ +\ Tsin\theta\ =\ 6$

$Rsin\alpha = \ 6 - 3.75sin53.13$

$Rsin\alpha\ =\ 3$-------------------------i

$(\rightarrow)\ Rcos\alpha\ =\ Tcos\theta$

$Rcos\alpha\ =\ 3.75cos53.13$

$Rcos\alpha = 2.238$------------------------ii

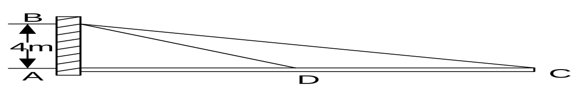
i/ii tan$\alpha = \frac{3}{2.238}$ $\alpha\ =\ 53.3°$

Put into 1; $Rsin53.3\ = 3$

$R\ =\ 3.74N$

The reaction at A is 3.74N at 53.28° to the beam

2. A uniform beam AC of mass 8kg and length 8m is hinged at A and maintained in equilibrium by two strings attached to it at points C and D as shown below. The tension in BC is twice that in BD, $\overline{AB} = 4m$, $\overline{AD} = {}^{3}/_{4}AC$, **UNEB 2008 No.16**



Find;

(i) Tension in the string BC

(ii) Magnitude and direction of the resultant force at the hinge

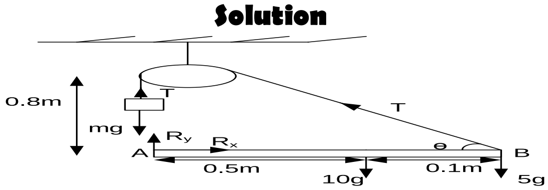
Solution

2. A rod of length 0.6m long and mass 10kg is hinged at A. its center of mass is 0.5m from A, alight inextensible string attached at B passes over a fixed pulley 0.8m above A and supports a mass M hanging freely. If a mass of 5kg is attached at B so as to keep the rod in a horizontal position, find the

(i) Value of m

(ii) Reaction at the hinge **UNEB 1999 No.14**

Solution



$tan\theta = \frac{0.8}{0.6} = \frac{4}{3},\ \ sin\theta = \frac{4}{5},\ \ cos\theta = \frac{3}{5}$

For 2kg mass: $T = mg \ldots\ldots (i)$

For the beam: Taking moments about A

$$0.6xTsin\theta = 5gx0.6 + 10gx0.5 \ldots\ldots. (ii)$$

$$0.6xTx\frac{4}{5} = 5gx0.6 + 10gx0.5$$

$$T = \frac{50}{3}g = 163.33N$$

$$T = g$$

$163.33 = 9.8m$

$$m = 16.67kg$$

$(\uparrow)\ Tsin\theta + R_Y = 10g + 5g$

$$163.33x\frac{4}{5} + R_Y = 15x9.8$$

$$R_Y = 16.336N$$

$(\rightarrow)\ Tcos\theta = R_X$

$$163.33x\frac{3}{5} = R_X$$

$$R_X = 97.998N$$

$$R = \sqrt{(R_X)^2 + (R_Y)^2} = \sqrt{(97.998)^2 + (16.336)^2}$$

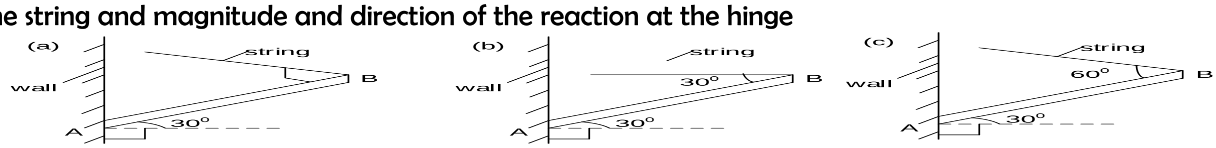
$$R = 99.35N$$

$$\alpha = tan^{-1}\left(\frac{R_Y}{R_X}\right) = tan^{-1}\left(\frac{16.336}{97.998}\right) = 9.34°$$

Reaction at B is $99.35N$ at $9.34°$ to the beam.

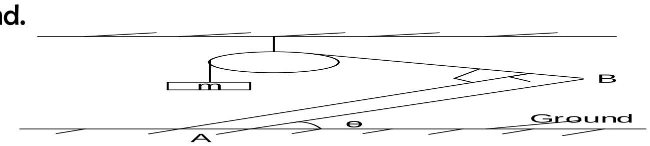
BEAMS HINGED AND MAINTAINED AT AN ANGLE

1. Each of the following diagrams shows a uniform rod of mass 5kg and length 6m freely hinged at A to the vertical wall. A string attached to B keeps the rod in equilibrium. For each case, find the tension in the string and magnitude and direction of the reaction at the hinge



Solution

2. A uniform rod AB of mass 5kg is smoothly hinged on the ground at point A. The rod making an angle θ with the horizontal ground is kept in equilibrium by a light inelastic string attached to point B. The string which makes $90°$ with the rod passes over a smooth fixed pulley and carries a stationary mass m of 2kg at its other end.

