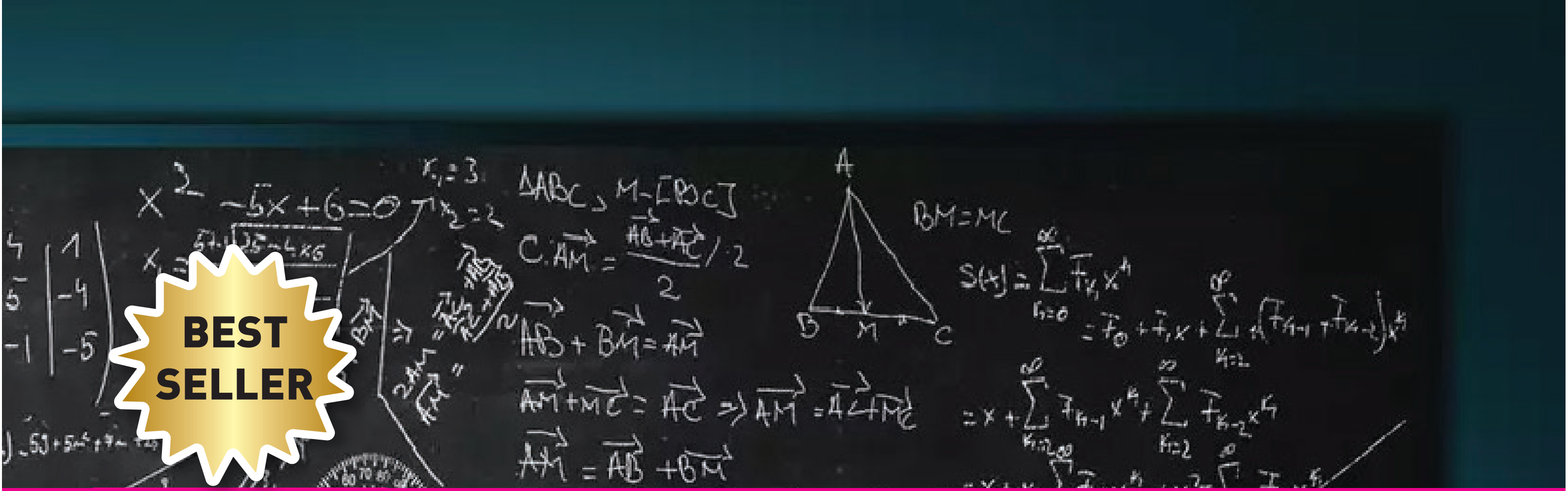
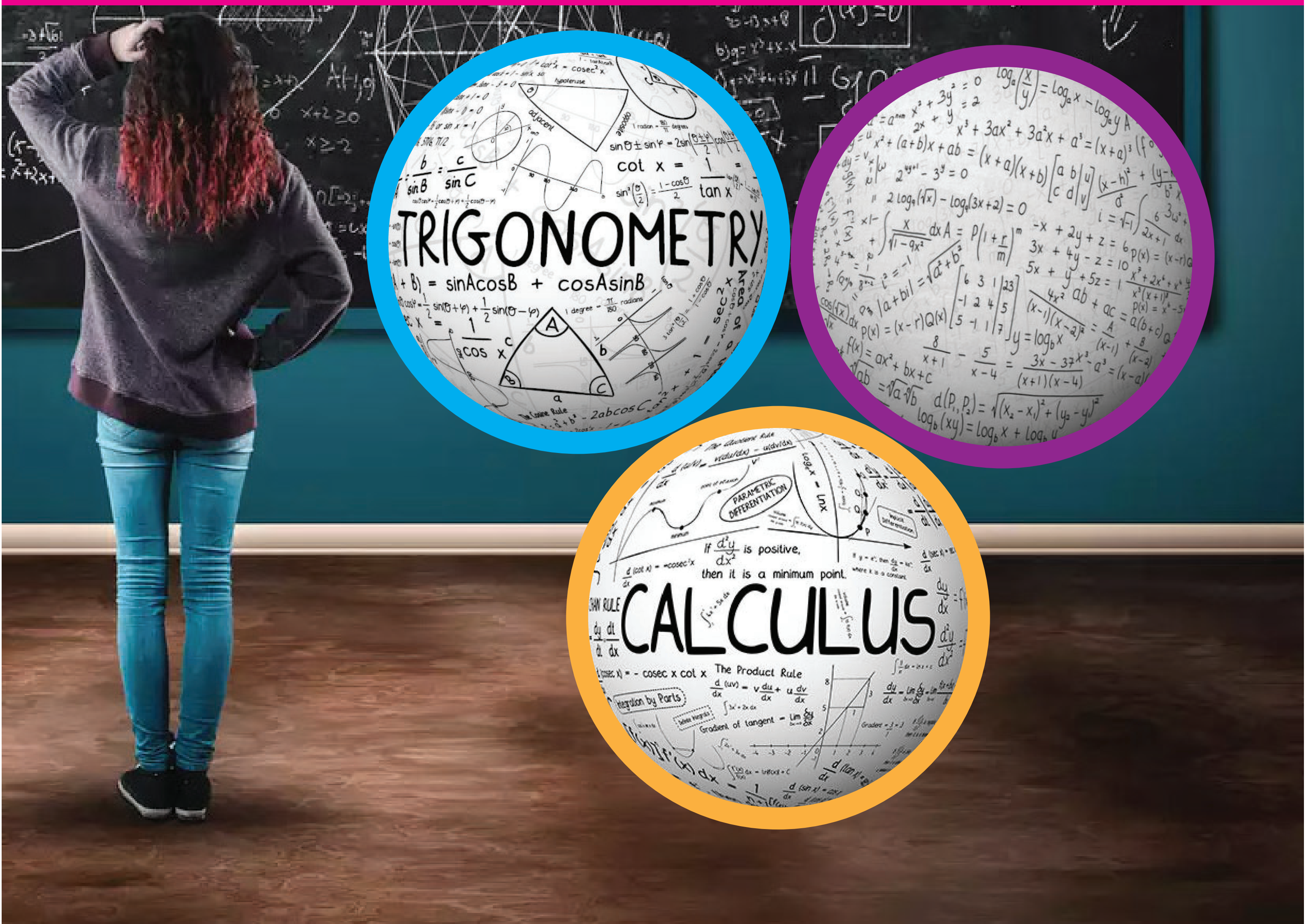


Principles of

PURE MATHEMATICS

KAWUMA FAHAD



IMPORTANT NOTICE

Principles of

Pure Mathematics

KAWUMA FAHAD

+256 772 032 053/ +256 704 145 545

fahadkawuma25@gmail.com

Published by:

Scofield International

+256 782 421 905

Kampala, Uganda

"Quality is our DNA"

PREFACE

Principles of Pure Mathematics is the culmination of years of experience and research right from the time when I was still a high school student to the time during completion of my post graduate studies in engineering from one of the best universities in China.

This book is intended to provide an excellent training in problem-solving and help the reader work logically on the mathematical principles. It in great detail consists of five sections: Algebra, Coordinate geometry, Vectors, Trigonometry and Calculus in relation to the syllabus stipulated by NCDC.

The knowledge of Pure Mathematics forms basis for a number of scenarios in Applied Mathematics. Some proofs in the section of mechanics require knowledge of trigonometry while some questions in numerical methods require knowledge of integration, especially when asked to find the error made in using the trapezium rule which requires finding the exact value of the integral. Also, topics like centre of mass require the knowledge of finding the area under the curve and solids of revolution. It is therefore important for the reader to understand that a number of scenarios of applied mathematics are built on the concepts of this pure mathematics book.

The worked examples in each section have been carefully selected to meet the demands of a wide range of students and teachers. At the end of each topic is a self-evaluation exercise with answers. This is to help the readers widen their experience and build their confidence in problem solving. Some problems require more thought and application and might appear to be quite more demanding for average learners. They should therefore not rule out guidance from their teachers in such situations.

At the end of each section are examination past paper questions to help learners get exposed to the way questions are examined in various topics or subsections by UNEB. My decision not to group them according to their respective topics or subsections is to challenge the experience of the learners in identifying the topic or subsection where the question is coming from.

It is therefore my sincere hope that students, teachers as well as general readers find this book a good, reliable and an indispensable guide to Pure Mathematics.

Finally, all misfortunes, if any in this book are purely my responsibility since it is difficult to claim perfection. I will be glad for any comments or compliments that will be directed to me. For no one is a monopolist of knowledge and no scientific theory is born in vacuum. Every scientist builds on the work of his predecessors.

I take this opportunity to thank all those who have suggested to improve this book. I am indebted to all the individuals who undertook the laborious task of assisting with proof reading and for the invaluable suggestions made throughout the preparation of this book.

CONTENTS

Algebra

Chapter 1 Indices, Logarithms and Surds

INDICES

a^n means $a \times a \times a \times ... \times a$ (n factors)
n is the index (plural indices) while a is the base.
An index is also called a power or an exponent

Basic rules of indices

When m and n are positive rational numbers:

multiplication: $a^m \times a^n = a^{m+n}$
division: $a^m \div a^n = a^{m-n}$
raising to a power: $(a^m)^n = a^{mn}$
zero index: $a^0 = 1$
negative index: $a^{-m} = \frac{1}{a^m}$
Fractional index: $a^{\frac{1}{n}} = \sqrt[n]{a}$
$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$

The basic laws can be illustrated as follows:

$$a^3 \times a^2 = a^{3+2} = a^5$$
$$a^7 \div a^3 = a^{7-3} = a^4$$
$$(a^3)^2 = a^{3\times 2} = a^6$$
$$7^0 = 1$$
$$5^{-2} = \frac{1}{5^2} = \frac{1}{25}$$
$$8^{\frac{1}{3}} = \sqrt[3]{8} = 2$$
$$\sqrt[3]{a^6} = a^{\frac{6}{3}} = a^2$$

Example 1

Evaluate (i) $\left(\frac{81}{256}\right)^{\frac{3}{4}}$ (ii) $\left(\frac{25}{49}\right)^{-\frac{1}{2}}$

Solution

(i) $\left(\frac{81}{256}\right)^{\frac{3}{4}} = \frac{(81)^{\frac{3}{4}}}{(256)^{\frac{3}{4}}} = \frac{(3^4)^{\frac{3}{4}}}{(4^4)^{\frac{3}{4}}} = \frac{3^3}{4^3} = \frac{27}{64}$

(ii) $\left(\frac{25}{49}\right)^{-\frac{1}{2}} = \left(\frac{49}{25}\right)^{\frac{1}{2}} = \frac{(49)^{\frac{1}{2}}}{(25)^{\frac{1}{2}}} = \frac{(7^2)^{\frac{1}{2}}}{(5^2)^{\frac{1}{2}}} = \frac{7}{5}$

Example 2

Simplify $\frac{27^{n+2} - 6\,.\,3^{3n+3}}{3^n.\,9^{n+2}}$

Solution

$$\frac{27^{n+2} - 6\,.\,3^{3n+3}}{3^n.\,9^{n+2}} = \frac{(3^3)^{n+2} - 2\,.\,3^1.\,3^{3n+3}}{3^n.\,(3^2)^{n+2}}$$
$$= \frac{3^{3(n+2)} - 2\,.\,3^{2n+4}}{3^n.\,3^{2(n+2)}}$$
$$= \frac{3^{3n}.\,3^6 - 2\,.\,3^{3n}.\,3^4}{3^{3n}.\,3^4}$$
$$= \frac{3^{3n}(3^6 - 2\,.\,3^4)}{3^{3n}.\,3^4}$$
$$= \frac{3^4(3^2 - 2)}{3^4}$$
$$= 7$$

Example 3

Simplify $\frac{x}{y^{\frac{1}{2}} + x^{\frac{1}{2}}} + \frac{x}{y^{\frac{1}{2}} - x^{\frac{1}{2}}}$

Solution

$$\frac{x}{y^{\frac{1}{2}} + x^{\frac{1}{2}}} + \frac{x}{y^{\frac{1}{2}} - x^{\frac{1}{2}}} = \frac{x\left(y^{\frac{1}{2}} - x^{\frac{1}{2}}\right) + x\left(y^{\frac{1}{2}} + x^{\frac{1}{2}}\right)}{\left(y^{\frac{1}{2}} + x^{\frac{1}{2}}\right)\left(y^{\frac{1}{2}} - x^{\frac{1}{2}}\right)}$$
$$= \frac{xy^{\frac{1}{2}} - x^{\frac{3}{2}} + xy^{\frac{1}{2}} + x^{\frac{3}{2}}}{y - x}$$
$$= \frac{2x\sqrt{y}}{y - x}$$

Example 4

Show that $17\left(1 - \frac{1}{17^2}\right)^{\frac{1}{2}} = n\sqrt{2}$ where n is an integer

Solution

$$17\left(1 - \frac{1}{17^2}\right)^{\frac{1}{2}} = \left[17^2\left(1 - \frac{1}{17^2}\right)\right]^{\frac{1}{2}}$$
$$= (17^2 - 1)^{\frac{1}{2}}$$
$$= (288)^{\frac{1}{2}}$$
$$= (2 \times 144)^{\frac{1}{2}}$$
$$= (2 \times 12^2)^{\frac{1}{2}}$$
$$= 12\sqrt{2}$$

Example 5

Simplify the following

(a) $\frac{2^{n-3} \times 8^{n+1}}{2^{2n-1} \times 4^{2-n}}$ (b) $\frac{\left(a^{\frac{1}{3}} \times b^{\frac{1}{2}}\right)^{-6}}{\sqrt[4]{a^8 b^9}}$

Solution

(a) $\frac{2^{n-3} \times 8^{n+1}}{2^{2n-1} \times 4^{2-n}} = \frac{2^{n-3} \times (2^3)^{n+1}}{2^{2n-1} \times (2^2)^{2-n}}$
$$= \frac{2^{n-3} \times 2^{3n+3}}{2^{2n-1} \times 2^{4-2n}}$$
$$= \frac{2^{n-3+(3n+3)}}{2^{2n-1+(4-2n)}}$$
$$= \frac{2^{4n}}{2^3} = 2^{4n-3}$$

(b) $$\frac{\left(a^{\frac{1}{3}}\times b^{\frac{1}{2}}\right)^{-6}}{\sqrt[4]{a^8b^9}} = \frac{a^{\frac{1}{3}\times -6}\times b^{\frac{1}{2}\times -6}}{(a^8b^9)^{\frac{1}{4}}}$$
$$= \frac{a^{-2}\times b^{-3}}{a^2b^{\frac{9}{4}}}$$
$$= a^{-2-2}\times b^{-3-\frac{9}{4}}$$
$$= a^{-4}b^{-\frac{21}{4}}$$
$$= \frac{1}{a^4b^{\frac{21}{4}}}$$

Example 6

Simplify

$$\frac{\sqrt{1-x}\frac{1}{2}(1+x)^{-\frac{1}{2}}+\frac{1}{2}(1-x)^{-\frac{1}{2}}\sqrt{1+x}}{1-x}$$

<u>Solution</u>

$$\frac{\sqrt{1-x}\frac{1}{2}(1+x)^{-\frac{1}{2}}+\frac{1}{2}(1-x)^{-\frac{1}{2}}\sqrt{1+x}}{1-x}$$
$$= \frac{\frac{1}{2}(1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}+\frac{1}{2}(1-x)^{-\frac{1}{2}}(1+x)^{\frac{1}{2}}}{1-x}$$
$$= \frac{\frac{1}{2}(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}[(1+x)^{-1}+(1-x)^{-1}]}{1-x}$$
$$= \frac{1}{2}(1-x)^{-\frac{1}{2}}(1+x)^{\frac{1}{2}}\left[\frac{1}{1+x}+\frac{1}{1-x}\right]$$
$$= \frac{1}{2}(1-x)^{-\frac{1}{2}}(1+x)^{\frac{1}{2}}\left[\frac{1-x+1+x}{(1+x)(1-x)}\right]$$
$$= (1-x)^{-\frac{3}{2}}(1+x)^{-\frac{1}{2}}$$
$$= [(1-x)^3(1+x)]^{-\frac{1}{2}}$$
$$= [(1-x)^2(1-x)(1+x)]^{-\frac{1}{2}}$$
$$= [(1-x)^2(1-x^2)]^{-\frac{1}{2}}$$
$$= (1-x)^{-1}(1-x^2)^{-\frac{1}{2}}$$
$$= \frac{1}{(1-x)\sqrt{1-x^2}}$$

LOGARITHMS

Logarithm is another word to mean index or power i.e. if $y = a^x$, then we define x as logarithm of y to base a ($\log_a y$). If $y = a^x$, then $x = \log_a y$

This can be used to convert from 'index form' to 'logarithmic form' and vice versa.

Logarithms to the base e, written $\ln x$ or $\log_e x$, are called natural logarithms.

Logarithms to the base 10, written $\log x$ or $\log_{10} x$ are called common logarithms.

The logarithm of a positive number N to the base a is defined as the power of a which is equal to N. Thus if

$$a^x = N$$

then x is the logarithm of N to the base a, written

$$x = \log_a N$$

Thus

$$a^{\log_a N} = N$$

Since we have $a^1 = a$ and $a^0 = 1$, it follows that

$$\boldsymbol{\log_a a = 1}$$
$$\boldsymbol{\log_a 1 = 0}$$

for all $a(\neq 0)$

Logarithm of a negative number

To evaluate $\log_a(-4)$ for some base $a > 0$, we need to solve the equivalent statement

$$x = \log_a(-4) \Leftrightarrow a^x = -4$$

However, the value of a^x where $a > 0$, will always be positive, therefore there is no value of x for which $a^x = -4$. This means that we cannot evaluate the logarithm of a negative number.

Basic laws of logarithms

The laws for the manipulation of logarithms are derived directly from the laws of indices.

1. $\boldsymbol{\log_a bc = \log_a b + \log_a c}$

Let $\log_a b = x$ and $\log_a c = y$

$$b = a^x \text{ and } c = a^y$$
$$bc = a^x.a^y$$
$$bc = a^{x+y}$$
$$\log_a bc = x + y = \log_a b + \log_a c$$

2. $\boldsymbol{\log_a\left(\frac{b}{c}\right) = \log_a b - \log_a c}$

Let $\log_a b = x$ and $\log_a c = y$

$$b = a^x \text{ and } c = a^y$$
$$\frac{b}{c} = \frac{a^x}{a^y}$$
$$\frac{b}{c} = a^{x-y}$$
$$\log_a bc = x - y = \log_a b - \log_a c$$

3. $\boldsymbol{\log_a b^n = n\log_a b}$

Let $\log_a b = x, \Rightarrow b = a^x$

$$b^n = (a^x)^n$$
$$b^n = a^{nx}$$
$$\log_a b^n = nx$$
$$\log_a b^n = n\log_a b$$

4. $\boldsymbol{\log_a b = \frac{\log_c b}{\log_c a}}$

Let $\log_a b = x$

$$a^x = b$$

Introducing $\log_c$ on both sides

$$\log_c a^x = \log_c b$$
$$x\log_c a = \log_c b$$
$$x = \frac{\log_c b}{\log_c a}$$
$$\log_a b = \frac{\log_c b}{\log_c a}$$

This rule is also called the change of base rule

5. $\log_a b = \frac{1}{\log_b a}$

Let $\log_a b = x$

$$a^x = b$$

Introducing $\log_b$ on both sides

$$\log_b a^x = \log_b b$$
$$x \log_b a = 1$$
$$x = \frac{1}{\log_b a}$$
$$\log_a b = \frac{1}{\log_b a}$$

Notes:

1. $\log\frac{1}{a} = \log a^{-1} = -\log a$
2. Logarithm of a negative number does not exist.

Example 7

Given that $\log_3 x = p$ and $\log_{18} x = q$, show that $\log_6 3 = \frac{q}{p-q}$

Solution

$$\log_{18} x = \frac{\log_3 x}{\log_3 18}$$
$$q = \frac{p}{\log_3 6 \times 3}$$
$$q = \frac{p}{\log_3 6 + \log_3 3}$$
$$q = \frac{p}{\log_3 6 + 1}$$
$$q(\log_3 6 + 1) = p$$
$$q \log_3 6 + q = p$$
$$q \log_3 6 = p - q$$
$$\log_3 6 = \frac{p-q}{q}$$
$$\log_6 3 = \frac{q}{p-q}$$

Example 8

If $a^2 + b^2 = 23ab$, show that $\log a + \log b = 2\log\left(\frac{a+b}{5}\right)$

Solution

From $(a+b)^2 = a^2 + 2ab + b^2$

$$\Rightarrow a^2 + b^2 = (a+b)^2 - 2ab$$
$$\therefore (a+b)^2 - 2ab = 23ab$$
$$(a+b)^2 = 25ab$$
$$\frac{(a+b)^2}{25} = ab$$
$$\left(\frac{a+b}{5}\right)^2 = ab$$

Introducing logarithm to the base 10 on each side,

$$\log\left(\frac{a+b}{5}\right)^2 = \log ab$$
$$2\log\left(\frac{a+b}{5}\right) = \log a + \log b$$

Example 9

Show that $\log_a b \,.\log_b c \,.\, \log_c a = 1$

Solution

$$\log_a b = \frac{\log_c b}{\log_c a}$$
$$\log_a b \times \log_c a = \log_c b$$
$$\log_a b \times \log_c a = \frac{1}{\log_b c}$$
$$\therefore \log_a b \times \log_c a \times \log_b c = 1$$

Example 10

Show that $\log_a(a+b)^2 = 2 + \log_a\left(1 + \frac{2b}{a} + \frac{b^2}{a^2}\right)$

Solution

$$\log_a(a+b)^2 = \log_a(a^2 + 2ab + b^2)$$
$$= \log_a a^2\left(1 + \frac{2b}{a} + \frac{b^2}{a^2}\right)$$
$$= \log_a a^2 + \log_a\left(1 + \frac{2b}{a} + \frac{b^2}{a^2}\right)$$
$$= 2\log_a a + \log_a\left(1 + \frac{2b}{a} + \frac{b^2}{a^2}\right)$$
$$= 2 + \log_a\left(1 + \frac{2b}{a} + \frac{b^2}{a^2}\right)$$

Example 11

If $\log_a b = \log_b c = \log_c a$, show that $a = b = c$

Solution

Introducing logarithm to the base a

$$\log_b c = \frac{\log_a c}{\log_a b} \text{ and } \log_c a = \frac{\log_a a}{\log_a c} = \frac{1}{\log_a c}$$
$$\log_a b = \frac{\log_a c}{\log_a b} = \frac{1}{\log_a c}$$
$$\log_a b = \frac{\log_a c}{\log_a b}$$
$$(\log_a b)^2 = \log_a c \ldots.(i)$$
$$\frac{\log_a c}{\log_a b} = \frac{1}{\log_a c}$$
$$(\log_a c)^2 = \log_a b \ \ldots(ii)$$

Dividing (i) and (ii);

$$\frac{(\log_a c)^2}{(\log_a b)^2} = \frac{\log_a b}{\log_a c}$$
$$(\log_a c)^3 = (\log_a b)^3$$
$$\log_a c = \log_a b$$
$$\Rightarrow c = b$$

Now $\log_a b = \frac{\log_a c}{\log_a b}$

$$\log_a b = 1 \Leftrightarrow \log_a c = \log_a b$$
$$a^1 = b$$
$$\Rightarrow a = b$$
$$\therefore a = b = c$$

Example 12

If u, v, s, t are all positive, show that

$$\log\left(\frac{u}{v}\right).\log\left(\frac{s}{t}\right) = \log\left(\frac{u}{s}\right).\log\left(\frac{v}{t}\right) + \log\left(\frac{u}{t}\right).\log\left(\frac{s}{v}\right)$$

the logarithms all being to the same base

Solution

$$\log\left(\frac{u}{v}\right).\log\left(\frac{s}{t}\right) = [\log u - \log v][\log s - \log t]$$

$= \log u \log s - \log u \log t - \log v \log s + \log v \log t$

$= \log u \log s - \log u \log t - \log v \log s + \log v \log t$
$\quad + \log u \log v - \log u \log v + \log s \log t - \log s \log t$

$= \log u \log s - \log u \log v + \log u \log v - \log v \log s$
$\quad + \log v \log t - \log s \log t + \log s \log t - \log u \log t$

$= \log u\,[\log s - \log v] + \log v\,[\log u - \log s]$
$\quad + \log t\,[\log v - \log s] + \log t\,[\log s - \log u]$

$$= \log u\left[\log\left(\frac{s}{v}\right)\right] + \log v\left[\log\left(\frac{u}{s}\right)\right] + \log t\left[\log\left(\frac{v}{s}\right)\right] + \log t\left[\log\left(\frac{s}{u}\right)\right]$$

$$= \log u\left[\log\left(\frac{s}{v}\right)\right] + \log v\left[\log\left(\frac{u}{s}\right)\right] - \log t\left[\log\left(\frac{s}{v}\right)\right] - \log t\left[\log\left(\frac{u}{s}\right)\right]$$

$$= \log\left(\frac{s}{v}\right)[\log u - \log t] + \log\left(\frac{u}{s}\right)[\log v - \log t]$$

$$= \log\left(\frac{s}{v}\right).\log\left(\frac{u}{t}\right) + \log\left(\frac{u}{s}\right).\log\left(\frac{v}{t}\right)$$

Example 13

If $x = \log_a bc$, $y = \log_b ca$ and $z = \log_c ab$, prove that

$$x + y + z = xyz - 2$$

Solution

$x = \log_a bc \Rightarrow a^x = bc$ … (i)

$y = \log_b ca \Rightarrow b^y = ca$ … (ii)

$z = \log_c ab \Rightarrow c^z = ab$ …. (iii)

From (i), $c = \frac{a^x}{b}$

Substituting for c in (ii);

$$b^y = \frac{a^x}{b} \times a$$
$$b^{y+1} = a^{x+1}$$
$$b = a^{\frac{x+1}{y+1}}$$

Now from (iii);

$$c^z = ab$$
$$\left(\frac{a^x}{b}\right)^z = ab$$
$$\frac{a^{xz}}{b^z} = ab$$
$$a^{xz} = ab^{z+1}$$
$$a^{xz} = a\left[a^{\frac{x+1}{y+1}}\right]^{z+1}$$
$$a^{xz} = a(a)^{\frac{(x+1)(z+1)}{y+1}}$$
$$a^{xz} = a^{1+\frac{(x+1)(z+1)}{y+1}}$$

Now since the bases are the same, we can equate the powers

$$1 + \frac{(x+1)(z+1)}{y+1} = xz$$
$$y + 1 + xz + x + z + 1 = xz(y+1)$$
$$x + y + z + xz + 2 = xyz + xz$$
$$x + y + z = xyz - 2$$

Example 14

By putting $\alpha = \log a$, $\beta = \log b$, $\gamma = \log c$ in the identity $\alpha(\beta - \gamma) + \beta(\gamma - \alpha) + \gamma(\alpha - \beta) = 0$, show that

$$\left(\frac{b}{c}\right)^{\log a}.\left(\frac{c}{a}\right)^{\log b}.\left(\frac{a}{b}\right)^{\log c} = 1$$

where the logarithms are taken to any base

Solution

$\log a\,(\log b - \log c) + \log b\,(\log c - \log a)$
$\quad + \log c\,(\log a - \log b) = 0$

$$\log a\left[\log\left(\frac{b}{c}\right)\right] + \log b\left[\log\left(\frac{c}{a}\right)\right] + \log c\left[\log\left(\frac{a}{b}\right)\right] = 0$$

$$\log\left(\frac{b}{c}\right)^{\log a} + \log\left(\frac{c}{a}\right)^{\log b} + \log\left(\frac{a}{b}\right)^{\log c} = 0$$

$$\log\left[\left(\frac{b}{c}\right)^{\log a}.\left(\frac{c}{a}\right)^{\log b}.\left(\frac{a}{b}\right)^{\log c}\right] = 0$$

Let N be any base to which the logarithms are taken

$$\left(\frac{b}{c}\right)^{\log a}.\left(\frac{c}{a}\right)^{\log b}.\left(\frac{a}{b}\right)^{\log c} = N^0$$

$$\left(\frac{b}{c}\right)^{\log a}.\left(\frac{c}{a}\right)^{\log b}.\left(\frac{a}{b}\right)^{\log c} = 1$$

Equations in which the unknown occurs as an index

Example 15

Solve the following equations:

(a) $\log_x 3 + \log_x 27 = 2$ (b) $\log_3 x + 3\log_x 3 = 4$

Solution

(a) $\log_x 3 + \log_x 27 = 2$

$$\log_x(3 \times 27) = 2$$
$$\log_x 81 = 2$$
$$x^2 = 81$$
$$x = 9$$

(b) $\log_3 x + 3\log_x 3 = 4$ can be written as

$$\log_3 x + 3\left(\frac{1}{\log_3 x}\right) = 4$$
$$(\log_3 x)^2 + 3 = 4\log_3 x$$

Let $\log_3 x = y$,

$$y^2 + 3 = 4y$$
$$y^2 - 4y + 3 = 0$$
$$y^2 - y - 3y + 3 = 0$$
$$y(y-1) - 3(y-1) = 0$$
$$(y-1)(y-3) = 0$$
$$y = 1 \text{ or } y = 3$$

When $y = 1$, $\log_3 x = 1 \Rightarrow x = 3^1 = 3$

When $y = 3$, $\log_3 x = 3 \Rightarrow x = 3^3 = 27$

Example 16

Solve for x, $\log_x 9 + \log_{x^2} 3 = 2.5$

Solution

$$\log_x 9 + \log_{x^2} 3 = 2.5$$
$$\log_x 3^2 + \frac{\log_x 3}{\log_x x^2} = 2.5$$
$$2\log_x 3 + \frac{\log_x 3}{2} = 2.5$$
$$4\log_x 3 + \log_x 3 = 5$$
$$5\log_x 3 = 5$$
$$\log_x 3 = 1$$
$$x = 3$$

Example 17

Given that $\log_2 x + 2\log_4 y = 4$, show that $xy = 16$. Hence solve for x and y the simultaneous equations:

$$\log_{10}(x+y) = 1$$
$$\log_2 x + 2\log_4 y = 4$$

Solution

$$\log_2 x + \frac{2\log_2 y}{\log_2 4} = 4$$
$$\log_2 x + \frac{2\log_2 y}{\log_2 2^2} = 4$$
$$\log_2 x + \frac{2\log_2 y}{2} = 4$$
$$\log_2 x + \log_2 y = 4$$
$$\log_2 xy = 4$$
$$xy = 2^4$$
$$xy = 16$$

Now,

$$x + y = 10$$

From $xy = 16$, $x = \frac{16}{y}$

$$\frac{16}{y} + y = 10$$
$$16 + y^2 = 10y$$
$$y^2 - 10y + 16 = 0$$
$$y^2 - 8y - 2y + 16 = 0$$
$$y(y-8) - 2(y-8) = 0$$
$$(y-8)(y-2) = 0$$
$$y = 8 \text{ or } y = 2$$

When $y = 8$, $x = \frac{16}{8} = 2$

When $y = 2$, $x = \frac{16}{2} = 8$

$$\therefore (x, y) = (8, 2) \text{ or } (2, 8)$$

Example 18

Solve the equation $2^{x^2} = 16^{x-1}$

Solution

$$2^{x^2} = (2^4)^{x-1}$$
$$2^{x^2} = 2^{4x-4}$$
$$x^2 = 4x - 4$$
$$x^2 - 4x + 4 = 0$$
$$(x-2)^2 = 0$$
$$x = 2$$

Example 19

Find x from the equation $3^{2x} = 5^{x+1}$

Solution

Taking logarithms to base 10 on both sides

$$\log 3^{2x} = \log 5^{x+1}$$
$$2x\log 3 = (x+1)\log 5$$
$$2x\log 3 = x\log 5 + \log 5$$
$$x(2\log 3 - \log 5) = \log 5$$
$$x = \frac{\log 5}{2\log 3 - \log 5} = \frac{0.6990}{0.2552} = 2.74$$

Example 20

Find, without using tables or calculator, the value of x, given that

$$\frac{2^{3x+7}}{4^{2x-2}} = \frac{8^{x-3}}{32^{5-x}}$$

Solution

$$\frac{2^{3x+7}}{2^{2(2x-2)}} = \frac{2^{3(x-3)}}{2^{5(5-x)}}$$
$$\frac{2^{3x+7}}{2^{4x-4}} = \frac{2^{3x-9}}{2^{25-5x}}$$
$$2^{(3x+7)-(4x-4)} = 2^{(3x-9)-(25-5x)}$$
$$2^{-x+11} = 2^{8x-34}$$
$$-x + 11 = 8x - 34$$
$$9x = 45$$
$$x = 5$$

Example 21

Solve the equation $3^{x^2} = 9^{x+4}$

Solution

$$3^{x^2} = 3^{2(x+4)}$$
$$x^2 = 2(x+4)$$
$$x^2 - 2x - 8 = 0$$
$$x^2 - 4x + 2x - 8 = 0$$
$$x(x-4) + 2(x-4) = 0$$
$$(x-4)(x+2) = 0$$
$$x = 4 \text{ or } x = -2$$

Example 22

Solve the equation $2^{3x+1} = 5^{x+1}$

Solution

Taking logarithms to base 10 on both sides

$$\log 2^{3x+1} = \log 5^{x+1}$$
$$(3x+1)\log 2 = (x+1)\log 5$$
$$3x\log 2 + \log 2 = x\log 5 + \log 5$$
$$x(3\log 2 - \log 5) = (\log 5 - \log 2)$$
$$x = \frac{\log 5 - \log 2}{3\log 2 - \log 5} = \frac{0.3979}{0.2041} = 1.95$$

Example 23

Solve the equation $2^{2x+1} - 5(2^x) + 2 = 0$

Solution

$$2(2^x)^2 - 5(2^x) + 2 = 0$$

Let $2^x = y$,

$$2y^2 - 5y + 2 = 0$$
$$2y^2 - y - 4y + 2 = 0$$
$$y(2y-1) - 2(2y-1) = 0$$
$$(2y-1)(y-2) = 0$$
$$y = \frac{1}{2} \text{ or } y = 2$$

When $y = \frac{1}{2}$, $2^x = \frac{1}{2} = 2^{-1} \Rightarrow x = -1$

When $y = 2$, $2^x = 2^1 \Rightarrow x = 1$

Example 24

Solve the equation $5^{2x} - 5^{x+1} + 4 = 0$

Solution

$$(5^x)^2 - 5(5^x) + 4 = 0$$

Let $5^x = y$

$$y^2 - 5y + 4 = 0$$
$$y^2 - y - 4y + 4 = 0$$
$$y(y-1) - 4(y-1) = 0$$
$$(y-1)(y-4) = 0$$
$$y = 1 \text{ or } y = 4$$

When $y = 1$, $5^x = 1$

$$5^x = 5^0$$
$$x = 0$$

When $y = 4$, $5^x = 4$

$$\log 5^x = \log 4$$
$$x \log 5 = \log 4$$
$$x = \frac{\log 4}{\log 5} = 0.8614$$

Example 25

Solve for t in: $4^{(2t+1)} + 4^{(t+3)} = 16\frac{1}{4}$

Solution

$$4^{2t} \times 4^1 + 4^t \times 4^3 = \frac{65}{4}$$
$$4(4^{2t}) + 64(4^t) = \frac{65}{4}$$
$$4(4^t)^2 + 64(4^t) = \frac{65}{4}$$
$$\text{Let } 4^t = y \ \Rightarrow 4y^2 + 64y = \frac{65}{4}$$
$$16y^2 + 256y - 65 = 0$$
$$16y^2 + 260y - 4y - 65 = 0$$
$$4y(4y+65) - (4y+65) = 0$$
$$(4y+65)(4y-1) = 0$$

Either $y = -\frac{65}{4}$ or $y = \frac{1}{4}$

When $y = -\frac{65}{4}$, $4^t = -\frac{65}{4}$, hence value of t does not exist.

When $y = \frac{1}{4}$, $4^t = \frac{1}{4} = 4^{-1}$, $t = -1$

$$\therefore t = -1$$

Example 26

Solve the equations

$$2^{x+y} = 8$$
$$3^{2x-y} = 27$$

Solution

$$2^{x+y} = 2^3$$
$$x + y = 3 \ \ldots. \text{(i)}$$
$$3^{2x-y} = 3^3$$
$$2x - y = 3 \ \ldots. \text{(ii)}$$

Adding (i) and (ii);

$$3x = 6$$
$$x = 2$$

Substituting for x in (i);

$$4 - y = 3$$
$$y = 1$$

Example 27

Solve the simultaneous equations

$$5^{x+2} + 7^{y+1} = 3468$$
$$7^y = 5^x - 76$$

Solution

From the first equation

$$5^x \times 5^2 + 7^y \times 7^1 = 3468$$
$$25(5^x) + 7(7^y) = 3468$$

Substituting for 7^y

$$25(5^x) + 7(5^x - 76) = 3468$$
$$25(5^x) + 7(5^x) - 532 = 3468$$
$$32(5^x) = 4000$$
$$5^x = 125$$
$$5^x = 5^3$$
$$x = 3$$

From $7^y = 5^x - 76$

$$7^y = 5^3 - 76$$
$$7^y = 49$$
$$7^y = 7^2$$
$$y = 2$$

Example 28

Solve the simultaneous equations

$$\log_2 x + 2\log_4 y = 4$$
$$x + 12y = 52$$

Solution

$$\log_2 x + 2\left(\frac{\log_2 y}{\log_2 4}\right) = 4$$
$$\log_2 x + 2\left(\frac{\log_2 y}{2}\right) = 4$$
$$\log_2 x + \log_2 y = 4$$
$$\log_2 xy = 4$$
$$xy = 2^4$$
$$xy = 16$$
$$x = \frac{16}{y}$$

$$= \frac{\sqrt{2}\left(\sqrt{2}+\sqrt{3}+\sqrt{5}\right)}{4}$$
$$= \frac{2+\sqrt{6}+\sqrt{10}}{4}$$

Example 37

Find, without using tables or a calculator, the exact value of
$$\frac{\left(2+\sqrt{3}\right)^2}{2-\sqrt{3}}+\frac{\left(2-\sqrt{3}\right)^2}{2+\sqrt{3}}$$

<u>Solution</u>

$$\frac{\left(2+\sqrt{3}\right)^2}{2-\sqrt{3}}+\frac{\left(2-\sqrt{3}\right)^2}{2+\sqrt{3}}=\frac{\left(2+\sqrt{3}\right)^3+\left(2-\sqrt{3}\right)^3}{\left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right)}$$
$$\left(2+\sqrt{3}\right)^3 = 2^3+3(2)^2\left(\sqrt{3}\right)+3(2)\left(\sqrt{3}\right)^2+\left(\sqrt{3}\right)^3$$
$$= 8+12\sqrt{3}+18+3\sqrt{3}$$
$$= 26+15\sqrt{3}$$
$$\left(2-\sqrt{3}\right)^3 = 2^3+3(2)^2\left(-\sqrt{3}\right)+3(2)\left(-\sqrt{3}\right)^2$$
$$+\left(-\sqrt{3}\right)^3$$
$$= 8-12\sqrt{3}+18-3\sqrt{3}$$
$$= 26-15\sqrt{3}$$
$$\frac{\left(2+\sqrt{3}\right)^3+\left(2-\sqrt{3}\right)^3}{\left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right)}=\frac{26+15\sqrt{3}+26-15\sqrt{3}}{2^2-3}$$
$$= 52$$

Equations involving surds

Example 38

Solve the equation $\sqrt{x}-\dfrac{6}{\sqrt{x}}=1$

<u>Solution</u>

Multiplying through by $\sqrt{x}$
$$x-6=\sqrt{x}$$
Squaring both sides
$$(x-6)^2=x$$
$$x^2-12x+36=x$$
$$x^2-13x+36=0$$
$$x^2-9x-4x+36=0$$
$$x(x-9)-4(x-9)=0$$
$$(x-9)(x-4)=0$$
$$\text{either } x=9 \text{ or } x=4$$
We need to verify the correctness of our solutions

When $x=9$, L.H.S $=3-\frac{6}{3}=3-2=1=$ R.H.S

When $x=4$, L.H.S $=2-\frac{6}{2}=2-3=-1\neq$ R.H.S
$$\therefore x=9$$

Example 39

Solve the equation
$$\sqrt{4-x}-\sqrt{6+x}=\sqrt{14+2x}$$

<u>Solution</u>

Squaring both sides we have
$$4-x-2\sqrt{(4-x)(6+x)}+6+x=14+2x$$
$$-2\sqrt{(4-x)(6+x)}=4+2x$$
$$-\sqrt{(4-x)(6+x)}=2+x$$
Squaring both sides, we now have
$$(4-x)(6+x)=4+4x+x^2$$
$$24-2x-x^2=4+4x+x^2$$
$$2x^2+6x-20=0$$
$$x^2+3x-10=0$$
$$x^2+5x-2x-10=0$$
$$x(x+5)-2(x+5)=0$$
$$(x+5)(x-2)=0$$
$$\text{either } x=-5 \text{ or } x=2$$
Verifying the solutions;

When $x=-5$, L.H.S $=\sqrt{9}-\sqrt{1}=2$

R.H.S $=\sqrt{4}=2=$ L.H.S

When $x=2$, L.H.S $=\sqrt{2}-\sqrt{8}=-\sqrt{2}$

R.H.S $=\sqrt{18}=3\sqrt{2}\neq$ L.H.S
$$\therefore x=-5$$

Example 40

Find the values of x which satisfy the equation
$$2\sqrt{x+5}-\sqrt{2x+8}=2$$

<u>Solution</u>

Squaring both sides
$$4(x+5)-4\sqrt{(x+5)(2x+8)}+2x+8=4$$
$$4x+20-4\sqrt{(x+5)(2x+8)}+2x+8=4$$
$$6x+24=4\sqrt{(x+5)(2x+8)}$$
$$3x+12=2\sqrt{(x+5)(2x+8)}$$
Squaring both sides
$$9x^2+72x+144=4(x+5)(2x+8)$$
$$9x^2+72x+144=4(2x^2+18x+40)$$
$$9x^2+72x+144=8x^2+72x+160$$
$$x^2-16=0$$
$$(x-4)(x+4)=0$$
$$\text{either } x=4 \text{ or } x=-4$$
Verifying the solutions;

When $x=4$, L.H.S $=2\sqrt{9}-\sqrt{16}=2=$ R.H.S

When $x=-4$, L.H.S $=2\sqrt{1}-\sqrt{0}=2=$ R.H.S
$$\therefore x=4 \text{ or } x=-4 \text{ i.e. } x=\pm 4$$

Self-Evaluation exercise

1. Evaluate

   (i) $\dfrac{8^{\frac{2}{3}}+4^{\frac{3}{2}}}{16^{\frac{3}{4}}}$

   (ii) $\dfrac{\sqrt{a^3}\times\sqrt[3]{b^2}}{\sqrt[4]{a^6}\times\sqrt[6]{b^{-2}}}$ when $b=3$

   [Ans: (i) $3/2$ (ii) 3]

2. If $\log_a n=x$ and $\log_c n=y$, where $n\neq 1$, prove that

$$\frac{x-y}{x+y}=\frac{\log_b c-\log_b a}{\log_b c+\log_b a}$$

Verify this result, without using any tables, when $a=4$, $b=2$, $c=8$, $n=4096$

3. If $\log_a\left(1+\frac{1}{8}\right)=l$, $\log_a\left(1+\frac{1}{15}\right)=m$ and $\log_a\left(1+\frac{1}{24}\right)=n$, show that $\log_a\left(1+\frac{1}{80}\right)=l-m-n$
4. Solve for x (i) $3^{2x-1}=5^x$ (ii) $7^{4x+2}=9^{3x-1}$ [Ans: (i) 1.87 (ii) -5.11]
5. Solve the equation $2^{x^2}=\frac{1}{4}8^x$ [Ans: 1, 2]
6. Find x if $9^{x^2}=3^{5x-2}$ [Ans: $\frac{1}{2}$, 2]
7. Solve the equation $5^{2x}-5^{1+x}+6=0$ [Ans: 0.431, 0.683]
8. Solve the equation $4^{2x}=2^{6x-1}$ [Ans: $\frac{1}{2}$]
9. Find x if $\log_x 8-\log_{x^2}16=1$ [Ans: 2]
10. Find x if $\log_x 3+\log_3 x=2.5$ [Ans: $\sqrt{3}$, 9]
11. Solve the simultaneous equations $2^{x+y}=6$, $3^{x-y}=4$ [Ans: $x=1.92$, $y=0.66$]
12. Solve the equation $2^x.3^{1-x}=6$ [Ans: -1.71]
13. If $p^2=qr$ show that $\log_q p+\log_r p=2\log_q p\log_r p$.
14. Solve the equation $\log_3 x+\log_x 3=\frac{10}{3}$ [Ans: 27, $\sqrt[3]{3}$]
15. Solve for x: $\log_{10}\left(\frac{x^2+24}{x}\right)=1$ [Ans: 4, 6]
16. Solve for x: $2^x\times 3^{x+1}=5^{2x+1}$ [Ans: -0.358]
17. Solve the equation $\log_{10}(x^2+9)-2\log_{10}x=1$ [Ans: 1]
18. Solve the equations: $3^{2x+y}=12$, $2^{x-y}=4$ [Ans: $x=1.42$, $y=-0.58$]
19. Solve the equation: $2^{2+2x}+3\times 2^x-1=0$ [Ans: -2]
20. Without using tables, show that
$$\frac{\log\sqrt{27}+\log\sqrt{8}-\log\sqrt{125}}{\log 6-\log 5}=\frac{3}{2}$$
21. Find x from the equation: $3^x-3^{-x}=6.832$ [Ans: 1.768]
22. Solve the equation $2^{2x+8}-32(2^x)+1=0$ [Ans: -4]
23. Solve the simultaneous equations
$$2^{x+y}=6^y,\ 3^x=6(2^y)$$
[Ans: 2.71, 1.71]
24. Simplify the expression $5\times 4^{3n+1}-20\times 8^{2n}$ [Ans: 0]
25. Find x from the equation $9^x-12(3^x)+27=0$ [Ans: 1, 2]
26. Solve the equation: $4^x+2=3\times 2^x$ [Ans: 0, 1]
27. If $2\log_8 N=p$, $\log_2 2N=q$, $q-p=4$, find N [Ans: 512]
28. Simplify $9^{2n+2}\times 6^{2n-3}\div(3^{5n}\times 6\times 4^{n-2})$ [Ans: 3^n]
29. Express in its simplest form, $\log_2 64-\log_2 16$ [Ans: 2]
30. Given that $\log_x u+\log_x v=p$ and $\log_x u-\log_x v=q$, prove that $u=x^{\frac{1}{2}(p+q)}$ and find a similar expression for v. [Ans: $v=x^{\frac{1}{2}(p-q)}$]
31. Solve the equation $\log_5 x=16\log_x 5$ [Ans: 625 or $1/625$]
32. Find the values of y which satisfy the equation:
$$(8^y)^y.\frac{1}{32^y}=4$$
[Ans: 2]
33. Given that $\log_9 x=p$ and $\log_{\sqrt{3}} y=q$, express xy and $\frac{x^2}{y}$ as powers of 3.
[Ans: $xy=3^{2p+\frac{q}{2}}$, $\frac{x^2}{y}=3^{4p-\frac{q}{2}}$]
34. Solve for x in the equation $e^{2x}+e^x-6=0$ [Ans: ln 2]
35. Find x and y given that $e^x+3e^y=3$ and $e^{2x}-9e^{2y}=6$, expressing each answer as a logarithm to base e.
[Ans: $x=\ln\left(\frac{5}{2}\right)$, $y=-\ln 6$]
36. If $2\log_y x+2\log_x y=5$, show that $\log_y x$ is either $\frac{1}{2}$ or 2. Hence find all pairs of values of x and y which satisfy simultaneously the equation above and the equation $xy=27$.
[Ans: $(x,y)=(3,9)$ or $(9,3)$]
37. Prove that, if $x=\log_{10}(a-by)-\log_{10}a$, where a and b are constants, then $y=\frac{a}{b}(1-10^x)$
Find the value of y when $a=4$, $b=2$ and $x=-2.065$ [Ans: 1.983]
38. Given that $2^{x+1}-5^y=131$, $2^{x-4}+5^{y-2}=13$, find x and y.
[Ans: $x=7$, $y=3$]
39. Find, without the use of tables or a calculator, the value of x, given that
$$\frac{2^{x+5}}{8^x}=\frac{4^{x-1}}{2^{2x-1}}$$
[Ans: $x=3$]
40. Solve for x, $9^x-4\times 3^x+3=0$ [Ans: 0, 1]

Chapter 2 Simultaneous Equations

Linear equations in one variable

Solving linear equations in one variable is a simple task as we perform a few mathematical operations on either side of the equation.

Example 1

Solve the equations

(a) $2x + 3(x - 1) = 4x + 12$

(b) $\frac{x+5}{5} = \frac{x-1}{6}$

Solution

(a) $2x + 3x - 3 = 4x + 12$

$$5x = 4x + 12$$
$$x = 12$$

(b) $\frac{x+5}{5} = \frac{x-1}{6}$

$$6(x + 5) = 5(x - 1)$$
$$6x + 30 = 5x - 5$$
$$x = -35$$

Simultaneous equations

When only one unknown quantity has to be found, only one equation is needed to provide a solution.

If two unknown quantities are involved in a problem we need two equations connecting them. Then, between the two equations we can eliminate one of the unknowns, producing just one equation containing just one unknown. This then ready for solution.

Solution of three linear equations

For three unknown quantities we need three equations. Then one unknown can be eliminated. One way to eliminate an unknown quantity is to add or subtract two of the equations and then go on to eliminate the second unknown in a similar way.

Example 2

Solve the equations

$$x + y - z = 4$$
$$2x + z = 7$$
$$3x - 2y = 5$$

Solution

$$x + y - z = 4 \quad \text{…. (i)}$$
$$2x + z = 7 \text{ …. (ii)}$$
$$3x - 2y = 5 \quad \text{…. (iii)}$$

As z appears only in equations (i) and (ii), we can eliminate z from these two equations

(i) + (ii) gives

$$3x + y = 11 \quad \text{…. (iv)}$$

Now bring in (iii)

$$3x - 2y = 5 \quad \text{… (iii)}$$

(iv) – (iii)

$$3y = 6$$
$$y = 2$$

Substituting for $y = 2$ in (iii) gives

$$3x - 4 = 5$$
$$3x = 9$$
$$x = 3$$

Now using $x = 3$ in (ii) gives

$$6 + z = 7$$
$$z = 1$$

Therefore the solution of the three simultaneous equations is

$$x = 3, y = 2, z = 1$$

Example 3

Solve the equations

$$x - y + 2z = 0$$
$$2x + y + z = 3$$
$$3x - y + z = 6$$

Solution

$$x - y + 2z = 0 \quad \text{…. (i)}$$
$$2x + y + z = 3 \quad \text{…. (ii)}$$
$$3x - y + z = 6 \text{ ….. (iii)}$$

The easiest letter to eliminate from two pairs of equations is y

(i) + (ii) gives

$$3x + 3z = 3$$

Dividing by 3 gives $\quad x + z = 1 \quad$ …. (iv)

(ii) + (iii) gives

$$5x + 2z = 9 \quad \text{… (v)}$$

Now we can either eliminate x or z from (iv) and (v);

$5 \times$ (iv) – (v) gives

$$3z = -4$$
$$z = -\frac{4}{3}$$

Using $z = -\frac{4}{3}$ in (iv) gives;

$$x - \frac{4}{3} = 3$$
$$x = \frac{7}{3}$$

Then using $x = \frac{7}{3}$ and $z = -\frac{4}{3}$ in (ii) gives;

$$\frac{14}{3} + y - \frac{4}{3} = 3$$
$$y = -\frac{1}{3}$$

Therefore the solution is $x = \frac{7}{3}, y = -\frac{1}{3}, z = -\frac{4}{3}$

$$y(y-3)+(y-3)=0$$
$$(y-3)(y+1)=0$$
$$y=3 \text{ or } y=-1$$

Since $x=y-1$, when $y=3$, $x=2$, and when $y=-1$, $x=-2$

The solution is $x=2, y=3; x=-2, y=-1$

Example 9

Solve the simultaneous equations

$$x+\sqrt{y}=9$$
$$x^2-y=9$$

Solution

From the first equation

$$x=9-\sqrt{y}$$

Substituting for x in the second equation

$$(9-\sqrt{y})^2-y=9$$
$$81-18\sqrt{y}+y-y=9$$
$$18\sqrt{y}=72$$
$$\sqrt{y}=4$$
$$y=16$$

Now from $x=9-\sqrt{y}$

$$x=9-\sqrt{16}=4$$

Self-Evaluation exercise

1. Solve the following simultaneous equations
   (a) $x+2y=3, x^2-xy+5y^2+2y=7$
   [Ans: (a) $x=1, y=1; x=\frac{29}{11}, y=\frac{2}{11}$]
   (b) $2x+y=1, x^2+xy+3x-y=4$
   [Ans: $x=1, y=-1; x=5, y=-9$]
   (c) $2x-3y=1, x^2+xy-4y^2=2$
   [Ans: $x=2, y=1; x=11, y=7$]
   (d) $x+2y=7, x^2+2y^2=17$
   [Ans: $x=3, y=2; x=\frac{5}{3}, y=\frac{8}{3}$]
2. Solve for a and b the simultaneous equations
$$a^2+b^2=\frac{13}{4}, \quad ab=-\frac{3}{2}$$
[Ans: $a=\frac{3}{2}, b=-1; a=1, b=-\frac{3}{2}, a=-1, b=\frac{3}{2}, a=-\frac{3}{2}, b=1$]
3. Solve the system of equations
$$x+y-z=9$$
$$3x+4y+3z=2$$
$$4x+5y+3z=5$$
[Ans: $x=-8, y=11, z=-6$]
4. Solve the simultaneous equations
$$x+3y-z=13$$
$$3x+y-z=11$$
$$x+y-3z=11$$
[Ans: $x=2, y=3$ and $z=-2$]
5. Solve the simultaneous equations
$$6x+4y-z=3$$
$$x+2y+4z=-2$$
$$5x+4y=0$$
[Ans: $x=4, y=-5, z=1$]
6. Solve the simultaneous equations
$$x+y+z=2$$
$$4x+y=4$$
$$-x+3y+2z=8$$
[Ans: $x=0, y=4, z=-2$]
7. Solve the simultaneous equations
$$-x+3y+24z=17$$
$$2x+6y+14z=6$$
$$x-y-z=2$$
[Ans: $x=10, y=-7, z=2$]
8. Solve the simultaneous equations
$$x+y-2z=7$$
$$2x-3y-2z=0$$
$$x-2y=-1$$
[Ans: $x=1, y=2, z=-2$]
9. Solve the simultaneous equations
$$x-y+z=3$$
$$4x+2y+z=6$$
$$x+y+z=2$$
[Ans: $x=2, y=-1, z=0$]
10. By row reducing the appropriate matrix to echelon form, solve the system of linear equations
    (a) $2x+6y+z=0$
    $x-2y+z=-10$
    $4x+3y+z=1$
    [Ans: $x=2, y=1, z=-10$]
    (b) $x+y-z=4$
    $2x+z=7$
    $3x-2y=5$
    [Ans: $x=3, y=2, z=1$]

Chapter 3 Quadratic Equations & Expressions

Any equation of the form $\boldsymbol{ax^2+bx+c=0}$ is called a quadratic equation and the values of x, which satisfy the equation, are called roots or solutions of the quadratic equation.
We shall discuss the three methods of obtaining the roots of the quadratic equation.

1. Solution by factorising

Consider the quadratic equation $3x^2+7x-6=0$
We get the product of the coefficient of x^2 and the constant term i.e. $3\times -6=-18$
Next, we find two factors of this product (-18) which add up to give the coefficient of x (7). These factors are 9 and -2
Now we split the middle term using these factors and then factorise the equation i.e.

$$3x^2-2x+9x-6=0$$
$$x(3x-2)+3(3x-2)=0$$
$$(3x-2)(x+3)=0$$

Now if the product of two quantities is zero then one, or both, of those quantities must be zero.

$$3x-2=0 \text{ or } x+3=0$$
$$x=\tfrac{2}{3} \text{ or } x=-3$$

The values $\frac{2}{3}$ and -3 are called the *roots* of that equation

Example 1

Find the roots of the equation $x^2+6x-7=0$

Solution

$1\times -7=-7$, we can use -1 and 7 as factors

$$x^2-x+7x-7=0$$
$$x(x-1)+7(x-1)=0$$
$$(x-1)(x+7)=0$$
$$x-1=0 \text{ or } x+7=0$$
$$x=1 \text{ or } x=-7$$

The roots of the equation are 1 and -7

Note:

It is usually best to collect the terms on the side where x^2 term is positive, for example

$$2-x^2=5x \text{ becomes } 0=x^2+5x-2$$
i.e. $x^2+5x-2=0$

Example 2

Solve the equation $4x-x^2=3$

Solution

$$0=x^2-4x+3$$
$$x^2-4x+3=0$$
$$x^2-x-3x+3=0$$
$$x(x-1)-3(x-1)=0$$
$$(x-1)(x-3)=0$$
$$x-1=0 \text{ or } x-3=0$$
$$x=3 \text{ or } x=1$$

Losing a solution

Quadratic equations sometimes have a common factor containing the unknown quantity. It is very tempting in such cases to divide by the common factor, but doing this results in the loss of part of the solution, as the following example shows.

First solution	*Second solution*
$x^2-5x=0$	$x^2-5x=0$
$x(x-5)=0$	$x-5=0$ (Dividing by x)
$x=0$ or $x-5=0$	$x=5$
$x=0$ or 5	The solution $x=0$ has been lost

Although dividing an equation by a numerical common factor is correct and sensible, dividing by a common factor containing the unknown quantity results in the loss of a solution.

2. Solution by completing the square

When there are no obvious factors, another method is needed to solve the equation. One such method involves adding a constant to the x^2 term and x term, to make a perfect square. This technique is called *completing the square.*

Consider $\quad x^2-2x$
Adding 1 gives $\quad x^2-2x+1$
Now $x^2-2x+1=(x-1)^2$ which is a perfect square.
Adding the number 1 was not a guess, it was found by using the fact that

$$x^2+2ax+\boxed{a^2}=(x+a)^2$$

We see from this that the number to be added is always

(half the coefficient of x)^2

Hence x^2+6x requires 3^2 to be added to make a perfect square

$$x^2+6x+9=(x+3)^2$$

To complete the square when the coefficient of x^2 is not 1, we first take out the coefficient of x^2 as a factor e.g.

$$2x^2+x=2\left(x^2+\frac{1}{2}x\right)$$

Now we add $\left(\frac{1}{2}\times\frac{1}{2}\right)^2$ inside the bracket, giving

$$2\left(x^2+\frac{1}{2}x+\frac{1}{16}\right)=2\left(x+\frac{1}{4}\right)^2$$

Take extra care when the coefficient of x^2 is negative e.g.

$$-x^2+4x=-(x^2-4x)$$

Then $$-(x^2-4x+4)=-(x-2)^2$$

$$-x^2+4x-4=-(x-2)^2$$

Example 3

Solve the equation $x^2-4x-2=0$, giving the solution in surd form.

Solution

$$x^2-4x-2=0$$

Now factors can be found so we isolate the two terms with x in,

$$x^2-4x=2$$

Add $\left\{\frac{1}{4}\times(-4)\right\}^2$ to both sides

$$x^2-4x+4=2+4$$

$$(x-2)^2=6$$

$$x-2=\pm\sqrt{6}$$

$$x=2\pm\sqrt{6}$$

$$\therefore x=2+\sqrt{6}\ \text{ or } x=2-\sqrt{6}$$

Example 4

Find the roots of the equation $2x^2-3x-3=0$

Solution

$$2x^2-3x=3$$

$$x^2-\frac{3}{2}x=\frac{3}{2}$$

$$x^2-\frac{3}{2}x+\frac{9}{16}=\frac{3}{2}+\frac{9}{16}$$

$$\left(x-\frac{3}{4}\right)^2=\frac{33}{16}$$

$$x-\frac{3}{4}=\pm\frac{\sqrt{33}}{4}$$

$$x=\frac{3\pm\sqrt{33}}{4}$$

$$x=\frac{3+\sqrt{33}}{4}=2.186 \text{ or } x=\frac{3-\sqrt{33}}{4}=-0.686$$

3. The formula for solving a quadratic equation

Solving a quadratic equation by completing the square is rather tedious. If the method is applied to a general quadratic equation, a formula can be derived which can then be used to solve any particular equation.

Using a, b and c to represent any numbers we have the general quadratic equation

$$ax^2+bx+c=0$$

Using the method of completing the square for this equation

$$ax^2+bx=-c$$

$$x^2+\frac{b}{a}x=-\frac{c}{a}$$

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\boldsymbol{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

Example 5

Find, by using the formula, the roots of the equation $2x^2-7x-1=0$ giving them correct to 3 decimal places.

Solution

Comparing with $ax^2+bx+c=0$ gives $a=2$, $b=-7$, $c=-1$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{7\pm\sqrt{49-4(2)(-1)}}{4}$$

$$=\frac{7\pm\sqrt{57}}{4}$$

$$x=\frac{7+\sqrt{57}}{4}=3.637 \text{ or } x=\frac{7-\sqrt{57}}{4}=-0.137$$

The discriminant between the roots of a quadratic equation

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The formula does not only enable us to solve quadratic equations but also to investigate the dependence of the roots on the relative values of a, b and c. In particular, the type of roots which arise depend on the quantity b^2-4ac whose square root is involved in the equation. This quantity is called the discriminant of the equation and it is often denoted by D.

$$D=b^2-4ac$$

If $b^2-4ac>0$, then the square root will be a real number and we shall obtain two real distinct roots of the equation.

If $b^2-4ac=0$, so is its square root, and both roots of the equation will be real and equal. They will both be equal to $-b/2a$.

If $b^2-4ac<0$, the square root involved in the equation is that of a negative number. Such a square root cannot be a real number.

We shall see later that it is a complex number. In this case we say that the equation has no real roots or the equation has complex roots.

Therefore, the discriminant gives the nature of the roots.

$$0 \geq 7\lambda^2 - 24\lambda - 16$$
$$7\lambda^2 - 24\lambda - 16 \leq 0$$
$$7\lambda^2 - 28\lambda + 4\lambda - 16 \leq 0$$
$$7\lambda(\lambda - 4) + 4(\lambda - 4) \leq 0$$
$$(\lambda - 4)(7\lambda + 4) \leq 0$$

Critical values of λ are 4 or $-\frac{4}{7}$

	$\lambda \leq -\frac{4}{7}$	$-\frac{4}{7} \leq \lambda \leq 4$	$\lambda \geq 4$
$\lambda - 4$	$-$	$+$	$+$
$7\lambda + 4$	$-$	$-$	$+$
$(\lambda - 4)(7\lambda + 4)$	$+$	$-$	$+$

$(\lambda - 4)(7\lambda + 4) \leq 0$ when $-\frac{4}{7} \leq \lambda \leq 4$

The relation between the roots of a quadratic equation and the coefficients

If the equation $ax^2 + bx + c = 0$ has roots α and $\beta,$ then its equivalent equation will be;

$(x - \alpha)(x - \beta) = 0$, as it gives $x = \alpha$ or $x = \beta$

$$x^2 - \beta x - \alpha x + \alpha\beta = x^2 + \frac{b}{a}x + \frac{c}{a}$$
$$x^2 - (\alpha + \beta)x + \alpha\beta = x^2 + \frac{b}{a}x + \frac{c}{a}$$

By comparing the coefficients on both sides, we obtain

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

where $\alpha + \beta$ is the sum of roots and $\alpha\beta$ is the product of roots.

Hence the equation $ax^2 + bx + c = 0$ can be written in the form;

$$\boldsymbol{x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0}$$

Example 11

Write down the sum and product of the roots of the following equations;

(a) $3x^2 - 2x - 7 = 0$

(b) $5x^2 + 11x + 3 = 0$

Solution

(a) $x^2 - \frac{2}{3}x - \frac{7}{3} = 0$

sum of roots $= -\left(-\frac{2}{3}\right) = \frac{2}{3}$

product of roots $= -\frac{7}{3}$

(b) $x^2 + \frac{11}{5}x + \frac{3}{5} = 0$

sum of roots $= -\frac{11}{5}$

product of roots $= \frac{3}{5}$

Example 12

Find the quadratic equation whose roots are $\frac{3}{4}$ and $-\frac{1}{2}$

Solution

Sum of roots $= \frac{3}{4} + \left(-\frac{1}{2}\right) = \frac{1}{4}$ and product of roots

$= \frac{3}{4} \times \left(-\frac{1}{2}\right) = -\frac{3}{8}$

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$
$$x^2 - \left(\frac{1}{4}\right)x + \left(\frac{-3}{8}\right) = 0$$
$$8x^2 - 2x - 3 = 0$$

Example 13

If α and β are the roots of the equation $ax^2 + bx + c = 0$, obtain in terms of a, b and c the values of

(a) $\alpha^2 + \beta^2$ (b) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (c) $\alpha^3 + \beta^3$ (d) $\alpha - \beta$

Solution

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$
$$\alpha + \beta = -\frac{b}{a} \qquad \alpha\beta = \frac{c}{a}$$

(a) $\alpha^2 + \beta^2 = \alpha^2 + 2\alpha\beta + \beta^2 - 2\alpha\beta$

$= (\alpha + \beta)^2 - 2\alpha\beta$

$= \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)$

$= \frac{b^2}{a^2} - \frac{2c}{a}$

$\alpha^2 + \beta^2 = \frac{b^2 - 2ac}{a^2}$

(b) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$

$= \frac{b^2 - 2ac}{a^2} \div \frac{c}{a}$

$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{b^2 - 2ac}{ac}$

(c) $(\alpha + \beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$

$= \alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta)$

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$= (\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta]$

$\alpha^3 + \beta^3 = \left(-\frac{b}{a}\right)\left[\frac{b^2}{a^2} - \frac{3c}{a}\right] = -\frac{b}{a}\left(\frac{b^2 - 3ac}{a^2}\right)$

$\alpha^3 + \beta^3 = \frac{3abc - b^3}{a^3}$

(d) $\alpha - \beta = \sqrt{(\alpha - \beta)^2}$

$= \sqrt{\alpha^2 - 2\alpha\beta + \beta^2}$

$= \sqrt{(\alpha + \beta)^2 - 2\alpha\beta - 2\alpha\beta}$

$= \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$

$\alpha - \beta = \sqrt{\frac{b^2}{a^2} - \frac{4c}{a}} = \sqrt{\frac{b - 4ac}{a^2}} = \frac{\sqrt{b^2 - 4ac}}{a}$

Example 14

If one root of the equation $px^2 + qx + r = 0$ is three times the other root, show that $3q^2 = 16pr$.

Solution

Let one root be α, then the other will be 3α

$$\alpha + 3\alpha = -\frac{q}{p}$$

$$= 2\lambda^2 c(b^2 - 2c)$$

The equation is in the form

$$x^2 - (\text{sum of roots})x + \text{product of roots} = 0$$

$$x^2 - \lambda b^2 x + 2\lambda^2 c(b^2 - 2c) = 0$$

To show that the roots are always real, we have to show that the discriminant, $D \geq 0$ i.e. is positive

$$D = (\lambda b^2)^2 - 4[2\lambda^2 c(b^2 - 2c)]$$

$$D = \lambda^2 (b^2)^2 - 8\lambda^2 c(b^2 - 2c)$$

$$D = \lambda^2[(b^2)^2 - 8cb^2 + 16c^2]$$

$$D = \lambda^2[(b^2)^2 - 2(4c)b^2 + (4c)^2]$$

$$D = \lambda^2(b^2 - 4c)^2$$

$$D = [\lambda(b^2 - 4c)]^2$$

It is clear that $D = [\lambda(b^2 - 4c)]^2 \geq 0$, thus the roots are always real.

Example 19

Show that if the equations $x^2 + bx + c = 0, x^2 + px + q = 0$ have a common root, then $(c - q)^2 = (b - p)(cp - bq)$

Solution

Let the common root be α

$$\alpha^2 + p\alpha + q = \alpha^2 + b\alpha + c$$

$$(p - b)\alpha = c - q$$

$$\alpha = \frac{c - q}{p - b}$$

Now substituting for α in one of the two equations

$$\left(\frac{c - q}{p - b}\right)^2 + b\left(\frac{c - q}{p - b}\right) + c = 0$$

$$\frac{(c - q)^2}{(p - b)^2} + \frac{(bc - bq)}{(p - b)} + c = 0$$

$$(c - q)^2 + (bc - bq)(p - b) + c(p - b)^2 = 0$$

$$(c - q)^2 + (p - b)[bc - bq + c(p - b)] = 0$$

$$(c - q)^2 + (p - b)[bc - bq + cp - bc] = 0$$

$$(c - q)^2 + (p - b)[cp - bq] = 0$$

$$(c - q)^2 = -(p - b)[cp - bq]$$

$$(c - q)^2 = (b - p)(cp - bq)$$

Maximum and minimum values of quadratic expressions

The method of completing the square, used to solve any equation in the form $ax^2 + bx + c = 0$ can be used to find the maximum or minimum value of the expression $ax^2 + bx + c$.

Example 20

Find the minimum value of the expression $x^2 + 3x + 4$

Solution

By completing the square;

$$x^2 + 3x + 4 = x^2 + 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 4$$

$$= \left[x^2 + 3x + \left(\frac{3}{2}\right)^2\right] - \frac{9}{4} + 4$$

$$= \left[x + \frac{3}{2}\right]^2 + \frac{7}{4}$$

Now $\left[x + \frac{3}{2}\right]^2$ cannot be negative for any value of x, i.e $\left[x + \frac{3}{2}\right]^2 \geq 0$

Thus $x^2 + 3x + 4$ is always positive and will have a minimum value of $\frac{7}{4}$ when $x + \frac{3}{2} = 0$ i.e. when $x = -\frac{3}{2}$

Example 21

Find the maximum value of $5 - 2x - 4x^2$

Solution

Let us first rewrite $5 - 2x - 4x^2$ as $-4x^2 - 2x + 5$

$$-4x^2 - 2x + 5 = -4\left(x^2 + \frac{1}{2}x\right) + 5$$

$$= -4\left(x^2 + \frac{1}{2}x + \frac{1}{16}\right) + \frac{4}{16} + 5$$

$$= -4\left(x + \frac{1}{4}\right)^2 + \frac{21}{4}$$

$$= \frac{21}{4} - 4\left(x + \frac{1}{4}\right)^2$$

$$\text{Now } 4\left(x + \frac{1}{4}\right)^2 \geq 0$$

Thus $5 - 2x - 4x^2$ has a maximum value of $\frac{21}{4}$

Example 22

Find by completing the square, the greatest value of the function $f(x) = 1 - 6x - x^2$

Solution

$$1 - 6x - x^2 = -x^2 - 6x + 1$$

$$= -[x^2 + 6x] + 1$$

$$= -[x^2 + 6x + 3^2 - 3^2] + 1$$

$$= -[x^2 + 6x + 9 - 9] + 1$$

$$= -[x^2 + 6x + 9] + 9 + 1$$

$$= -(x + 3)^2 + 10$$

$$= 10 - (x + 3)^2$$

Since $(x + 3)^2$ is the square of a real number, it cannot be negative, it is zero when $= -3$, otherwise it is positive.

$10 - (x + 3)^2$ is therefore always less than or equal to 10. Thus, the greatest value is 10

Example 23

Show that $3x^2 + 6x + 20$ is always positive

Solution

$$3x^2 + 6x + 20 = 3\left(x^2 + 2x + \frac{20}{3}\right)$$

$$= 3\left(x^2 + 2x + 1 - 1 + \frac{20}{3}\right)$$

$$= 3\left[(x + 1)^2 + \frac{17}{3}\right]$$

which, being the sum of two positive quantities, is always positive.

Example 24

Express $\dfrac{2x^2 + 8x + 7}{x^2 + 4x + 5}$ in the form $a - \dfrac{b}{(x + c)^2 + d}$

and state the values of a, b, c and d

Solution

$$\begin{array}{r|l} & \quad 2 \\ \hline x^2+4x+5 & 2x^2+8x+7 \\ & -2x^2+8x+10 \\ \hline & \qquad\quad -3 \end{array}$$

$$\frac{2x^2+8x+7}{x^2+4x+5} = 2 - \frac{3}{x^2+4x+5}$$

Now $x^2+4x+5 = x^2+4x+4-4+5$

$$= (x+2)^2+1$$

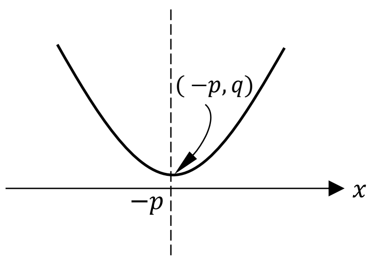
$$\frac{2x^2+8x+7}{x^2+4x+5} = 2 - \frac{3}{(x+2)^2+1}$$

$$a=2, b=3, c=2, d=1$$

Graphical representation of maximum and minimum values of a quadratic function

If $f(x) = a(x+p)^2+q$, and $a>0$, then $f(x)$ has a least value of q, when $x=-p$ as shown below.

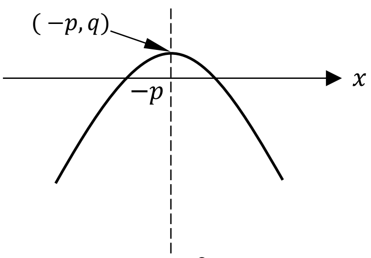
Line of symmetry



$a > 0$

If $a < 0$, $f(x)$ has a greatest value of q, when $x = -p$ as shown below.

Line of symmetry



$a < 0$

Note:

The graph of $y = f(x)$ has a line of symmetry at $x=-p$

Self-Evaluation exercise

1. If the roots of the equation $3x^2-5x+1=0$ are α, β, find the values of

   (a) $\alpha\beta^2+\alpha^2\beta$ (b) $\alpha^2-\alpha\beta+\beta^2$ (c) $\alpha^3+\beta^3$ (d) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$

   [Ans: (a) $\frac{5}{9}$ (b) $\frac{16}{9}$ (c) $\frac{80}{27}$ (d) $\frac{80}{9}$]

2. The equation $4x^2+8x-1=0$ has roots α, β. Find the values of

   (a) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$ (b) $(\alpha-\beta)^2$ (c) $\alpha^3\beta+\alpha\beta^3$ (d) $\frac{1}{\alpha^2\beta}+\frac{1}{\alpha\beta^2}$

   [Ans: (a) 72 (b) 5 (c) $-\frac{9}{8}$ (d) -32]

3. If α and β are the roots of the equation $3x^2-7x-1=0$ find the values of

   (a) $(\alpha-\beta)^2$ (b) $\alpha^2+\beta^2$ (c) $\alpha^4+\beta^4$

   [Ans: (a) $\frac{61}{9}$ (b) $\frac{55}{9}$ (c) $\frac{3007}{81}$]

4. If α and β are the roots of the equation $5x^2-3x-1=0$, form the equations with integral coefficients which have roots

   (a) $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$

   (b) $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$

   [Ans: (a) $x^2-19x+25=0$ (b) $25x^2+72x-5=0$]

5. The roots of the equation $x^2+6x+q=0$ are α and $\alpha-1$. Find the value of q.

   [Ans: $\frac{35}{4}$]

6. The roots of the equation $x^2-px+8=0$ are α and $\alpha+2$. Find two possible values of p.

   [Ans: ± 6]

7. Find the condition that the roots of the equation $px^2+qx+r=0$ should be (i) equal in magnitude and opposite in sign, (ii) reciprocals.

   [Ans: (i) $q=0$ (ii) $p=r$]

8. One root of the equation $px^2+qx+r=0$ is twice the other root. Show that $2q^2-9rp=0$.

9. Find the values of λ for which the equation

   $10x^2+4x+1 = 2\lambda x(2-x)$ has equal roots

   [Ans: 3, $-\frac{1}{2}$]

10. If the equation $a^2x^2+6abx+ac+8b^2=0$ has equal roots, prove that the roots of the equation $ac(x+1)^2 = 4b^2x$ are also equal.

11. The roots of the equation $x^2+ax+b=0$ are α, β. Find the equation whose roots are $p\alpha+q\beta$, $p\beta+q\alpha$. If the original equation is $x^2-4x-5=0$, find the values of $\frac{p}{q}$ in order that the new equation shall have one zero root.

    [Ans: $x^2+a(p+q)x+b(p^2+q^2)+(a^2-2b)pq=0$; 5, $\frac{1}{5}$]

12. Form the equation whose roots are the cubes of the roots of the equation $x^2-3x+4=0$, without solving the equation, giving the numerical values of the coefficients of the new equation.

    [Ans: $x^2+9x+64=0$]

13. Show that the roots of the equation $2bx^2+2(a+b)x+3a=2b$ are real when a and b are real.
    If one of this equation is twice the other, prove that either $a=2b$ or $4a=11b$

14. The equation $x^2+2px+p^2+q^2=r^2$ has real roots. Show that $r^2 \geq q^2$

15. Find the values of λ for which the roots of the equation

(b) Write down the values of $\alpha+\beta$ and $\alpha\beta$

(c) Form an equation with integral coefficients whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$

(d) Prove that $\alpha-\beta=\sqrt{5}$

[Ans: (a) $2<k<6$ (b) $-7, 11$ (c) $11x^2-27x+11=0$]

42. (a) If α^2 and β^2 are the roots of $x^2-21x+4=0$ and α and β are both positive, find:

(i) $\alpha\beta$;

(ii) $\alpha+\beta$;

(iii) the equation with roots $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$

(b) If $\alpha+\beta=5$ and $\alpha\beta=2$, calculate $\frac{1}{\alpha}+\frac{1}{\beta}$ and hence determine the values of m and n such that $x^2+mx+n=0$ has roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$

[Ans: (a) (i) 2 (ii) 5 (iii) $4x^2-21x+1=0$ (b) $\frac{5}{2}$, $m=-\frac{5}{2}$; $n=\frac{1}{2}$]

43. Given that α and β are the roots of the equation $2x^2+x+2=0$,

(a) evaluate $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$,

(b) find an equation whose roots are $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$

(c) show that $27\alpha^4=11\alpha+10$

[Ans: (a) $-11/4$ (b) $4x^2+11x+9=0$]

44. The real roots of the equation $x^2+6x+c=0$ differ by $2n$ where n is real and non-zero. Show that $n^2=9-c$. Given that the roots also have opposite signs, find the set of possible values of n.

[Ans: $n>3$ or $n<-3$]

45. The equation $ax^2+bx+c=0$ and $bx^2+ax+c=0$, where $a\neq b$, $c\neq 0$, have a common root. Prove that $a+b+c=0$

46. The roots of the quadratic equation $x^2-px+q=0$ are α and β. Form, in terms of p and q, the quadratic equation whose roots are $\alpha^3+p\alpha^2$, $\beta^3+p\beta^2$.