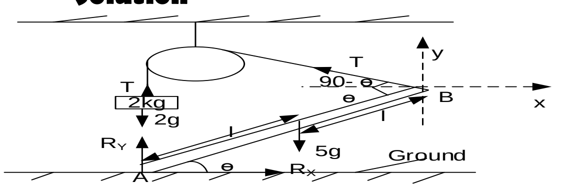


Show that

a. $cos\theta = \frac{4}{5}$

b. the magnitude of the reaction at the hinge is $\frac{49}{5}\sqrt{13}N$

Solution



For 2kg mass: $T = 2g \ldots\ldots(i)$

For the beam: Taking moments about A

$$Tx2l = 5gxlcos\theta \ldots\ldots(ii)$$

$$2gx2l = 5gxlcos\theta$$

$$cos\theta = \frac{4}{5}$$

$$(\uparrow)\ Tsin(90-\theta) + R_Y = 5g$$

$$Tcos\theta + R_Y = 5g$$

$$2gx\frac{4}{5} + R_Y = 5g$$

$$R_Y = \frac{17}{5}g = \frac{17}{5}x9.8 = \frac{833}{25}$$

$$(\rightarrow)\ Tcos(90-\theta) = R_X$$

$$Tsin\theta = R_X$$

$$2gx\frac{3}{5} = R_X$$

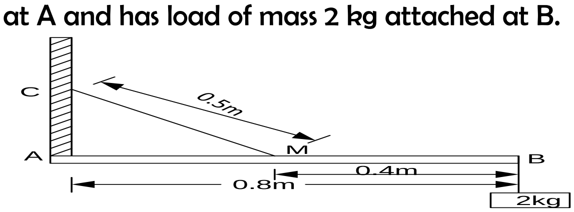
$$R_X = \frac{6}{5}g = \frac{6}{5}x9.8 = \frac{294}{25}$$

$$R = \sqrt{\left(\frac{294}{25}\right)^2 + \left(\frac{833}{25}\right)^2} = \sqrt{\frac{60025x13}{625}} = \frac{49}{5}\sqrt{13}$$

Exercise: 23D

1. A uniform rod AB of length 3m and mass 8kg is freely hired to vertical wall at A. A string BC of length 4m attached to B and to a point C on the wall, keeps the rod in equilibriums. If C is 5 m vertically above A, find the; **UNEB 2019 No.2**
   (i) Tension in the string
   (ii) Magnitude of the normal reaction at A
   An(31.36N, 18.7816N)

2. The figure below shows a uniform beam of length 0.8 metres and 1 kg. The beam is hinged at A and has load of mass 2 kg attached at B.



The beam is held in a horizontal position by a light inextensible string of length 0.5 metres. The string joins the mid-point M of the beam to a point C vertically above A. Find the:

(a) tension in the string.

(b) magnitude and direction of the force exerted by the hinge. **UNEB 2018 No.13**

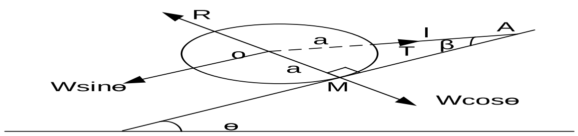
An(81.6667N, 68.21N at 16.70° with AB)

3. A non uniform rod AB of mass 10kg has its centre of gravity at a distance $^1/_4\,AB$ from B. The rod is smoothly hinged at A. It is maintained in equilibrium at $60°$ above the horizontal by a light inextensible string tied at B and at right angle to AB. Calculate the magnitude and direction of the reaction at A. **UNEB 2017 No. 13 An(85.75N, at 68.2° to horizontal)**

BEAMS ON INCLINED PLANES

1. a sphere of radius a and weight W rests on a smooth inclined plane supported by a string of length l with one end attached to a point on the surface of the sphere and the other end fastened to a point on the plane. If the angle of inclination of the plane to the horizontal be θ. Prove that the tension of the string is $\frac{W(a+l)sin\theta}{\sqrt{l^2+2al}}$

Solution

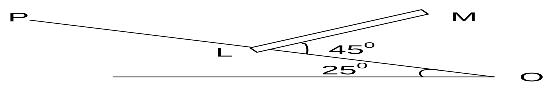


$$AM^2 = OA^2 - OM^2$$
$$AM = \sqrt{(+l)^2 - a^2}$$
$$AM = \sqrt{l^2 + 2al}$$
$$cos\beta = \frac{\sqrt{l^2 + 2al}}{l + a}$$

Also $Wsin\theta = Tcos\beta$

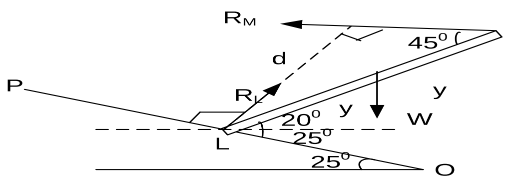
$$T = \frac{Wsin\theta}{cos\beta} = \frac{W(a+l)in\theta}{\sqrt{l^2 + 2al}}$$

2. A uniform rod LM of weight W rests with L on a smooth plane PO of inclination 25° as shown in the diagram below



The angle between LM and the plane is 45°. What force parallel to PO applied at M will keep the rod in equilibrium? **UNEB 2007 No.13 b**

Solution



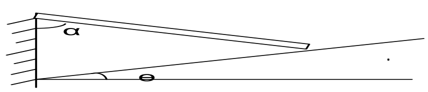
Taking moments about L

$$R_M x d = W x y cos20$$

But $d = 2ysin45$

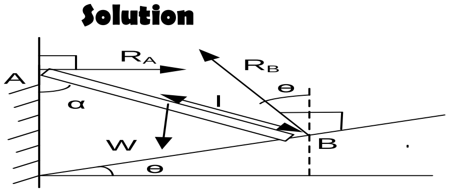
$$R_M x 2ysin45 = W x ycos20$$
$$R_M = 0.6645W$$

3. The diagram shows a uniform ladder resting in equilibrium with its top end against a smooth vertical wall and its base on a smooth inclined plane. The plane makes an angle of θ with the horizontal and the ladder makes an angle of α with the wall.