[Ans: $x^2+(5pq-2p^3)x+q^3+2p^2q^2=0$]

47. If the equation $x^2-qx+r=0$ has roots $\alpha+2$, $\beta-1$, where α, β are the real roots of the equation $2x^2-bx+c=0$, and $\alpha\geq\beta$, find q and r in terms of b and c. In the case $\alpha=\beta$, show that $q^2=4r+9$

[Ans: $q=\frac{b}{2}+1$, $r=\frac{c}{2}+\frac{b}{4}-\frac{3\sqrt{b^2-8c}}{4}-2$]

48. If the roots of the equation $x^2-bx+c=0$ are $\sqrt{\alpha}$ and $\sqrt{\beta}$. Show that

$$\alpha^2+\beta^2=\left(b^2-2c-\sqrt{2}c\right)\left(b^2-2c+\sqrt{2}c\right)$$

Chapter 4 Polynomials and the Remainder Theorem

A polynomial in x, a variable, is an expression of the form

$$c_0x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_{n-1}x + c_n$$

where n is a positive integer

and $c_0, c_1, c_2, \ldots\ldots, c_{n-1}, c_n$ are constants.

The **degree** of n, the highest power of x.

The **constant** term is c_n.

For example, $2x^7 + 3x^5 - x^4 + 6x + 4$ is a polynomial of degree 7 with constant term 4.

Operations on polynomials

Addition: Add corresponding terms (powers of x)

$$\begin{array}{rl} f(x) = & 3x^4 - 5x^3 \qquad\quad + x - 4 \\ g(x) = & \qquad\quad 4x^3 - 3x^2 + 4x + 3 \\ \hline f(x) + g(x) = & 3x^4 - x^3 - 3x^2 + 5x - 1 \\ \hline \end{array}$$

Subtraction: Subtract corresponding terms

$$\begin{array}{rl} f(x) = & 3x^4 - 5x^3 \qquad\quad + x - 4 \\ g(x) = & \qquad\quad 4x^3 - 3x^2 + 4x + 3 \\ \hline f(x) - g(x) = & 3x^4 - 9x^3 + 3x^2 - 3x - 7 \\ \hline \end{array}$$

Multiplication: This can be set out like a 'long multiplication'. Leave spaces for 'missing terms'

$$\begin{array}{ll} f(x) & \qquad 3x^3 \qquad - 2x + 4 \\ g(x) & \qquad\qquad x^2 \qquad - 3 \\ \hline f(x) \times x^2 & 3x^5 - 2x^3 + 4x^2 \\ f(x) \times -3 & \qquad - 9x^3 \qquad + 6x - 12 \\ \hline f(x) \times g(x) & 3x^5 - 11x^3 + 4x^2 + 6x - 12 \\ \hline \end{array}$$

Division: This can be set out like a 'long division'. Leave spaces for 'missing terms'

$$\begin{array}{r|l} & x^2 + 4x + 11 \\ \hline x^2 - 4x + 3 & x^4 \qquad - 2x^2 + 3x - 6 \\ - & x^4 - 4x^3 + 3x^2 \\ \hline & \quad 4x^3 - 5x^2 + 3x \\ & - \; 4x^3 - 16x^2 + 12x \\ \hline & \qquad 11x^2 - 9x - 6 \\ & \qquad - \; 11x^2 - 44x + 33 \\ \hline & \qquad\qquad 35x - 39 \end{array}$$

$$\frac{x^4 - 2x^2 + 3x - 6}{x^2 - 4x + 3} = x^2 + 4x + 11 + \frac{35x - 39}{x^2 - 4x + 3}$$

This approach can be extended to the division of a polynomial $f(x)$ by a polynomial $g(x)$ of degree less than or equal to the degree of $f(x)$. If the division gives quotient $Q(x)$ and remainder $R(x)$, then

$$\frac{f(x)}{g(x)} \equiv Q(x) + \frac{R(x)}{g(x)}$$

$$f(x) \equiv g(x)Q(x) + R(x)$$

where $R(x)$ is of lower degree than $g(x)$

[In particular, if $g(x)$ is a quadratic function, then $R(x)$ is of the form $Ax + B$]

The Remainder Theorem:

If a polynomial $f(x)$ is divided by $(x - a)$, then the remainder is $f(a)$.

$$f(x) = (x - a)Q(x) + R$$

$$f(a) = (a - a)Q(x) + R$$

$$f(a) = R$$

The Factor Theorem:

If $(x - a)$ is a factor of $f(x)$, then $f(a) = 0$ i.e. the remainder is zero.

Conversely, if $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$.

This may be used to find the factors of a polynomial. Factors of the constant term are usually tested first.

If it is suspected that $(x - a)$ is a repeated factor:

(a) 'take out' the factor, either by inspection or long division to give

$$f(x) \equiv (x - a)g(x)$$

(b) test $(x - a)$ as a factor of $g(x)$

Special factors

$$a^2 - b^2 = (a - b)(a + b)$$

$$a^2 \pm 2ab + b^2 = (a \pm b)^2$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Example 1

Find the remainder when $3x^5 - x^2 + 1$ is divided by $(x + 2)$

Solution

Let $f(x) = 3x^5 - x^2 + 1$

$f(-2) = 2(-2)^5 - (-2)^2 + 1 = -96 - 4 + 1 = -99$

Example 2

When the cubic polynomial $x^3 + ax^2 - 3x + 4$ is divided by $x - 3$, the remainder obtained is twice the remainder obtained when the polynomial is divided by $x - 2$. Find the value of a.

Solution

Let $f(x) = x^3 + ax^2 - 3x + 4$

$f(3) = 3^3 + a(3^2) - 3(3) + 4$

$$3 + c = -4$$
$$c = -7$$
$$p = b = -4$$
$$q = c = -7$$
$$\therefore p = 1 \text{ and } q = -7$$

Example 13

Given that $x^4 - 6x^3 + 10x^2 + ax + b$ is a perfect square, find the values of a and b.

Solution

Let $x^4 - 6x^3 + 10x^2 + ax + b \equiv (x^2 + Bx + C)^2$

$$(x^2 + Bx + C)^2 = x^4 + Bx^3 + Cx^2 + Bx^3 + B^2x^2 + BCx + Cx^2 + BCx + C^2$$
$$= x^4 + 2Bx^3 + (B^2 + 2C)x^2 + 2BCx + C^2$$
$$x^4 - 6x^3 + 10x^2 + ax + b \equiv x^4 + 2Bx^3 + (B^2 + 2C)x^2 + 2BCx + C^2$$

Comparing coefficients:

$$2B = -6$$
$$B = -3$$
$$B^2 + 2C = 10$$
$$(-3)^2 + 2C = 10$$
$$9 + 2C = 10$$
$$C = \frac{1}{2}$$
$$a = 2BC$$
$$a = 2(-3)\left(\frac{1}{2}\right) = -3$$
$$b = C^2$$
$$b = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$
$$\therefore a = -3, \qquad b = \frac{1}{4}$$

Repeated roots of a polynomial

Suppose now that a polynomial $f(x)$ has a repeated factor $(x - a)$ such that

$$f(x) \equiv (x - a)^2 \mathrm{g}(x)$$

Differentiating,

$$f'(x) = (x - a)^2 \mathrm{g}'(x) + 2(x - a)\mathrm{g}(x)$$
$$= (x - a)[(x - a)\mathrm{g}'(x) + 2\mathrm{g}(x)]$$

Hence, if $f(x)$ has a repeated factor $(x - a)$, then $(x - a)$ is also a factor of $f'(x)$.

$$f(a) = (a - a)^2 \mathrm{g}(x) = 0$$
$$f'(a) = (a - a)[(a - a)\mathrm{g}'(x) + 2\mathrm{g}(x)] = 0$$

Furthermore, $(x - a)^2$ is a factor of a polynomial $f(x)$ if and only if $f(a) = f'(a) = 0$.

Example 14

Given that the polynomial $f(x) = x^3 + 3x^2 - 9x + k$ has a repeated linear factor, find the possible values of k.

Solution

Differentiating; $f'(x) = 3x^2 + 6x - 9$

$$= 3(x^2 + 2x - 3) = 3(x - 1)(x + 3)$$

$\therefore$ the repeated factor of $f(x)$ is $(x - 1)$ or $(x + 3)$

If $(x - 1)$ is a factor of $f(x)$, then $f(1) = 0$

$$(1)^3 + 3(1)^2 - 9(1) + k = 0$$
$$1 + 3 - 9 + k = 0$$
$$k = 5$$

If $(x + 3)$ is a factor of $f(x)$, then $f(-3) = 0$

$$(-3)^3 + 3(-3)^2 - 9(-3) + k = 0$$
$$-27 + 27 + 27 + k = 0$$
$$k = -27$$

Thus the possible values of k are 5 and -27

Example 15

Find the roots of the equation $4x^3 + 12x^2 - 15x + 4 = 0$ given that it has a repeated root.

Solution

Let $f(x) = 4x^3 + 12x^2 - 15x + 4$

then $f'(x) = 12x^2 + 24x - 15$

$$= 3(4x^2 + 8x - 5)$$
$$= 3(4x^2 + 10x - 2x - 5)$$
$$= 3(2x - 1)(2x + 5)$$

Any repeated root of the equation $f(x) = 0$ is also a root of the equation $f'(x) = 0$.

$$3(2x - 1)(2x + 5) = 0$$
$$x = \tfrac{1}{2} \text{ or } x = -\tfrac{5}{2}$$

the repeated root must be either $-\frac{5}{2}$ or $\frac{1}{2}$

$$f\left(\tfrac{1}{2}\right) = 4\left(\tfrac{1}{2}\right)^3 + 12\left(\tfrac{1}{2}\right)^2 - 15\left(\tfrac{1}{2}\right) + 4 = 0$$

$(2x - 1)$ is a repeated root of $f(x)$

$$f\left(-\tfrac{5}{2}\right) = 4\left(-\tfrac{5}{2}\right)^3 + 12\left(-\tfrac{5}{2}\right)^2 - 15\left(-\tfrac{5}{2}\right) + 4 = 54 \neq 0$$

$(2x + 5)$ is not a repeated factor of $f(x)$

$$4x^3 + 12x^2 - 15x + 4 \equiv (2x - 1)^2(Ax + B)$$
$$= (4x^2 - 4x + 1)(Ax + B)$$
$$= 4Ax^3 + 4Bx^2 - 4Ax^2 - 4Bx + Ax + B$$
$$= 4Ax^3 + (4B - 4A)x^2 + (A - 4B)x + B$$

Comparing coefficients:

$$A = 1, B = 4$$
$$\therefore 4x^3 + 12x^2 - 15x + 4 = (2x - 1)^2(x + 4)$$
$$x = \tfrac{1}{2}, x = \tfrac{1}{2}, x = -4$$

Self-Evaluation exercise

1. When $(x^4 + kx^2 + 4x + 2)$ is divided by $(x + 3)$, the remainder is 8. Find the value of k.

   [Ans: -7]

2. If $f(x)$ denotes the polynomial $2x^3 - 3x^2 - 8x - 3$, find the remainders when $f(x)$ is divided by
   (i) $x - 1$
   (ii) $x + 3$
   (iii) $2x + 1$

   [Ans: (i) -12 (ii) -60 (iii) 0]

(b) Given that $(x+3)$ is a factor of $f(x)$, find the value of b

[Ans: $a=-20$, $b=-6$]

24. $f(x)=x^3+(p+1)x^2-18x+q$, where p and q are integers.
   (a) Given that $(x-4)$ is a factor of $f(x)$, show that $16p+q+8=0$
   (b) Given that $(x+p)$ is also a factor of $f(x)$, and that $p>0$, show that $p^2+18p+q=0$
   Hence find the value of p and the corresponding value of q. Also find the third factor.

[Ans: $p=2$, $q=-40$; $x+5$]

25. The polynomials x^3+4x^2-2x+1 and x^3+3x^2-x+7 leave the same remainder when divided by $x-p$. Find the possible values of p

[Ans: $3, -2$]

26. Given that $f(x)=4x^4+12x^3-5x^2-21x+10$, find by inspection two solutions of the equation $f(x)=0$. Hence factorise $f(x)$ and solve the equation completely.

[Ans: $x=1, -2, \frac{1}{2}, -\frac{5}{2}$]

27. Find the roots of the equation $x^3-6x^2-63x-108=0$ given that it has a repeated root.

[Ans: $-3,\ -3, 12$]

28. Given that $P(x)=8x^3-12x^2-18x+k$, find the values of k such that the equation $P(x)=0$ has a repeated root. Give the roots of the equation in each case.

[Ans: $k=-5$: $-\frac{1}{2}, -\frac{1}{2}, \frac{5}{2}$; $k=27$: $\frac{3}{2}, -\frac{3}{2}$]

29. Find all real values of k for which the equation $x^3-3kx^2+2k+2=0$ has repeated roots and, for each such k, solve the equation completely.

[Ans: $k=-1$: $-3, 0, 0$; $k=1$: $-1, 2, 2$]

30. Given that $f(x)=2x^4+ax^3+bx^2-8x+c$, find the real coefficients a, b and c when the following conditions are satisfied:
   (a) $(x+2)$ is a factor of $f(x)$ and $f'(x)$
   (b) when $f(x)$ is divided by $(x-2)$ the remainder is 16
   Factorise $f(x)$ completely

[Ans: $a=3$, $b=-9$, $c=12$; $(x+2)^2(x-1)(2x-3)$]

31. What is the value of a if $2x^2-x-6$, $3x^2-8x+4$ and $ax^3-10x-4$ have a common factor?

[Ans: 3]

32. Given that $(x-1)$ and $(x-2)$ are factors of $6x^4+ax^3-17x^2+bx-4$, find a and b, and any remaining factors.

[Ans: $a=-9$, $b=24$; $(6x^2+9x-2)$]

33. Given that $x^3=a(x+1)^3+b(x+1)^2+c(x+1)+d$, find the values of a, b, c and d.

[Ans: $a=1$, $b=-3$, $c=3$, $d=-1$]

34. A cubic polynomial gives remainders $(5x+4)$ and $(12x-1)$ when divided by x^2-x+2 and (x^2+x-1) respectively. Find the polynomial.

[Ans: x^3-2x^2+8x+2]

35. A cubic polynomial gives remainders $(13x-2)$ and $(-1-7x)$ when divided by x^2-x-3 and x^2-2x+5 respectively. Find the polynomial.

[Ans: $3x^3-5x^2+6x+4$]

36. Given that $P(x)=2x^4+mx^3-nx^2-7x+k$ is divisible by $(x-2)$ and $(x+3)$ and leaves a remainder of -18 when divided by $(x+1)$
   (a) Solve for m, n and k
   (b) Hence, find all linear factors of $P(x)$

[Ans: (a) $m=\frac{18}{5}$, $n=\frac{39}{5}$, $k=\frac{78}{5}$ (b) $(x-2)$, $(x+3)$]

37. $P(x)=x^3+mx^2+nx+k$ is divisible by x^2-4 and leaves a remainder of 30 when divided by $(x-3)$. Solve for m, n and k and hence fully factorise $P(x)$ into its three linear factors.

[Ans: $m=3$, $n=-4$, $k=-12$; $(x-2)(x+2)(x+3)$]

38. If the polynomial $P(x)=x^2+ax+1$ is a factor of $T(x)=2x^3-16x+b$, find the values of a and b.

[Ans: $a=3$, $b=-6$; $a=-3$, $b=6$]

39. When a polynomial, $P(x)$, is divided by $x-\alpha$, it leaves a remainder of α^3 and when it is divided by $x-\beta$ it leaves a remainder of β^3. Find the remainder when $P(x)$ is divided by $(x-\alpha)(x-\beta)$

[Ans: $(\alpha^2+\alpha\beta+\beta^2)x-\alpha\beta(\alpha+\beta)$]

40. (a) Prove that if x^3+mx+n is divisible by $(x-k)^2$, then $\left(\frac{m}{3}\right)^3+\left(\frac{n}{2}\right)^2=0$
   (b) Prove that if x^3+mx+n and $3x^2+m$ have a common factor $(x-k)$ then $4m^3+27n^2=0$

41. Prove that $P(x)=x^n-a^n$ is divisible by $(x-a)$ for all integer values of n

Chapter 5 Partial Fractions

Rational expression

An expression of the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x) \neq 0$ are polynomials in x is called a rational expression.

The expressions $\frac{5x-2}{x^2+3x+2}$, $\frac{3x^2+2x-1}{x^2+x-2}$, $\frac{x+1}{x^2-1}$ are examples of rational expressions.

A rational function which may be expressed as a sum of separate fractions is said to be resolved into its **partial fractions**.

Consider the sum of $\frac{7}{x-2}$ and $\frac{5}{x-1}$, we simplify it as follows:

$$\frac{7}{x-2}+\frac{5}{x-1}=\frac{7(x-1)+5(x-2)}{(x-2)(x-1)}=\frac{7x-7+5x-10}{(x-2)(x-1)}=\frac{12x-17}{(x-2)(x-1)}$$

Conversely the process of writing the given fraction $\frac{12x-17}{(x-2)(x-1)}$ as $\frac{7}{x-2}+\frac{5}{x-1}$ is known as splitting into partial fractions or expressing as partial fractions.

Note:

Expressing an algebraic fraction in terms of its partial fractions is rarely of interest in itself; it is however often an important means to other ends.

Two common uses of partial fractions include the binomial series and integration, which are discussed in the respective chapters of this book.

Proper fraction

A proper fraction is one in which the degree of the numerator is less than the degree of the denominator.

The expressions $\frac{3x+1}{x^2+4x+3}$, $\frac{7x^2+9}{x^3+x^2-5}$ are examples of proper fractions.

Type 1: Linear factors, none of which is repeated

If a linear factor $ax+b$ is a factor of the denominator $q(x)$, then the corresponding to this factor, associate a simple fraction $\frac{A}{ax+b}$, where A is a constant $(A \neq 0)$

i.e. when the factors of the denominator of the given fraction are all linear none of which is repeated, we write the partial fractions as follows:

$$\frac{x+3}{(x+5)(2x+1)} \equiv \frac{A}{x+5}+\frac{B}{2x+1}$$

where A and B are constants to be determined

Example 1

Resolve into partial fractions $\frac{3x+7}{x^2-3x+2}$

Solution

The denominator x^2-3x+2 can be factorised into linear factors.

$$x^2-3x+2=x^2-x-2x+2=x(x-1)-2(x-1)$$
$$=(x-1)(x-2)$$

$$\text{Let } \frac{3x+7}{(x-1)(x-2)} \equiv \frac{A}{x-1}+\frac{B}{x-2}$$
$$=\frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

Equating the numerators

$$3x+7=A(x-2)+B(x-1)$$
$$3x+7=Ax-2A+Bx-B$$
$$3x+7=(A+B)x-2A-B$$

Equating the coefficients of like powers of x, we get

$$A+B=3 \text{ …. (i)}$$
$$-2A-B=7 \text{ …(ii)}$$

Adding (i) and (ii);

$$A=-10$$

Substituting for A in (i);

$$-10+B=3$$
$$B=13$$

$$\therefore \frac{3x+7}{x^2-3x+2} \equiv \frac{-10}{x-1}+\frac{13}{x-2} \equiv \frac{13}{x-2}-\frac{10}{x-1}$$

Note: The constants A and B can also be found by successively giving suitable values for x.

To find A, put $x=1$

$$3(1)+7=A(1-2)+B(0)$$
$$10=A(-1)$$
$$A=-10$$

To find B, put $x=2$,

$$3(2)+7=A(0)+B(2-1)$$
$$B=13$$

which yields the same result as before

Example 2

Express into partial fractions $\frac{x+4}{(x^2-4)(x+1)}$

Solution

The denominator $(x^2-4)(x+1)$ can be further factored into linear factors i.e.

$$(x^2-4)(x+1)=(x+2)(x-2)(x+1)$$

$$\text{Let } \frac{x+4}{(x^2-4)(x+1)} \equiv \frac{A}{x+2}+\frac{B}{x-2}+\frac{C}{x+1}$$
$$=\frac{A(x-2)(x+1)+B(x+2)(x+1)+C(x+2)(x-2)}{(x+2)(x-2)(x+1)}$$

$$x+4=A(x-2)(x+1)+B(x+2)(x+1)$$
$$+C(x+2)(x-2)$$

To find A, put $x=-2$,

$$-2+4 = A(-4)(-1) + B(0) + C(0)$$
$$2 = 4A$$
$$A = \frac{1}{2}$$

To find B, put $x = 2$,
$$2+4 = A(0) + B(4)(3) + C(0)$$
$$6 = 12B$$
$$B = \frac{1}{2}$$

To find C, put $x = -1$,
$$-1+4 = A(0) + B(0) + C(1)(-3)$$
$$3 = -3C$$
$$C = -1$$
$$\frac{x+4}{(x^2-4)(x+1)} \equiv \frac{1/2}{(x+2)} + \frac{1/2}{(x-2)} + \frac{-1}{(x+1)}$$
$$\frac{x+4}{(x^2-4)(x+1)} \equiv \frac{1}{2(x+2)} + \frac{1}{2(x-2)} - \frac{1}{x+1}$$

Example 3

Express $\dfrac{x+1}{2x^3-5x^2+2x}$ in partial fractions

Solution

$$2x^3-5x^2+2x = x(2x^2-5x+2) = x(x-2)(2x-1)$$

Let $\dfrac{x+1}{2x^3-5x^2+2x} \equiv \dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{C}{2x-1}$
$$\equiv \frac{A(x-2)(2x-1) + Bx(2x-1) + Cx(x-2)}{x(x-2)(2x-1)}$$
$$x+1 \equiv A(x-2)(2x-1) + Bx(2x-1) + Cx(x-2)$$

Put $x = 0$;
$$1 = 2A$$
$$A = \tfrac{1}{2}$$

Put $x = 2$;
$$3 = 6B$$
$$B = \tfrac{1}{2}$$

Put $x = \frac{1}{2}$;
$$\tfrac{3}{2} = C \times \tfrac{1}{2} \times \left(-\tfrac{3}{2}\right)$$
$$C = -2$$
$$\therefore \frac{x+1}{2x^3-5x^2+2x} \equiv \frac{1}{2x} + \frac{1}{2(x-2)} - \frac{2}{2x-1}$$

Example 4

Express $\dfrac{9x-72}{x^3-3x^2-18x+40}$ in partial fractions

Solution

We need to first factorise the expression using the factor theorem.

Let $f(x) = x^3 - 3x^2 - 18x + 40$

Trying the factors of 40 as values of x

$f(1) = 1 - 3 - 18 + 40 \neq 0 \Rightarrow (x-1)$ is not a factor

$f(2) = 8 - 12 - 36 + 40 = 0 \Rightarrow (x-2)$ is not a factor

$$\begin{array}{r|l}
 & x^2 - x - 20 \\
\hline
x-2 & x^3 - 3x^2 - 18x + 40 \\
 & -\ \ x^3 - 2x^2 \\
\hline
 & -x^2 - 18x + 40 \\
 & -\ -x^2 + 2x \\
\hline
 & -20x + 40 \\
 & -\ \ -20x + 40 \\
\hline
 & -\ \ -
\end{array}$$

$$\begin{aligned}
f(x) &= (x-2)(x^2-x-20) \\
&= (x-2)(x^2+4x-5x-20) \\
&= (x-2)[x(x+4)-5(x+4)] \\
&= (x-2)(x-5)(x+4)
\end{aligned}$$

Hence $x^3 - 3x^2 - 18x + 40 = (x-2)(x-5)(x+4)$

Let $\dfrac{9x-72}{(x-2)(x-5)(x+4)} \equiv \dfrac{A}{x-2} + \dfrac{B}{x-5} + \dfrac{C}{x+4}$
$$\frac{9x-72}{(x-2)(x-5)(x+4)} \equiv \frac{A(x-5)(x+4)+B(x-2)(x+4)+C(x-2)(x-5)}{(x-2)(x-5)(x+4)}$$
$$9x-72 \equiv A(x-5)(x+4) + B(x-2)(x+4)$$
$$+C(x-2)(x-5)$$

Put $x = 2$;
$$18-72 = A(-3)(6)$$
$$-54 = -18A$$
$$A = 3$$

Put $x = -4$;
$$-36-72 = C(-6)(-9)$$
$$-108 = 54C$$
$$C = -2$$

Put $x = 5$;
$$45-72 = B(3)(9)$$
$$-27 = 27B$$
$$B = -1$$
$$\frac{9x-72}{x^3-3x^2-18x+40} \equiv \frac{3}{x-2} - \frac{1}{x-5} - \frac{2}{x+4}$$

Type 2: Linear factors, some of which are repeated

If a linear factor $ax+b$ occurs n times as a factor of the denominator of the given fraction, then the corresponding to these factors associate the sum of n simple fractions,
$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_n}{(ax+b)^n}$$
where $A_1, A_2, A_3, \ldots, A_n$ are constants

Example 5

Resolve into partial fractions $\dfrac{9}{(x-1)(x+2)^2}$

Solution

Let $\dfrac{9}{(x-1)(x+2)^2} \equiv \dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2}$
$$= \frac{A(x+2)^2 + B(x-1)(x+2) + C(x-1)}{(x-1)(x+2)^2}$$

Improper fractions

An improper fraction is one in which the degree of the numerator is greater than or equal the degree of the denominator. i.e. the degree of $P(x) \geq$ the degree of $Q(x)$

If the rational function is an improper fraction

(a) Divide to obtain a quotient and a proper fraction

(b) Resolve the proper fraction into partial fractions as before.

Example 11

Resolve into partial fractions $\dfrac{x^2+x+1}{x^2-5x+6}$

Solution

Here the degree of the numerator is the same as the degree of the denominator

$$\begin{array}{r|l} & 1 \\ \hline x^2-5x+6 & x^2+x+1 \\ & -\ x^2-5x+6 \\ \hline & 6x-5 \end{array}$$

$$\frac{x^2+x+1}{x^2-5x+6} = 1 + \frac{6x-5}{x^2-5x+6}$$

$$x^2-5x+6 = x^2-3x-2x+6 = (x-3)(x-2)$$

Let $\dfrac{6x-5}{x^2-5x+6} \equiv \dfrac{A}{x-2} + \dfrac{B}{x-3} \equiv \dfrac{A(x-2)+B(x-3)}{(x-2)(x-3)}$

$$6x-5 \equiv A(x-3)+B(x-2)$$

Put $x=2$,

$$12-5=-A$$
$$A=-7$$

Put $x=3$,

$$18-5=B$$
$$B=13$$

$$\frac{6x-5}{x^2-5x+6} \equiv -\frac{7}{x-2}+\frac{13}{x-3}$$

$$\therefore \frac{x^2+x+1}{x^2-5x+6} = 1-\frac{7}{x-2}+\frac{13}{x-3}$$

Example 12

Resolve into partial fractions

$$\frac{4x^3+16x^2-15x+13}{(x+2)(2x-1)^2}$$

Solution

Since the degree of the numerator is equal to the degree of the denominator, the denominator is divided into the numerator

$(x+2)(2x-1)^2 = (x+2)(4x^2-4x+1)$
$= 4x^3+4x^2-7x+2$

$$\begin{array}{r|l} & 1 \\ \hline 4x^3+4x^2-7x+2 & 4x^3+16x^2-15x+13 \\ & -\ 4x^3+4x^2-7x+2 \\ \hline & 12x^2-8x+11 \end{array}$$

$$\frac{4x^3+16x^2-15x+13}{(x+2)(2x-1)^2} \equiv 1+\frac{12x^2-8x+11}{(x+2)(2x-1)^2}$$

Let $\dfrac{12x^2-8x+11}{(x+2)(2x-1)^2} \equiv \dfrac{A}{(x+2)}+\dfrac{B}{(2x-1)}+\dfrac{C}{(2x-1)^2}$

$$12x^2-8x+11 \equiv A(2x-1)^2+B(x+2)(2x-1) + C(x+2)$$

Put $x=-2$,

$$48+16+11=25A$$
$$75=25A$$
$$A=3$$

Put $x=-\frac{1}{2}$,

$$3-4+11=\tfrac{5}{2}C$$
$$20=5C$$
$$C=4$$

Put $x=0$,

$$11=A-2B+2C$$
$$11=3-2B+8$$
$$B=0$$

$$\frac{12x^2-8x+11}{(x+2)(2x-1)^2} \equiv \frac{3}{(x+2)}+\frac{4}{(2x-1)^2}$$

$$\therefore \frac{4x^3+16x^2-15x+13}{(x+2)(2x-1)^2} \equiv 1+\frac{3}{(x+2)}+\frac{4}{(2x-1)^2}$$

Example 13

Express $\dfrac{x^3+2x^2-x+3}{(x+2)(x-3)}$ in partial fractions

Solution

$$\begin{array}{r|l} & x+3 \\ \hline x^2-x-6 & x^3+2x^2-x+3 \\ & -\ x^3-x^2-6x \\ \hline & 3x^2+5x+3 \\ & -\ 3x^2-3x-18 \\ \hline & 8x+21 \end{array}$$

$$\frac{x^3+2x^2-x+3}{(x+2)(x-3)} = x+3+\frac{8x+21}{(x+2)(x-3)}$$

Let $\dfrac{8x+21}{(x+2)(x-3)} \equiv \dfrac{A}{x+2}+\dfrac{B}{x-3}$

$$\equiv \frac{A(x-3)+B(x+2)}{(x+2)(x-3)}$$

$$8x+21 \equiv A(x-3)+B(x+2)$$

Put $x=3$:

$$45=B(5)$$
$$B=9$$

Put $x=-2$:

$$5=A(-5)$$
$$A=-1$$

$$\therefore \frac{x^3+2x^2-x+3}{(x+2)(x-3)} \equiv x+3-\frac{1}{x+2}+\frac{9}{x-3}$$

Example 14

Express $\dfrac{x^3-3x^2+1}{x^2-x-2}$ in partial fractions

Solution

$$\begin{array}{r|l} & x-2 \\ \hline x^2-x-2 & x^3-3x^2 \qquad +1 \\ - & x^3-x^2-2x \\ \hline & -2x^2+2x+1 \\ - & -2x^2+2x+4 \\ \hline & \qquad\qquad -3 \end{array}$$

$$\frac{x^3-3x^2+1}{x^2-x-2} = x-2-\frac{3}{x^2-x-2}$$

$$x^2-x-2 = x^2+x-2x-2 = (x+1)(x-2)$$

Let $\dfrac{-3}{(x+1)(x-2)} \equiv \dfrac{A}{x+1}+\dfrac{B}{x-2} \equiv A(x-2)+B(x+1)$

$$-3 \equiv A(x-2)+B(x+1)$$

Put $x=-1$;

$$-3=-3A$$
$$A=1$$

Put $x=2$;

$$-3=3B$$
$$B=-1$$

$$\therefore \frac{x^3-3x^2+1}{x^2-x-2} \equiv x-2+\frac{1}{x+1}-\frac{1}{x-2}$$

Self-Evaluation exercise

Express the following into partial fractions

1. $\frac{3x-10}{(x-2)(x-4)}$ [Ans: $\frac{2}{x-2}+\frac{1}{x-4}$]
2. $\frac{x}{x^2-1}$ [Ans: $\frac{1}{2(x-1)}+\frac{1}{2(x+1)}$]
3. $\frac{5}{6-x-x^2}$ [Ans: $\frac{1}{3+x}+\frac{1}{2-x}$]
4. $\frac{5x^2-12x-5}{(x^2-1)(x-2)}$ [Ans: $\frac{6}{x-1}+\frac{2}{x+1}-\frac{3}{x-2}$]
5. $\frac{17x+11}{(x-2)(x+3)(x+1)}$ [Ans: $\frac{3}{x-2}-\frac{4}{x+3}+\frac{1}{x+1}$]
6. $\frac{3x+1}{(x+1)(x^2+1)}$ [Ans: $\frac{x+2}{x^2+1}-\frac{1}{x+1}$]
7. $\frac{x+2}{(2x-1)(x^2+1)}$ [Ans: $\frac{2}{2x-1}-\frac{x}{x^2+1}$]
8. $\frac{3x+1}{x(2x^2+1)}$ [Ans: $\frac{1}{x}+\frac{3-2x}{2x^2+1}$]
9. $\frac{x^2-10}{(x^2+3)(2x-1)}$ [Ans: $\frac{2x+1}{x^2+3}-\frac{3}{2x-1}$]
10. $\frac{x^2-13}{(x-1)^2(x+2)}$ [Ans: $\frac{2}{x-1}-\frac{4}{(x-1)^2}-\frac{1}{x+2}$]
11. $\frac{3x^2+7x+1}{x^3+2x^2+x}$ [Ans: $\frac{1}{x}+\frac{2}{x+1}+\frac{3}{(x+1)^2}$]
12. $\frac{2x^2-3x-2}{x^3-x^2}$ [Ans: $\frac{5}{x}+\frac{2}{x^2}-\frac{3}{x-1}$]
13. $\frac{x^2+23}{(x+1)^3(x-2)}$ [Ans: $\frac{1}{x-2}-\frac{1}{x+1}-\frac{2}{(x+1)^2}-\frac{8}{(x+1)^3}$]
14. $\frac{2x^2-5x-5}{(2x^2+5)(4x-5)}$ [Ans: $\frac{x}{2x^2+5}-\frac{1}{4x-5}$]
15. $\frac{x+2}{(x-2)(x^2-x+2)}$ [Ans: $\frac{1}{x-2}-\frac{x}{x^2-x+2}$]
16. $\frac{x^2+1}{x^2-1}$ [Ans: $1+\frac{1}{x-1}-\frac{1}{x+1}$]
17. $\frac{x^2}{x^2-x-2}$ [Ans: $1+\frac{4}{3(x-2)}-\frac{1}{3(x+1)}$]
18. $\frac{x(x-2)}{(3x-1)(x-1)}$ [Ans: $\frac{1}{3}+\frac{5}{6(3x-1)}-\frac{1}{2(x-1)}$]
19. $\frac{x^3}{x^2-4}$ [Ans: $x+\frac{2}{x-2}+\frac{2}{x+2}$]
20. $\frac{x^2-x}{(x^2+3)(x^2+2)}$ [Ans: $\frac{x+3}{x^2+3}-\frac{x+2}{x^2+2}$]
21. $\frac{3x^3+2x^2+2x-3}{(x^2+2)(x+1)^2}$ [Ans: $\frac{2x-1}{x^2+2}+\frac{1}{x+1}-\frac{2}{(x+1)^2}$]
22. $\frac{2x^4-2x^3+x}{(2x-1)^2(x-2)}$ [Ans: $\frac{x}{2}+1-\frac{1}{4(2x-1)}-\frac{1}{4(2x-1)^2}+\frac{2}{x-2}$]
23. $\frac{x^6-x^5-4x^2+x}{x^4+3x^2+2}$ [Ans: $x^2-x-3+\frac{3}{x^2+1}+\frac{3x}{x^2+2}$]
24. $\frac{7x-1}{6-5x+x^2}$ [Ans: $\frac{20}{x-3}-\frac{13}{x-2}$]
25. $\frac{7x^2-25x+6}{(x^2-2x-1)(3x-2)}$ [Ans: $\frac{x-5}{x^2-2x-1}+\frac{4}{3x-2}$]
26. $\frac{x^2+x+1}{x^2+2x+1}$ [Ans: $1-\frac{1}{x+1}+\frac{1}{(x+1)^2}$]
27. Express $\frac{x^2-x-1}{x^3-8}$ into partial fractions

    [Ans: $\frac{1}{12(x-2)}+\frac{11x+8}{12(x^2+2x+4)}$]
28. Express $\frac{5x^3+2x^2+5x}{x^4-1}$ in partial fractions

    [Ans: $\frac{3}{x-1}+\frac{2}{x+1}+\frac{1}{x^2+1}$]
29. Express in partial fractions $\frac{1+x^2}{(1+x)(1+x^3)}$

    [Ans: $\frac{2}{3(1+x)^2}+\frac{1}{3(1-x+x^2)}$]
30. Use the remainder theorem to find the three factors of x^4+3x^2-4 and hence resolve

    $$\frac{2x^3-x^2-7x-14}{x^4+3x^2-4}$$

    into partial fractions

    [Ans: $\frac{1}{x+1}-\frac{2}{x-1}+\frac{3x+2}{x^2+4}$]

Chapter 6 Binomial Theorem

Binomial

A binomial is an algebraic expression of two terms (bi) which are connected by the operation '$+$' or '$-$'. For example, $x + 2y$, $x - y$, $x^3 + 4y$, $a + b$, etc. are binomials.

The theorem about the expansion of a power of two terms is called the binomial theorem.

Pascal's triangle:

Let us consider the expansion of the

$$(a+b)^1 = a + b$$

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$

$$(a+b)^3 = (a+b)^2(a+b) = (a^2+2ab+b^2)(a+b) = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = (a+b)^3(a+b) = (a^3+3a^2b+3ab^2+b^3)(a+b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a+b)^5 = (a+b)^4(a+b) = (a^4+4a^3b+6a^2b^2+4ab^3+b^4)(a+b)$$
$$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^4$$

The multiplications have been done and like terms collected (recall from multiplication of polynomials)

If we extract the coefficients of a and b, we obtain the Pascal's triangle

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Next line obtained by rule **1 6 15 20 15 6 1**

. . . .

- It can be observed that the coefficients of the various terms of the expansion $(a+b)^n$ for $n = 1, 2, 3, \ldots$. form a pattern.
- The first and last numbers are 1 each. The other numbers are obtained by adding the left and right numbers in the previous row.
- $1,\ 1 + 4 = 5, 4 + 6 = 10, 6 + 4 = 10, 4 + 1 = 5,\ 1$
- Reading from either end of each row, the coefficients are the same
- There are $(n + 1)$ terms
- Each term is of degree n
- The powers of a are descending while the powers of b are ascending

Example 1

Expand $(2 + x)^5$ in powers of x

Solution

Using the coefficients from the Pascal's triangle

$$(2+x) = 2^5 + 5\,.\,2^4\,.\;x + 10\,.\;2^3x^2 + 10\,.\;2^2x^3 + 5\,.\,2x^4 + x^5$$
$$= 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$$

Example 2

Expand $(2 + x)^4$ and use your expansion to find

(a) $(2.1)^4$

(b) $(1.9)^4$

Solution

$$(2+x)^4 = 2^4 + 4(2)^3x + 6(2)^2x^2 + 4(2)x^3 + x^4$$
$$= 16 + 32x + 24x^2 + 8x^3 + x^4$$

(a) $(2.1)^4 = (2 + 0.1)^4 = 16 + 32(0.1) + 24(0.1)^2 + 8(0.1)^3 + (0.1)^4$

$$= 16 + 3.2 + 0.24 + 0.008 + 0.0001$$
$$\therefore (2.1)^4 = 19.4481$$

(b) $(1.9)^4 = (2 - 0.1)^4$

$$= 16 + 32(-0.1) + 24(-0.1)^2 + 8(-0.1)^3 + (-0.1)^4$$
$$= 16 - 3.2 + 0.24 - 0.008 + 0.0001$$
$$= 13.0321$$
$$\therefore (1.9)^4 = 13.0321$$

Example 3

Write down the expansion of $\left(1 + \frac{1}{4}x\right)^4$. Taking the first three terms of the expansion, put $x = 0.1$, and find the value of $(1.025)^4$, correct to three decimal places.

Particular terms of the expansion

By considering the general term of a binomial expansion, a term involving a particular power of x may be found.

Example 13

Calculate the value of the term independent of x in the expansion of $\left(x-\frac{3}{x^2}\right)^{15}$

Solution

The general term of the expansion $(a+b)^n$ is

$$T_{r+1} = {}^nC_r a^{n-r}b^r$$

For the expansion $\left(x-\frac{3}{x^2}\right)^{15}$

$$T_{r+1} = {}^{15}C_r(x)^{15-r}\left(-\frac{3}{x^2}\right)^r = {}^{15}C_r(x)^{15-r}(-3)^r(x^{-2r})$$

$$T_{r+1} = {}^{15}C_r x^{15-3r}(-3)^r$$

For the term independent of x, the index/power of x must be zero.

$$15-3r=0$$
$$r=5$$

The 6th term is the term independent of x and has the value

$$T_6 = {}^{15}C_5(-3)^5 = -729729$$

Example 14

Find the coefficient of x^5 in the expansion of $\left(x+\frac{1}{x^3}\right)^{17}$

Solution

$$T_{r+1} = {}^nC_r a^{n-r}b^r = {}^{17}C_r(x)^{17-r}\left(\frac{1}{x^3}\right)^r$$
$$= {}^{17}C_r x^{17-r}x^{-3r}$$
$$= {}^{17}C_r x^{17-4r}$$

Let T_{r+1} be the term containing x^5

$$\Rightarrow 17-4r=5$$
$$4r=12$$
$$r=3$$
$$T_{r+1}=T_4 = {}^{17}C_3 x^{17-4(3)} = 680x^5$$
$$\therefore \text{The coefficient of } x^5 = 680$$

Example 15

Find the constant term in the expansion of $\left(\sqrt{x}-\frac{2}{x^2}\right)^{10}$

Solution

$$T_{r+1} = {}^{10}C_r\left(\sqrt{10}\right)^{10-r}\left(-\frac{2}{x^2}\right)^r$$
$$= {}^{10}C_r x^{\frac{10-r}{2}}\frac{(-2)^r}{x^{2r}}$$
$$= {}^{10}C_r(-2)^r x^{\frac{10-r}{2}-2r}$$
$$= {}^{10}C_r(-2)^r x^{\frac{10-5r}{2}}$$

If T_{r+1} is the constant term, then

$$\frac{10-5r}{2}=0$$
$$10-5r=0$$
$$r=2$$

The constant term is the 3rd term,

$$T_3 = {}^{10}C_2(-2)^2x^0 = \frac{10\times 9}{2\times 1}\times 4 = 180$$

Example 16

Find, and simplify the middle term in the expansion, in ascending powers of x, of $(3-5x)^8$

Solution

The middle term of $(a+bx)^n$ is term containing $x^{\frac{n}{2}}$ (n even)

The middle term is the 5th term and contains $(-5x)^4$

$$T_5 = {}^8C_4(3)^4(-5x)^4 = \frac{(8)(7)(6)(5)}{(1)(2)(3)(4)}\times 3^4\times(-5)^4\times x^4$$
$$= 3543750x^4$$

The binomial theorem for any rational index

If n is any rational value, positive or negative, and $-1<x<1$, then the binomial theorem is

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$$

Note:

The RHS is an infinite series since $n(n-1)(n-2)$ ... will never become zero. The condition $|x|<1$ ensures that this series converges.

The general term is $\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r$

The first term must be 1. If it is not, it must be written in the form

$(a+x)^n = a^n\left(1+\frac{x}{a}\right)^n$ and the expansion can be applied provided that $-1<\frac{x}{a}<1$

Example 17

Find the first four terms of $(1-2x)^{-\frac{3}{2}}$ and state the range for which the expansion is valid.

Solution

$$(1-2x)^{-\frac{3}{2}} = 1+\left(-\frac{3}{2}\right)(-2x)+\frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{2!}(-2x)^2$$
$$+\frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{3!}(-2x)^3+\cdots$$
$$= 1+3x-\frac{15}{2}x^2+\frac{35}{2}x^3+\cdots$$

This expansion is valid for $-1<-2x<1$ i.e. $\frac{1}{2}>x>-\frac{1}{2}$ or $-\frac{1}{2}<x<\frac{1}{2}$

Example 18

Obtain the first five terms in the expansion of $(1+x)^{\frac{1}{2}}$. Hence evaluate $\sqrt{1.03}$ to 5 significant figures

Solution

$$(1+x)^{\frac{1}{2}} = 1+\left(\frac{1}{2}\right)x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3$$

Putting $x = 0.2 = \frac{1}{5}$,

$$\left(1+\frac{3}{125}\right)^{\frac{1}{3}} = \left(\frac{128}{125}\right)^{\frac{1}{3}} = \left(\frac{2^7}{5^3}\right)^{\frac{1}{3}} = \frac{(2^6\times 2^1)^{\frac{1}{3}}}{5} = \frac{4}{5}\sqrt[3]{2}$$

But from the series,

$$\left(1+\frac{3}{125}\right)^{\frac{1}{3}} = 1 + (0.2)^3 - (0.2)^6 + \frac{5}{3}(0.2)^9 + \cdots$$
$$= 1 + 0.008 - 0.000064 + 0.000000853$$
$$= 1.0079369$$

Thus,

$$\frac{4}{5}\sqrt[3]{2} = 1.0079369$$
$$\sqrt[3]{2} = \tfrac{5}{4}(1.0079369) = 1.25992 \text{ (5 d.p)}$$

Example 23

Given that the first three terms in the expansion in ascending powers of x of $(1-8x)^{\frac{1}{4}}$ are the same as the first three terms in the expansion of $\frac{(1+ax)}{(1+bx)}$, find the values of a and b. Hence find an approximation to $(0.6)^{\frac{1}{4}}$ in the form $\frac{p}{q}$, where p and q are integers.

<u>Solution</u>

$$(1-8x)^{\frac{1}{4}} = 1 + \tfrac{1}{4}(-8x) + \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}(-8x)^2 + \cdots$$
$$= 1 - 2x - \tfrac{3}{32}(64x^2) + \cdots$$
$$= 1 - 2x - 6x^2 + \cdots$$

$$\frac{1+ax}{1+bx} = (1+ax)(1+bx)^{-1}$$
$$= (1+ax)\left[1 + (-1)bx + \frac{(-1)(-2)}{2!}(bx)^2 + \cdots\right]$$
$$= (1+ax)[1 - bx + b^2x^2 + \cdots]$$
$$= 1 + ax - bx - abx^2 + b^2x^2 + \cdots$$
$$= 1 + (a-b)x + (b^2 - ab)x^2 + \cdots$$

Since the first three terms of the expansion are the same

$$a - b = -2 \ \ldots \text{(i)}$$
$$b^2 - ab = -6 \ \ldots \text{(ii)}$$
$$b(b-a) = -6$$
$$b(2) = -6$$
$$b = -3$$

From (i);

$$a + 3 = -2$$
$$a = -5$$

Thus $(1-8x)^{\frac{1}{4}} \approx \frac{1-5x}{1-3x}$

$$(0.6)^{\frac{1}{4}} = (1-0.4)^{\frac{1}{4}} = \left(1 - 8(0.05)\right)^{\frac{1}{4}}$$

Substituting $x = 0.05$, we have

$$(1-0.4)^{\frac{1}{4}} \approx \frac{1-5(0.05)}{1-3(0.03)} = \frac{0.75}{0.85} = \frac{15}{17}$$

Hence $(0.6)^{\frac{1}{4}}$ is approximately equal to $\frac{15}{17}$

Example 24

Assuming that x is small so that terms in x^3 and higher powers may be neglected, find a quadratic approximation to $\sqrt{\left(\frac{1-x}{1+2x}\right)}$.

<u>Solution</u>

$$\sqrt{\left(\frac{1-x}{1+2x}\right)} = (1-x)^{\frac{1}{2}}(1+2x)^{-\frac{1}{2}}$$

$$(1-x)^{\frac{1}{2}} = 1 + \frac{1}{2}(-x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(-x)^2 + \cdots$$
$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$$

$$(1+2x)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(2x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(2x)^2 + \cdots$$
$$= 1 - x + \frac{3}{2}x^2 + \cdots$$

Neglecting terms in $x^3$and higher powers,

$$\sqrt{\left(\frac{1-x}{1+2x}\right)} \approx \left(1 - \frac{1}{2}x - \frac{1}{8}x^2\right)\left(1 - x + \frac{3}{2}x^2\right)$$
$$\approx 1 - x + \frac{3}{8}x^2 - \frac{1}{2}x + \frac{1}{2}x^2 - \frac{1}{8}x^2$$

Hence $\sqrt{\left(\frac{1-x}{1+2x}\right)} \approx 1 - \frac{3}{2}x + \frac{15}{8}x^2$

Example 25

Express $\frac{\sqrt{1+x}}{1-x}$ in ascending powers of x up to and including the term in x^3. By substituting $x = \frac{1}{4}$, show that $\sqrt{5} \approx \frac{4557}{2048}$

<u>Solution</u>

$$\frac{\sqrt{1+x}}{1-x} = (1+x)^{\frac{1}{2}}(1-x)^{-1}$$

$$(1+x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!} + \cdots$$
$$= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 + \cdots \text{ for } |x| < 1$$

$$(1+x)^{-1} = 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!} + \cdots$$
$$= 1 + x + x^2 + x^3 + \cdots \ \text{ for } |x| < 1$$

$$\frac{\sqrt{1+x}}{1-x} = \left(1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3\right)(1 + x + x^2 + x^3)$$
$$= 1 + x + x^2 + x^3 + \tfrac{1}{2}x + \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3 - \tfrac{1}{8}x^2 - \tfrac{1}{8}x^3 + \tfrac{1}{16}x^3 + \cdots$$
$$= 1 + \tfrac{3}{2}x + \tfrac{11}{8}x^2 + \tfrac{23}{16}x^3 + \cdots$$

Hence $$\frac{\sqrt{1+x}}{1-x} = 1 + \frac{3}{2}x + \frac{11}{8}x^2 + \frac{23}{16}x^3$$

When $x = \frac{1}{4}$,

$$\frac{\sqrt{5/4}}{3/4} = 1 + \tfrac{3}{2}\left(\tfrac{1}{4}\right) + \tfrac{11}{8}\left(\tfrac{1}{4}\right)^2 + \tfrac{23}{16}\left(\tfrac{1}{4}\right)^3 + \cdots$$
$$\tfrac{2}{3}\sqrt{5} = 1 + \tfrac{3}{8} + \tfrac{11}{128} + \tfrac{23}{1024} + \cdots$$
$$\sqrt{5} \approx \tfrac{3}{2}\left(\tfrac{1519}{1024}\right) = \tfrac{4557}{2048}$$

Example 26

Find the expansion of $(1 - x + 2x^2)^{\frac{1}{2}}$ up to and including the term in x^4.

Solution

$(1 - x + 2x^2)$ must be expressed in the form $(1 + z)^4$ where $z = -x + 2x^2$.

$$[1 + (-x + 2x^2)]^{\frac{1}{2}}$$
$$= 1 + \frac{1}{2}(-x + 2x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(-x + 2x^2)^2$$
$$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(-x + 2x^2)^2$$
$$+ \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}(-x + 2x^2)^4 + \cdots$$
$$= 1 + \frac{1}{2}(-x + 2x^2) - \frac{1}{8}(-x + 2x^2)^2$$
$$+ \frac{1}{16}(-x + 2x^2)^3 - \frac{5}{128}(-x + 2x^2)^4 + \cdots$$

Now,

$(-x + 2x^2) = x^2 - 4x^3 + 4x^4$

$(-x + 2x^2)^3 = (-x)^3 + 3(-x)^2(2x^2) + \cdots$

$= -x^3 + 6x^4 + \cdots$

$(-x + 2x^2)^4 = (-x)^4 + \cdots$

Therefore,

$$(1 - x + 2x^2)^{\frac{1}{2}}$$
$$= 1 - \frac{1}{2}x + x^2 - \frac{1}{8}(x^2 - 4x^3 + 4x^4)$$
$$+ \frac{1}{16}(-x^3 + 6x^4 + \cdots) - \frac{5}{128}(x^4 + \cdots)$$
$$= 1 - \frac{1}{2}x + \left(1 - \frac{1}{8}\right)x^2 + \left(\frac{1}{2} - \frac{1}{16}\right)x^3$$
$$+ \left(-\frac{1}{2} + \frac{3}{8} - \frac{5}{128}\right)x^4 + \cdots$$
$$= 1 - \frac{1}{2}x + \frac{7}{8}x^2 + \frac{7}{16}x^3 - \frac{21}{128}x^4 + \cdots$$

Example 27

Expand $\frac{1}{1+x+2x^2}$ in ascending powers of x up to and including the term in x^3.

Solution

Let $y = x + 2x^2$

$$\frac{1}{1 + x + 2x^2} = \frac{1}{1 + y} = (1 + y)^{-1}$$
$$= 1 + (-1)y + \frac{(-1)(-2)}{2!}y^2 + \frac{(-1)(-2)(-3)}{3!}y^3 + \cdots$$
$$= 1 - y + y^2 - y^3 + \cdots$$

$y^2 = (x + 2x^2) = x^2 + 4x^3 + 4x^4$

$y^3 = (x + 2x^2)^3 = (x)^3 + 3(x)^2(2x^2) + \cdots = x^3 + \cdots$

Substituting for y gives;

$$\frac{1}{1+x+2x^2} = 1 - x - 2x^2 + x^2 + 4x^3 - x^3 + \cdots$$
$$= 1 - x - x^2 + 3x^3 + \cdots$$

Example 28

Express $\frac{3+x}{(2-x)(1+2x)}$ in partial fractions, and hence, or otherwise, obtain the first three non-zero terms in the expansion of this expression in ascending powers of x.

State the range of values for which the expansion is valid.

Solution

$$\text{Let } f(x) = \frac{3+x}{(2-x)(1+2x)} \equiv \frac{A}{2-x} + \frac{B}{1+2x}$$
$$\equiv \frac{A(1+2x) + B(2-x)}{(2-x)(1+2x)}$$

Equating numerators:

$$3 + x \equiv A(1 + 2x) + B(2 - x)$$

Put $x = 2$:

$$5 = 5A$$
$$A = 1$$

Putting $x = -\frac{1}{2}$:

$$\frac{5}{2} = \frac{5}{2}B$$
$$B = 1$$

$$\text{Hence } \frac{3+x}{(2-x)(1+2x)} \equiv \frac{1}{2-x} + \frac{1}{1+2x} \equiv \frac{1}{2\left(1-\frac{x}{2}\right)} + \frac{1}{1+2x}$$
$$\equiv \frac{1}{2}\left(1 - \frac{x}{2}\right)^{-1} + (1 + 2x)^{-1}$$

By the binomial expansion;

$$\frac{1}{2}\left(1 - \frac{x}{2}\right)^{-1} = \frac{1}{2}[1 + (-1)\left(-\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(-\frac{x}{2}\right)^2$$
$$+ \frac{(-1)(-2)(-3)}{3!}\left(-\frac{x}{3}\right)^3 + \cdots$$
$$= \frac{1}{2}\left[1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \cdots\right]$$
$$= \frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$$

valid for $-1 < \frac{x}{2} < 1 \Rightarrow -2 < x < 2$

$$(1 + 2x)^{-1} = 1 + (-1)(2x) + \frac{(-1)(-2)}{2!}(2x)^2$$
$$+ \frac{(-1)(-2)(-3)}{3!}(2x)^2 + \cdots$$
$$= 1 - 2x + 4x^2 - 8x^3 + \cdots$$

valid for $-1 < 2x < 1 \Rightarrow -\frac{1}{2} < x < \frac{1}{2}$

Adding gives:

$$f(x) = \frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 + 1 - 2x + 4x^2 - 8x^3$$
$$= \frac{3}{2} - \frac{7}{4}x + \frac{33}{8}x^2 + \cdots$$

Validity is given when both conditions are valid – the smaller interval is taken.

Series is valid for $-\frac{1}{2} < x < \frac{1}{2}$

Example 29

Expand $\frac{16x^2+8x}{(1+x)(1+3x)(1+5x)}$ in ascending powers of x, up to and including the term in x^3.

Solution

All factors of the denominator are linear – need numerators A, B and C.

$$\text{Let } f(x) = \frac{16x^2+8x}{(1+x)(1+3x)(1+5x)} \equiv \frac{A}{1+x} + \frac{B}{1+3x} + \frac{C}{1+5x}$$
$$\equiv \frac{A(1+3x)(1+5x) + B(1+x)(1+5x) + C(1+x)(1+3x)}{(1+x)(1+3x)(1+5x)}$$

Equating the numerators:

$$16x^2 + 8x = A(1 + 3x)(1 + 5x) + B(1 + x)(1 + 5x)$$
$$+C(1 + x)(1 + 3x)$$

Put $x = -1$: $16 - 8 = A(-2)(-4)$

$$8 = 8A$$
$$A = 1$$

Put $x = -\frac{1}{3}$:

$$\frac{16}{9} - \frac{8}{3} = B\left(\frac{2}{3}\right)\left(-\frac{2}{3}\right)$$
$$-8 = -4B$$
$$B = 2$$

Put $x = -\frac{1}{5}$:

$$\frac{16}{25} - \frac{8}{5} = C\left(\frac{4}{5}\right)\left(\frac{2}{5}\right)$$
$$-24 = 8C$$
$$C = -3$$

$$f(x) = \frac{1}{1+x} + \frac{2}{1+3x} - \frac{3}{1+5x}$$
$$= (1+x)^{-1} + 2(1+3x)^{-1} - 3(1+5x)^{-1}$$
$$= (1 - x + x^2 - x^3 + \cdots)$$
$$+ 2(1 - 3x + (3x)^2 - (3x)^3 + \cdots)$$
$$- 3(1 - (5x) + (5x)^2 - (5x)^3 + \cdots)$$
$$= 1 - x + x^2 - x^3 + 2 - 6x + 18x^2 - 54x^3 - 3 + 15x$$
$$- 75x^2 + 375x^3 + \cdots$$
$$f(x) = 8x - 56x^2 + 320x^3 + \cdots$$

Example 30

Expand $f(x) = \frac{4x^3 - 7x + 3}{(2-x)(1+x^2)}$ in partial fractions.

Expand $f(x)$ in ascending powers of x as far as, and including, the term in x^3.

For what values of x is this expansion valid?

Solution

Look at the denominator: one factor is linear and needs a numerator A, one is quadratic and needs numerator $Bx + C$.

$$f(x) = \frac{4x^2 - 7x + 3}{(2-x)(1+x^2)} \equiv \frac{A}{2-x} + \frac{Bx + C}{1+x^2}$$
$$\equiv \frac{A(1+x^2) + (Bx+C)(2-x)}{(2-x)(1+x^2)}$$

Equating the numerators:

$$4x^2 - 7x + 3 \equiv A(1+x^2) + (Bx+C)(2-x)$$

Equating the numerators:

$$4x^2 - 7x^2 + 3 \equiv A(1+x^2) + (Bx+C)(2-x)$$

Put $x = 2$:

$$16 - 14 + 3 = A(1+4)$$
$$5A = 5$$
$$A = 1$$

Put $x = 0$:

$$3 = A(1) + C(2)$$
$$3 = 1 + 2C$$
$$C = 1$$

Put $x = 1$:

$$4 - 7 + 3 = A(1+1) + (B + C)(1)$$
$$0 = 2A + B + C$$
$$0 = 2 + B + 1$$
$$B = -3$$

Alternatively, compare coefficients of x^2:

$$4 = A - B \Rightarrow B = -3$$

$$\text{Hence } f(x) \equiv \frac{1}{2-x} + \frac{1-3x}{(1+x^2)}$$

$$\frac{1}{2-x} = \frac{1}{2\left(1-\frac{x}{2}\right)} = \frac{1}{2}\left(1 - \frac{x}{2}\right)^{-1} = \frac{1}{2}\left(1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \cdots\right)$$

valid $-2 < x < 2$

$$\frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 + \cdots$$

valid for $-1 < x < 1$

$$f(x) = \frac{1}{2}\left(1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \cdots\right)$$
$$+ (1 - 3x)(1 - x^2 + x^4 + \cdots)$$
$$= \frac{1}{2} + \frac{x}{4} + \frac{x^3}{16} + 1 - 3x - x^2 + 3x^3 + \cdots$$
$$= \frac{3}{2} - \frac{11}{4}x - \frac{7}{8}x^2 + \frac{49}{16}x^3 + \cdots$$

The series is valid when $-2 < x < 2$ and $-1 < x < 1$. Taking the smaller range, $-1 < x < 1$

Example 31

Given that $g(x) = \frac{5-5x}{(1+x^2)(3-x)}$, express $g(x)$ in partial fractions. Hence or otherwise, show that the expansion of $g(x)$ as a series in ascending powers of x, up to and including the term in x^4 is

$$\frac{5}{3} - \frac{10}{9}x - \frac{55}{27}x^2 + \frac{80}{81}x^3 + \frac{485}{243}x^4$$

Solution

$$\text{Let } \frac{5-5x}{(1+x^2)(3-x)} \equiv \frac{Ax + B}{1+x^2} + \frac{C}{3-x}$$
$$\equiv \frac{(Ax+B)(3-x) + C(1+x^2)}{(1+x^2)(3-x)}$$

Equating the numerators:

$$5 - 5x \equiv (Ax + B)(3-x) + C(1+x^2)$$

Putting $x = 3$:

$$5 - 15 = C(1+9)$$
$$10C = -10$$
$$C = -1$$

Putting $x = 0$:

$$5 = B(3) + C$$
$$5 = 3B - 1$$
$$B = 2$$

Putting $x = 1$:

$$0 = (A + B)(2) + C(2)$$
$$0 = 2A + 4 - 2$$
$$A = -1$$

$$\text{Hence } \frac{5-5x}{(1+x^2)(3-x)} = \frac{2-x}{1+x^2} - \frac{1}{3-x}$$

$$\frac{2-x}{1+x^2} = (2-x)(1+x^2)^{-1} = (2-x)(1 - x^2 + x^4 + ..)$$
$$= 2 - 2x^2 + 2x^4 - x + x^3 + ..$$
$$= 2 - x - 2x^2 + x^3 + 2x^4 + ..$$

$$\frac{(1+x)^{\frac{3}{2}}-\left(1+\frac{1}{2}x\right)^{3}}{\sqrt{1-x}}\simeq -\frac{3}{8}x^2$$

12. Obtain the expansion of $\frac{3-4x}{1-3x+2x^2}$ in ascending powers of x as far as the term in x^4.
[Ans: $3+5x+9x^2+17x^3+33x^4$]

13. Write down the first four terms in the expansion in ascending powers of x of $(1+4x)^{\frac{1}{2}}$, and simplify the coefficients. Hence by putting $x=-\frac{1}{100}$, calculate $\sqrt{6}$ correct to four decimal places.
[Ans: $1+2x-2x^2+4x^3$; 2.4495]

14. If terms containing x^4 and higher powers of x can be neglected, show that
$$\frac{2}{(x+1)(x^2+1)}\approx 2(1-x)$$

15. Show that
$$\frac{12}{(3+x)(1-x)^2}\approx 4+\frac{20}{3}x+\frac{88}{9}x^2$$
provided that x is small enough to neglect powers higher than 2

16. In the questions below, assume that x is so small that terms in x^3 and higher powers may be neglected. Hence find a quadratic approximation to the given function, stating the values of x for which your answer is valid.
(a) $\sqrt[3]{\frac{8+x}{1-3x}}$
(b) $\frac{(1+4x)^{\frac{1}{4}}}{(1+5x)^{\frac{1}{5}}}$
(c) $\frac{1}{(1+x)(3-x)}$
(d) $\frac{1}{(1-x)(1+2x)^2}$
(e) $\frac{1}{(1-2x)\sqrt{1-x}}$
[Ans: (a) $2+\frac{25}{12}x+\frac{1175}{288}x^2$, $|x|<\frac{1}{3}$ (b) $1+\frac{1}{2}x^2$, $|x|<\frac{1}{5}$ (c) $\frac{1}{3}-\frac{2}{9}x+\frac{7}{27}x^2$, $|x|<1$ (d) $1-3x+9x^2$, $|x|<\frac{1}{2}$ (e) $1+\frac{5}{2}x+\frac{43}{8}x^2$, $|x|<\frac{1}{2}$]

17. Use partial fractions to find the first non-zero terms in the expansion of the given function in ascending powers of x. State the values of x for which the expansion is valid.
(a) $\frac{3-5x}{(1-3x)(1+x)}$
(b) $\frac{4x}{(1-x)(3+x)}$
(c) $\frac{1+x}{(1+x^2)(1-x)}$
(d) $\frac{4-x}{(1-x)^2(1+2x)}$
(e) $\frac{2}{(1+x)(1+x^2)}$
(f) $\frac{8(2x-1)}{(x-2)^2(x^2+2)}$
[Ans: (a) $3+x+11x^2+25x^3+83x^4$, $|x|<\frac{1}{3}$ (b) $\frac{4}{3}x+\frac{8}{9}x^2+\frac{28}{27}x^3+\frac{80}{81}x^4+\frac{244}{243}x^5$, $|x|<1$ (c) $1+2x+x^2+x^4+2x^5$, $|x|<1$ (d) $4-x+12x^2-11x^3+38x^4$, $|x|<\frac{1}{2}$ (e) $2-2x+2x^4-2x^5+2x^8$, $|x|<1$ (f) $-1+x+\frac{7}{4}x^2+\frac{1}{2}x^3-\frac{3}{16}x^4$, $|x|<\sqrt{2}$]

18. Expand $(1+2x)^{\frac{1}{4}}$ in ascending powers of x as far as the term in x^3, stating the values of x for which the expansion is valid. Hence obtain approximate values of (a) $\sqrt[4]{1.4}$ (b) $\sqrt[4]{1.08}$.
[Ans: $1+\frac{1}{2}x-\frac{3}{8}x^2+\frac{7}{16}x^3$, $|x|<\frac{1}{2}$; (a) 1.0885 (b) 1.019428]

19. Given that the first three terms in the expansion in ascending powers of x of $(1+x+x^2)^n$ are the same as the first three terms in the expansion of $\left(\frac{1+ax}{1-3ax}\right)^3$, find the non-zero values of n and a. Show that the coefficients of x^3 in the two expansions differ by 7.5.
[Ans: $n=6$, $a=\frac{1}{2}$]

20. Show that if x is so small so that x^4 and higher powers can be neglected then $\frac{1+2x+3x^2}{(1-x)(1+x^2)}$ can be expressed in the form $A+Bx+Cx^2+Dx^3$ and find A, B, C, D.
[Ans: $1+3x+5x^2+3x^3$]

21. If x is so small that x^3 and higher powers of x may be neglected, find the values of a and b such that
$$\sqrt{1+4x}\approx\frac{1+ax}{1+bx}$$
By letting $x=0.04$, find an approximation to $\sqrt{29}$ in the form p/q where p and q are integers.
[Ans: $a=3$, $b=1$; $70/13$]

22. Expand $\sqrt{4-x}$ as a series in ascending powers of x up to and including the terms in x^2. If terms in $x^n, n\geq 3$, can be neglected, find the quadratic approximation to $\sqrt{\frac{4-x}{1-2x}}$. State the range of values of x for which this approximation is valid.
[Ans: $2-\frac{1}{4}x-\frac{1}{64}x^2$; $2+\frac{7}{4}x+\frac{175}{64}x^2$, $|x|<\frac{1}{2}$]

23. Expand the function $(2-x)\sqrt{1+2x+2x^2}$ in ascending powers of x as far as the term in x^3.
[Ans: $2+x-\frac{3}{2}x^3$]

24. Show that $\frac{1}{\sqrt{1-x}}-\sqrt{1+x}=\frac{x^2}{2}+\frac{x^3}{4}$ if x^4 and higher powers of x may be neglected.

25. If x^4 and higher powers of x can be neglected, show that
$$\sqrt{\frac{1-x}{1+x+x^2}}=1-x+\frac{1}{2}x^3$$

Chapter 7 Inequalities

A statement involving the symbols '>', '<', '≥', '≤' is called an inequality. For example, $5 > 3$, $x \leq 4$, $x + y \geq 9$. An inequality may contain more than one variable and it can be linear, quadratic or cubic etc. For example, $3x - 2 < 0$ is a linear inequality in one variable, $2x + 3y \geq 4$ is a linear inequality in two variables and $x^2 + 3x + 2 < 0$ is a quadratic inequality in one variable.

Rules for manipulating inequalities

If a, b, c, d and k are numbers such that $a > b$ and $c > d$, then:

(a) $a \pm k > b \pm k$

(b) $ak > bk$ for $k > 0$ (positive)
$ak < bk$ for $k < 0$ (negative)

(c) $a + c > b + d$

A similar set of results arise for $<$

Note: We cannot make any deductions about $a - c$ and $b - d$ or ac and bd or $a \div c$ and $b \div d$

Solution of an inequality

The value(s) of the variable(s) which makes the inequality a true statement is called its solutions. The set of all solutions of an inequality is called the solution set of the inequality. For example, $x - 1 \geq 0$, has infinite number of solutions as all real values grater than or equal to one make it a true statement. The inequality $x^2 + 1 < 0$ has no solution in **R** as no real value of x makes it a true statement.

To solve an inequality, we can

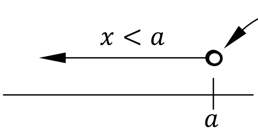
(i) add (or subtract) the same quantity to (from) both sides without changing the sign of inequality

(ii) multiply (or divide) both sides by the same positive quantity without changing the sign of inequality. However, if both sides if the inequality are multiplied (or divided) by the same negative inequality the sign of the inequality is reversed.

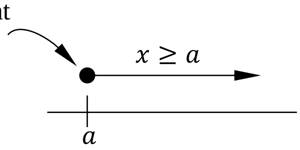
1. Linear inequalities in one unknown

These can be solved using the rules of inequalities. The solution set can be illustrated on a number line.
Note the symbols used:

shows end point not included



shows end point included



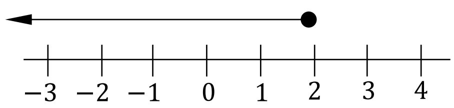
Example 1

Find the solution set of $8 - x \geq 5x - 4$

Solution

$$8 + 4 \geq 5x + x$$
$$12 \geq 6x$$
$$2 \geq x$$
$$\text{or } x \leq 2$$

This is illustrated as



2. Linear inequalities in two unknowns

They are best solved graphically.
The corresponding equality gives the boundary line. This is drawn as:

(a) a continuous line if the inequality is ≥ or ≤

(b) a dotted line if the inequality is > or <

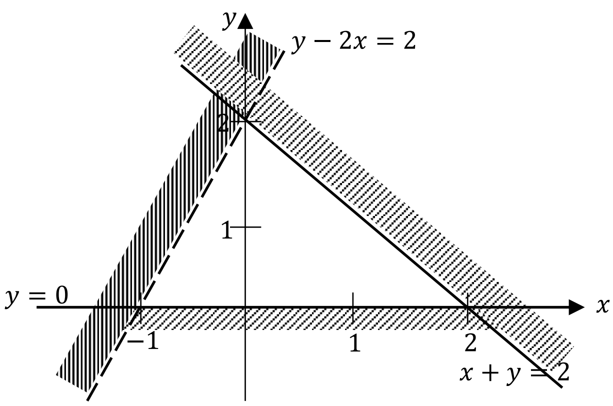
A convenient point is chosen to identify on which side of the line the inequality applies. The solution set of the inequality is usually left unshaded.

Example 2

Solve $y \geq 0$, $x + y \leq 2$ and $y - 2x < 2$

Solution

Draw the lines $y = 0$ and $x + y = 2$ both continuous and the line $y - 2x = 2$ (dotted).



Test the point $(0, 1)$. The unshaded region gives the solution set.

Critical values of x are 1, 2 and 4

Let $f(x) = \frac{(x-4)(x-1)}{x-2}$

	$x<1$	$1<x<2$	$2<x<4$	$x>4$
$x-4$	$-$	$-$	$-$	$+$
$x-1$	$-$	$+$	$+$	$+$
$x-2$	$-$	$-$	$+$	$+$
$f(x)$	$-$	$+$	$-$	$+$

True if $x > 4$ or $1 < x < 2$

Example 7

Solve the inequality

$$\frac{1}{x-4} > \frac{1}{3-x}$$

Solution

$$\frac{1}{x-4} - \frac{1}{3-x} > 0$$

$$\frac{(3-x)-(x-4)}{(x-4)(3-x)} > 0$$

$$\frac{7-2x}{(x-4)(3-x)} > 0$$

Critical values of x are 3, 3.5, 4

Let $f(x) = \frac{7-2x}{(x-4)(3-x)}$

	$x<3$	$3<x<3.5$	$3.5<x<4$	$x>4$
$7-2x$	$+$	$+$	$-$	$-$
$x-4$	$-$	$-$	$-$	$+$
$3-x$	$+$	$-$	$-$	$-$
$f(x)$	$-$	$+$	$-$	$+$

$$\therefore \frac{1}{x-4} > \frac{1}{3-x} \text{ when } 3 < x < 3.5 \text{ or } x > 4$$

Example 8

Find the sets of values of x for which $|x-3| > 2|x+1|$

Solution

Squaring,

$$x^2 - 6x + 9 > 4x^2 + 8x + 4$$

$$0 > 3x^2 + 14x - 5$$

$$0 > 3x^2 + 15x - x - 5$$

$$0 > 3x(x+5) - (x+5)$$

$$0 > (3x-1)(x+5)$$

Critical values are $x = -5$ and $x = \frac{1}{3}$

Let $f(x) = (3x-1)(x+5)$

	$x<-5$	$-5<x<\frac{1}{3}$	$x>\frac{1}{3}$
$3x-1$	$-$	$-$	$+$
$x+5$	$-$	$+$	$+$
$f(x)$	$+$	$-$	$+$

$$\therefore |x-3| > 2|x+1| \text{ when } -5 < x < \frac{1}{3}$$

Example 9

Find the set of values of x for which $f(x) > \frac{1}{2}$ where $f(x) = \frac{x(x-2)}{(x+3)}$

Solution

$$\frac{x(x-2)}{(x+3)} > \frac{1}{2}$$

$$\frac{x(x-2)}{(x+3)} - \frac{1}{2} > 0$$

$$\frac{2x(x-2)-(x+3)}{2(x+3)} > 0$$

$$\frac{2x^2 - 4x - x - 3}{2(x+3)} > 0$$

$$\frac{2x^2 - 5x - 3}{2(x+3)} > 0$$

$$\frac{2x^2 - 6x + x - 3}{2(x+3)} > 0$$

$$\frac{2x(x-3)+(x-3)}{2(x+3)} > 0$$

$$\frac{(x-3)(2x+1)}{2(x+3)} > 0$$

Critical values of x are -3, $-\frac{1}{2}$ and 3

Let $g(x) = \frac{(x-3)(2x+1)}{2(x+3)}$

	$x<-3$	$-3<x<-\frac{1}{2}$	$-\frac{1}{2}<x<3$	$x>3$
$x+3$	$+$	$+$	$-$	$-$
$2x+1$	$-$	$-$	$-$	$+$
$x-3$	$+$	$-$	$-$	$-$
$g(x)$	$-$	$+$	$-$	$+$

$$\therefore \frac{x(x-2)}{(x+3)} > \frac{1}{2} \text{ when } -3 < x < -\frac{1}{2} \text{ or } x > 3$$

Example 10

For what values of x is $f(x) = x^3 - 12x^2 + 39x - 28 < 0$?

Solution

$$f(x) = x^3 - 12x^2 + 39x - 28$$

Using the factor theorem, trying the factors of 28,

$$f(1) = (1)^3 - 12(1)^2 + 39(1) - 28 = 0$$

$x-1$ is a factor of $f(x)$

Now using long division to obtain the remaining factors,

$$\begin{array}{r l}
 & x^2 - 11x + 28 \\
x-1 \,\big) & x^3 - 12x^2 - 39x - 28 \\
- & x^3 - x^2 \\
 & -11x^2 - 39x - 28 \\
- & -10x^2 + 11x \\
 & 28x - 28 \\
- & 28x - 28 \\
 & - \quad -
\end{array}$$

Chapter 8 Arithmetic and Geometric Progressions

ARITHMETIC PROGRESSION (A.P)

An arithmetic progression (abbreviated as A.P) is a sequence of numbers in which each term, except the first, is obtained by adding a fixed number to the immediately preceding term. This fixed number is called the **common difference,** which is generally denoted by d. Usually we denote the first term by a and the last term as l.

For example, 1, 3, 5, 7, …. is an A.P with common difference 2

The general term or the nth term of the A.P is given by

$$u_n = a + (n-1)d$$

The nth term from the last is given by

$$u_n = l - (n-1)d$$

Example 1

Show that (x^2+xy+y^2), (z^2+xz+x^2) and (y^2+yz+z^2) are consecutive terms of an A.P, if x, y and z are in A.P.

Solution

The terms will be be in A.P if

$$(z^2+xz+x^2)-(x^2+xy+y^2) = (y^2+yz+z^2)-(z^2+xz+x^2)$$

$$z^2+xz-xy-y^2 = y^2+yz-xz-x^2$$

$$x^2+2xz+z^2-y^2 = y^2+yz+xy$$

$$(x+z)^2-y^2 = y(x+y+z)$$

$$(x+z-y)(x+z+y) = y(x+y+z)$$

$$x+z-y = y$$

$$z-y = y-x$$

which is true since x, y, z are in A.P

Example 2

Find three numbers in arithmetical progression such that their sum is 27 and their product is 504.

Solution

Let the three numbers in arithmetic progression be $a-d$, a, $a+d$

$$a-d+a+a+d = 27$$

$$3a = 27$$

$$a = 9$$

$$a(a-d)(a+d) = 504$$

$$a(a^2-d^2) = 504$$

Since $a = 9$,

$$81-d^2 = 56$$

$$d^2 = 25$$

$$d = \pm 5$$

Hence the required numbers are 4, 9 and 14

Example 3

The product of three numbers in A.P. is 224, and the largest number is 7 times the smallest. Find the numbers.

Solution

Let the three numbers in A.P. be $a-d$, a, $a+d$ $(d>0)$

Now

$$(a-d)a(a+d) = 224$$

$$a(a^2-d^2) = 224 \quad \text{… (i)}$$

Now, since the largest number is 7 times the smallest,

$$a+d = 7(a-d)$$

$$a+d = 7a-7d$$

$$8d = 6a$$

$$d = \frac{3a}{4}$$

Substituting for d in (i);

$$a\left(a^2-\frac{9a^2}{16}\right) = 224$$

$$\frac{7a^3}{16} = 224$$

$$a^3 = 512$$

$$a = 8$$

and

$$d = \frac{3a}{4} = \frac{3}{4}\times 8 = 6$$

Hence, the three numbers are 2, 8, 14

Sum of an Arithmetic progression

In the series

$$a, a+d, a+2d, a+3d, \ldots.$$

the coefficient of a in any term is one less than the number of the term in the series. Thus $a+3d$ is the fourth term.