Prove that, $tan\alpha = 2tan\theta$

Solution



Taking moments about B

$$R_A x 2l\ cos\alpha = W x l sin\alpha$$
$$R_A = \frac{Wtan\alpha}{2} \ldots\ldots(i)$$

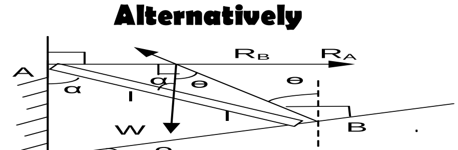
(↑) $R_B cos\theta = W \ldots..(ii)$

(→) $R_B sin\theta = R_A \ldots.(iii)$

$(iii) \div (ii)$ $\frac{R_B sin\theta}{R_B co\theta} = \frac{R_A}{W}$

$$Wtan\theta = R_A$$
$$Wtan\theta = \frac{Wtan\alpha}{2}$$
$$tan\alpha = 2tan\theta$$

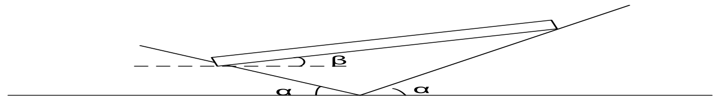
Alternatively



Using cotangent rule for triangle

$$l\ cot\theta - l\ cot90° = 2l\ cot\alpha$$
$$cot\theta = 2cot\alpha$$
$$\frac{1}{tan\theta} = \frac{2}{tan\alpha}$$
$$tan\alpha = 2tan\theta$$

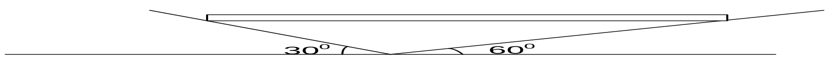
4. A rod AB of length l has its centre of gravity at a point G where $AG = \frac{1}{4}l$. The rod rests in equilibrium in a vertical plane at an angle β to the horizontal with its ends in contact with two inclined planes whose line of intersection is perpendicular to the rod. If the planes are smooth and equally inclined at angle α to the horizontal



Show that $2tan\alpha\ tan\beta = 1$ and reaction on each plane is $\frac{w}{1+cos\alpha}$

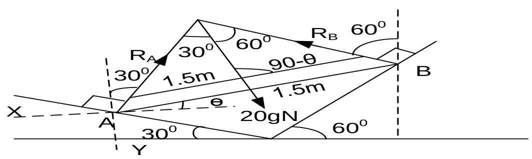
Solution

5. A uniform rod 3m long and of mass 20kg is placed on two smooth planes inclined at $30°$ and $60°$ to the horizontal.



Find the reaction on each plane and the inclination of the rod to the horizontal when it is in equilibrium

Solution



(↑) $R_Acos30 + R_Bcos60 = 20g$

$R_B = 40g - R_A\sqrt{3} \ldots\ldots(i)$

(→) $R_Bsin60 = R_Asin30$

$R_B\sqrt{3} = R_A \ldots.(ii)$

$R_B = 40g - R_B\sqrt{3}x\sqrt{3}$

$R_B = 98N$

$R_A = 98\sqrt{3} = 169.74N$

Using cotangent rule for triangle

$1.5cot30 - 1.5cot60 = 3cot(90 - \theta)$

$$1.5\sqrt{3} - 1.5\frac{1}{\sqrt{3}} = 3tan\theta$$

$\theta = 30°$

Exercise 23E

CHAPTER 14: SIMPLE HARMONIC MOTION

This is a periodic motion of a body whose acceleration is directly proportional to the displacement from a fixed point and is directed towards the fixed point.

$$a \; \alpha - x$$
$$a = -\omega^2 x$$

The negative signs means the acceleration and the displacement are always in opposite direction.

MAXIMUM ACCELERATION

$a_{max} = -\omega^2 r$ where **r** is the acceleration

Force, F

$$F = ma = m\omega^2 x$$

Maximum Force, F_{max}

$$F = ma_{max} = m\omega^2 r$$

VELOCITY IN TERMS OF DISPLACEMENT

Velocity of a body executing S.H.M can be expressed as a function of displacement x. this is obtained from the acceleration

$a = -\omega^2 x$

$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$

but $\frac{dx}{dt} = v$

$a = \frac{dv}{dx}$

$v \cdot \frac{d}{dx} = -\omega^2 x$

$vdv = -\omega^2 xdx$

integrating both sides

$\int v\, dv = -\omega^2 \int x\, dx$

$\frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + C$ ....[1]

Where C is a constant of integration

At momentary rest $v = 0$, $x = r$(amplitude)

$$\frac{0^2}{2} = -\frac{\omega^2 r^2}{2} + C$$

$C = \frac{\omega^2 r^2}{2}$

Put into [1]

$\frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + \frac{\omega^2 r^2}{2}$

$v^2 = \omega^2 r^2 - \omega^2 x^2$

$\boxed{v^2 = \omega^2(r^2 - x^2)}$

Velocity is maximum when x =0

$v^2 = \omega^2 r^2$

$\boxed{v_{max} = \omega r}$

DISPLACEMENT AT ANY TIME ,t

$$\frac{dx}{dt} = v$$
$$\frac{dx}{dt} = \omega\sqrt{(r^2 - x^2)}$$
$$\int \frac{dx}{\sqrt{(r^2 - x^2)}} = \int \omega\, dt$$
$$sin^{-1}\frac{x}{r} = \omega t + \varepsilon$$
$$x = rsin(\omega t + \varepsilon)$$

When timing begins at the centre, $t = 0, x = 0$

$\boxed{x = rsi\omega t}$ particle moves away from centre

When timing begins at the amplitude, $t = 0, x = r$

$\boxed{x = rcos\omega t}$ particle moves towards the centre

Period T

This is the time taken for one complete oscillation.

$$T = \frac{distance}{speed}$$

$= \frac{2\pi}{v}$ but $v = r\omega$

$$T = \frac{2\pi r}{\omega r}$$
$$T = \frac{2\pi}{\omega}$$

Examples

1. A particles moves in a straight line with S.H.M about mean position O with a periodic time of $\frac{\pi}{2}$ s. Find the magnitude of the acceleration of the particle when 1m from O

Solution

2. A Particle moving with S.H.M has velocities of $4ms^{-1}$ and $3ms^{-1}$ at distances of 3m and 4m respectively from equilibrium position. Find
   i) amplitude , ii) period ,

Solution

(i) $v = 4ms^{-1}, x = 3m$ and
Using $v^2 = \omega^2(r^2 - x^2)$
$4^2 = \omega^2(r^2 - 3^2)$
$16 = \omega^2(r^2 - 9)$------- (1)
Also $v = 3ms^{-1}, x = 4m$
$3^2 = \omega^2(r^2 - 4^2)$
$9 = \omega^2(r^2 - 16)$------- (2)
Equation 1 divide by 2
$$\frac{16}{9} = \frac{\omega^2(r^2 - 9)}{\omega^2(r^2 - 16)}$$
$16(r^2 - 16) = 9(r^2 - 9)$
$r^2 = 25$
$r = 5m$; Amplitude $= 5m$

(ii) Using eqn(1)
$16 = \omega^2(5^2 - 9)$
$\omega^2 = 1$
$\omega = 1rads^{-1}$
But $T = \frac{2\pi}{\omega} = \frac{2x\frac{22}{7}}{1} = 6.28second$

3. A particle moves with s.h.m about a mean position O. The particle is initially projected from O with speed $\frac{\pi}{6}$ m/s and just reaches a point A, 2m form O.
   (a) Find how far the particle is from O, 3 seconds after projection
   (b) How many seconds after projection is the particle a distance of 1m from O
   (i) For the first time (ii) Second time (iii) Third time

Solution

(a) At equilibrium position, $v_{max} = \omega r$
$$\frac{\pi}{6} = \omega x2$$
$$\omega = \frac{\pi}{12}\ rad/s$$
$x = rsin\omega t$ since particle move away from O
$$x = 2sin\left(\frac{\pi}{12}x3\right) = 1.414m$$
$$x = rsin\omega t$$
$$1 = 2sin\frac{\pi}{12}t$$
$$\frac{\pi}{12}t = sin^{-1}(0.5)$$
$$\frac{\pi}{12}t = 30°, 150°, 210°$$
$$t = 2s, 10s, 14s$$

4. A particle is released from rest at point A, 1m from a second point O. The particle accelerates towards O and moves with S.H.M of time period 12s and O as the centre of oscillation
   (a) Find how far the particle is from O, 1 seconds after release
   (b) How many seconds after release is the particle at the midpoint of OA
   (i) For the first time (ii) Second time

Solution

5. A particle of mass 2kg moving with simple harmonic motion along the x-axis, is attracted towards the origin O by a force of 32x newton's. Initially the particle is at rest at x=20. Find the
   (a) Amplitude and period of the oscillation
   (b) Velocity of the particle at any time, t> 0
   (c) Speed when $t = \frac{\pi}{4}s$ **Uneb 2016 No16**

Solution

(a) $F = m\omega^2 x$
$$32x = 2\omega^2 x$$
$$\omega = 4\ rad/s$$
$$T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = 1.571s$$
$$v^2 = \omega^2(r^2 - x^2)$$
$$0 = 4^2(r^2 - 20^2)$$
$$r = 20m$$

(b) $x = rcos\omega t$
$$v = \frac{d}{dt}(rcos\omega t)$$
$$v = -r\omega sin\omega t = -20x4sin4t$$
$$v = -80in4t$$
(c) Speed$= -80sin4x\frac{\pi}{4} = om/s$

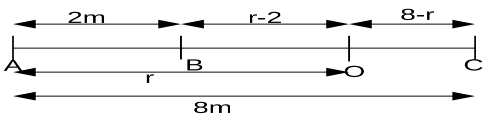
6. The points A, O, B, C lie in that order on a straight line with $AO = OC = 6c$ and $OB = 5cm$. A particle performs S.H.M of period 3s and amplitude 6cm between A and C. Find the time taken for the particle to travel from A to B.

Solution

UNEB

1. A particle passes through 3 points A, B and C in that order with velocity $0m/s$, $2m/s$ and $-1m/s$ respectively. The particle is moving with S.H.M in a straight line. What is the amplitude and period of the motion If $\overline{AB} = 2m$ and $\overline{AC} = 8m$. **Uneb 1988 NO.7.**

Solution



At B: $v = 2ms^{-1}, x = r - 2$

Using $v^2 = \omega^2(r^2 - x^2)$

$2^2 = \omega^2(r^2 - (r-2)^2)$

$1 = \omega^2(-1)$------- (1)

At C: $v = -1ms^{-1}, x = 8 - r$

$(-1)^2 = \omega^2(r^2 - (8-r)^2)$

$1 = \omega^2(16r - 64)$------- (2)

Equation 1 divide by 2

$$\frac{1}{1} = \frac{\omega^2(r-1)}{\omega^2(16r-64)}$$

$(16r - 64) = (r - 1)$

$r = 4.2$

; Amplitude $= 4.2m$

Using $1 = \omega^2(r-1)$

$1 = \omega^2(4.2 - 1)$

$$\omega^2 = \frac{1}{3.2}$$

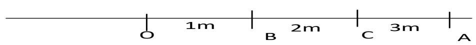
$\omega = 0.56rads^{-1}$

But $T = \frac{2\pi}{\omega} = \frac{2x\frac{22}{7}}{0.56} = 11.24second$

2. A particle is performing S.H.M motion with centre O, amplitude 6m and period 2π. Points B and C lie between O and A with $\overline{OB} = 1m$, $\overline{OC} = 3m$ and $\overline{OA} = 6m$. Find the least time taken while travelling from **Uneb 2004 NO.1**