If then the series consists of n terms and l denotes the last or nth term

$$l = a+(n-1)d$$

To obtain the sum S_n of n terms of the series we have

$$S_n = a+(a+d)+(a+2d)+\ldots+(l-d)+l$$

If we now write the series in the reverse order,

$$S_n = l+(l-d)+\ldots+(a+2d)+(a+d)+a$$

Adding and noticing that the sums of terms in corresponding positions are all $a+l$ we have

$$2S_n = (a+l)+(a+l)+\ldots \text{ to } n \text{ terms}$$

Hence

$$S_n = \frac{n}{2}(a+l)$$

But $l = a+(n-1)d$

$$\boldsymbol{S_n = \frac{n}{2}[2a+(n-1)d]}$$

Example 11

If a, b, c, d are in G.P, prove that $a^2 - b^2, b^2 - c^2, c^2 - d^2$ are also in G.P

Solution

Let r be the common ratio of the given G.P. Then

$$\frac{b}{a} = \frac{c}{b} = \frac{d}{c} = r$$

$b = ar, c = br = ar^2, d = cr = ar^3$

Now,

$$a^2 - b^2 = a^2 - a^2r^2 = a^2(1 - r^2)$$
$$b^2 - c^2 = a^2r^2 - a^2r^4 = a^2r^2(1 - r^2)$$
$$c^2 - d^2 = a^2r^4 - a^2r^6 = a^2r^4(1 - r^2)$$
$$\frac{b^2 - c^2}{a^2 - b^2} = \frac{a^2r^2(1 - r^2)}{a^2(1 - r^2)} = r^2$$
$$\frac{c^2 - d^2}{b^2 - c^2} = \frac{a^2r^4(1 - r^2)}{a^2r^2(1 - r^2)} = r^2$$
$$\therefore \frac{b^2 - c^2}{a^2 - b^2} = \frac{c^2 - d^2}{b^2 - c^2} = r^2$$

Hence, $a^2 - b^2, b^2 - c^2, c^2 - d^2$, are in a G.P

Example 12

Find three numbers in geometric progression such that their sum is 39 and their product is 729.

<u>Solution</u>

Let the required numbers be $\frac{a}{r}$, a and ar.

$$\frac{a}{r} \times a \times ar = 729$$
$$a^3 = 729$$
$$a = 9$$
$$\frac{9}{r} + 9 + 9r = 39$$
$$9 + 9r + 9r^2 = 39r$$
$$9r^2 - 30r + 9 = 0$$
$$3r^2 - 10r + 3 = 0$$
$$3r^2 - 9r - r + 3 = 0$$
$$3r(r - 3) - (r - 3) = 0$$
$$(r - 3)(3r - 1) = 0$$
$$r = 3 \text{ or } \frac{1}{3}$$

The required numbers are 3, 9 and 27

Example 13

If p, q and r are three successive terms of a geometric progression, show that $\log p, \log q$ and $\log r$ are three successive terms of an arithmetic progression. (p, q, and r are > 0)

<u>Solution</u>

$$\frac{q}{p} = \frac{r}{q}$$

Introducing log on both sides

$$\log\frac{q}{p} = \log\frac{r}{q}$$
$$\log q - \log p = \log r - \log q$$

Hence $\log p$, $\log q$ and $\log r$ are in arithmetic progression.

Sum of a Geometric Progression

In the series,

$$a, ar, ar^2, ar^3, \ldots$$

the index of r in any term is one less than the number of the term in the series. Thus ar^3 is the fourth term. The last or nth term of the series is given by

$$l = ar^{n-1}$$

To obtain the sum S_n of n terms of the series, we have

$$S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1}$$

Multiplying throughout by r

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$$

If we subtract, all the terms on the right hand side except a and ar^n cancel in pairs.

Hence

$$S_n - rS_n = a - ar^n$$
$$\boldsymbol{S_n = \frac{a(1 - r^n)}{1 - r}}$$

when $r > 1$

$$\boldsymbol{S_n = \frac{a(r^n - 1)}{r - 1}}$$

Example 14

The first three terms of a geometric progression are $k - 3$, $2k - 4$, $4k - 3$ in that order. Find the value of k and the sum of the first eight terms of the progression.

<u>Solution</u>

Since the terms are in G.P, then

$$\frac{2k - 4}{k - 3} = \frac{4k - 3}{2k - 4}$$
$$(2k - 4)^2 = (k - 3)(4k - 3)$$
$$4k^2 - 16k + 16 = 4k^2 - 15k + 9$$
$$k = 7$$

The first three terms of the progression are 4, 10, 25

$a = 4, r = \frac{10}{4} = \frac{5}{2} = 2.5$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$S_8 = \frac{4[(2.5)^8 - 1]}{2.5 - 1} = 4066.3438$$

Example 15

The first and last terms of a geometric series are 2 and 2048 respectively. The sum of the series is 2730. Find the number of terms and the common ratio.

<u>Solution</u>

Let the number of terms be n and the common ratio r.

nth term, $u_n = ar^{n-1}$

$$2r^{n-1} = 2048$$
$$r^{n-1} = 1024 \ldots \text{(i)}$$
$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{5}} = \frac{5}{4}$$

Now $S_\infty - S_n < 10^{-6}$

$$\frac{5}{4} - \frac{5}{4}[1 - 5^{-n}] < 10^{-6}$$

$$5^{-n} < 10^{-6}$$

Introducing log to both sides

$$\log 5^{-n} < \log 10^{-6}$$

$$-n \log 5 < -6$$

$$n > \frac{-6}{-\log 5}$$

$$n > 8.58$$

Thus $n \geq 9$ i.e. 9 terms or more need to be taken.

Combined A.Ps and G.Ps

Example 20

If a, b, c are three consecutive terms of an A.P and x, y, z are three consecutive terms of a G.P. Then prove that

$$x^{b-c}.\ y^{c-a}.\ z^{a-b} = 1$$

Solution

We have a, b, c as three consecutive terms of A.P. Then

$$b - a = c - b = d$$

$$c - a = 2d$$

$$a - b = -d$$

$$b - c = -d$$

Now

$$x^{b-c}.\ y^{c-a}.\ z^{a-b} = x^{-d}.\ y^{2d}.z^{-d}$$

Since x, y, z are in G.P

$$\frac{y}{x} = \frac{z}{y}$$

$$y = \sqrt{xz}$$

$$x^{b-c}.\ y^{c-a}.\ z^{a-b} = x^{-d}.\ (\sqrt{xz})^{2d}.z^{-d}$$

$$= (xz)^{-d} \times (xz)^{d}$$

$$= (xz)^{-d+d}$$

$$= (xz)^0$$

$$\therefore x^{b-c}.\ y^{c-a}.\ z^{a-b} = 1$$

Example 21

The third, sixth and seventh terms of a geometric progression (whose common ratio is neither 0 nor 1) are in arithmetic progression. Prove that the sum of the first three terms is equal to the fourth term.

Solution

$u_3 = ar^2$, $u_6 = ar^5$, $u_7 = ar^6$

ar^2, ar^5, ar^6 are in A.P, then

$$ar^5 - ar^2 = ar^6 - ar^5$$

$$ar^2(r^3 - 1) = ar^5(r - 1)$$

From binomial expansion, $r^3 - 1 = (r-1)(r^2 + r + 1)$

$$ar^2(r-1)(r^2 + r + 1) = ar^5(r - 1)$$

$$a(r^2 + r + 1) = ar^3$$

$$ar^2 + ar + a = ar^3$$

$$\therefore S_3 = u_4$$

Example 22

If the 2nd, 5th and 9th terms of a non-constant A.P. are in G.P, find the common ratio of the G.P

Solution

$a + d$, $a + 4d$, $a + 8d$ are in a G.P

$$\Rightarrow r = \frac{a + 4d}{a + d} = \frac{a + 8d}{a + 4d}$$

$$(a + 4d)^2 = (a + d)(a + 8d)$$

$$a^2 + 8ad + 16d^2 = a^2 + 9ad + 8d^2$$

$$8d^2 = ad$$

$$8d = a$$

$$r = \frac{8d + 4d}{8d + d} = \frac{12d}{9d} = \frac{4}{3}$$

Example 23

Three positive numbers form an increasing G.P. If the middle term in this G.P. is doubled, the new numbers are in A.P. Find the common ratio of the G.P

Solution

Let a, ar, ar^2 be the terms in the G.P, then

a, $2ar$, ar^2 are in A.P.

$$2ar - a = ar^2 - 2ar$$

$$4ar = a + ar^2$$

$$4r = 1 + r^2$$

$$r^2 - 4r + 1 = 0$$

$$r^2 - 4r + 4 = 3$$

$$(r - 2)^2 = 3$$

$$(r - 2) = \pm\sqrt{3}$$

$$r = 2 \pm \sqrt{3}$$

$$r = 2 - \sqrt{3} \text{ or } r = 2 + \sqrt{3}$$

Since the G.P is an increasing G.P, $r > 1$

$$\therefore r = 2 + \sqrt{3}$$

Simple and compound interest

If a sum of money of money P (the principal) is invested at a simple interest of r per cent. per annum, the amount A (principal plus interest) after n years is given by

$$A = P\left(1 + \frac{nr}{100}\right)$$

for the interest for one year is $\frac{Pr}{100}$ and for n years $\frac{nPr}{100}$. The various amounts after one, two, three, … years therefore form an arithmetic progression.

If, on the other hand, the same principal is invested at compound interest of r per cent. per annum, the interest being added annually, the amount after one year is $P\left(1 + \frac{r}{100}\right)$, and this is the principal for the second year.

Hence after two years the amount is

$$P\left(1 + \frac{r}{100}\right)\left(1 + \frac{r}{100}\right) \text{ or } P\left(1 + \frac{r}{100}\right)^2$$

and so on. Thus after n years the amount will be given by

$$A = P\left(1 + \frac{r}{100}\right)^n$$

Example 27

A family decides to save some money in an account that pays 9% annual compound interest calculated at the end of each year. They put $2500 into the account at the beginning of each year. All interests are added to the account and no withdrawals are made. How much money will they have in the account on the day after they have made their tenth payment?

Solution

The problem is best looked at from the last payment of $2500 which has just made and which has not earned any interest.

The previous payment has earned one lot of 9% and so is now worth 2500×1.09

The previous payment has earned two years' worth of compound interest and is worth 2500×1.09^2

The process can be continued for all the other payments and the various amounts of interest that each has earned. They form a geometric progression.

Last payment 1st payment

$$2500 + 2500 \times 1.09 + 2500 \times 1.09^2 + \ldots + 2500 \times 1.09^9$$

The total amount saved can be calculated using the formula

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{10} = \frac{2500(1.09^{10} - 1)}{1.09 - 1} = 37982.32$$

The family will save about $37982.32

Self-Evaluation exercise

1. The nth term of an arithmetic progression (A.P) is denoted by u_n, and the sum of the first n terms is denoted by S_n.
   (a) In a certain A.P, $u_5 + u_{16} = 44$ and $S_{18} = 3S_{10}$. Calculate the value of the first term and the common difference.
   (b) In another A.P, $u_1 = 1$. Given that u_7, u_{11} and u_{17} are in geometric progression, find the value of each.
   [Ans: (a) $a = 3$, $d = 3$ (b) 1, 1, 1 or 4, 6, 9]
2. If it is given that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are three consecutive terms of an arithmetic series. Show that a^2, b^2 and c^2 are also three consecutive terms of an arithmetic series.
3. A man invests £100 at the beginning of each year for ten years. The rate of compound interest is 9% per annum. Calculate the total value of the investment at the end of the ten full years.
   [Ans: £1656.03]
4. The fourth, seventh and sixteenth terms of an A.P. are in geometric progression. If the first six terms of the A.P. have a sum of 12, find the common difference of the A.P. and the common ratio of the G.P.
   [Ans: 2, 3]
5. The third, fifth and seventeenth terms of an A.P. are in geometric progression. Find the common ratio of the G.P.
   [Ans: 6]
6. The third term of a geometric progression is 2, and the fifth is 18. Find two possible values of the common ratio, and the second term in each case.
   [Ans: ± 3, $\pm\frac{2}{3}$]
7. The third term of a geometrical progression is 2, and the fifth is 18. Find two possible values of the common ratio, and the second term in each case.
   [Ans: ± 3, $\pm\frac{2}{3}$]
8. Three numbers, $n - 2$, n, $n + 3$, are consecutive terms of a geometric progression. Find n, and the term after $n + 3$.
   [Ans: 6, $13\frac{1}{2}$]
9. Find the ratio of the sum of the first 10 terms of the series
   $$\log x + \log x^2 + \log x^4 + \log x^8 + \ldots$$
   to the first term.
   [Ans: 1023]
10. A man pays a premium of £100 at the beginning of every year to an Insurance company on the understanding that at the end of fifteen years he can receive back the premiums which he has paid with 5% compound interest. What should he receive? (Give your answer correct to 3 s.f)
    [Ans: £2270]
11. A man earned in a certain year £2000 from a certain source and his annual earnings from this time continued to increase at the rate of 5%. Find to the nearest £ the whole amount he received from this source in this year and the next seven years. Give your answer correct to three significant figures.
    [Ans: £19100]
12. Show that, if $\log a$, $\log b$, $\log c$ are consecutive terms of an arithmetic progression, then a, b, c are in geometric progression.
13. The eight term of an arithmetic progression is twice the third term, and the sum of the first eight terms is 39. Find the first three terms of the progression, and show that its sum to n terms is $\frac{3}{8}n(n + 5)$
    [Ans: $\frac{9}{4}$, 3, $\frac{15}{4}$]
14. Prove that $\log a + \log ax + \log ax^2 + \ldots$ to n terms is $n \log a + \frac{1}{2}n(n - 1)\log x$
15. If $1 + 2x + 4x^2 + \ldots = \frac{3}{4}$, find the value of x
    [Ans: $-\frac{1}{6}$]
16. A man saved $66000 in 20 years. In each succeeding year after the first year, he saved $200 more than what

Chapter 9 Proof by Mathematical Induction

Mathematical induction is a method of proving a given (or suspected) result for positive integers.
This method is often used to prove the formula for the sum of n terms of a series.
To prove by induction

1. Show that the result is true for $n = 1$
2. Assume the validity of the result for n equal to some arbitrary but fixed natural number, say k
3. Show that the result is also true for $n = k + 1$
4. Conclude that the result holds for all natural numbers.

The $\sum$ notation
It is useful to have a short way of writing expressions like

$$1^2 + 2^2 + 3^3 + \cdots + n^2$$

This is done by writing

$$\sum_{1}^{n} r^2 = 1^2 + 2^2 + \cdots + n^2$$

Example 1
Prove by induction that $n^2 + n$ is even for all natural numbers.

Solution

Let $P(n) = n^2 + n$
Put $n = 1$,

$$P(1) = 1^2 + 1 = 2, \text{ which is even}$$

$P(1)$ is true
Let us assume that the statement is true for $n = k$, i.e. $P(k)$ is even.

$$P(k) = k^2 + k$$

To prove $P(k + 1)$ is true

$$\begin{aligned}(k+1)^2 + (k+1) &= k^2 + 2k + 1 + k + 1 \\ &= k^2 + 2k + k + 2 \\ &= (k^2 + k) + 2(k+1) \\ &= \text{an even number} + 2(k+1) \\ &= \text{sum of two even numbers} \\ &= \text{an even number}\end{aligned}$$

$$\therefore P(k+1) \text{ is true.}$$

Thus, if $P(-k)$ is true, then $P(k + 1)$ is also true
$\therefore$ By the principle of induction, $n^2 + n$ is even for all natural numbers

Example 2
Show by induction that

$$S_n = \sum_{r=1}^{n} [a + (r-1)d] = \frac{1}{2}n[2a + (n-1)d]$$

Solution

For $n = 1$,

$$L.H.S = a + (1-1)d = a$$

$$R.H.S = \frac{1}{2}(1)[2a + (1-1)d] = a$$

Since $L.H.S = R.H.S$, the result is true for $n = 1$
Assume the result is true for $n = k$,

$$S_k = \frac{1}{2}k[2a + (k-1)d]$$

Add the next term, the $(k + 1)th$ term giving

$$\begin{aligned}S_{k+1} &= \frac{1}{2}k[2a + (k-1)d] + [a + (k+1) - 1] \\ &= \frac{1}{2}[2ak + k^2d - kd + 2a + 2k] \\ &= \frac{1}{2}[2a(k+1) + kd(k+1)] \\ S_{k+1} &= \frac{1}{2}(k+1)[2a + kd]\end{aligned}$$

This is S_n with n replaced by $(k + 1)$, thus if the result is true for k, it is true for $(k + 1)$.
$\therefore$ By the principle of mathematical induction, the formula is true for all n.

Example 3
Prove by induction that

$$\frac{1}{1\,.\,2} + \frac{1}{2\,.\,3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

for all positive integers n

Solution

For $n = 1$,

$$L.H.S = \frac{1}{1(1+1)} = \frac{1}{2}$$

$$R.H.S = \frac{1}{1+1} = \frac{1}{2}$$

Since L.H.S = R.H.S, the statement is true for $n = 1$
Assume true for $n = k$,

$$\Rightarrow \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

For $n = k + 1$,

$$\begin{aligned}\frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k(k+2)+1}{(k+1)(k+2)} \\ &= \frac{k^2 + 2k + 1}{(k+1)(k+2)}\end{aligned}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$
$$= \frac{k+1}{k+2}$$

$\therefore$ it is true for $k+1$

Thus if it is true for $n = k$, it is also true for $n = k+1$

$\therefore$ by mathematical induction, the result is true or all positive integers n.

Example 4

Prove by mathematical induction

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

for all natural numbers

Solution

For $n = 1$,

$$L.H.S = 1$$
$$R.H.S = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

Since L.H.S = R.H.S, the statement is true for $n = 1$

Now assume that the statement is true for $n = k$,

$$\Rightarrow 1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

For $n = k+1$,

$$1 + 2 + 3 + \cdots + k + k + 1 = \frac{k(k+1)}{2} + k + 1$$
$$= \frac{k(k+1) + 2(k+1)}{2}$$
$$= \frac{(k+1)(k+2)}{2}$$

$\therefore$ it is true for $k+1$

Thus if it is true for k, then it is also true for $k+1$

By the principle of mathematical induction, the statement is true for all natural numbers.

Example 5

Prove by induction

$$1\,.\,2 + 2\,.\,3 + 3\,.\,4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

for all natural numbers

Solution

For $n = 1$,

$$L.H.S = 1(1+1) = 2$$
$$R.H.S = \frac{1(1+1)(1+2)}{3} = \frac{1(2)(3)}{3} = 2$$

Since L.H.S = R.H.S, it is true for $n = 1$.

Now assume that the statement is true for $n = k$

$$\Rightarrow 1\,.\,2 + 2\,.\,3 + 3\,.\,4 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}$$

For $n = k+1$,

$$1.2 + 2.3 + 3.4 + \cdots + k(k+1) + (k+1)(k+2)$$
$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$
$$= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$$
$$= \frac{(k+1)(k+2)(k+3)}{3}$$

$\therefore$ it is true for $k+1$

Thus if it is true for $n = k$, it is also true for $n = k+1$

By the principle of mathematical induction, the statement is true for all natural numbers.

Example 6

Prove by mathematical induction

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all natural numbers

Solution

Putting $n = 1$,

$$L.H.S = 1^2$$
$$R.H.S = \frac{1(1+1)[2(1)+1]}{6} = \frac{(2)(3)}{6} = 1$$

Since L.H.S = R.H.S, the statement is true for $n = 1$

Now assume that the statement is true for $n = k$

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

To prove that $P(k+1)$ is true;

$$[1^2 + 2^2 + 3^2 + \cdots + k^2] + (k+1)^2$$
$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$
$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$
$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$
$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$
$$= \frac{(k+1)[2k^2 + 4k + 3k + 6]}{6}$$
$$= \frac{(k+1)[2k(k+2) + 3(k+2)]}{6}$$
$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

Thus if it is true for $n = k$, it is also true for $n = k+1$

By the principle of mathematical induction, the statement is true for all natural numbers

Example 7

Prove by the principle of mathematical induction that

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n! = (n+1)! - 1$$

for all natural numbers n

Solution

For $n = 1$,

L.H.S = $1 \times 1! = 1$, R.H.S = $(1+1)! - 1 = 2! - 1 = 1$

Since L.H.S = R.H.S, it is true for $n = 1$

Assume true for $n = k$,

$$\Rightarrow 1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + k \times k! = (k+1)! - 1$$

For $n = k + 1$,

$1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + k \times k! + (k+1) \times (k+1)!$

$$= (k+1)! - 1 + (k+1) \times (k+1)!$$
$$= (k+1)!\,[1 + (k+1)] - 1$$
$$= (k+2)(k+1)! - 1$$
$$= (k+2)! - 1$$

It is also true for $k + 1$

By the principle of mathematical induction, the statement is true for all natural numbers.

Example 8

Prove by mathematical induction that $2^{3n} - 1$ is divisible by 7, for all natural numbers n.

Solution

Let $P(n) = 2^{3n} - 1$

For $n = 1$,

$P(1) = 2^{3(1)} - 1 = 8 - 1 = 7$, which is divisible by7

$P(1)$ is true

Now assume that the statement is true for $n = k$

$$P(k) = 2^{3k} - 1$$
$$\Rightarrow \frac{2^{3k} - 1}{7} = A$$

where A is an integer

$$2^{3k} - 1 = 7A$$
$$2^{3k} = 7A + 1$$

Now to prove $P(k+1)$is true, consider

$$P(k+1) = 2^{3(k+1)} - 1$$
$$= 2^{3k} \times 2^3 - 1$$
$$= 8(2^{3k}) - 1$$
$$= 8(7A + 1) - 1$$
$$= 8(7A) + 8 - 1$$
$$= 8(7A) + 7$$
$$= 7(8A + 1)$$

which is divisible by 7, $P(k+1)$ is true

Thus if $P(k)$ is true, then $P(k+1)$ is true.

By the principle of induction, $2^{3n} - 1$ is divisible by 7 for all natural numbers n

Example 9

Prove that the number, $a_n = 4^n + 5$, is divisible by 3 for all positive integral values of n

Solution

For $n = 1$,

$$a_1 = 4^1 + 5 = 9$$

a_1 is divisible by 3 hence the statement is true for $n = 1$

Assuming true for $n = k$,

i.e. $a_k = 4^k + 5$ is divisible by 3

$$\Rightarrow \frac{4^k + 5}{3} = A$$
$$4^k + 5 = 3A$$
$$4^k = 3A - 5$$

Now to prove that a_{k+1} is true (divisible by 3);

$$a_{k+1} = 4^{k+1} + 5$$
$$= 4^k . 4 + 5$$
$$= 4(3A - 5) + 5$$
$$= 4(3A) - 20 + 5$$
$$= 4(3A) - 15$$
$$= 3(4A - 5)$$

which is divisible by 3, hence a_{k+1} is true

By the principle of mathematical induction, a_n is divisible by 3 for all positive integer values n

Example 10

By the method of induction, show that $10^n + 3\,.\,4^{n+2} + 5$ is divisible by 9 for all positive values of n

Solution

Let $P(n) = 10^n + 3\,.\,4^{n+2} + 5$

$$P(1) = 10^1 + 3\,.\,4^{1+2} + 5 = 10 + 3(64) + 5 = 207$$
$$\frac{207}{9} = 23$$

$P(1)$ is divisible by 9, hence $P(1)$ is true

Assume $P(k)$ is divisible by 9

$$P(k) = 10^k + 3\,.\,4^{k+2} + 5$$
$$\Rightarrow \frac{10^k + 3\,.\,4^{k+2} + 5}{9} = A$$

where A is a positive natural number

$$10^k + 3\,.\,4^{k+2} + 5 = 9A$$
$$10^k = 9A - 3\,.\,4^{k+2} - 5$$

For $n = k + 1$,

$$P(k+1) = 10^{k+1} + 3\,.\,4^{k+3} + 5$$
$$= 10(10^k) + 3\,.\,4(4^{k+2}) + 5$$
$$= 10[9A - 3\,.\,4^{k+2} - 5] + 12(\,4^{k+2}) + 5$$
$$= 90A - 30(4^{k+2}) - 50 + 12(4^{k+2}) + 5$$
$$= 90A - 18(4^{k+2}) - 45$$
$$= 9(10A - 2(4^{k+2}) - 5)$$

$\therefore P(k+1)$ is divisible by 9

Thus if $P(k)$ is true, then $P(k+1)$ is also true

Hence, by induction, the number $10^n + 3\,.\,4^{n+2} + 5$ must be divisible by 9 for all positive integers.

Example 11

Show that for all positive integer values of n, $5^{2n} + 3n - 1$ is an integer multiple of 9.

Solution

Let $P(n) = 5^{2n} + 3n - 1$

For $n = 1$,

$$P(1) = 5^{2(1)} + 3(1) - 1 = 27$$

which is a multiple of 9, hence true for $n = 1$

Assume true for $n = k$

$$\Rightarrow 5^{2k} + 3k - 1 = 9A$$

where A is some integer

$$5^{2k} = 9A - 3k + 1$$

For $n = k + 1$,

$$P(k+1) = 5^{2(k+1)} + 3(k+1) - 1$$
$$= 25(5^{2k}) + 3k + 2$$
$$= 25(9A - 3k + 1) + 3k + 2$$
$$= 25(9A) - 75k + 25 + 3k + 2$$
$$= 25(9A) - 72k + 27$$
$$= 9(25A - 8k + 3)$$

which is a multiple of 9, hence $P(k+1)$is true

Example 12

Prove by induction that $a^n - b^n$ is divisible by $(a - b)$ for all natural numbers.

Solution

Let $P(n) = a^n - b^n$

For $n = 1$,

$$P(1) = a^1 - b^1 = a - b$$

which is divisible by $a - b$ hence $P(1)$ is true

Now assume the statement is true for $n = k$, i.e. $a^k - b^k$ is divisible by $a - b$

$$\Rightarrow \frac{a^k - b^k}{a - b} = C$$

where C is a natural number (or an integer)

$$a^k - b^k = C(a - b)$$
$$a^k = b^k + C(a - b)$$

Now to prove $P(k+1)$ is true i.e. prove $a^{k+1} - b^{k+1}$ is divisible by $a - b$

$$a^{k+1} - b^{k+1} = a^k \,.\, a - b^k \,.\, b$$
$$= a[b^k + C(a - b)] - b(b^k)$$
$$= a(b^k) + aC(a - b) - b(b^k)$$
$$= b^k(a - b) + aC(a - b)$$
$$= (a - b)[b^k + aC]$$

which is divisible by $(a - b)$, hence $P(k+1)$ is true

By the principle of mathematical induction, $a^n - b^n$ is divisible by $a - b$ for all natural numbers.

Example 13

Prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integral values of n.

Solution

For $n = 1$,

$$R.H.S = (1)x^{1-1} = x^0 = 1$$

$$L.H.S = \frac{d}{dx}(x^1) = \frac{d}{dx}(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x} = 1$$

Since L.H.S = R.H.S, it is true for $n = 1$,

Assume true for $n = k$,

$$\Rightarrow \frac{d}{dx}(x^k) = kx^{k-1}$$

Now prove true for $n = k + 1$ i.e. prove that $\frac{d}{dx}(x^{k+1}) = (k+1)x^k$

$$\frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x^k \,.\, x)$$

Using the product rule:

$$\frac{d}{dx}(x^k.\ x) = x^k(1) + x\left[\frac{d}{dx}(x^k)\right]$$
$$= x^k + x(kx^{k-1}) \text{ by assumption}$$
$$= x^k + kx^k$$
$$= x^k(1 + k)$$
$$= (k+1)x^k$$

Thus if $\frac{d}{dx}(x^k) = kx^{k-1}$, then $\frac{d}{dx}(x^{k+1}) = (k+1)x^k$.

By mathematical induction, the statement is true for all positive integral values of n

Self-Evaluation exercise

Prove, by induction, that the given statements are true for all integral positive values of n

1. The sum of the first n terms of the series $1\,.\,3 + 2\,.\,4 + 3\,.\,5 + \cdots + r(r+2)$ is $\frac{1}{6}n(n+1)(2n+7)$
2. $\frac{1}{1\,.\,3} + \frac{1}{3\,.\,5} + \frac{1}{5\,.\,7} + \cdots + \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$
3. $\sum_{r=1}^{n} \frac{r}{2^r} = 2 - \frac{n+2}{2^n}$
4. $\sum_{r=2}^{n} \frac{1}{r^2 - 1} = \frac{3}{4} - \frac{2n+1}{2n(n+1)}$
5. $\sum_{r=1}^{n} r(r+1)(r+2) = \frac{n}{4}(n+1)(n+2)(n+3)$
6. $(1 \times 4) + (2 \times 5) + (3 \times 6) + \cdots + n(n+3) = \frac{1}{6}n(n+1)(n+5)$
7. $1 + 3 + 5 + \cdots + 2n - 1 = n^2$
8. $n(n+1)(n+2)$ is an integer multiple of 6
9. $7^{2n+1} + 1$ is an integer multiple of 8
10. $n^3 + 3n^2 - 10n$ is divisible by 3
11. $3^{2n} - 1$ is a multiple of 8
12. $7^n + 4^n + 1$ is divisible by 6
13. $\sum_{r=1}^{n} \cos(2r-1)\,x = \frac{\sin 2nx}{2 \sin x}$
14. $\sum_{r=1}^{n} \frac{(r+4)}{2^r r(r+1)(r+2)} = \frac{1}{2} - \frac{1}{2^n(n+1)(n+2)}$
15. $(2n+1)(2n-1)$ is an odd number
16. $2 + 4 + 6 + 9 + \cdots + 2n = n(n+1)$
17. $1 + 4 + 7 + \cdots + 3n - 2 = \frac{n(3n-1)}{2}$
18. $4 + 8 + 12 + \cdots + 4n = 2n(n+1)$
19. $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$
20. $5^{2n} - 1$ is divisible by 24
21. 10^{2n-1} is divisible by 11
22. The sum $S_n = n^3 + 3n^2 + 5n + 3$ is divisible by 3
23. $7^{2n} + 16n - 1$ is divisible by 64

24. $2^n > n$

25. Prove by induction that for all positive integers n,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Deduce that

$$(n+1)^3 + (n+2)^3 + \cdots + (2n)^3 = \frac{1}{4}n^2(3n+1)(5n+3)$$

26. Prove that if n is a positive integer, $10^n - 1$ is divisible by 9. Hence prove that a necessary and sufficient condition for a positive integer to be divisible by 9 is that the sum of its digits is divisible by 9.

27. Prove that if n is any positive integer,

$$1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + n(n+1)(n+2)$$
$$= \frac{1}{3}n(n+1)(n+2)(n+3)$$

28. Prove by induction that $n(n+1)(2n+1)$ is a multiple of 6 for all natural numbers.

29. Show that $11^{2n} - 1$ is always exactly divisible by 120 when n is a positive integer.

30. Show that $3^{4n+2} + 2\,.\,4^{3n+1}$ is exactly divisible by17 if n is a positive integer.

31. Use the method of induction to prove that $6^n - 1$ is divisible by 5 for all positive integral values of n

32. Prove that $8^n - 7n + 6$ is divisible by 7 for all positive integral values.

33. Show that, for all positive integral values of n, $7^n + 2^{2n+1}$ is divisible by 3

34. Prove that $7^{2n} + (2^{3n-3})(3^{n-1})$ is divisible by 25 for any natural number n.

35. Using mathematical induction prove that for every integer $n \geq 1$, $2\,.\,7^n + 3\,.\,5^n - 5$ is divisible by 24

36. Use mathematical induction to show that $25^{n+1} - 24n + 5735$ is divisible by 576 for all $n \geq 1$.

37. Prove, using the principle of mathematical induction, that

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

38. Prove by mathematical induction that $n^3 - n$ is divisible by 6 for all $n \geq 2$.

39. Prove by mathematical induction, that $81 \times 3^{2n} - 2^{2n}$ is divisible by 5 for all natural numbers.

Chapter 10 Permutations and Combinations

Fundamental principles of counting

We shall start by discussing two fundamental principles i.e. principle of addition and principle of multiplication. These two principles will enable us understand permutations and combinations and form the base for permutations and combinations.

Fundamental principle of multiplication: If there are two jobs such that one of them can be completed in m ways, and when it has been completed in any one of these m ways, a second job can be completed in n ways; then the two jobs in succession can be completed in $m \times n$ ways.

Example 1

In a class, there are 15 boys and 20 girls. The teacher wants to select a boy and a girl to represent the class in a function. In how many ways can the teacher make this selection?

Solution

Here the teacher is to perform two jobs:

(i) Selecting a boy among 15 boys, and
(ii) Selecting a girl among 20 girls

The first of these can be performed in 15 ways and the second in 20 ways.

Therefore, by the fundamental principle of multiplication, the required number of ways is $15 \times 20 = 300$

Fundamental principle of addition: If there are two jobs such that they can be performed independently in m and n ways respectively, then either of the two jobs can be performed in $(m + n)$ ways.

Example 2

In a class, there are 20 boys and 10 girls. The teacher wants to select either a boy or a girl to represent the class in a function. In how many ways can the teacher make this selection?

Solution

Here the teacher is to perform either of the following two jobs:

(i) selecting a boy among 20 boys, (or)
(ii) selecting a girl among 10 girls

The first of these can be performed in 20 ways and the second in 10 ways. Therefore, by fundamental principle of addition, either of the two jobs can be performed in $(20 + 10) = 30$ ways

Thus, the teacher can make selection of boy/girl in 30 ways.

Example 3

A room has 10 doors. In how many ways can a man enter the room through one door and come out through a different door?

Solution

Clearly, a person can enter the room through any one of the ten doors. So, there are ten ways of entering into the room.

After entering into the room, the man can come out through any one of the remaining 9 doors. So, he can come out through a different door in 9 ways.

Hence, the number of ways in which a man can enter a room through one door and come out through a different door

$$= 10 \times 9 = 90$$

Example 4

How many words (with or without meaning) of three distinct letters of the English alphabets are there?

Solution

Here we have to fill up three places by **distinct** letters of the English alphabet. Since there are 26 letters of the English alphabet, the first place can be filled by any of these letters. So, there are 26 ways of filling up the first place.

Now, the second place can be filled up by any of the remaining 25 letters.

So, there are 25 ways of filling up the second place.

After filling up the first two places, only 24 letters are left to fill up the third place. So, the third place can be filled in 24 ways. Hence, the required number of words

$$= 26 \times 25 \times 24 = 15600$$

Example 5

How many three-digit numbers can be formed by using the digits 1, 2, 3, 4, 5.

Solution

We have to determine the total number of three-digit numbers formed by using the digits 1, 2, 3, 4, 5.

Clearly, the repetition of digits is allowed.

A three-digit number has three places i.e. one's, ten's and hundred's.

The one's place can be filled in 5 ways.

Similarly, each of the ten's and hundred's place can be filled in 5 ways.

$\therefore$ Total number of required numbers

$$= 5 \times 5 \times 5 = 125$$

Example 6
How many five-figure odd numbers can be made from the digits 1, 2, 3, 4, 5, if no digit is repeated?

Solution

For a number to be odd, we must have 1, 3 or 5 at the one's place. So, there are three ways of filling the one's place.
Since no digit is repeated, the remaining four places can be filled in 4, 3, 2 and 1 ways respectively.
Hence, total number of odd numbers
$$= 4 \times 3 \times 2 \times 1 \times 3 = 72$$

Example 7
There are 6 multiple choice questions in an examination. How many sequences of answers are possible, if the first three questions have 4 choices each and the next three have 5 each?

Solution

Here we have to perform 6 jobs of answering 6 multiple choice questions.
Each of the first three questions can be answered in 4 ways and each of the next three can be answered in 5 ways.
So, the total number of different sequences
$$= 4 \times 4 \times 4 \times 5 \times 5 \times 5 = 8000$$

Example 8
In how many ways can 5 persons sit in a car, 2 including the driver in the front seat and 3 in the back seat, if 2 particular persons do not know driving?

Solution

Let us mark the 5 seats by the letters A, B, C, D and E eith A as driver's seat.
Since 2 particular persons out of the 5 do not know driving, there are 3 choices for seat A, 4 choices for seat B, 3 choices for seat D and 1 choice for seat E
Therefore, the total number of arrangements
$$= 3 \times 4 \times 3 \times 2 \times 1 = 72$$

Example 9
How many three-digit numbers greater than 600 can be formed by using the digits 4, 5, 6, 7, 8?

Solution

Clearly, repetition of digits is allowed. Since a three-digit number greater than 600 will have 6, 7 or 8 at hundred's place. So, hundred's place can be filled in 3 ways.
Each of the ten's and one's place can be filled in 5 ways.
Hence, total number of required numbers
$$= 3 \times 5 \times 5 = 75$$

Example 10
How many numbers divisible by 5 and lying between 5000 and 6000 can be formed from the digits 5, 6, 7, 8 and 9?

Solution

Clearly, a number between 5000 and 6000 must have 5 at thousand's place.
Since the number is divisible by 5 it must have 5 at one's place.
Now, each of the remaining places (i.e. hundred's and ten's) can be filled in 5 ways.
Hence the total number of required numbers
$$= 1 \times 5 \times 5 \times 1 = 25$$

Example 11
How many three-digit odd numbers can be formed by using the digits 4, 5, 6, 7, 8, 9 if:
(a) the repetition of digits is not allowed?
(b) the repetition of digits is allowed?

Solution

For a number to be odd, we must have 5, 7 or 9 at the one's place. So, there are three ways of filling the one's place.
(a) Since the repetition of digits is not allowed, the ten's place can be filled with any of the remaining 5 digits in 5 ways.
Now, four digits are left. So, hundred's place can be filled in 4 ways.
So, required number of numbers
$$= 3 \times 5 \times 4 = 60$$
(b) Since the repetition of digits is allowed, so each of the ten's and hundred's place can be filled in 6 ways.
Hence required number of numbers
$$= 3 \times 6 \times 6 = 108$$

Example 12
How many numbers are there between 500 and 1000 which have exactly one of their digits as 8?

Solution

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9$$
For the number to be between 500 and 1000, it is a three-digit number with 5, 6, 7, 8 and 9 as the possible digits to occupy the hundred's place.
(i) If the digit 8 occupies the hundred's place, the digit in the ten's place can be filled in 9 ways and that in one's place also 9 ways.
$$\text{Number of ways} = 1 \times 9 \times 9 = 81$$
(ii) If the digit 8 occupies the ten's place, then the digit in the hundred's place can be filled in 4 ways and that in one's place 9 ways.
$$\text{Number of ways} = 4 \times 1 \times 9 = 36$$
(iii) If the digit 8 occupies the one's place, then the digit in the hundred's place can be filled in 4 ways and that in one's place 9 ways
$$\text{Number of ways} = 4 \times 1 \times 9 = 36$$
Total numbers between 500 and 1000
$$= 81 + 36 + 36 = 153$$

We have already defined $0! = 1$. This can be concluded as follows.

we know that $\quad {}^nP_r = \dfrac{n!}{(n-r)!}$

Putting $r = n$,

$$ {}^nP_n = \frac{n!}{(n-n)!} $$
$$ \Rightarrow n! = \frac{n!}{0!} $$
$$ 0! = \frac{n!}{n!} = 1 $$
$$ \therefore \mathbf{0! = 1} $$

Example 16

Write down all the permutations of the vowels A, E, I, O, U in English alphabets taking 3 at a time and starting with E.

Solution

The permutations of vowels A, E, I, O, U

EAI, EIA, EIO, EOI, EOU, EUO, EAO, EOA, EIU, EUI, EAU, EUA

Clearly, there are 12 permutations

Example 17

Evaluate ${}^8P_3$

Solution

$$ {}^8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{(8\times 7\times 6)\times 5!}{5!} $$
$$ = 8\times 7\times 6 $$
$$ = 336 $$

Example 18

Given ${}^nP_5 = 42\,{}^nP_3$, find the value of n.