(i) A to B (ii) A to C

Solution



$\omega = \frac{2\pi}{2\pi} = 1s$

$x = rcos\omega t$

$1 = 6cos1xt$

$t = 1.403s$

(ii) $x = rcos\omega t$

$3 = 6cos1xt$

$t = 1.047s$

Exercise 24A

1. A Particle moving with S.H.M about a mean position O has velocities of 5ms^{-1} and 8ms^{-1} at distances of 16m and 12m respectively from O. Find; **Uneb 1993 NO.5**
   i) amplitude ,
   ii) period **An(** 4π s)
2. A particle is describing S.H.M in a straight line directed towards a fixed point O. When it's distance from O is 3m, its velocity is 25m/s and acceleration is 75ms^{-2}. Determine the
   i) period and amplitude of oscillation
   ii) Time taken by particle to reach O
   iii) Velocity of the particle as it passes through O **Uneb 1999 NO.11 Nov/Dec**

   An($\frac{2\pi}{5}s$, $\mathbf{5.83m, 0.108s, 29.15m/s}$)
   (i))
3. The velocity of a particle at any time t is given by an equation $v(t) = -a\omega\ sin\omega t + b\omega cos\omega t$
   (i) Find the expression for the displacement x at any time given that $x = 0$ when time $t = 0$
   (ii) Show that the motion of the particle is simple harmonic

   Uneb 2009 NO.8 An($x = acos\omega t + bsin\omega t, \ddot{x} = -\boldsymbol{\omega^2 x}$)
4. A Particle is moving with S.H.M. When the particle is 15m from equilibrium , its speed is 6ms^{-1} and when the particle is 13m from the equilibrium, its speed is 9ms^{-1} . Find the amplitude of the motion. **Uneb 2019 NO.8 An(16.4256m)**
5. A particles moves with S.H.M about mean position O with a periodic time of $\frac{2\pi}{3}$ s. When the particle is 0.8m from one extreme end, its speed is 3.6ms^{-1}. Find the amplitude of the motion. **Uneb 2020 No.4 An(**1.3)

HOOKE'S LAW AND ELASTIC STRINGS

Hooke's law states that the tension in a stretched string is proportional to the extension from its natural (un-stretched) length

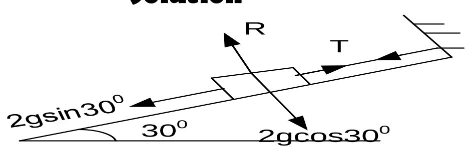
$$\boxed{T = \lambda\frac{e}{l}}$$

λ is modulus of elasticity of the string

Examples

1. An elastic string of natural length 4m and modulus 25N. find
   (i) The tension in the string when the extension is 20cm
   (ii) The extension of the string when the tension is 6N
   Solution

2. A smooth surface inclined at 30° to the horizontal has a body A of mass 2kg that is held at rest on the surface by a light elasti c string which has one end attached to A and the other to a fixed point on the surface 1.5m away from A up to the line of greatest slope. If the modulus of elasticity of the string is 2gN, find its natural length.
   Solution

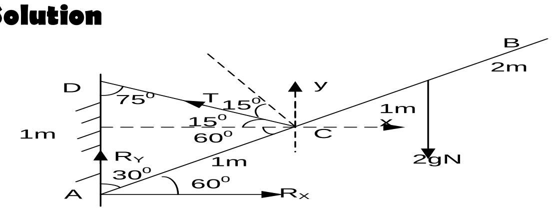


$T = \lambda\frac{e}{l}$

$2gsi30 = 2gx\frac{1.5 - l}{l}$

$l = 1m$

3. A uniform rod AB of length 4m and mass 2kg rests at 30° to the smooth vertical wall and is supported with end A in contact with the wall by an elastic string connecting to a point C on the rod to a point D on the wall vertically above A. if the natural length of the string is 0.4m and the distance AC and AD are 1m. find the;
   (i) Tension in the string and reaction at A **(ii)** Modulus of elasticity
   Solution



Taking moments about A: $1xTcos15 = 2gx2cos60$

$T = 20.2914N$

$(\uparrow)\ Tsin15 + R_Y = 2g$

$20.2914sin15 + R_Y = 2g$

$R_Y = 14.3482\text{N}$

$(\rightarrow)\ Tcos15 = R_X$

$R_X = 20.2914os15 = 19.6N$

$R = \sqrt{(R_X)^2 + (R_Y)^2} = \sqrt{(14.3482)^2 + (19.6)^2}$

$R = 24.291N$

Using sine rule: $\frac{l^1}{sin30} = \frac{1}{sin75}$

$l^1 = 0.518m$

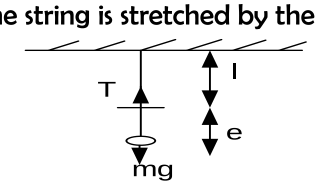
$T = \lambda\frac{e}{l}$

$75.729 = \lambda\frac{0.518 - 0.4}{0.4}$

$\lambda = 257.58N$

EQUILIBRIUM OF A SUSPENDED BODY

When an elastic string has one end fixed and a mass attached to its other end so that the mass is suspended in equilibrium, the string is stretched by the force due to the mass.



$T = mg$

$\boxed{mg = \lambda\frac{e}{l}}$ At equilibrium

Examples

1. A light elastic string of natural length 65cm has one end fixed and a mass of 500g freely suspended from the other. Find the modulus of elasticity of the string if the total length of the string in the equilibrium position is 85cm

Solution

$$mg = \lambda\frac{e}{l}$$

$$0.5x9.8 = \lambda x\frac{0.85 - 0.65}{0.65}$$

$$\lambda = 15.93N$$

2. A light elastic spring of natural length 1.5m has one end fixed and a mass of 400g freely suspended from the other. The modulus of the spring is 44.1N.
   (i) Find the extension of the spring when the body hangs in the equilibrium
   (ii) The mass is pulled vertically downwards a distance of 10cm and released, find the acceleration of the body when released.

Solution

POTENTIAL ENERGY STORED IN AN ELASTIC STRING

Work done $= average\ force\ x\ extnsion\ = \lambda\frac{e^2}{2l}$

Examples

1. An elastic string of natural length 6.4m and modulus 55N. Find the work done in stretching it from 6.4m to a length of 6.8m.

Solution

Work done $= \lambda\frac{e^2}{2l}$

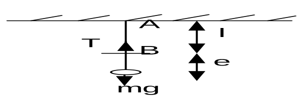
Work done $= 55x\frac{0.4^2}{2x6.4} = 0.688J$

2. An elastic string of natural length 1.2m and modulus of elasticity 8N is stretched until the extending force is 6N. Find the extension and work done. **Uneb 1998 No.6**

Solution

3. A light elastic string of natural length 1.2m has one end fixed and a mass of 5kg freely suspended from the other. The modulus of the string is such that a 5kg mass hanging vertically would stretch the string by 15cm. The mass is held at A and allowed to fall vertically, how far from A will it come to rest.

Solution



$$mg = \lambda\frac{e}{l}$$

$$5x9.8 = \lambda\frac{0.15}{1.2}$$

$$\lambda = 392N$$

At $u = 0m/s, s = 1.2m, g = 9.8ms^{-2}$

$v^2 = u^2 + 2gs = 0^2 + 2x9.8x12$

$v = 23.52m/s$

$w = k.e + p.e$

$$\lambda\frac{e^2}{2l} = \frac{mv^2}{2} + mge$$

$$392x\frac{e^2}{2x1.2} = \frac{5x23.52}{2} + 5x9.8e$$

$163.333e^2 - 49e - 58.5 = 0$

$e = 0.769m \quad o\ e = -0.468m$

Depth $= 1.2 + 0.769 = 1.969$m

Exercise 25

1. An elastic string of natural length 1m and modulus 20N. Find the tension in the string when the extension is 20cm. **An(4N)**
2. A body of mass 4kg lies on a smooth horizontal surface and is connected to a point O on the surface by a light elastic string of natural length 64cm and modulus 25N. When the body moves in a horizontal circular path about O with constant speed of vm/s, the extension in the string is 36cm. Find v. **An(1.875m/s)**
3. A body of mass 5kg lies on a smooth horizontal surface and is connected to a point O on the

surface by a light elastic string of natural length 2m and modulus 30N. When the body moves in a horizontal circular path about O with constant speed of 3m/s, find the extension in the string. **An(1m)**

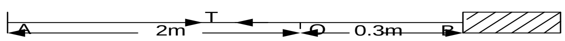
4. An elastic string of natural length 2m and modulus 10N. Find the energy store when it is extended to a length of 3m **An(2.5J)**

SIMPLE HARMONIC MOTION IN STRINGS

Examples

1. One end of a light elastic string of natural length 2m and modulus 10N is fixed to a point A on a smooth horizontal surface. A body of mass 200g is attached to the other end of the string and is held at rest at point B on the surface causing the spring to extend by 30cm. show that when released, the body will move with S.H.M. State its amplitude and find the max speed.

Solution



$$T = \lambda\frac{x}{l} = 10\frac{x}{2} = 5x$$

$F = ma$

$5x = -0.2a$

$a = -25x$

Since $a = -\omega^2$

$\omega^2 = 25$

$\omega = 5rad\ s^{-1}$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{5}s \qquad and \quad r = 0.3m$$

$$v_{max} = \omega r = 5x0.3 = 1.5m/s$$

2. A particle of mass m is attached by means of light strings AP and BP of the same natural length a metres and modulus of elasticity mgN and 2mgN respectively, to points A and B on a smooth horizontal surface. The particle is released form the midpoint of $\overline{AB}$ where $\overline{AB} = 3a$. Show that the subsequent motion is simple harmonic with period $T = \left(\frac{4\pi^2 a}{3g}\right)^{1/2}$. **Uneb 2001, N.o14b**

Solution

3. A particle of mass 1.5kg lies on a smooth horizontal table and is attached to two light elastic strings fixed at points P and Q 12m apart. The strings are of natural length 4m and 5m and their modulus are λ and 2.5λ respectively.
   (a) Show that the particle stays in equilibrium at a point R mid-way between P and Q
   (b) If the particle is held at some point S in the line PQ with $PS = 4.8m$ and then released. Show that the particle performs S.H.M and find the
      (i) Period of oscillation
      (ii) Velocity when the particle is 5.5m from P **Uneb 2008, N.13**

Solution

(a) At equilibrium $\lambda\frac{e_1}{4} = 2.5\lambda\frac{e_2}{5}$

$$e_1 = 2e_2 \ldots\ldots (i)$$

$$e_1 + e_2 + 4 + 5 = 12$$

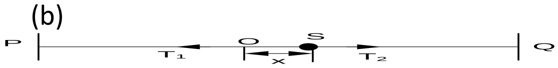
$$2e_2 + e_2 + 9 = 12$$

$$e_2 = 1$$

$$e_1 = 2$$

At mid-point R; $e_1 + 4 = e_1 + 5 = 6$

(b)