Solution

$$ \frac{n!}{(n-5)!} = 42\frac{n!}{(n-3)!} $$
$$ \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)!}{(n-5)!} = 42\frac{n(n-1)(n-2)(n-3)!}{(n-3)!} $$
$$ (n-3)(n-4) = 42 $$
$$ n^2 - 7n + 12 = 42 $$
$$ n^2 - 7n - 30 = 0 $$
$$ n^2 - 10n + 3n - 30 = 0 $$
$$ n(n-10) + 3(n-10) = 0 $$
$$ (n-10)(n+3) = 0 $$
$$ n = 10 \text{ or } n = -3 $$

Since n cannot be negative, $n = 10$

Example 19

If ${}^5P_r = {}^6P_{r-1}$, find r

Solution

$$ \frac{5!}{(5-r)!} = \frac{6!}{[6-(r-1)]!} $$
$$ \frac{5!}{(5-r)!} = \frac{6\times 5!}{(7-r)!} $$
$$ \frac{1}{(5-r)!} = \frac{6}{(7-r)(6-r)(5-r)!} $$
$$ 1 = \frac{6}{(7-r)(6-r)} $$
$$ 42 - 7r - 6r + r^2 = 0 $$
$$ r^2 - 13r + 36 = 0 $$
$$ r^2 - 9r - 4r + 36 = 0 $$
$$ r(r-9) - 4(r-9) = 0 $$
$$ (r-9)(r-4) = 0 $$
$$ r = 9 \text{ or } r = 4 $$

${}^5P_r$ is meaningful for $r \le 5$

$$ \therefore r = 4 $$

Example 20

In how many ways can five children stand in a queue?

Solution

The number of ways in which 5 persons can stand in a queue is the same as the number of arrangements of 5 different things taken all at a time.

Hence the required number of ways

$$ = {}^5P_5 = 5! = 120 $$

Example 21

How many different signals can be made by hoisting 6 differently coloured flags one above the other, when any number of them may be hoisted at one time?

Solution

The signals can be made by using at a time one or two or three or four of five or six flags.

The total number of signals when r flags are used at a time from 6 flags is equal to the number of arrangements of 6, taking r at a time i.e. ${}^6P_r$

Hence by the fundamental principle of addition, the total number of different signals

$$ = {}^6P_1 + {}^6P_2 + {}^6P_3 + {}^6P_4 + {}^6P_5 + {}^6P_6 $$
$$ = 6 + (6)(5) + (6)(5)(4) + (6)(5)(4)(3) $$
$$ +(6)(5)(4)(3)(2) + (6)(5)(4)(3)(2)(1) $$
$$ = 6 + 30 + 120 + 360 + 720 + 720 $$
$$ = 1956 $$

Example 22

Find the number of different 4-letter words with or without meanings, that can be formed from the letters of the word ‘***NUMBER***’

Solution

There are 6 letters in the word ‘NUMBER’

So, the number of 4 letter words

= the number of arrangements of 6 letters taken 4 at a time

$$ = {}^6P_4 $$
$$ = 360 $$

Example 23

How many different words can be formed with the letters of the word ***ORDINATE*** so that vowels occupy odd places?

Solution

In the word ORDINATE, there are 4 different vowels and 4 different consonants.

Four vowels can be placed at 4 odd places in ${}^4P_4 = 4! = 24$ different ways

Then 4 consonants can be placed at 4 even places in ${}^4P_4 = 4! = 24$ different ways

Hence required number of words $= 24 \times 24 = 576$

Example 24

How many 6-digit numbers can be formed from the digits 0, 1, 3, 5, 7, 9 which are divisible by 10 and no digit is repeated?

Solution

The given digits are 0, 1, 3, 5, 7 and 9, which are 6 in number. We are required to form 6-digit number which are divisible by 10 and no digit is repeated is repeated.

As the number is divisible by 10, so it must have a 0 at one's place, therefore, one's place can be filled up in only one way.

The remaining 5 places can be filled up by the remaining 5 digits in ${}^5P_5$ ways.

∴ the required number of numbers $= {}^5P_5 = 5! = 120$

Example 25

How many numbers greater than 50000 can be formed by using the digits 0, 2, 3, 5 and 6, each digit is used only in each number?

Solution

The given digits are 0, 2, 3, 5 and 6, which are 5 in number. As the number 50000 has five digits and the numbers greater than 50000 are to be formed by using each of the given digit only once, the numbers of only five digits are to be formed.

Ten thousand's place can be filled up by any of the digits 5 or 6 in 2 different ways.

Remaining 4 places can be filled up by the remaining digits in ${}^4P_4$ ways.

∴ The required number of numbers $= 2 \times {}^4P_4 = 48$

Example 26

In how many ways can 3 mathematics books, 4 history books, 3 chemistry books and 2 biology books can be arranged on a shelf so that all the books on the same subject are together?

Solution

First, we consider books of a particular subject as one unit. Thus, there are 4 units and these units can be arranged in 4! ways.

Now in each of these arrangements, mathematics books can be arranged among themselves in 3! ways, history books in 4! ways, chemistry books in 3! ways and biology books in 2! ways.

Therefore, the total number of arrangements

$$= 4! \times 3! \times 4! \times 3! \times 2!$$
$$= 24 \times 6 \times 24 \times 6 \times 2$$
$$= 41472$$

Example 27

A family of 4 brothers and 3 sisters is to be arranged in a row, for a photograph. In how many ways can they be seated, if

(a) all the sisters sit together
(b) all the sisters are not together

Solution

(a) Since the 3 sisters are inseparable, consider them as one single unit.
This together with 4 brothers make 5 persons who can be arranged themselves in 5! ways.
In each of these permutations, the 3 sisters can be rearranged among themselves in 3! ways.
Hence the total number of arrangements required
$$= 5! \times 3! = 120 \times 6 = 720$$

(b) The number of arrangements of all the persons without any restriction $= 7! = 5040$
Number of arrangements in which all the sisters sit together $= 720$
∴ Number of arrangements required
$$= 5040 - 720 = 4320$$

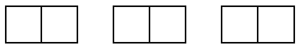
Example 28

Three married couples are to be seated in a row having six seats in a cinema hall. If spouses are to be seated next to each other, in how many ways can they be seated?

Find also the number of ways of their seating if all the ladies sit together and all gents sit together.

Solution

Let us denoted three married couples by C_1, C_2 and C_3 where each couple is considered as one unit as shown below.



C_1 C_2 C_3

The number of ways in which the spouses can be seated next to each other $= 3!$

As each couple can be seated in 2! ways,

The number of seating arrangements so that the spouses are seated next to each other

$$= 3! \times 2! \times 2! \times 2! = 6 \times 2 \times 2 \times 2 = 48$$

Further, if three ladies sit together and all gents sit together then a unit of ladies and a unit of gents can be arranged in 2! ways. Also the ladies can be arranged among themselves in

3! ways and the gents can be arranged among themselves in 3! ways.
Thus, the total number of arrangements of seating all ladies and all gents together $= 2! \times 3! \times 3! = 72$

Example 29

In how many ways can 5 boys and 3 girls be seated in a row so that no two girls are together?

Solution

Let us first seat the 5 boys. This can be done in ${}^5P_5$ i.e. $5! = 120$ ways.

$$\square\,\mathrm{B}\,\square\,\mathrm{B}\,\square\,\mathrm{B}\,\square\,\mathrm{B}\,\square\,\mathrm{B}\,\square$$

Now no two girls are together if they are seated only at the places marked '$\square$'. There are 6 such places and the 3 girls can be seated in ${}^6P_3 = 120$ ways.
Hence, by fundamental principle of multiplication, the total number of ways $= 120 \times 120 = 1440$

Example 30

Find the number of different words that can be formed from the letters of the word '***TRIANGLE***' so that no two vowels are together?

Solution

The word 'TRIANGLE' has 5 consonants and 3 vowels.
Let us first arrange 5 consonants. This can be done in ${}^5P_5$ ways or 5! ways. Mark these consonants by C as shown below.

$$\square\,\mathrm{C}\,\square\,\mathrm{C}\,\square\,\mathrm{C}\,\square\,\mathrm{C}\,\square\,\mathrm{C}\,\square$$

Now no two vowels are together if these are arranged only at the places '$\square$'. There are 6 such places and the 3 vowels can be arranged at these places in ${}^6P_3$ ways.
Hence, the number of ways of arranging the letters of the word 'TRIANGLE' so that no vowels are together

$$= 5! \times {}^6P_3 = 14400$$

Example 31

7 candidates are to be examined – 2 in mathematics and the remaining in different subjects. In how many ways can they be seated in a row so that the two examinees in mathematics may not sit together?

Solution

When there is no restriction, the total number of ways in which 7 candidates can sit $= {}^7P_7 = 5040$.
When two candidates of mathematics sit together, we consider them as one candidate. Now the total candidates become 6, and they can be seated in ${}^6P_6 = 6!$ ways. But 2 mathematics students can be arranged among themselves in 2! ways. Thus the number of ways in which mathematics students sit together $= 2! \times 6! = 2 \times 720 = 1440$
Hence, the number of ways in which mathematics students do not sit together $= 5040 - 1440 = 3600$

Example 32

Find the number of different (eight letter) words can be formed out of the letters of the word ***DAUGHTER*** so that
(a) the word starts with D and ends with R
(b) position of letter H remains unchanged
(c) relative position of vowels and consonants remains unaltered
(d) no two vowels are together
(e) all vowels occur together
(f) all vowels never occur together

Solution

The given word consists of 8 different letters out of which 3 are vowels and 5 are consonants.

(a) If the words have to start with D and end with R, then we can arrange remaining 6 places in
$${}^6P_6 = 6! = 720 \text{ ways.}$$
(b) If position of H remains unchanged, the remaining 7 letters can be arranged in 7 places in
$${}^7P_7 = 7! = 5040 \text{ ways}$$
(c) The relative position of vowels and consonants remains unaltered means that vowel can take the place of vowel and consonant can take place of consonant. Now the 3 vowels can be arranged among themselves in $3! = 6$ ways and the 5 consonants can be arranged among themselves in $5! = 120$ ways.
Thus the total number of words that can be formed
$$= 6 \times 120 = 720 \text{ ways}$$
(d) First let us arrange the consonants in a row. This can be done in ${}^5P_5 = 5! = 120$ ways
$$\square\,\mathrm{C}\,\square\,\mathrm{C}\,\square\,\mathrm{C}\,\square\,\mathrm{C}\,\square\,\mathrm{C}\,\square$$
Now no two vowels are together if they are put at places marked '$\square$'. The 3 vowels can fill up these places in ${}^6P_3 = 120$ ways.
Hence, the total number of words
$$= 120 \times 120 = 14400$$
(e) To find the number of arrangements where the three vowels A, E, O all occur together, temporarily considering this as one block, we can arrange this and 5 consonants in ${}^6P_6 = 6!$ ways.
The three vowels can be arranged themselves in 3! ways. Hence the number of words in which 3 vowels occur together
$$= 6! \times 3! = 7206 = 4320$$
(f) The total number of (8-digit) words formed out of letters of given word $= {}^8P_8 = 8! = 40320$
Hence the number of words formed in which all vowels are never together
$= $ total number of words formed – the number of words formed in which all vowels are together
$$= 40320 - 4320 = 36000$$

Permutations of objects not all distinct

The number of mutually distinguishable permutations of n things, taken all at a time, of which p are alike and of one kind, q alike of second, r of the third, such that $p+q+r=n$, is $\frac{n!}{p!q!r!}$

Example 33

Find the number of permutations of the letters of the word ***HEIDELBERG***

Solution

Here we are given 9 letters of which there are 3 E's and the remaining 7 are different.

Required number of permutations $=\frac{10!}{3!}=604800$

Example 34

How many arrangements can be made with the letters of the word "***MATHEMATICS***"?

Solution

There are 11 letters in the word 'MATHEMATICS' of which two are M's, two are A's, two are T's and all other are distinct.

∴ required number of arrangements $=\frac{11!}{2!\times2!\times2!}=4989600$

Example 35

How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the odd places?

Solution

There are 4 odd digits 1, 1, 3, 3 and 4 odd places (1st, 3rd, 5th, 7th)

So odd digits can be arranged in odd places in $\frac{4!}{2!2!}$ ways

The remaining 3 even digits 2, 2, 4 can be arranged in 3 even places (2nd, 4th, 6th) in $\frac{3!}{2!}$ ways.

Hence, the required number of numbers

$$=\frac{4!}{2!\,2!}\times\frac{3!}{2!}=6\times3=18$$

Example 36

In how many of the distinct permutations of the letters in ***MISSISSIPPI*** do the four ***I***'s not come together?

Solution

The given word has 11 letters, four I's, four S's, two P's and one M.

$$\text{Total number of permutations }=\frac{11!}{4!\,4!\,2!}=34650$$

When all the four I's come together, then consider these four I's as one letter and 7 others – four S's, two P's, one M

So the number of permutations in which I's come together

$$=\frac{8!}{4!\,2!}=840$$

∴ The number of permutations in which the four I's do not come together $=34650-840=33810$

Example 37

In how many ways can the letters of the word ***PERMUTATIONS*** be arranged such that

(a) there is no restriction,
(b) P comes before S,
(c) words start with P and end with S,
(d) I's are together,
(e) all vowels are together,
(f) P comes before S and there are always 4 letters between P and S,
(g) there are four letters between P and S?

Solution

The given word has 12 letters – two T's and 10 different letters

(a) Total number of arrangements is $\frac{12!}{2!}=6\times11!$
$=239500800$

(b) Out of above arrangements, P comes before S in half the arrangements
Hence the required number of arrangements $=3\times11!$
$=119750400$

(c) As position of P and S is fixed, remaining 10 letters (two T's and eight other different letters) can be arranged in $\frac{10!}{2!}=5\times9!$ ways
$=1814400$

(d) Considering two T's as a block, we have to arrange 11 different things, which can be done in 11! ways
$=39916800$

(e) Considering the five vowels in given letter – E, U, A, I, O as a block, we have 8 objects having 2 alike objects (T's). So this can be arranged in $\frac{8!}{2!}=4\times7!$ ways
Now within the block, 5 different vowels can be arranged in 5! ways.
Hence, the required number of arrangements $=4\times7!\times5!=2419200$

(f) The number of ways in which P comes before S and there are exactly four letters between P and S is given by

1	2	3	4	5	6	7	8	9	10	11	12
P					S						
	P					S					
		P					S				
			P					S			
				P					S		
					P					S	
						P					S

There are 7 such ways in which P comes before S and there are exactly 4 letters between P and S.

$$= \frac{10!}{4!\,6!} = 210$$

(b) If two particular books are always selected. This means two books are selected out of the remaining 8 books.

$\therefore$ required number of ways $= {}^{8}C_2 = \frac{8!}{2!6!} = 28$

(c) If two particular books are never selected
This means four books are selected out of the remaining 8 books.

$\therefore$ required number of ways $= {}^{8}C_4 = \frac{8!}{4!4!} = 70$

Example 52

In how many ways can players for a cricket team of 11 (5 batsmen, 3 all-rounders, 2 bowlers, 1 wicket keeper) be selected from a group of 25 players containing 10 batsmen, 8 bowlers, 5 all-rounders and 2 wicket keepers?

Solution

The selection of the team is divided into 4 phases:

i. selection of 5 batsmen out of 10. This can be done in ${}^{10}C_5$ ways
ii. selection of 3 all-rounders out of 5. This can be done in ${}^{5}C_3$ ways.
iii. selection of 2 bowlers out of 8. This can be done in ${}^{8}c_2$ ways
iv. selection of 1 wicket keeper out of 12. This can be done in ${}^{2}C_1$ ways.

$\therefore$ The team can be selected in ${}^{10}C_5 \times {}^{5}C_3 \times {}^{8}c_2 \times {}^{2}C_1$ ways

$$= 252 \times 10 \times 28 \times 2 = 141120 \text{ ways}$$

Example 53

Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour, assuming that the balls are of the same colour are distinguishable?

Solution

Since balls of the same colour are distinguishable, therefore, we have 6 different red balls, 5 different white balls and 5 different blue balls.

We are to make selection of 9 balls, consisting of 3 balls of each colour.

The number of ways of selecting 3 red balls from 6 different red balls $= {}^{6}C_3 = 20$

The number of ways of selecting 3 white balls from 5 different white balls $= {}^{5}C_3 = 10$

The number of ways of selecting 3 blue balls from 5 different blue balls $= {}^{5}C_2 = 10$

$\therefore$ The required number of ways of selecting 9 balls

$$= 20 \times 10 \times 10 = 2000$$

Example 54

How many committees of five persons with a chairperson can be selected from 12 persons?

Solution

First, we select a chairperson. Any one person out 12 persons can be selected as chairperson. So, there are 12 ways of selecting a chairperson. As committees of 5 persons are to be selected from 12 persons, so we have to select 4 or more persons from the remaining 11 persons and this can be done in ${}^{11}C_4$ ways.

$\therefore$ The required number of committees that can be formed

$$= 12 \times {}^{11}C_4 = 12 \times 330 = 3960$$

Example 55

A boy has 3 library tickets and 8 books of his interest in the library. Of these 8 books, he does not want to borrow Principles of Applied Mathematics unless Principles of Pure Mathematics is also borrowed. In how many ways can he choose the three books to be borrowed?

Solution

We have the following mutually exclusive possibilities

i. Boy borrows Principles of Applied Mathematics.
When the boy borrows Principles of Applied Mathematics, then he borrows Principles of Pure Mathematics also. So, he can borrow just one more book out of the remaining 6 books.
The number of possible choices is ${}^{6}C_1$
ii. Boy does not borrow Principles of Applied Mathematics
When the boy does not borrow Principles of Applied Mathematics, then he can borrow any 3 books out of the remaining 7 books.
The number of possible choices is ${}^{7}C_3$
Hence, the total number of possible ways

$$= {}^{6}C_1 + {}^{7}C_3 = 6 + 35 = 41$$

Example 56

From a class of 25 students, 10 are to be chosen for an excursion party. There are three students who decide that either all of them will join or none of them will join. In how many ways can the excursion party be chosen?

Solution

We have the following two mutually exclusive possibilities:

i. *When three particular students join the party*
When three particular students join the party, we have to choose 7 more students out of the remaining 22 students. This can be done in ${}^{22}C_7$ ways

ii. *When three particular students do not join the party*
When three particular students do not join the party, the10 students out of the remaining 22 students. This can be done in ${}^{22}C_{10}$ ways

containing 6 questions. He is not permitted to attempt more than 5 questions from either section. Find the number of different ways of selecting the questions.

Solution

The different mutually exclusive possibilities are:

i. 2 from section A and 5 from section B
Number of ways $= {}^6C_2 \times {}^6C_5 = 15 \times 6 = 90$

ii. 3 from section A and 4 from section B
Number of ways $= {}^6C_3 \times {}^6C_4 = 20 \times 15 = 300$

iii. 4 from section A and 3 from section B
Number of ways $= {}^6C_4 \times {}^6C_3 = 15 \times 20 = 300$

iv. 5 from section A and 2 from section B
Number of ways $= {}^6C_5 \times {}^6C_2 = 6 \times 15 = 90$

∴ Total number of ways of selecting these questions

$$= 90 + 300 + 300 + 90 = 780$$

Example 62

Out of 6 boys and 4 girls, a committee of 5 is to be formed. In how many ways can this be done if

(a) at least 2 girls are included?

(b) at most 2 girls are included?

Solution

(a) including at least two girls

The different mutually exclusive possibilities are:

i. 2 girls and 3 boys
Number of ways $= {}^4C_2 \times {}^6C_3 = 6 \times 20 = 120$

ii. 3 girls and 2 boys
Number of ways $= {}^4C_3 \times {}^6C_2 = 4 \times 15 = 60$

iii. 4 girls and 1 boy
Number of ways $= {}^4C_4 \times {}^6C_2 = 1 \times 6 = 6$

∴ Total number of ways $= 120 + 60 + 6 = 186$

(b) Including at most 2 girls, the different mutually exclusive possibilities are

i. 5 boys and no girl
Number of ways $= {}^6C_5 = 6$

ii. 1 girl and 4 boys
Number of ways $= {}^4C_1 \times {}^6C_4 = 4 \times 15 = 60$

iii. 2 girls and 3 boys
Number of ways $= {}^4C_2 \times {}^6C_3 = 6 \times 20 = 120$

∴ Total number of ways $= 6 + 60 + 120 = 186$

Example 63

A box contains two white, three black and four red balls. In how many ways can three balls be drawn from the box, if at least one black ball is to be included in the draw

Solution

Box contains 3 black balls and 6 non-black balls (2 white and 4 red). The different possibilities are

i. 1 black ball, 2 non-black balls

ii. 2 black balls, 1 non-black ball

iii. 3 black balls

∴ The required number of ways

$$= {}^3C_1 \times {}^6C_2 + {}^3C_2 \times {}^6C_1 + {}^3C_3$$
$$= 3 \times 15 + 3 \times 6 + 1 = 64$$

Example 64

A team of 8 players is to be chosen from a group of 12 players. One of the 8 is then to be elected as captain and another as vice-captain. In how many ways can this be done?

Solution

The number of ways of choosing 8 players out of 12 players

$$= {}^{12}C_8 = 495$$

Now out of the 8 players, any one player can be elected as a captain, so there are 8 ways of electing a captain. After electing a captain, 7 players are left and any one out of these can be elected as vice-captain, so there are 7 ways of selecting a vice captain.

∴ The required number of ways $= 495 \times 8 \times 7 = 27720$

Selection of r objects from a group containing n objects some of which are similar

Example 65

Find the number of different selections of 3 and 4 letters from the word ***NUMBERING***

Solution

The word NUMBERING has 8 different letters (NUMBERIG) and one double letter (NN)

(a) different selection of 3 letters

i. No doubles (All letters different)
NUMBERIG
Number of ways $= {}^8C_3 = 56$

ii. 1 double and 1 different letter
NN – UMBERIG
Number of ways $= {}^7C_1 = 7$

Total number of different selections $= 56 + 7 = 63$

(b) Different selection of 4 letters

i. No doubles (All letters different)
NUMBERIG
Number of ways $= {}^8C_4 = 70$

ii. 1 double and 1 different letter
NN – UMBERIG
Number of ways $= {}^7C_2 = 21$

Total number of different selections $= 70 + 21 = 91$

Example 66

Find the number of different selections of 4 letters from the word ***STATISTICS***

Solution

The word STATISTICS contains 5 single letters, 3 double letters and 2 treble letters.

The single letters are STAIC

The double letters are SS, TT, II

The treble letters are SSS, TTT

Selections of any size from a group

The number of possible selections of any size that can be made from a group of unlike things deserves special consideration.

Example 75

How many different selections can be made from four letters A, B, C, D?

Solution

Method 1:

Number of selections of 1 letter = ${}^4C_1 = 4$
Number of selections of 2 letters = ${}^4C_2 = 6$
Number of selections of 3 letters = ${}^4C_3 = 4$
Number of selections of 4 letters = ${}^4C_4 = 1$
Total number of possible selections $= 4 + 6 + 4 + 1 = 15$
With a larger number of objects to select from, the above method can be hectic.

Method 2:

In any given selection, the letter A is either included or not included i.e. there are 2 ways of dealing with this letter.
Similarly, the letter B is either included or not included and so there are 2 ways of dealing with this letter.
Extending this to all the four letters, we see that there are $2 \times 2 \times 2 \times 2$ ways of dealing with the letters, but this includes the case in which none of the letters is included, and this is not a selection.
Thus number of selections $= 2^4 - 1 = 15$, as obtained in method 1.
In general, there are $2^r - 1$ selections which can be made from r unlike items

Group containing repeated items

If the group from which selections are to be made includes some repeated items. Suppose, for example that a group of letters includes 3 A's. These can be dealt with in 4 ways i.e. either no A's, 1A, 2A's or 3A's are included in a particular selection. The different letters can be considered as before

Example 76

How many different selections can be made from the letters of the word ***OSMOSIS***?

Solution

There are 3S's, 2O's and 2 other different letters
The S's can be dealt with in 4 ways
The O's can be dealt with in 3 ways
The M and I can be dealt with in 2 ways each
Total number of selections $= 4 \times 3 \times 2^2 - 1 = 47$

Example 77

How many different selections can be made from the letters of the word ***INABILITY***?

Solution

$$INABILITY = I_3NABLTY$$

Number of selections = $4 \times 2^6 - 1 = 255$

Self-Evaluation exercise

1. If ${}^nC_{12} = {}^nC_8$, find r
[Ans: 20]
2. If ${}^nP_r = 840,\ {}^nC_r = 35$, find the values of n and r
[Ans: $n = 7, r = 4$]
3. A book club offers a choice of 20 books of which a member choose six. Find the number of different ways in which a member may make his choice
Given that 12 of the 20 books on offer are novels and that the other 8 are biographies, find the number of different ways in which a member chooses 6 so that
(a) he has 3 novels and 3 biographies
(b) he has at least 4 biographies
[Ans: 38760 (a) 12320 (b) 5320]
4. Nine people are going to travel in two taxis. The larger has five seats, and the smaller has four. In how many ways can the party be split up? [Ans: 126]
5. Twelve people are to travel by three cars, each of which holds four. Find the number of ways in which the party may be divided if two people refuse to travel in the same car. [Ans: 252]
6. In how many ways can a party of five people be selected from six men and four women so that there are always more men than women in the party?
[Ans: 186]
7. There are 10 articles, 2 of which are alike and the rest all different. In how many ways can selection of 5 articles be made?
[Ans: 182]
8. We wish to select 6 persons from 8, but if the person A is chosen, then B must be chosen. In how many ways can selections be made?
[Ans: 22]
9. In an examination, a student has to answer 4 questions out of 5 questions; questions 1 and 2 are however compulsory. Determine the number of ways in which the student can make the choice.
[Ans: 3]
10. A bag contains six white marbles and five red marbles. Find the number of ways in which four marbles can be drawn from the bag if
(a) they can be of any colour
(b) two must be white and two red
(c) they must all be of the same colour
[Ans: (a) 330 (b) 25 (c) 20]
11. In how many ways can a football team of 11 players be selected from 16 players to
(a) include 2 particular players?

Chapter 11 Complex Numbers

Introduction

The number system that we are aware of today is the gradual development from natural numbers to integers, from integers to rational numbers and from rational numbers to real numbers.

If we consider the following polynomial equations (i) $x-1=0$, (ii) $x+1=1$, (iii) $x^2-3=0$, we see that all of them have solutions in the real number system. However, this real number system is not sufficient to solve equations of the form $x^2+9=0$ i.e. there does not exist any real number which satisfies $x^2=-9$. The mathematical need to have solutions for equations of the above form led us to extend the real number system to a new kind of system that allows the square root of negative numbers.

Let us consider solution of a simple quadratic equation $x^2+16=0$. Its solutions are $x=\pm 4\sqrt{-1}$. We assume that square root of -1 is denoted by the symbol i, called the imaginary unit. Thus for any two real numbers a and b, we can form a new number $a+ib$. This number $a+ib$ is called a complex number.

The complex number system

A complex number is of the form $a+ib$ where a and b are real numbers and i is called the imaginary unit, having the property that $i^2=-1$. If $z=a+ib$, then a is called the real part of z, denoted by $Re(z)$ and b is called the imaginary part of z and is denoted by $Im(z)$.

Some examples of complex numbers are $3-2i$, $\sqrt{2}+3i$

Note that 3 is the real part and -2 is the imaginary part and so on.

Two complex numbers $a+ib$ and $c+id$ are equal if and only if $a=c$ and $b=d$ i.e. the corresponding real parts are equal and the corresponding imaginary parts are equal.

The real numbers can be considered as a subset of the set of complex numbers with $b=0$. Hence the complex numbers $0+i0$ and $-2+i0$ represent the real numbers 0 and -2 respectively. If $a=0$ the complex number $0+ib$ or ib is called a **pure imaginary number**.

We further observe that higher powers (multiples of i) can be reduced to ± 1 or $\pm i$.

Since $i^2=-1$

$$i^3=i^2.i=-i$$
$$i^4=i^3.i=-i.i=-i^2=1$$
$$i^5=i^4.i=i$$

and so on

Example 1

Write the following as complex numbers

(a) $\sqrt{-35}$ (b) $3-\sqrt{-7}$

Solution

(a) $\sqrt{-35}=\sqrt{(-1)\times 35}=\sqrt{-1}\,.\sqrt{35}=i\sqrt{35}$

(b) $3-\sqrt{-7}=3-\sqrt{-1}\times\sqrt{7}=3-i\sqrt{7}$

Example 2

Show that $i^9+2i^{11}+i^{13}=0$

Solution

$$\begin{aligned} i^9+2i^{11}+i^{13} &= i^8.i+2i^{10}.i+i^{12}.i \\ &= (i^2)^4.i+2(i^2)^5.i+(i^2)^6.i \\ &= (-1)^4.i+2(-1)^5.i+(-1)^6 i \\ &= i-2i+i=0 \end{aligned}$$

Example 3

Simplify $(2+i)^4-(2-i)^4$

Solution

$$\begin{aligned} &(2+i)^4-(2-i)^4 \\ &= 2^4+4\,.2^3 i+6\,.2^2 i^2+4\,.2i^3+i^4 \\ &\quad -(2^4-4\,.2^3 i+6.\,2^2 i^2-4\,.2i^3+i^4) \\ &= 64i+16i^3 \\ &= 64i-16i \\ &= 48i \end{aligned}$$

Negative of a complex number

If $z=a+ib$ is a complex number then the negative of z is denoted by $-z$ and it is defined as $-z=-a+i(-b)$

Basic algebraic operations

Addition

$$(a+ib)+(c+id)=(a+c)+i(b+d)$$

Subtraction

$$(a+ib)-(c+id)=(a-c)+i(b-d)$$

To perform the operations with complex numbers we can proceed as in the algebra of real numbers replacing i^2 by -1 whenever it occurs.

Multiplication

$$\begin{aligned}(a+ib)(c+id) &= ac+iad+ibc+i^2bd \\ &= (ac-bd)+i(ad+bc)\end{aligned}$$

Conjugate of a complex number

If $z = a + ib$, then the conjugate of z is denoted by $\overline{z}$ or z^* and is defined by

$$\overline{z} = a - ib$$

Division

In simplifying the division of two complex numbers, we multiply the numerator and denominator by the conjugate of the denominator.

$$\frac{a+ib}{c+id} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id}$$

$$\frac{a+ib}{c+id} = \frac{ac+bd}{c^2+d^2} + i\left(\frac{bc-ad}{c^2+d^2}\right)$$

Operations with the conjugate

Addition:

$$z + z^* = (a+ib) + (a-ib) = 2a$$

Subtraction:

$$z - z^* = (a+ib) - (a-ib) = 2ib$$

Multiplication:

$$zz^* = (a+ib)(a-ib) = a^2 + b^2$$

Division:

$$\frac{z}{z^*} = \frac{a+ib}{a-ib} = \frac{(a+ib)(a+ib)}{(a-ib)(a+ib)}$$

$$= \left(\frac{x^2-y^2}{x^2+y^2}\right) + i\left(\frac{2xy}{x^2+y^2}\right)$$

Properties of the conjugate of a complex number

1. $z\bar{z} = (a+ib)(a-ib) = a^2 + b^2$ which is a non-negative real number.
2. Conjugate of $\overline{z}$ is z i.e. $\overline{\overline{z}} = z$
3. If z is real, i.e. $b = 0$ then $z = \bar{z}$

   Conversely, if $\overline{z} = z$, i.e. $a + ib = a - ib$, then $b = -b$ $\Rightarrow 2b = 0, \therefore b = 0 \Rightarrow z$ is real.

   Thus z is real $\Leftrightarrow$ the imaginary part is 0
4. Let $z = a + ib$, then $\overline{z} = a - ib$

   $$a = Re(z) = \frac{z+\bar{z}}{2}$$

   Similarly,

   $$b = Im(z) = \frac{z-\bar{z}}{2i}$$
5. The conjugate of the sum of two complex numbers z_1, z_2 is the sum of their conjugates i.e.

   $$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

   Proof:

   Let $z_1 = a + ib$ and $z_2 = c + id$, then

   $z_1 + z_2 = (a+ib) + (c+id) = (a+c) + i(b+d)$

   $\overline{z_1 + z_2} = (a+c) - i(b+d)$

   $\overline{z_1} = a - ib, z_2 = c - id$

   $\overline{z_1} + \overline{z_2} = (a-ib) + (c-id) = (a+c) - i(b+d)$

   $= \overline{z_1 + z_2}$

   Similarly, it can be proved that the conjugate of the difference of two complex numbers z_1, z_2 is the difference of their conjugates i.e.

   $$\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$$
6. The conjugate of the product of two complex numbers z_1, z_2 is the product of two complex numbers z_1, z_2 is the product of their conjugates i.e.

   $$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$$

   Proof:

   Let $z_1 = a + ib$ and $z_2 = c + id$, then

   $z_1 z_2 = (a+ib)(c+id) = (ac-bd) + i(ad+bc)$

   $\overline{z_1 z_2} = (ac-bd) - i(ad+bc)$

   $\overline{z_1} = a - ib, \overline{z_2} = c - id$

   $\overline{z_1}\,\overline{z_2} = (a-ib)(c-id) = (ac-bd) - i(ad+bc)$

   $= \overline{z_1 z_2}$
7. The conjugate of the quotient of two complex numbers z_1, z_2 $(z_2 \neq 0)$ is the quotient of their conjugates i.e.

   $$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

   The proof of this property is left for the readers to do it on their own.
8. $\overline{z^n} = (\overline{z})^n$

Example 4

Find the complex conjugate of (i) $2 + i\sqrt{7}$ (ii) $-4 - 9i$

<u>**Solution**</u>

By definition, the complex conjugate is obtained by reversing the sign of the imaginary part of the complex number. Hence the required conjugates are

(i) $2 - i\sqrt{7}$ (ii) $-4 + 9i$

Example 5

Express the following in the form $a + ib$

(a) $(3+2i) + (-7-i)$

(b) $(8-6i) - (2i-7)$

(c) $(2-3i)(4+2i)$

(d) $\frac{5+5i}{3-4i}$

<u>**Solution**</u>

(a) $(3+2i) + (-7-i) = 3 + 2i - 7 - i = -4 + i$

(b) $(8-6i) - (2i-7) = 8 - 6i - 2i + 7 = 15 - 8i$

(c) $(2-3i)(4+2i) = 8 + 4i - 12i - 6i^2 = 14 - 8i$

(d) $\frac{5+5i}{3-4i} = \frac{5+5i}{3-4i} \times \frac{3+4i}{3+4i} = \frac{15+20i+15i-20}{3^2+4^2}$

$$= \frac{-5+35i}{25} = -\frac{1}{5} + \frac{7}{5}i$$

Example 6

What is the conjugate of $\dfrac{\sqrt{5+12i} + \sqrt{5-12i}}{\sqrt{5+12i} - \sqrt{5-12i}}$?

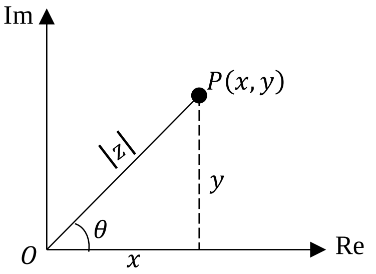
On the Argand diagram, each complex number is represented by a line of a certain length in a particular direction. Thus each complex number is shown as a vector on the Argand diagram.