$F = ma$

$T_2 - T_1 = ma$

$$2.5\lambda\left(\frac{1-x}{5}\right) - \lambda\left(\frac{2+x}{4}\right) = 15\ddot{x}$$

$$\ddot{x} = -\frac{\lambda}{2}x$$

It is in the form of $a = -\omega^2 x$ hence S.H.M

$$\omega^2 = \frac{\lambda}{2} \qquad \therefore \omega = \left(\frac{\lambda}{2}\right)^{1/2}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(\frac{\lambda}{2}\right)^{1/2}} = \left(\frac{8\pi^2}{\lambda}\right)^{1/2}$$

$$v^2 = \omega^2(r^2 - x^2)$$

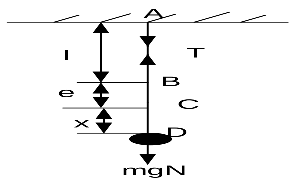
when 5.5m from P, $x = 0.5m$

$$v = \sqrt{\frac{\lambda}{2}(1.2^2 - 0.5^2)} = \sqrt{0.595\lambda}m/s$$

ELASTIC STRINGS OR SPRINGS HANGING VERTICALLY

1. A particle of mass m is suspended by a string from a fixed point A and has natural length l. If the string is extended from B and C where $BC = e$ and this extension is due to weight of the body (mg), $CD = x$ is the length a particle is pulled vertically downward

Solution



At equilibrium; $T = mg$

$$T = \lambda\frac{e}{l}$$

When pulled a distance, x: $T - T_1 = ma$

$$\lambda\frac{e}{l} - \lambda\left(\frac{e+x}{l}\right) = m\ddot{}$$

$$-\lambda\frac{x}{l} = m\ddot{x}$$

$$\ddot{x} = -\frac{\lambda}{ml}x$$

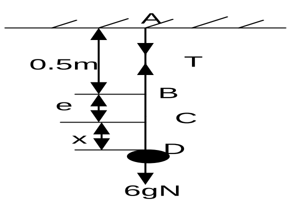
It is in the form of $a = -\omega^2 x$ hence S.H.M

$$\omega^2 = \frac{\lambda}{ml} \quad \therefore \omega = \sqrt{\frac{\lambda}{ml}}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{\lambda}{ml}}} = 2\pi\sqrt{\frac{ml}{\lambda}}$$

2. A light elastic spring of natural length 50cm and modulus 20gN, hangs vertically with its upper end fixed and a body of mass 6kg attached to its lower end. The body initially rests in equilibrium and then pulled down a distance of 25cm and released.
   (i) Show that the ensuing motion will simple harmonic and
   (ii) find the period of motion and the maximum speed of the body.

Solution



(i) At equilibrium; $T = mg$

$$6g = 20gx\frac{e}{0.5}$$

$$e = 0.15m$$

When pulled a distance, x:

$$T - T_1 = ma$$

$$6g - \lambda\left(\frac{e+x}{0.5}\right) = 6\ddot{x}$$

$$6g - 20g\left(\frac{0.15+x}{0.5}\right) = 6\ddot{x}$$

$$\ddot{x} = -\frac{196}{3}x$$

It is in the form of $a = -\omega^2 x$ hence S.H.M

(ii) $\omega^2 = \frac{196}{3}$

$$\omega = \left(\frac{196}{3}\right)^{1/2} = 8.083rad/s$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{8.083} = 0.773s$$

$$v_{max} = \omega r = 8.083x0.25 = 2.021m/s$$

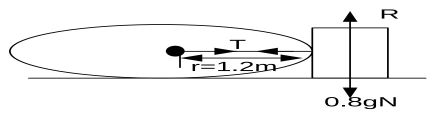
CHAPTER 15: CIRCULAR MOTION

CIRCULAR MOTION ON A SMOOTH HORIZONTAL SURFACE

Examples

1. A particle of mass 0.8kg is attached to one end of a light in extensible string of length 1.2m, the other end is fixed to a point P on a smooth horizontal table. The particles is set moving in a circular path. If the speed of the particle is 16m/s.
   (i) Determine the tension in the string and reaction on the table.
   (ii) If the string snaps, when the tension in the string exceeds 100N, find the greatest angular velocity at which the particle can travel

Solution



(i) $T = F = \frac{m\,v^2}{r} = \frac{0.8x\;16^2}{1.2} = 170.667N$

$R = 0.8gN = 0.8x9.8 = 7.84N$

(ii) $T = m\omega^2 r$

$100 = 0.8x\omega^2 x1.2$

$\omega = 10.21rads^{-1}$

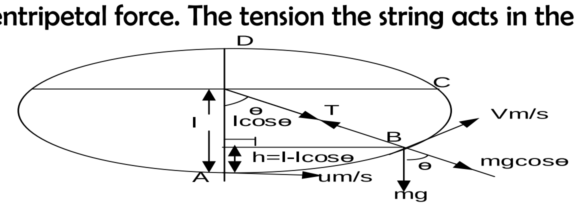
2. A ball is tied on an elastic string of natural length 30m to a fixed point on a smooth horizontal table upon which a ball is describing a circle around a point at a constant speed. If the modulus of elasticity of the string is 100 times the weight of the ball and the number of revolution per minute is 20. Show that the extension in the string is approximately 4.7m

Solution

MOTION IN A VERTICAL CYCLE

PARTICLE IN FOURTH QUADRANT

Consider a body of mass m attached to a string (light rod) of length l and whirled in a vertical circle with a constant speed V. If there is no air resistance to the motion, then the net force towards the centre is the centripetal force. The tension the string acts in the same way as the reaction R



At equilibrium at B: $T - mgcos\theta = \frac{m\,v^2}{l}$

$T = \frac{m\,v^2}{l} + mgcos\theta$..........(1)

But $v^2 = u^2 + 2as$

$a = -g, s = h = l - lcos\theta$

$\boxed{v^2 = u^2 - 2g(l - lcos\theta)} \ldots\ldots(2)$

When the particle comes momentarily to rest at some point A, then $v = 0$

$0 = u^2 - 2g(l - lcos\theta)$

$\boxed{cos\theta = 1 - \frac{u^2}{2gl}} \ldots.(3)$

if the particle is attached to a rod, it can complete circles when $v > 0$ and $\theta = 180°$