Any complex number $z = x + iy$ may be represented on an Argand diagram by

either (a) the point $P(x, y)$

or (b) the position vector $\overline{OP}$

The modulus of z, $|z|$, is the length of OP. The argument of z, $\arg z$, is the angle θ between OP and positive real axis, where $-\pi < \theta \le \pi$

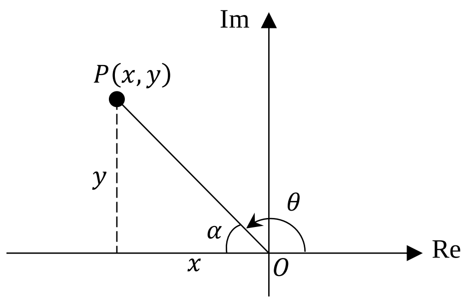


$$|z| = \sqrt{x^2 + y^2}$$

$$\arg z = \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

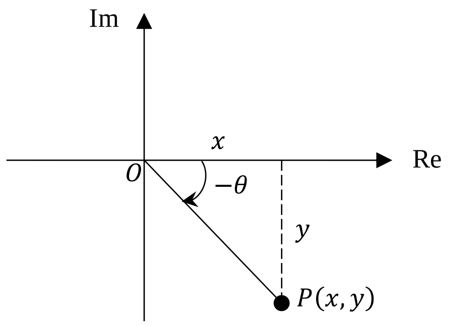
Note: When finding $\arg z$ illustrate the point on an Argand diagram to ensure the correct solution of $\tan^{-1}\left(\frac{y}{x}\right)$

For $z = -x + yi$



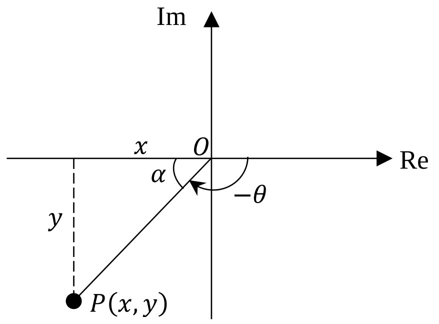
$$\arg z = \theta = \pi - \alpha = \pi - \tan^{-1}\frac{y}{x}$$

For $z = x - yi$



$$\arg z = -\theta = -\tan^{-1}\frac{y}{x}$$

For $z = -x - yi$



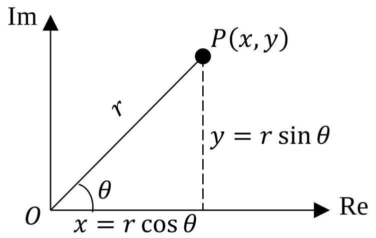
$$\arg z = -\theta = -(\pi - \alpha) = -\left(\pi - \tan^{-1}\frac{y}{x}\right)$$

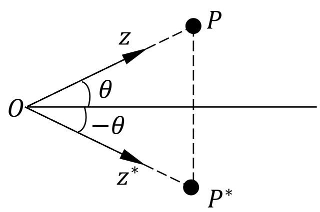
Polar form (modulus-argument form)

The polar form of a complex number is

$$z = r(\cos\theta + i\sin\theta)$$

where $r = OP$ and $x\hat{O}P$





$|z| = r$, where $r \ge 0$

$\arg z = \theta$, where $-\pi < \theta \le \pi$

$$z^* = r(\cos\theta - i\sin\theta) = r(\cos(-\theta) + i\sin(-\theta))$$

$|z^*| = r$ and $\arg z^* = -\theta$

Results:

1. For any two complex numbers z_1 and z_2
   (a) $|z_1 z_2| = |z_1||z_2|$
   (b) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

Proof:

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then $|z_1| = r_1$, $\arg z_1 = \theta_1$; $|z_2| = r_2$, $\arg z_2 = \theta_2$

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \\ &\quad + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \end{aligned}$$

$\therefore |z_1 z_2| = r_1 r_2 = |z_1||z_2|$ and

$$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$$

$$a^2 - b^2 = -5 \text{ ... (i)}$$
$$2ab = 12 \text{ ... (ii)}$$

Either find a and b by inspection

or $b = \frac{6}{a}$ from (ii)

In (i); $a^2 - \frac{36}{a^2} = -5$

$$a^4 + 5a^2 - 36 = 0$$
$$(a^2 + 9)(a^2 - 4) = 0$$

a is real $\Rightarrow a^2 \neq -9$, $a^2 = 4$, $a = \pm 2$

In (ii), when $a = 2$, $b = 3$, when $a = -2$, $b = -3$

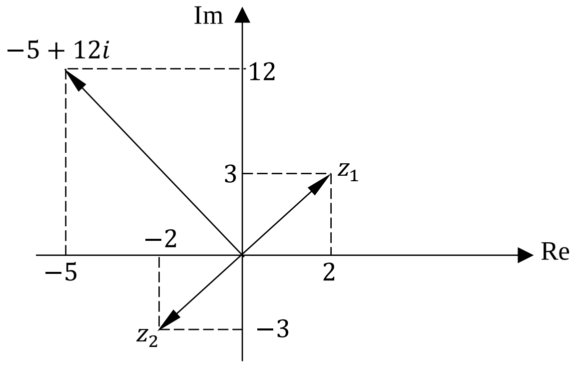
$$z_1 = 2 + 3i, z_2 = -2 - 3i$$

(b) $|-5 + 12i| = \sqrt{5^2 + 12^2} = 13$

$|z_1| = \sqrt{2^2 + 3^2} = \sqrt{13}$

$|z_2| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13}$

$$\arg(-5 + 2i) = \pi - \tan^{-1}\frac{12}{5} = 1.966 \text{ radians}$$
$$\arg(2 + 3i) = \tan^{-1}\frac{3}{2} = 0.983 \text{ radians}$$
$$\arg(-2 - 3i) = -\left(\pi - \tan^{-1}\frac{3}{2}\right) = -2.159 \text{ radians}$$



Example 19

Find the modulus and argument of $z_1 = 1 + i$, and $z_2 = \sqrt{3} - i$. Hence, or otherwise, write down $|z_1^{14}|$, $|z_2^3|$, $|z_1 z_2|$ and $\arg\left(\frac{z_1}{z_2}\right)$

<u>Solution</u>

$|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\arg z = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$\left|\sqrt{3} - i\right| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$

$\arg z_2 = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

$|z_1^{14}| = |z_1|^{14} = \left(\sqrt{2}\right)^{14} = 128$

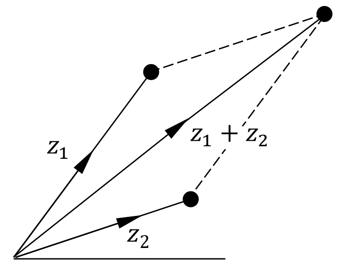
$|z_2^3| = |z_2|^3 = 2^3 = 8$

$|z_1 z_2| = |z_1||z_2| = 2\sqrt{2}$

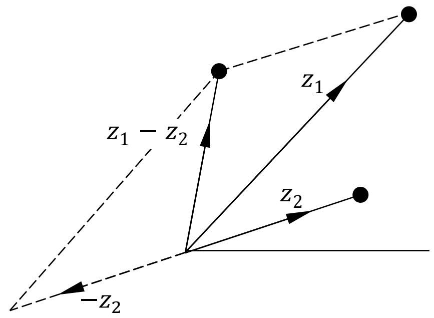
$$\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2 = \frac{\pi}{4} - \left(-\frac{\pi}{6}\right) = \frac{5\pi}{12}$$

Geometric representation of operations

Addition: $z_1 + z_2$



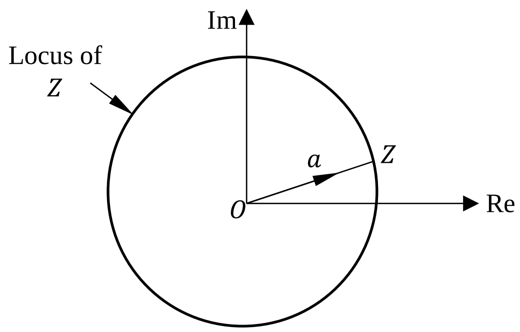
Subtraction: $z_1 - z_2$



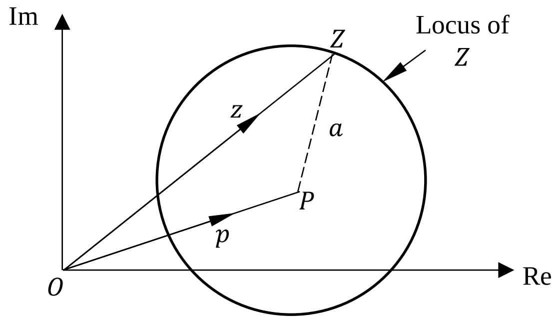
Locus of a complex number

If z is a **variable** complex number, represented by the position vector $\overrightarrow{OZ}$, then the **locus** of Z under certain conditions can be sketched. Some of the common loci are illustrated below.

The locus of Z when $|z| = a$ is a circle, centre O radius a



The locus of Z when $|z - p| = a$, where p is a fixed complex number, is a circle, centre P, radius a



Solutions of polynomial equations

Consider the equation $x^2 - 4x + 7 = 0$

The discriminant is $b^2 - 4ac = (-4)^2 - 4(7)(1) = -12$ which is negative.

Thus, the roots of this quadratic equation are not real. The roots are given by

$$\frac{4 \pm \sqrt{-12}}{2} = \frac{4 \pm 2i\sqrt{3}}{2} = 2 \pm i\sqrt{3}$$

Thus we see that the roots $2 + i\sqrt{3}$ and $2 - i\sqrt{3}$ are conjugate to each other.

Cube roots of unity

Let x be the cube root of unity, then $x = (1)^{\frac{1}{3}}$

$$x^3 = 1$$
$$x^3 - 1 = 0$$

Now we can simplify $x^3 - 1$ in two ways i.e. using the binomial expansion or using long division.

(a) using long division

Since $x = 1$ is the root of the equation, $x - 1$ is a factor

$$\begin{array}{rl} & x^2 + x + 1 \\ \hline x - 1 \,) & x^3 - 1 \\ - & x^3 - x^2 \\ \hline & x^2 - 1 \\ - & x^2 - x \\ \hline & x - 1 \\ - & x - 1 \\ \hline & - \quad - \end{array}$$

$$\therefore (x^3 - 1) = (x - 1)(x^2 + x + 1)$$

(b) Using the binomial expansion

$$(x - 1)^3 = x^3 - 3x^2 + 3x - 1$$
$$(x - 1)^3 = x^3 - 1 - 3x(x - 1)$$
$$x^3 - 1 = (x - 1)^3 + 3x(x - 1)$$
$$x^3 - 1 = (x - 1)[(x - 1)^2 + 3x]$$
$$x^3 - 1 = (x - 1)[x^2 - 2x + 1 + 3x]$$
$$\therefore (x^3 - 1) = (x - 1)(x^2 + x + 1)$$

Now we solve

$$(x - 1)(x^2 + x + 1) = 0$$
$$x - 1 = 0 \text{ or } x^2 + x + 1 = 0$$

Hence $x = 1$and $x = \frac{-1 \pm \sqrt{1-(4)(1)(1)}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$

$\therefore$ cube roots of unity are 1, $\frac{-1+\sqrt{3}i}{2}$, $\frac{-1-\sqrt{3}i}{2}$

Here again, the two complex roots $\frac{1}{2}\left(-1 + \sqrt{3}i\right)$ and $\frac{1}{2}\left(-1 - \sqrt{3}i\right)$ are conjugate to each other.

From the above two examples one can infer that in an equation with real coefficients, imaginary roots occur in pairs (i.e. one is the conjugate of the other). This paved way for the following theorem.

If the complex number $p + qi$ is a root of a polynomial equation with real coefficients, then its conjugate, $p - qi$, is also a root.

If ω is used to denote the complex root $\frac{1}{2}\left(-1 + \sqrt{3}i\right)$, then

$$\omega^2 = \frac{1}{4}\left(-1 + \sqrt{3}i\right)^2 = \frac{1}{4}\left(1 - 2\sqrt{3}i - 3\right)$$
$$= \frac{1}{2}\left(-1 - \sqrt{3}i\right)$$

and this is the second complex cube root of unity.

Hence we can write the three cube roots of unity in the form

$$1, \omega, \omega^2 \text{ where } \omega = \frac{1}{2}\left(-1 + \sqrt{3}i\right)$$

It should be noted that (by definition) $\omega^3 = 1$, that $\omega^4 = \omega \times \omega^3 = \omega$, $\omega^5 = \omega^2 \times \omega^3 = \omega^2$, etc.

and that

$$1 + \omega + \omega^2 = 1 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 0$$
$$1 + \omega + \omega^2 = 0$$

These relations are often useful in working out some problems.

Example 26

If ω is one of the complex cube roots of unity, show that $(1 + \omega^2) = \omega$

<u>Solution</u>

From $1 + \omega + \omega^2 = 0$

$$1 + \omega^2 = -\omega$$
$$(1 + \omega^2)^4 = (-\omega)^4 = \omega^4 = \omega^3 \times \omega = 1 \times \omega = 1$$

Example 27

If ω is a complex cube root of unity, form the quadratic equation whose roots are ω and $1/\omega$.

<u>Solution</u>

$$\text{Product of roots} = \omega \times \frac{1}{\omega} = 1$$
$$\text{Sum of roots} = \omega + \frac{1}{\omega} = \frac{\omega^2 + 1}{\omega} = \frac{-\omega}{\omega} = -1$$
$$x^2 - (\text{sum})x + (\text{product}) = 0$$

Hence the required quadratic equation is

$$x^2 + x + 1 = 0$$

Example 28

If ω is a complex cube root of unity and if $x = a + b$, $y = a\omega + b\omega^2$, $z = a\omega^2 + b\omega^4$, show that $x^2 + y^2 + z^2 = 6ab$

<u>Solution</u>

$x^2 = (a + b)^2 = a^2 + 2ab + b^2$

$y^2 = (a\omega + b\omega^2)^2 = a^2\omega^2 + 2ab\omega^3 + b^2\omega^4$

$\qquad = a^2\omega^2 + 2ab + b^2\omega$

$z^2 = (a\omega^2 + b\omega^4)^2 = a^2\omega^4 + 2ab\omega^6 + b^2\omega^8$

Since the remainder is zero, then $1 + i$ is a root

$$z^4 + 3z^2 - 6z + 10 = (z^2 + 2z + 5)(z^2 - 2z + 2) = 0$$

$$z^2 + 2z + 5 = 0$$

Completing the square

$$z^2 + 2z + 1 = -4$$

$$(z + 1)^2 = -4$$

$$z + 1 = \pm\sqrt{-4}$$

$$z + 1 = \pm 2i$$

$$z = -1 \pm 2i$$

$\therefore$ The other roots are $1 - i, -1 + 2i, -1 - 2i$

De Moivre's theorem and its applications

When n is a positive integer, De Moivre's theorem states that

$$\boldsymbol{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$$

This can be proved by mathematical induction as follows;

For $n = 1$,

L.H.S $= (\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$

R.H.S $= \cos\theta + i\sin\theta$

Since L.H.S = R.H.S, it is true for $n = 1$

Assume that the statement is true for $n = k$

$$\Rightarrow (\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$$

Now for $n = k + 1$,

$$\begin{aligned}(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta) \\ &= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta) \\ &= \cos k\theta\cos\theta - \sin k\theta\sin\theta \\ &\quad + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta) \\ &= \cos(k+1)\theta + i\sin(k+1)\theta\end{aligned}$$

If then the theorem is true when $n = k$, it is also true when $n = k + 1$, hence by mathematical induction

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Negative integer

Let m be a negative integer and equal to $-n$ (n is a positive integer)

$$\begin{aligned}(\cos\theta + i\sin\theta)^m &= (\cos\theta + i\sin\theta)^{-n} \\ &= \frac{1}{(\cos\theta + i\sin\theta)^n} \\ &= \frac{1}{\cos n\theta + i\sin n\theta} \text{ (By theorem)} \\ &= \frac{\cos n\theta - i\sin n\theta}{(\cos n\theta + i\sin n\theta)(\cos n\theta + i\sin n\theta)} \\ &= \frac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta + \sin^2 n\theta} \\ &= \cos n\theta - i\sin n\theta \\ &= \cos(-n)\theta + i\sin(-n)\theta \\ &= \cos m\theta + i\sin m\theta\end{aligned}$$

$$(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$$

This shows that De Moivre's theorem is also valid when n is a negative integer.

Fraction (irrational integer)

Let n be a fraction and equal to $\frac{p}{q}$, where q is a positive integer and p is any integer.

Consider $\left[\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right]^q = \cos\theta + i\sin\theta$

Therefore $\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$ is such that its qth power is $\cos\theta + i\sin\theta$.

Hence $\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$ is one of the values of $(\cos\theta + i\sin\theta)^{\frac{1}{q}}$

Raise each of these quantities to the pth power.

$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^p$ is one of the values of $\left[(\cos\theta + i\sin\theta)^{\frac{1}{q}}\right]^p$

i.e. $\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$ is one of the values of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$.

Properties

1. $(\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta)$
   $= \cos n\theta - i\sin n\theta$
2. $(\cos\theta - i\sin\theta)^n = [\cos(-\theta) + i\sin(-\theta)]^n$
   $= \cos(-n\theta) + i\sin(-n\theta)$
   $= \cos n\theta - i\sin n\theta$
3. $(\sin\theta + i\cos\theta)^n = \left[\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right]^n$
   $= \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right)$

Example 34

Simplify

$$\frac{(\cos 2\theta + i\sin 2\theta)^3(\cos 3\theta - i\sin 3\theta)^{-3}}{(\cos 4\theta + i\sin 4\theta)^{-6}(\cos\theta + i\sin\theta)^8}$$

Solution

The given expression

$$\begin{aligned}&= \frac{(\cos\theta + i\sin\theta)^6(\cos\theta + i\sin\theta)^9}{(\cos\theta + i\sin\theta)^{-24}(\cos\theta + i\sin\theta)^8} \\ &= (\cos\theta + i\sin\theta)^{6+9+24-8} \\ &= (\cos\theta + i\sin\theta)^{31} \\ &= \cos 31\theta + i\sin 31\theta\end{aligned}$$

Example 35

Simplify: $\frac{(\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^5}$

Solution

$$\sin\theta + i\cos\theta = i(-i\sin\theta + \cos\theta) = i(\cos\theta - i\sin\theta)$$

$$\frac{(\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^5} = \left[\frac{(\cos\theta + i\sin\theta)^4}{i^5(\cos\theta - i\sin\theta)^5}\right]$$

$$32i\sin^5\theta = z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$$
$$32i\sin^5\theta = \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$
$$32i\sin^5\theta = (2i\sin 5\theta) - 5(2i\sin 3\theta) + 10(2i\sin\theta)$$
$$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$$

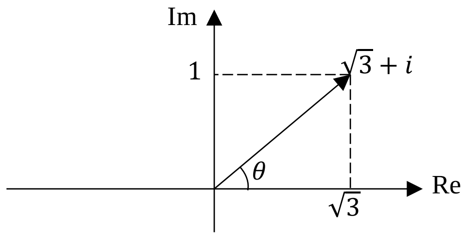
Example 40

Find $\left(\sqrt{3}+i\right)^5$ using De Moivre's theorem.

<u>Solution</u>

Let $z = \sqrt{3} + i = r(\cos\theta + i\sin\theta)$

$$r = \left|\sqrt{3}+i\right| = \sqrt{3+1} = 2$$



$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

Therefore, we have $z = \sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$$\left(\sqrt{3}+i\right)^5 = 2^5\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^5$$

Using De Moivre's theorem

$$\left(\sqrt{3}+i\right)^5 = 32\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$
$$= 32\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$
$$= -16\sqrt{3} + 16i$$

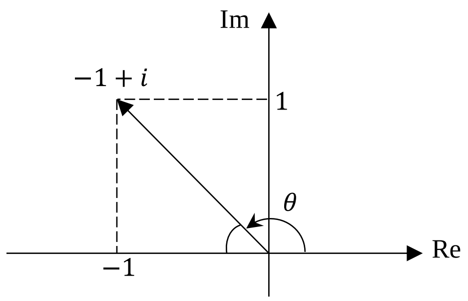
Example 41

Find $(-1+i)^{-4}$ using De Moivre's theorem

<u>Solution</u>

Let $z = -1 + i = r(\cos\theta + i\sin\theta)$

$$r = |-1+i| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$



and $\theta = \pi - \tan^{-1}\left(\frac{1}{1}\right) = \frac{3\pi}{4}$

Thus $z = -1 + i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$$(-1+i)^{-4} = \left(\sqrt{2}\right)^{-4}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]^{-4}$$

Using De Moivre's theorem

$$(-1+i)^{-4} = \frac{1}{\left(\sqrt{2}\right)^4}[\cos(-3\pi) + i\sin(-3\pi)]$$
$$= \frac{1}{4}[-1 + 0i]$$
$$= -\frac{1}{4}$$

Roots of a complex number

A number ω is called an nth root of a complex number z, if $\omega^n = z$ and we write $\omega = z^{\frac{1}{n}}$

Working rule to find the nth roots of a complex number

1. Write the given number in polar form
2. Add $2k\pi$ to the argument
3. Apply De Moivre's theorem (bring the power to inside)
4. Put $k = 0, 1, \ldots, n-1$

Illustration

Let $z = r(\cos\theta + i\sin\theta)$

$$= r[\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)]$$
$$z^{\frac{1}{n}} = \left[r(\cos(2k\pi + \theta) + i\sin(2k\pi + \theta))\right]^{\frac{1}{n}}$$
$$= r^{\frac{1}{n}}\left[\cos\left(\frac{2k\pi + \theta}{n}\right) + i\sin\left(\frac{2k\pi + \theta}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots, (n-1)$

Only these values of k will give n different values of $z^{\frac{1}{n}}$ provided $z \neq 0$

Note:

1. The number of nth roots of a non-zero complex number is n
2. The moduli of these roots are the same non-negative real numbers.
3. The argument of these n roots are equally spaced. That is if θ is the principal value of arg z i.e. $-\pi \leq \theta \leq \pi$ then the arguments of other roots of $z$are obtained by adding respectively $\frac{2\pi}{n}, \frac{4\pi}{n}, \ldots, \frac{\theta}{n}$
4. If k is given integral values greater than or equal to n, these n values are repeated and no fresh root is obtained.

The nth roots of unity

$$1 = (\cos 0 + i\sin 0) = \cos 2k\pi + i\sin 2k\pi$$

nth roots of unity $= 1^{\frac{1}{n}} = (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{n}}$

$$= \left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right)$$

where $k = 0, 1, 2, \ldots, n-1$

$\therefore$ The nth roots of unity are $\cos 0 + i\sin 0$, $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, $\cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}$, ……..…., $\cos(n-1)\frac{2\pi}{n} + i\sin(n-1)\frac{2\pi}{n}$

[Ans: (a) $1+3i$ (b) $-i$ (c) $-10+10i$ (d) 1]

3. Find the real values of x and y for which the following equations are satisfied.
   (a) $(1-i)x+(1+i)y=1-3i$
   (b) $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$
   (c) $\sqrt{x^2+3x+8}+(x+4)i=y(2+i)$

   [Ans: (a) $x=2, y=-1$ (b) $x=3, y=-1$ (c) $x=-7$, $y=-3$ and $x=-\frac{8}{3}, y=\frac{4}{3}$]
4. For what values of x and y are the numbers $-3+ix^2y$ and x^2+y^2+4i complex conjugate of each other?

   [Ans: $x=\pm1, y=-4$ and $x=\pm2i, y=1$]
5. Find the modulus and argument of $\frac{1+2i}{3+4i}$

   [Ans: $1/\sqrt{5}$, 0.1798]
6. Find the modulus and argument of $z_1=\frac{2-i}{3i-1}$, $z_2=\frac{i-3}{2+i}$ and of z_1+z_2

   [Ans: (i) $1/\sqrt{2}, \frac{5\pi}{4}$ (ii) $\sqrt{2}, \frac{3\pi}{4}$ (iii) $\frac{1}{2}\sqrt{10}$, 2.82]
7. Given that $f(z)=\frac{7-z}{1-z^2}$ where $z=1+2i$, show that $|z|=2|f(z)|$
8. Find the real and imaginary parts of the complex number z when $\frac{z}{z+1}=1+2i$

   [Ans: $-1, \frac{1}{2}$]
9. If $z=x+yi$ and $\overline{z}$ is the conjugate of z, find the values of x and y such that
$$\frac{1}{z}+\frac{2}{\overline{z}}=1+i$$
   [Ans: $\frac{3}{10}, \frac{9}{10}$]
10. Express in the form $a+ib$
    (a) $\frac{3-i}{2+i}$
    (b) $\frac{4+3i}{2-i}$
    (c) $\left(\frac{1+i}{1-i}\right)^2$

    [Ans: (a) $1-i$ (b) $1+2i$ (c) $-1+10i$]
11. If $(x+yi)^2=a+bi$, show that $x^2-y^2=a$, $2xy=b$. Hence evaluate $\sqrt{8+6i}$

    [Ans: $\pm(3+i)$]
12. Find two real numbers x and y so that
$$x(3+4i)-y(1+2i)+5=0$$
    [Ans: $x=-5, y=-10$]
13. If $(x+yi)^3=a+bi$, show that
$$a^2+b^2=(x^2+y^2)^3$$
14. If $z=x+yi$, show that $|z|^2=z\overline{z}$. Show that $\left|\frac{1}{z}\right|=\frac{1}{|z|}$
15. If $z=\bar{z}$, find the locus of the point represented by z

    [Ans: $y=0$]
16. If $|z-2|=|z+2|$, find the locus of the point represented by z

    [Ans: $x=0$]
17. If $|z-3i|=2|z-3|$, find the locus of the point represented by z.

    [Ans: $x^2+y^2-8x+2y+9=0$]
18. Find the modulus of the complex number
$$\frac{(2-3i)(3+4i)}{(6+4i)(16-8i)}$$
    [Ans: $\frac{5}{34}$]
19. P represents the variable complex number z. Find the locus of P, if
    (a) $Im\left[\frac{2z+1}{iz+1}\right]=-2$
    (b) $Re\left(\frac{z-1}{z+i}\right)=1$
    (c) $|2z-3|=2$
    (d) $\arg\left(\frac{z-1}{z+3}\right)=\frac{\pi}{2}$

    [Ans: (a) $x+2y=2$ (b) $x+y+1=0$ (c) $4x^2+4y^2-12x+5=0$ (d) $x^2+y^2+2x-3=0$]
20. Given that $z=\sqrt{3}+i$, find the modulus and argument of
    (a) z^2, (b) $\frac{1}{z}$

    Show in an Argand diagram the points representing the complex numbers z, z^2 and $\frac{1}{z}$

    [Ans: (a) $4\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ (b) $\frac{1}{2}\left[\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right]$]
21. Show that if a, b, c, d and $\frac{a+bi}{c+di}$ are real, then $ad=bc$. Hence show that if $z=x+yi$ and $\frac{z^2+2z}{z^2+4}$ is real, the point represented by z lies on the real axis or on a certain circle.

    [Ans: Circle has equation $x^2+y^2-4x-4=0$]
22. Use De Moivre's theorem to show that
$$\frac{(\cos 3\theta+i\sin 3\theta)^5(\cos\theta-i\sin\theta)^3}{(\cos 5\theta+i\sin 5\theta)^7(\cos 2\theta-i\sin 2\theta)^5}=\cos 13\theta-i\sin 13\theta$$
23. Use De Moivre's theorem to show that
$$\cos 4\theta=\cos^4\theta-6\cos^2\theta+\sin^4\theta$$
$$\sin 4\theta=4\cos^3\theta\sin\theta-4\cos\theta\sin^3\theta$$
24. One root of the equation $x^2-\lambda x-\mu=0$ is $2-i$. Find λ and μ.

    [Ans: $\lambda=4, \mu=-5$]
25. Given that z^* is the conjugate of z and $z=a+bi$ where a and b are real, find the possible values of z if $zz^*-2iz=7-4i$

    [Ans: $2+i$, $2-3i$]
26. If $1-\sqrt{3}i$ is a root of $2x^3-5x^2+10x-4=0$, find the other roots.

$$z^3 - 11z + 20 = 0$$
find the remaining roots.

[Ans: $2 - i, -4$]

51. Show that $1 + i$ is a root of the equation $x^4 + 3x^2 - 6x + 10 = 0$. Hence write down one quadratic factor of $x^4 + 3x^2 - 6x + 10$, and find all the other roots of the equation.

[Ans: $x^2 - 2x + 2, 1 \pm i, -1 \pm 2i$]

52. By using De Moivre's theorem, or otherwise, find the roots of the equation $z^4 + 4 = 0$.
Hence, or otherwise, express $z^4 + 4$ as a product of two quadratic polynomials in z with real coefficients.

[Ans: $\pm(1 + i), \pm(1 - i), (x^2 - 2x + 2)(x^2 + 2x + 2)$]

53. If z_1 and z_2 are complex numbers, solve the simultaneous equations $4z_1 + 3z_2 = 23$, $z_1 + iz_2 = 6 + 8i$, giving both answers in the form $x + yi$.

[Ans: $2 + 3i, 5 - 4i$]

54. Given that $z_1 = 3 + 2i$ and $z_2 = 4 - 3i$,
(i) find z_1z_2 and $\frac{z_1}{z_2}$, each in the form $a + ib$;
(ii) verify that $|z_1z_2| = |z_1||z_2|$

[Ans: (i) $18 - i, \frac{6}{25} + \frac{17}{25}i$]

55. Solve the equation $z^3 = 8$
(a) by finding one root by inspection and hence solving $z^3 - 8 = 0$ by an algebraic method,
(b) by expressing z in modulus-argument form and then using De Moivre's theorem.
Illustrate the roots on an Argand diagram

[Ans: $2, -1 \pm \sqrt{3}i$]

56. If $x = a + b$, $y = a\omega + b\omega^2$, $z = a\omega^2 + b\omega$, show that
(a) $xyz = a^3 + b^3$
(b) $x^3 + y^3 + z^3 = 3(a^3 + b^3)$
where ω is the complex cube root of unity.

57. Prove that if $\omega^3 = 1$, then
$$\frac{1}{1 + 2\omega} - \frac{1}{1 + \omega} + \frac{1}{2 + \omega} = 0$$

58. Find all the values of the following
(a) $(i)^{\frac{1}{3}}$
(b) $(8i)^{\frac{1}{3}}$
(c) $\left(-\sqrt{3} - i\right)^{\frac{2}{3}}$

[Ans: (a) $\frac{\sqrt{3}}{2} + \frac{1}{2}i, -\frac{\sqrt{3}}{2} + \frac{1}{2}i, -i$ (b) $\sqrt{3} + i, -\sqrt{3} + i, -2i$
(c) $-0.276 - 1.563i, 1.492 + 0.543i, -1.216 + 1.020i$]

59. Use De Moivre's theorem to show that
$$\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

60. Given that $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ for a positive value of n show that it is also true for n a negative integer.

61. If $a + ib$ is a root of the quadratic equation $x^2 + cx + d = 0$, show that $a^2 + b^2 = d$ and $2a + c = 0$.

62. Prove that the non-real cube roots of unity are $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

63. Show that for any complex numbers $z = z + iy$ and $w = a + bi$,
(a) $(z + w)^* = z^* + w^*$
(b) $(zw)^* = z^*w^*$
(c) $\left(\frac{z}{w}\right)^* = \frac{z^*}{w^*}$
(d) $(z - w)^* = z^* - w^*$
(e) $(z^2)^* = (z^*)^2$
(f) $(z^*)^* = z$

64. Verify that $z = -1 + \sqrt{3}i$ is a root of the equation $z^4 - 4z^2 - 16z - 16 = 0$ and hence find the other roots.

[Ans: $-1 - i\sqrt{3}, 1 \pm \sqrt{5}$]

65. If $z^n + z^{-n} = 2\cos(n\theta)$ show that $5z^4 - z^3 - 6z^2 - z + 5 = 0 \Rightarrow 10\cos^2\theta - \cos\theta - 8 = 0$.

66. Show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$, and hence show that the roots of $x(16x^4 - 20x^2 + 5) = 0$ are $0, \cos\frac{\pi}{10}, \cos\frac{3\pi}{10}, \cos\frac{7\pi}{10}, \cos\frac{9\pi}{10}$

67. Given that ω is a complex root of the equation $z^5 - 1 = 0$ and is such that it has the smallest positive argument, show that ω^2, ω^3 and ω^4 are the other complex roots.
(a) Hence show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$
(b) Factorise $z^5 - 1$ into real linear and quadratic factors. Hence deduce that
(i) $2\left(\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right)\right) = -1$
(ii) $4\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{4\pi}{5}\right) = -1$

68. Simplify
$$\frac{(\cos\theta + i\sin\theta)^9(\cos 3\theta + i\sin 3\theta)^{-5}}{(\cos(-2\theta) + i\sin(-2\theta))^4}$$

[Ans: $\cos 2\theta + i\sin 2\theta$]

69. Given that $z_1 = \frac{-2+i}{1-3i}$ and $z_2 = \frac{-3+i}{2+i}$, find
(a) $\arg\left(\frac{z_1}{z_2}\right)$
(b) $\left|\frac{z_1}{z_2}\right|$

[Ans: (a) $\frac{\pi}{2}$ (b) $\frac{1}{2}$]

70. Show that $z^n + z^{-n} = 2\cos n\theta$, where z is a complex number. Hence solve the equation $3z^4 + 2z^3 + z^2 + 2z + 3 = 0$

71. (a) If $(3 + i)z = 4(2 - i)$, express z and z^2 in the form $x + iy$ where x and y are real.
(a) If $a = \frac{3-i}{2+i}$, $b = \frac{7+i}{2-3i}$, find the complex number $\frac{a+b}{a-b}$
(b) Find all complex numbers z, such that $z^4 + 3z^3 - 2z^2 + 3z + 1 = 0$

Chapter 12 Examination Questions

SECTION A

1. A committee of seven people is to be selected from 4 men and 6 women. If the committee must have **at least** two men, determine the total possible number of ways of selecting the committee.
[2024, No. 1]

2. The population of a country increases in a geometric progression (G.P) by 2.75% per annum. Calculate the number of years it will take for the population to double.
[2024, No. 5]

3. Solve the inequality $\frac{7-2x}{(x+1)(x-2)} > 0$
[2023, No. 5]

4. Prove by induction that $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$
[2023, No. 1]

5. Solve the simultaneous equations
$$2\log_{10} y = \log_{10} 2 + \log_{10} x$$
$$2^y = 4^x$$
[2022, No. 1]

6. Solve the inequality $\frac{5-4x}{1-x} < 3$
[2022, No. 5]

7. (a) Express $Z = \frac{3+i}{1-i}$ in the form $a + bi$, where a and b are integers
(b) Find the argument of Z
[2020, No. 2]

8. In how many ways can the letters of the word BUNDESLIGA be arranged if;
(a) there is no restriction?
(b) the vowels must be together?
[2020, No. 6]

9. Show that the modulus of $\frac{(1-i)^6}{1+i} = 4\sqrt{2}$
[2019, No. 1]

10. Given that $\alpha + \beta = \frac{-1}{3}$ and $\alpha\beta = \frac{2}{3}$, form a quadratic equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.
[2019, No. 6]

11. Express the function $f(x) = x^2 + 12x + 32$, in the form $a(x+b)^2 + c$.
Hence find the minimum value of the function $f(x)$.
[2018, No. 4: Ans: -4]

12. Show that $2\log 4 + \frac{1}{2}\log 25 - \log 20 = 2\log 2$
[2018, No. 7]

13. The coefficients of the first three terms of the expansion of $\left(1 + \frac{x}{2}\right)^n$ are in an Arithmetic progression (AP). Find the value of n. [2017, No. 1: Ans: 8]

14. Solve for x in the equation $4^{2x} - 4^{x+1} + 4 = 0$
[2017, No. 4: Ans: $\frac{1}{2}$]

15. Without using mathematical tables or a calculator, find the value of
$$\frac{\left(\sqrt{5}+2\right)^2 - \left(\sqrt{5}-2\right)^2}{8\sqrt{5}}$$
[2016, No. 1: Ans: 1]

16. Given that $2x^2 + 7x - 4$, $x^2 + 3x - 4$ and $7x^2 + ax - 8$ have a common factor, find the:
(a) factors of $2x^2 + 7x - 4$ and $x^2 + 3x - 4$
(b) value of a in $7x^2 + ax - 8$
[2016, No. 5: Ans: $a = 26$]

17. The first term of an Arithmetic Progression (A.P) is equal to the first term of a Geometric Progression (G.P) whose common ratio is $1/3$ and the sum to infinity is 9. If the common difference of the A.P is 2, find the sum of the first ten terms of the A.P.
[2015, No. 1: Ans: 150]

18. Solve for x in:
$$\log_a(x+3) + \frac{1}{\log_x a} = 2\log_a 2$$
[2015, No. 3: Ans: $x = 1$]

19. Solve the simultaneous equations:
$$x - 2y - 2z = 0$$
$$2x + 3y + z = 1$$
$$3x - y - 3z = 3$$
[2014, No. 1: Ans: $x = -2, y = 3, z = -4$]

20. Solve the equation $\sqrt{2x+3} - \sqrt{x+1} = \sqrt{x-2}$
Verify your answer.
[2014, No. 6: Ans: $x = 3$]

21. Solve $\log_x 5 + 4\log_5 x = 4$
[2013, No. Ans: $\sqrt{5}$]

22. In a Geometric Progression (G.P), the difference between the fifth and the second term is 156. The difference between the seventh and the fourth term is 1404. Find the possible values of the common ratio.
[2013, No. 2: Ans: $r = \pm 3$]

23. Solve the simultaneous equations
$$3x - y + z = 3,$$
$$x - 2y + 4z = 3,$$
$$2x + 3y - z = 4.$$
[2012, No. 1: Ans: $x = 1, y = 1, z = 1$]

24. The sum of the first n terms of a Geometric Progression (G.P) is $\frac{4}{3}(4^n - 1)$. Find its n^{th} term as an integral power of 2. [2012, No. 5: Ans: 2^{2n}]

52. By row reducing the appropriate matrix to echelon form, solve the system of equations
$$x + 2y - 2z = 0$$
$$2x + y - 4z = -1$$
$$4x - 3y + z = 11$$
[Nov 1998, No. 4: Ans: $x = 3$, $y = 1$, $z = 2$]

53. Solve simultaneously
$$\frac{1}{x} - \frac{1}{y} = \frac{1}{6}$$
$$x(5 - x) = 2y$$
[Mar 1998, No. 1: Ans: $(x, y) = (9, -18)$ and $(2, 3)$]

54. Prove that $\log_6 x = \frac{\log_3 x}{1+\log_3 2}$. Hence given that $\log_3 2 = 0.631$, find without using tables or calculator $\log_6 4$ correct to 3 significant figures.
[Mar 1998, No. 2: Ans: 0.774]

55. Find the values of k for which the equation $\frac{x^2-x+1}{x-1}$ has repeated roots. What are the repeated roots?
[1997, No. 2: Ans: $k = -1$ or 3, repeated roots 0 and 2]

56. By reducing to echelon form, solve the simultaneous equations
$$x + y + z = 0$$
$$x + 2y + 2z = 2$$
$$2x + y + 3z = 4$$
[1997, No. 3: Ans: $x = -2$, $y = -1$, $z = 3$]

57. Solve $3(3^{2x}) + 2(3^x) - 1 = 0$
[1996, No. 1: Ans: -1]

58. Express as equivalent fraction with a rational denominator
$$\frac{\sqrt{2}}{\sqrt{2} + \sqrt{3} - \sqrt{5}}$$
[1996, No. 2: Ans: $\frac{3+\sqrt{6}+\sqrt{15}}{6}$]

59. Solve the inequality $\frac{x-1}{x-2} > \frac{x-2}{x+3}$
[1996, No. 3: Ans: $-3 < x < \frac{7}{6}$ or $x > 2$]

60. Find how many terms of the series $1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots$ must be taken so that the sum will differ from the sum to infinity by less than 10^{-6}
[1996, No. 4: Ans: 9]

61. Solve the simultaneous equations
$$2x - 5y + 2z = 14$$
$$9x + 3y - 4z = 13$$
$$7x + 3y - 2z = 3$$
[1996, No. 5: Ans: $x = 1$, $y = -4$, $z = -4$]

SECTION B

1. (a) Given that the polynomial $x^3 - 13x + p$ is exactly divisible by $x - 4$, find the value of p.
Hence solve the equation $x^3 - 13x + p = 0$.
(b) Solve the inequality
$$\frac{x^2 - x - 8}{x + 3} \geq \frac{x}{2}$$
[2024, No. 10]

2. (a) The point C n the complex plane corresponds to the complex number z such that $3|z - 2| = |z - 6i|$. Show that the locus of C is a circle.
(a) Find the square root of $-5 + 12i$
[2014, No. 14]

3. (a) Four different Mathematics books and six other different books are to be arranged on a shelf. In how many ways can the Mathematics books be arranged on the shelf?
(b) On a certain day, Fatuma drunk 6 bottles of the 9 bottles of soda available. On the next day she drunk 5 bottles of the 7 bottles of soda available. In how many ways could she have chosen the bottles of soda to drink in the two days.
(c) Given that ${}^{20}C_r = {}^{20}C_{r-2}$, find the value of r.
[2023, No. 12]

4. (a) Solve the equation $z^3 - 7Z^2 + 19Z - 13 = 0$
(b) Find the fourth roots of $8\left(-\sqrt{3} + i\right)$
[2023, No. 9]

5. Expand $\left(\frac{1+3x}{1-x}\right)^{\frac{1}{2}}$ up to the term in x^3
Hence substitute $x = \frac{1}{5}$ to evaluate $\sqrt{8}$ correct to two decimal places.
[2022, No. 12]

6. (a) Given the geometric progression (G.P.) 2, 6, 18, 54, … find the sum of the first ten terms of the G.P.
(b) In an arithmetic progression (A.P), the sum of the fifth and sixteenth terms is 44. The sum of the first 18 terms is three times the sum of the first ten terms. Determine the
 (i) value of the first term
 (ii) common difference of the A.P.
 (iii) sum of the first 30 terms of the A.P
[2022, No. 9]

7. (a) A polynomial $P(x)$ is given by $P(x) = (x + 2)(x - 1)Q(x) + (ax + b)$ where $Q(x)$ is the quotient and $ax + b$ is the remainder. When $P(x)$ is divided by $x - 1$, the remainder is 4 and when divided by $x + 2$, the remainder is 1. Find the values of a and b
(b) (i) Expand $(1 + x^4)^{-\frac{1}{2}}$ up to the fourth term
 (ii) Use the first two terms of the expansion to find the value of $\frac{1}{\sqrt{144.0144}}$ correct to two significant figures.
[2020, No. 9]

8. (a) Solve the simultaneous equations:
$$2x^2 - 5xy + 2y^2 = 0$$
$$x + y = 6$$

[2013, No. 9: Ans: (a) (i) $2[\cos 30° + i\sin 30°]$ (b) $[1, 7]$

22. Given the equation $x^3 + x - 10 = 0$,
    (a) show that $x = 2$ is a root of the equation
    (b) deduce the values of $\alpha + \beta$ and $\alpha\beta$ where α and β are other roots of the equation.
    Hence form a quadratic equation whose roots are $\alpha^2$and β^2.
    [2013, No. 10: Ans: (b) $-2, 5$; $x^2 + 6x + 25 = 0$]
23. If $z = \frac{(2-i)(5+12i)}{(1+2i)^2}$
    (a) find the:
    (i) modulus of z,
    (ii) argument of z
    (b) represent z on a complex plane
    (c) write z in the polar form

[2012, No. 9: Ans: (a) (i) 5.814 (ii) $-86.06°$ (c) $5.814(\cos 0.478\pi - i\sin 0.478\pi)$]

24. Solve for x in the following equations:
    (a) $9^x - 3^{x+1} = 10$
    (b) $\log_4 x^2 - 6\log_x 4 - 1 = 0$
    [2012, No. 15: Ans: (a) 1.465 (b) 16, 0.125]
25. (a) The first term of an Arithmetic Progression (A.P) is ½. The sixth term of the A.P. is four times the fourth term. Find the common difference of the A.P.
    (b) The roots of a quadratic equation $x^2 + px + q = 0$ are α and β. Show that the quadratic equation whose roots are $\alpha^2 - q\alpha$ and $\beta^2 - q\beta$ is given by $x^2 - (p^2 + pq - 2q)x + q^2(q + p + 1) = 0$
    [2011, No. 9: Ans: (a) $-\frac{3}{14}$]
26. (a) Form a quadratic equation having $-3 + 4i$ as one of its roots.
    (b) Given that $Z_1 = -1 + i\sqrt{3}$ and $Z_2 = -1 - i\sqrt{3}$
    (i) express $\frac{Z_1}{Z_2}$ in the form $a + i\sqrt{b}$, where a and b are real numbers.
    (ii) represent $\frac{Z_1}{Z_2}$ on an Argand diagram.
    (iii) find $\left|\frac{Z_1}{Z_2}\right|$
    [2011, No. 10: Ans: (a) $z^2 + 6z + 25 = 0$ (b) (i) $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$ (iii) 1]
27. (a) Expand $\sqrt{\left(\frac{1+x}{1-x}\right)}$ in ascending powers of x to a term in x^2
    (b) (i) Using the expansion of $(1 + x)^{\frac{1}{2}}$ up to the term in x^3, find the value of $\sqrt{1.08}$ to 4 decimal places.
    (ii) Express $\sqrt{1.08}$ in the form $\frac{a}{b}\sqrt{c}$. Hence evaluate $\sqrt{3}$ correct to 3 significant figures.