$v^2 = u^2 - 2g(-lcos\theta)$

$u^2 - 2g(l - lcos180) > 0$

$u^2 > 2g(l + l)$

$\boxed{u^2 > 4gl} \ldots\ldots(4)$

Put (2) into (1) : $T = \frac{m\,[u^2 - 2g(l - lcos\theta)]}{l} + mgcos\theta$

$T = \frac{mu^2 - 2mgl + 2mglcos\theta + mglcos\theta}{l}$

$\boxed{T = \frac{m}{l}(u^2 - 2gl + 3glcos\theta)} \ldots.(5)$

if the particle is attached to a string, it can complete circles when $T > 0$ and $\theta = 180°$

$T = \frac{m}{l}(u^2 - 2gl + 3glcos\theta)$

$\frac{m}{l}(u^2 - 2gl + 3glcos180) > 0$

$(u^2 - 2gl - 3gl) > 0$

$\boxed{u^2 > 5gl} \ldots\ldots(6)$

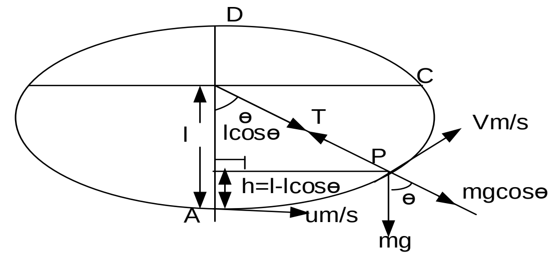
Examples

1. A particle P of mass 5kg is suspended from a fixed point O by a light inextensible string of length 1m. The particle is projected form its lowest position at the point A, with a horizontal speed of 4m/s. when angle $AOP = 60°$. Find ;

(a) Speed at P

(b) The tension in the string at P

Solution



$$T.M.E_A = T.M.E_P$$

$$5\left(\frac{1}{2}x4^2 + 9.8x0\right) = 5\left[\frac{1}{2}xv^2 + 9.8x(1 - cos60)\right]$$

$$80 = 5xv^2 + 49$$

$$v = 2.49m/s$$

$$T = \frac{m\,v^2}{l} + mgcos\theta$$

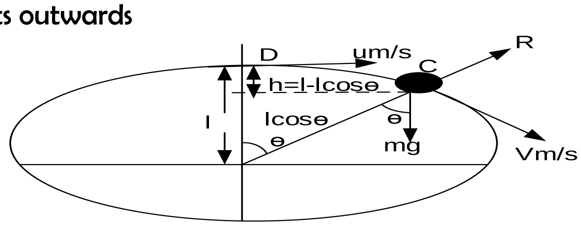
$$T = \frac{5x\,(2.49)^2}{1} + 5x9.8cos60 = 55.5N$$

2. A light rod of length 2m is pivoted at one end, O and has a particle of mass 8kg attached at the other end. The rod is held vertically with the particle at point A, directly below O and the particle is given an initial horizontal speed um/s. Find ;
   (a) An expression in terms of u and θ for the speed of the particle when at point B where $< AOB = \theta$
   (b) Restrictions on u^2 if the particle to perform complete oscillations

Solution

PARTICLE IN FIRST QUADRANT

Consider a body of mass rolled from the top of a sphere of radius l . The normal reaction R Acts outwards



At equilibrium at C: $mgcos\theta - R = \frac{m\,v^2}{l}$

$R = mgcos\theta - \frac{m\,v^2}{l}$..........(1)

But $v^2 = u^2 + 2ah$

$a = g, h = l - lcos\theta$

$\boxed{v^2 = u^2 + 2g(l - lcos\theta)}$ ... .(2)

Putting (2) into (1)

$$R = mgcos\theta - \frac{m\,[u^2 + 2g(l - lcos\theta)]}{l}$$

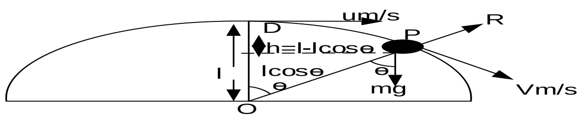
$$R = \frac{mglcos\theta - mu^2 - 2mgl + 2mglos\theta}{l}$$

$\boxed{R = \frac{m}{l}(3glcos\theta - 2gl - u^2)}$ ....(3)

Example

1. A particle P of mass 5kg is slightly disturbed from rest on the top of a smooth hemisphere, radius 4m and centre O, resting with its plane face on the horizontal ground.
   (a) Show that the particle leaves the surface of the hemisphere at the point P, where the angle between the radius PO and the upward vertical is $cos^{-1}\left(\frac{2}{3}\right)$
   (b) Find the speed at P

Solution



At equilibrium at P: $mgcos\theta - R = \frac{m\,v^2}{l}$

$R = mgcos\theta - \frac{m\,v^2}{l}$..........(1)

But $v^2 = u^2 + 2ah$

$u = 0,\ \ a = g, h = l - lcos\theta$

$\boxed{v^2 = 2g(l - lcos\theta)}$ ... .(2)

Putting (2) into (1) : $R = mgcos\theta - \frac{m\,v^2}{l}$

$$R = mgcos\theta - \frac{m\,[2g(l - lcos\theta)]}{l}$$

$$R = \frac{3mglcos\theta - 2mgl}{l}$$

When the particle leaves the surface of the hemisphere, $R = 0$

$$0 = \frac{3mglcos\theta - 2mgl}{l}$$
$$3cos\theta = 2$$
$$\theta = cos^{-1}\left(\frac{2}{3}\right)$$
$$v^2 = 2g(l - l\cos\theta)$$

$$v^2 = 2g\left(l - lx\frac{2}{3}\right)$$
$$v^2 = \frac{2}{3}gl = \frac{2}{3}x9.8x4 = 26.1333$$
$$v = 5.1121m/s$$