[2010, No. 9: Ans: (a) $1 + x + \frac{1}{2}x^2$ (b) (i) 1.0392 (ii) 1.73]

28. (a) Given that the complex number Z and its conjugate $\bar{Z}$, satisfy the equation $Z\bar{Z} + 3\bar{Z} = 34 - 12i$, find the values of Z.
    (b) Find the Cartesian equation of the locus of a point P represented by the equation
    $$\left|\frac{Z + 3}{Z + 2 - 4i}\right| = 1$$
    [2010, No. 14: Ans: (a) $(3 - 4i)$, $(-6 + 4i)$ (b) $8y + 2x = 11$]
29. (a) By using the Binomial theorem, expand $(8 - 24x)^{\frac{2}{3}}$ as far as the 4^{th} term. Hence evaluate $4^{\frac{2}{3}}$ to one decimal place.
    (b) Find the coefficient of x in the expansion of $\left(x + \frac{2}{x^2}\right)^{10}$
    [2009, No. 9: Ans: (a) $4 - 8x - 4x^2 - \frac{16}{3}x^3$, 2.5 (b) 960]
30. (a) Given that $\frac{ix}{1+iy} = \frac{3x+i4}{x+3y}$, find the values of x and y
    (b) If $Z = x + iy$, find the equation of the locus $\left|\frac{Z+3}{Z-1}\right| = 4$
    [2009, No. 12: Ans: (a) $(x, y) = (2, 1.5)$; $(-2, -1.5)$ (b) $x^2 + y^2 - \frac{38}{15}x + \frac{7}{15} = 0$]
31. (a) Find the binomial expansion of $\left(1 - \frac{x}{2}\right)^5$. Use your expansion to estimate $(0.875)^5$ to **four** decimal places.
    (b) A financial credit society gives a 2% compound interest per annum to its members. If Ochola deposits shs 100,000 at the beginning of every year starting with 2004, how much would he collect at the end of 2008 if there are no withdrawals within this period?
    [2008, No. 15: Ans: (b) shs. 530812]
32. (a) The function $f(x) = x^3 + px^2 - 5x + q$ has a factor $(x - 2)$ and has a value of 5 when $x = -3$. Find p and q.
    (b) The roots of the equation $ax^2 + bx + c = 0$ are α and β. Form the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.
    (c) Simplify: $\frac{\sqrt{3}-2}{(2\sqrt{3}+3)}$ in the form $p + q\sqrt{3}$ where p, q are rational numbers.
    [2007, No. 9: Ans: (a) $p = 3$, $q = -10$ (b) $acx^2 - (b^2 - 2ac)x + ac = 0$ (c) $4 - \frac{7}{3}\sqrt{3}$]
33. (a) What is the smallest number of terms of the Geometric Progression (G.P) 5, 10, 20, … that can give a sum greater that 500,000?
    (b) Prove by induction $\sum_{r=1}^{n}\frac{1}{r(r+1)} = \frac{n}{n+1}$
    (c) Solve simultaneously: $a^3 + b^3 = 26$ and $a + b = 2$
    [2007, No. 14: Ans: (a) 17 (c) $(a, b) = (-1, 3)$ or $(3, -1)$]

34. (a) Express the complex numbers $Z_1 = 4i$ and $Z = 2 - 2i$ in the trigonometric form $r(\cos\theta + i\sin\theta)$.
Hence or otherwise evaluate $\frac{Z_1}{Z_2^2}$
(b) Find the values of x and y in
$$\frac{x}{2+3i} - \frac{y}{3-2i} = \frac{6+2i}{1+8i}$$
[2006, No. 9: Ans: (a) $z_1 = 4(\cos 90° + i\sin 90°)$, $z_2 = 2\sqrt{2}(\cos(-45°) + i\sin(-45°))$, $-\frac{1}{2}$ (b) $x = 2.8, y = 0.4$]

35. (a) Expand $(a+b)^4$. Hence find $(1.996)^4$, correct to 3 decimal places
(b) A credit society gives out a compound interest of 4.5% per annum. Muggaga deposits shs 300,000 at the beginning of each year. How much money will he have at the end of 4 years, if there are no withdrawals during this period?
[2006, Ans: (b) Shs. 1,341,212.917]

36. (a) Determine the Binomial expansion of $\left(1 + \frac{x}{2}\right)^4$
Hence evaluate $(2.1)^4$ correct to 2 decimal places
(b) A geometric progression (G.P) has a common ratio $r < 1$, $u_1 = 15$ and $S_\infty = 22.5$, where S_∞ is the sum to infinity and u_1, the first term. Find the:
(i) value of r,
(ii) ratio of $u_2 : u_3$
[2005, No. 11: Ans: (b) (i) $\frac{1}{3}$ (ii) $3:1$]

37. (a) Solve the equations $\frac{4x-3y}{4} = \frac{2y-x}{3} = \frac{z+4y}{2}$ and $6x + 6y + 2z = 6$
(b) Given the polynomial $f(x) = Q(x)g(x) + R(x)$, where $Q(x)$ is the quotient, $g(x) = (x-\alpha)(x-\beta)$ and $R(x)$ the remainder, show that
$$R(x) = \frac{(x-\beta)f(\alpha) + (\alpha - x)f(\beta)}{\alpha - \beta}$$
when $f(x)$ is divided by $g(x)$
Hence find the remainder when $f(x)$ is divided by $x^2 - 9$, given that $f(x)$ divided by $(x-3)$ is 2 and when divided by $x + 3$ is -3.
[2005, No. 14: Ans: (a) $x = \frac{17}{15}$, $y = \frac{16}{15}$, $z = -\frac{18}{5}$ (b) $\frac{5x-3}{6}$]

38. (a) Find n if ${}^nP_4 = 30\,{}^nC_5$
(b) How many arrangements can be made from the letters of the name *MISSISSIPPI*,
(i) when all the letters are taken at a time?
(ii) if the two letters P begin every word
(c) Find the number of ways in which a senior six Mathematics student can choose one or more of the four girls in the Mathematics class to join a discussion group.
[2004, No. 9: Ans: (a) 8 (b) (i) 34650 (ii) 630 (c) 15]

39. (a) Without using tables or calculators, simplify
$$\frac{\left(\cos\frac{\pi}{17} + i\sin\frac{\pi}{17}\right)^8}{\left(\cos\frac{\pi}{17} + i\sin\frac{\pi}{17}\right)^9}$$
(b) Given that x and y are real, find the values of x and y which satisfy the equation:
$$\frac{2y+4i}{2x+y} - \frac{y}{x-i} = 0$$
[2004, No. 15: Ans: (a) -1 (b) $(x,y) = (-1,-2)$ or $(1,2)$]

40. (a) Given the inequalities $y > x - 5$ and $0 < y < \frac{6}{x}$, Illustrate graphically by shading out the unwanted regions.
(b) Solve the simultaneous equations
$$xy + 2x = 5$$
$$9x = y + 6$$
Illustrate your solutions on a graph
[2003, No. 10: Ans: (b) $(x,y) = (1,3)$ or $\left(-\frac{5}{9}, -1\right)$]

41. (a) Use De Moivres theorem to express $\tan 5\theta$ in terms of $\tan\theta$
(b) Solve the equation $z^3 + 1 = 0$
[2003, No. 12: Ans: (a) $\frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$ (b) -1, $\frac{1-i\sqrt{3}}{2}$]

42. (a) The tenth term of an arithmetic progression (A.P) is 29 and the fifteenth term is 44. Find the value of the common difference and first term.
Hence find the sum of the first 60 terms.
(b) A cable 10 m long is divided into ten pieces whose lengths are in a geometric progression. The length of the longest piece is 8 times the length of the shortest piece. Calculate to the nearest centimeter the length of the third piece.
[2002, No. 9: Ans: (a) 2, 3; 5430 (b) 45 cm]

43. (a) Find the equation whose roots are $-1 \pm i$, where, $i = \sqrt{-1}$.
(b) Find the sum of the first 10 terms of the series $1 + 2i - 4 - 8i + 16 + \cdots$ in the form $a + bi$, where a and b are constants and $i = \sqrt{-1}$
(c) Prove by induction that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
[2002, No. 13: Ans: (a) $z^2 + 2z + 2 = 0$ (b) $205 + 410i$]

44. (a) Use De Moivres' theorem or otherwise to simplify
$$\frac{(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta)}{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}}$$
(b) Express $\frac{i}{4+6i}$ in modulus-argument form
(c) Solve $(z + 2z^*)z = 5 + 2z$, where z^* is the complex conjugate of z

[2001, No. 9: Ans: (a) $\cos\left(\frac{5\theta}{2}\right) + i\sin\left(\frac{5\theta}{2}\right)$ (b) $0.1387[\cos(0.187\pi) + i\sin(0.187\pi)$ (c) $1 \pm 2i$]

45. (a) It can be proved by induction that, for all positive n

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$$

From this result, deduce that

$$(n+1)^3 + (n+2)^3 + \cdots (2n)^3 = \tfrac{1}{4}n^2(3n+1)(5n+3)$$

(b) A man deposits sh.800, 000 into his savings account on which interest is 15% per annum. If he makes no withdrawals, after how many years will his balance exceed sh. 8 million?

[2001, No. 10: Ans: (b) 16.5 years]

46. (a) The nth term of a series is $U_n = a3^n + bn + c$. Given that $U_1 = 4$, $U_2 = 13$ and $U_3 = 46$, find the values of a, b and c.

(b) If α and β are the roots of the equation $x^2 - px + q = 0$, find the equation whose roots are $\frac{\alpha^3-1}{\alpha}$ and $\frac{\beta^3-1}{\beta}$

[2000, No. 9: Ans: (a) $a = 2$, $b = -3$, $c = 1$ (b) $qx^2 - (p^2q - 2q^2 - p)x + q^3 - p^3 + 3pq + 1 = 0$]

47. (a) Prove by induction that $2^n + 3^{2n-3}$ is always divisible by 7 for $n \geq 2$

(b) Expand $\left(1 - \frac{x}{3}\right)^{\frac{1}{2}}$ as far as the term in x^2. Hence evaluate $\sqrt{8}$, correct to three decimal places.

[2000, No. 10: Ans: (b) $1 - \frac{x}{6} - \frac{x^2}{72} - \cdots$, 2.829]

48. (a) Solve the equation $2(3^{2x}) - 5(3^x) + 2 = 0$

(b) The equations of three planes P_1, P_2 and P_3 are $2x - y + 3z = 3$, $3x + y + 2z = 7$ and $x + 7y - 5z = 13$ respectively. Determine where the three planes meet.

[1999, No. 10: Ans: (a) ± 0.6309 (b) $(-2, 5, 4)$]

49. If z is a complex number, describe and illustrate on the Argand diagram the locus given by each of the following:

(i) $\left|\frac{z+i}{z-2}\right| = 3$ (ii) Arg $(z+3) = \frac{\pi}{6}$

[1999, No. 10: Ans: (i) $8x^2 + 8y^2 - 2y - 36x + 35 = 0$, centre $\left(\frac{9}{4}, \frac{1}{8}\right)$, $r = 0.8385$ (ii) $y = \frac{x\sqrt{3}}{3} + \sqrt{3}$]

50. (a) Given that $Z_1 = -i + 1$, $Z_2 = 2 + i$ and $Z_3 = 1 + 5i$, represent Z_2Z_3, $Z_2 - Z_1$ and $\frac{1}{Z_1}$ on the Argand diagram. Also show the representation of $\frac{Z_2Z_3}{Z_2-Z_1} + \frac{1}{Z_1}$

(b) Prove that for positive integer n, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.
Deduce that this formula is also true for negative values of n

[Nov 1998, No. 9]

51. (a) Solve $4^x - 2^{x+1} - 15 = 0$

(b) Five million shillings are invested each year at a rate of 15% interest. In how many years will it accumulate to more than sh. 50 million?

[Nov 1998, No. 10: Ans: (a) 2.3219 (b) 6 years]

52. (a) When $f(x) = x^3 - ax + b$ is divided by $x + 1$, the remainder is 2 and $x + 2$ is a factor of $f(x)$. Find a and b.

(b) If the roots of the equation $x^2 + 2x + 3 = 0$ are α and β, form the equation whose roots are $\alpha^2 - \beta$ and $\beta^2 - \alpha$

[Mar 1998, No. 9: Ans: (a) $a = 5$, $b = -2$ (b) $x^2 + 2 = 0$]

53. (a) Show the region represented by $|z - 2 + i| < 1$ on an Argand diagram

(b) Express the complex number $z = 1 - \sqrt{3}i$ in modulus argument form and hence find z^2 and $\frac{1}{z}$ in the form $a + bi$

[Mar 1998, No. 10: Ans: (b) $z = 2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$ $z^2 = -2 - 2\sqrt{3}i$, $\frac{1}{z} = \frac{1}{4} + \frac{i\sqrt{3}}{4}$]

54. (a) If $\log_b a = x$, show that $b = a^{\frac{1}{x}}$ and deduce that $\log_a b = \frac{1}{\log_b a}$

(b) Solve

(i) $\log_n 4 + \log_4 n^2 = 3$

(ii) $2^{2x-1} + \frac{3}{2} = 2^{x+1}$

[1997, No. 9: Ans: (b) (i) 2, 4 (ii) 0, 1.585]

55. (a) Given the complex numbers $z_1 = 1 - i$, $z_2 = 7 + i$ represent z_1z_2 and $z_1 - z_2$ on the Argand diagram. Determine the modulus and argument of $\frac{z_1-z_2}{z_1z_2}$

(b) If z is a complex number in the form $(a + bi)$, solve $\left(\frac{z-1}{z+1}\right)^2 = i$

[1997, No. 10: Ans: (a) $0.6325, -124.7°$ (b) $z = 1 \pm i\sqrt{2}$]

56. (a) Find x if $\log_x 8 - \log_{x^2} 16 = 1$

(b) The sum of p terms of an arithmetic progression is q and the sum of q terms is p; find the sum of $p + q$ terms.

[1996, No. 9: Ans: (a) 2 (b) $-(p+q)$]

57. (a) Given that $z = \sqrt{3} + i$, find the modulus and argument of (i) z^2 (ii) $\frac{1}{z}$

(iii) show in an Argand diagram the points representing complex numbers z, z^2 and $\frac{1}{z}$

(b) In an Argand diagram, P represents a complex number z such that $2|z - 2| = |z - 6i|$
Show that P lies on a circle; find

(i) the radius of this circle

(ii) the complex number represented by its centre

[1996, No. 10: Ans: (a) (i) 4, $\frac{\pi}{3}$ (ii) $\frac{1}{2}$, $\frac{-\pi}{6}$ (b) (i) $3x^2 - 16x + 3y^2 + 12y - 20 = 0$, 4.2164 units (ii) $\frac{8}{3} - 2i$]

(b) Express $2x^3 + 5x^2 - 4x - 3$ in the form $(x^2 + x - 2)Q(x) + Ax + B$; where $Q(x)$ is a polynomial in x and A and B are constants. Determine the values of A and B and the expression $Q(x)$.

[1993, No. 2: Ans: (a) $a = 4$, $b = -12$, $c = 9$, $(2p - 3)$, $(2p - 3)$ (b) $A = -3$, $B = 3$, $2x + 3$]

70. (a) (i) Show that $\ln 2^r$, $r = 1, 2, 3$ is an arithmetic progression.
    (ii) Find the sum of the first 10 terms of the progression
    (iii) Determine the least value of m for which the sum of the first $2m$ terms exceeds 883.7.

    (b) Given that the equations $y^2 + py + q = 0$ and $y^2 + my + k = 0$ have a common root. Show that $(q - k)^2 = (m - p)(pk - mq)$

    [1993, No. 3: Ans: (a) (ii) 38.12 (iii) 25]

71. Solve the simultaneous equations

$$z_1 + z_2 = 8$$
$$4z_1 - 3iz_2 = 26 + 8i$$

Using the values of z_1 and z_2, find the modulus and argument of $z_1 + z_2 - z_1 z_2$

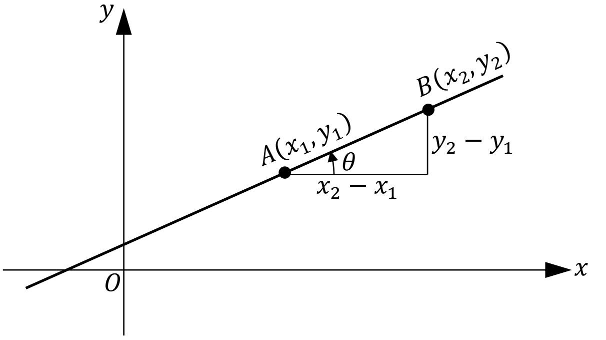
[1993; Ans: $z_1 = 8 + 2i$; $z_2 = -2i$]

Coordinate geometry

Chapter 13 The Straight Line

THE GRADIENT

Gradient of a straight line is a measure of its slope with respect to x-axis. Gradient is defined as
the increase in y divided by the increase in x between one point and another point on the line.

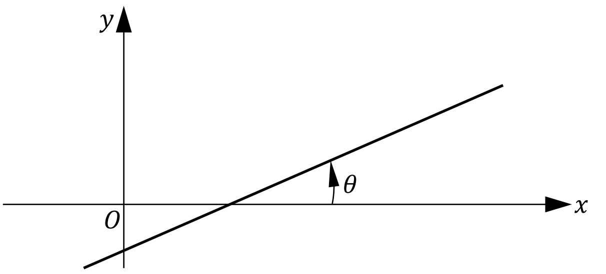


In general, the gradient of the line passing through $A(x_1, y_1)$ and $B(x_2, y_2)$ is

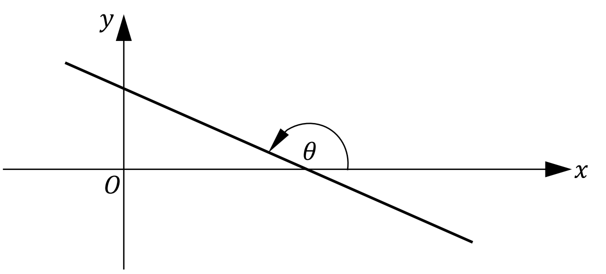
$$\frac{\text{the increase in } y}{\text{the increase in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

As the gradient of a straight line is the increase in y divided by the increase in x from one point on the line to another.
gradient measures the increase in y per unit increase in x, i.e. the rate of increase of y with respect to x.

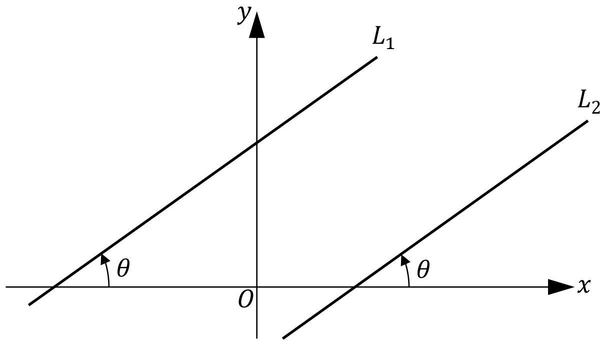
From the examples below, the gradient of a line may be positive or negative



A positive gradient indicates an 'uphill' slope with respect to the positive direction of the x-axis i.e. the line makes an acute angle with the positive sense of the x-axis.



Parallel lines

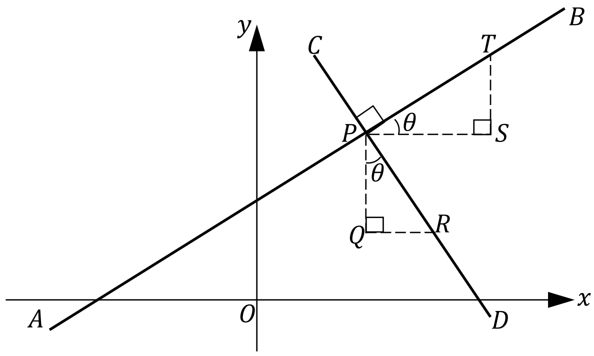


If L_1 and L_2 are parallel lines, they are equally inclined to the positive direction of the x-axis i.e.

parallel lines have equal gradients

Perpendicular lines

Consider the perpendicular lines AB and CD whose gradients are m_1 and m_2 respectively.



If AB, makes an angle θ with the x-axis, then CD makes an angle θ with the y-axis. Therefore triangles PQR and PST are similar.

Now the gradient of AB is $\frac{ST}{PS} = m_1$

and the gradient of CD is $-\frac{PQ}{QR} = m_2$, i.e. $\frac{PQ}{QR} = -m_2$

But $\frac{ST}{PS} = \frac{QR}{PQ}$ (triangles PQR and PST are similar)

therefore $m_1 = -\frac{1}{m_2}$ or $m_1 m_2 = -1$ i.e.

the product of the gradients of perpendicular lines is -1, or, if one line has gradient m, any line perpendicular to it has gradient $-\frac{1}{m}$

Intercept of a line on the axes

1. Intercept of a line on x-axis.

If a line cuts x-axis at a point $(a, 0)$, then a is called the intercept of the line on x-axis. $|a|$ is called the length of the

$$7y - 38 = -28x + 36$$
$$28x + 7y - 74 = 0$$

Example 6

Find the equation of the straight line which passes through the intersection of the straight lines $5x - 6y = 1$ and $3x + 2y + 5 = 0$ and is perpendicular to the straight line $3x - 5y + 11 = 0$

Solution

The straight line passing through the intersection of the given straight lines is

$$5x - 6y - 1 + \lambda(3x + 2y + 5) = 0$$
$$(5 + 3\lambda)x + (-6 + 2\lambda)y + (-1 + 5\lambda) = 0$$

Rewriting in the form $y = mx + c$

$$y = -\left(\frac{5 + 3\lambda}{-6 + 2\lambda}\right)x - \left(\frac{-1 + 5\lambda}{-6 + 2\lambda}\right)$$

This straight line is perpendicular to $3x - 5y + 11 = 0$

Product of the slopes of the perpendicular staight lines is -1

i.e. $m_1 m_2 = -1$

$$\Rightarrow -\left(\frac{5 + 3\lambda}{-6 + 2\lambda}\right)\left(\frac{3}{5}\right) = -1$$
$$15 + 9\lambda = -30 + 10\lambda$$
$$\lambda = 45$$
$$\Rightarrow 5x - 6y - 1 + 45(3x + 2y + 5) = 0$$
$$5x - 6y - 1 + 135x + 90y + 225 = 0$$
$$140x + 84y + 224 = 0$$
$$5x + 3y + 8 = 0$$

is the equation of the required staight line

Example 7

If O is the origin, A the point $(8, 0)$ and B the point $(0, 6)$, find the coordinates of the points P and Q, where the line $3x + 2y = c$ meets OA and AB respectively. If the area of the triangle OPQ is one-half that of the triangle OAB, find the value of c.

Solution

The line OA is the x-axis and its equation is $y = 0$. The coordinates of P are therefore given by the solution of the simultaneous equations $y = 0$ and $3x + 2y = c$

$$\Rightarrow 3x + 0 = c$$
$$x = \frac{c}{3}$$

Hence, P is the point $\left(\frac{c}{3}, 0\right)$

The line AB makes intercepts of 8 and 6 units on the coordinate axes and its equation is given by

$$\frac{x}{8} + \frac{y}{6} = 1$$
$$3x + 4y = 24$$

The coordinates of Q are given by the solution of the simultaneous equations $3x + 4y = 24$ and $3x + 2y = c$.

$$3x + 4y = 24 \quad \ldots [1]$$
$$3x + 2y = c \quad \ldots [2]$$

[1] – [2];

$$2y = 24 - c$$
$$y = 12 - \frac{c}{2}$$

Substituting for y in [1] gives;

$$x = \frac{2c}{3} - 8$$

The area of the triangle formed by the points $(0, 0)$, (x_1, y_1), (x_2, y_2) is $\frac{1}{2}(x_1y_2 - x_2y_1)$.

The area of the triangle OPQ is given by

$$\frac{1}{2}\left[\frac{c}{3}\left(12 - \frac{c}{2}\right) - \left(\frac{2c}{3} - 8\right)(0)\right] = 2c - \frac{c^2}{12}$$

The area of the triangle OAB is that of a right-angled triangle of base 8 and height 6 and hence is 24 units.

If $\Delta OPQ = \frac{1}{2}\Delta OAB$

$$2c - \frac{c^2}{12} = \frac{24}{2}$$
$$c^2 - 24 + 144 = 0$$
$$(c - 12)^2 = 0$$
$$c = 1$$

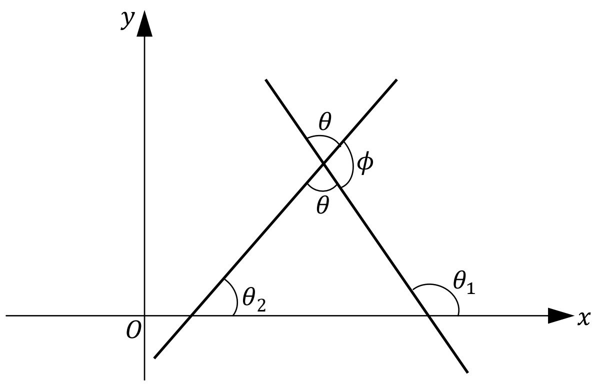
The gradient of a straight line

The gradient of a line is equal to the tangent of the angle between the line and the positive direction of the x-axis

i.e. $m = \tan\theta$

The angle between lines

Consider two lines which make angles θ_1 and θ_2 with the positive direction of the x-axis.



If the gradients of these lines are m_1 and m_2 respectively, then $m_1 = \tan\theta_1$, $m_2 = \tan\theta_2$

$$\theta + \theta_2 = \theta_1$$

One angle between the lines is θ, where $\theta = \theta_1 - \theta_2$

$$\tan\theta = \tan(\theta_1 - \theta_2)$$
$$\tan\theta = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1\tan\theta_2}$$

Hence the angle θ between two lines with gradients m_1 and m_2 is given by

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2}$$

If ϕ is the exterior angle between the lines, then

Solution

The equation of any straight line perpendicular to $4x+3y+1=0$ is $3x-4y+k=0$

If the line passes through $(-1,3)$, then

$$3(-1)-4(3)+k=0$$
$$\Rightarrow k=15$$

Substituting for the value of k gives

$$3x-4y+15=0$$

Example 13

Find the equation to the straight line which passes through the point $(4,-5)$ and is perpendicular to the straight line $3x+4y+5=0$.

Solution

$$3x+4y+5=0 \quad \dots [1]$$

Method I:

Any straight line perpendicular to [1] is given by

$$4x-3y+c=0 \;\dots [2]$$

The straight line [2] passes through the point $(4,-5)$ if

$$4(4)-3(-5)+c=0$$
$$c=-16-15=-31$$

The required equation is therefore

$$4x-3y=31$$

Method II:

Any straight line passing through the given point is

$$y-(-5)=m(x-4)$$

This straight line is perpendicular to [1] if the product of their slopes is -1

$$m\times -\frac{3}{4}=-1$$
$$m=\frac{4}{3}$$

The required equation is therefore

$$y+5=\frac{4}{3}(x-4)$$
$$4x-3y=31$$

Method III:

Any straight line is $y=mx+c$. It passes through the point $(4,-5)$, if

$$-5=4m+c \quad \dots [3]$$

It is perpendicular to [1] if

$$m\times -\frac{3}{4}=-1 \quad \dots [4]$$

Hence $m=\frac{4}{3}$ and then [3] gives $c=-\frac{31}{3}$

The required equation is therefore

$$y=\frac{4}{3}x-\frac{31}{3}$$
$$4x-3y=31$$

Example 14

Find the equation of a straight line whose y-intercept is -3 and which is

(a) parallel to the line joining the points $(-2,3)$ and $(4,-5)$

(b) perpendicular to the line joining the points $(0,-5)$ and $(-1,3)$

Solution

Here $c=y$-intercept $=-3$

(a) Let m be the slope of the required line

Since the required line is parallel to the line joining the points $A(-2,3)$ and $B(4,-5)$

$$m=\text{slope of } AB=\frac{-5-3}{4-(-2)}=\frac{-8}{6}=-\frac{4}{3}$$

The equation of the line is $y=-\frac{4}{3}x+(-3)$

$$4x+3y+9=0$$

(b) Let m be the slope of the required line

Since the required line is perpendicular to the line joining the points $A(0,-5)$ and $B(-1,3)$.

$$m\times(\text{slope of } AB)=-1$$
$$m\times\frac{3-(-5)}{-1-0}=-1$$
$$m\left(\frac{8}{-1}\right)=-1$$
$$m=\frac{1}{8}$$

The equation of the line is given by $y=\frac{1}{8}x+(-3)$

$$x-8y-24=0$$

Example 15

Find the value of k so that the line $2x+ky-9=0$ maybe (a) parallel to $3x-4y+7=0$ (b) perpendicular to $3y+2x-1=0$

Solution

Given $2x+ky-9=0 \dots [1]$

Slope of line [1] $=-\frac{\text{coeff. of } x}{\text{coeff. of } y}=-\frac{2}{k}$

(a) Given line is $3x-4y+7=0 \dots [2]$

Slope of line [2] $=-\frac{3}{-4}=\frac{3}{4}$

As the lines [1] and [2] are parallel,

$$-\frac{2}{k}=\frac{3}{4}$$
$$\Rightarrow k=-\frac{8}{3}$$

(b) Given line is $3y+2x-1=0$ i.e. $2x+3y-1=0 \dots [3]$

Slope of line [3] $=-\frac{2}{3}$

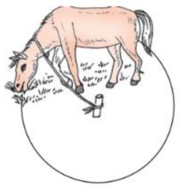
As the lines [1] and [3] are perpendicular,

$$\left(-\frac{2}{k}\right)\left(-\frac{2}{3}\right)=-1$$
$$4=-3k$$
$$\Rightarrow k=-\frac{4}{3}$$

Chapter 14 Locus

Introduction

Consider an animal tied with a rope to a post in the ground, what path will it describe, if it moves so as always to keep the rope taut? If you look at the tip of the seconds hand of a watch, you see that it completes one revolution in one minute and that in completing this revolution, it traces out the circumference of a circle. Will the tip always describe the same path? If so, why?



(a) In the first case, the animal moves along a circle if (i) the post is fixed, (ii) the animal moves so that the rope remains taut, i.e. its distance from the post does not change and it remains constant. Had these two conditions not been imposed on the motion of the animal, it would not have moved along a fixed path (the circle)

(b) The tip of the seconds hand describes a circle because (i) its one end is fixed, (ii) when it moves round the centre, the distance between the two ends remains constant. If the tip of the seconds hand of the watch were not bound to fulfill these conditions, it would not have described a fixed path (the circle).

Definition: If a point, which according to certain laws, describes a path and if every point on this path satisfies the given law, then the path is called the locus of the point.

Equation of a locus

The (Cartesian) equation of a locus is the equation in x and y that is satisfied by the coordinates of every point on the locus and not by the coordinates of any point outside the locus.

The coordinates (x, y) of the moving point which generates the locus are called current coordinates. The point covers all positions on the locus and is called the general point.

Method to find the equation of the locus of a point

(i) Let (x, y) be any point on the locus

(ii) Write the given geometrical condition (or conditions)

(iii) Express the given condition in mathematical form in terms of x, y and known constant (or constants) and simplify it, if necessary.

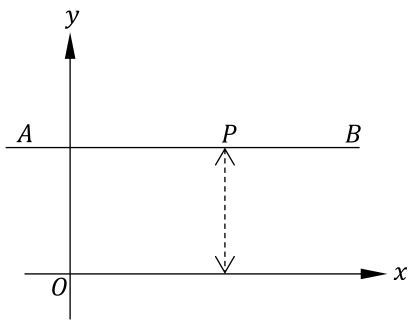
(iv) Eliminate the variable (or variables), if any

(v) The equation so obtained will be the equation of the required locus.

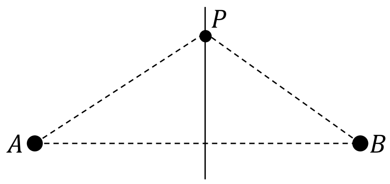
Note: Sometimes we take any point on the locus as (α, β) or (h, k) or (x_1, y_1) instead of (x, y). We write the given geometrical condition (or conditions) and express it in terms of α, β. Then we change α to x and β to y to get the required equation of the locus.

Examples of Loci

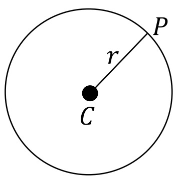
1. Let a point P move (in a plane) such that its distance from the x-axis is always equal to $b\ (> 0)$. The point P will trace out a straight line AB parallel to x-axis at a distance b above the x-axis. Therefore, the locus of the moving point P is the straight line AB.



2. Let A, B be two fixed points (in a plane) and a point P move in the plane such that its distances from the points A and B are equal. Obviously, all positions of the moving point P will lie on the perpendicular bisector of the segment AB. Therefore, the locus of the moving point P which remains at equal distances from A and B is the perpendicular bisector of the segment AB.



3. Let a point move in a plane such that its distance from a fixed point C (in the plane) is always equal to $r (> 0)$. Obviously, all positions of the moving point P will lie on the circumference of a circle with centre C and radius r. Therefore, the locus of the point P is the circle with centre C and radius r.

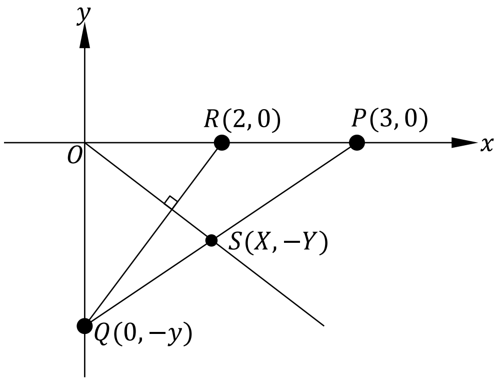


$$(x-3)^2+(y-6)^2=\frac{(3x+5y-4)^2}{34}$$
$$34(x^2+y^2-6x-12y+45)$$
$$=9x^2+25y^2+16+30xy-24x-40y$$
$$25x^2+9y^2-30xy-180x-368y+1514=0$$

Example 7

The points $R(2,0)$ and $P(3,0)$ lie on the x-axis and $Q(0,-y)$ lies on the y-axis. The perpendicular from the origin to RQ meets PQ at point $S(X,-Y)$. Determine the locus of S in terms of X and Y.

Solution



Gradient of $RQ=\frac{0-(-y)}{2-0}=\frac{y}{2}$

Gradient of $OS=\frac{-Y-0}{X-0}=\frac{-Y}{X}$

Since RQ and OS are perpendicular,

$$\frac{y}{2}\times\frac{-Y}{X}=-1$$
$$y=\frac{2X}{Y}$$

Now looking at PQ,

$$\text{Gradient } QS=\text{Gradient } QP$$
$$\frac{-Y+y}{X-0}=\frac{0+y}{3-0}$$
$$-3Y+3y=Xy$$
$$-3Y+3\left(\frac{2X}{Y}\right)=X\left(\frac{2X}{Y}\right)$$
$$-3Y^2+6X=2X^2$$
$$2X^2+3Y^2=6X$$

This is the equation of the locus of S

Example 8

A point P is such that the sum of squares of its distances from the two axes of coordinates is equal to the square of its distance from the line $x-y=1$. Find the equation of the locus of P.

Solution

Let $P(\alpha,\beta)$ be a variable point.

The equations of the coordinate axes are $y=0$ and $x=0$

The given line is $x-y-1=0$

According to the given condition;

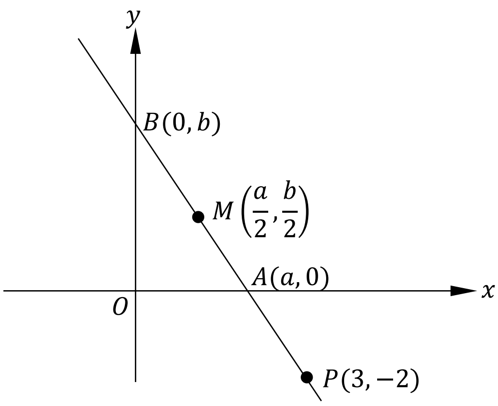
$$\left(\frac{|\beta|}{1}\right)^2+\left(\frac{|\alpha|}{1}\right)^2=\left(\frac{|\alpha-\beta-1|}{\sqrt{1^1+(-1)^2}}\right)^2$$
$$\beta^2+\alpha^2=\frac{(\alpha-\beta-1)^2}{2}$$
$$2\alpha^2+2\beta^2=\alpha^2+\beta^2+1-2\alpha\beta-2\alpha+2\beta$$
$$\alpha^2+\beta^2+2\alpha\beta+2\alpha-2\beta-1=0$$

$\therefore$ The locus of the point P is $x^2+y^2+2xy+2x-2y-1=0$

Example 9

A straight line passes through the point $(3,-2)$. Find the locus of the middle portion of the line intercepted between the axes.

Solution



Let the equation of the line be

$$\frac{x}{a}+\frac{y}{b}=1 \text{ ... [1]}$$

where a,b vary.

Since the line [1] passes through the point $(3,-2)$,

$$\frac{3}{a}-\frac{2}{b}=1 \text{ ... [2]}$$

The line [1] meets the axes in points $A(a,0)$ and $B(0,b)$, then AB is the portion of the line intercepted between the axes. Let M be mid-point of AB, then the coordinates of M are $\left(\frac{a}{2},\frac{b}{2}\right)$.

For the locus of M, put $\frac{a}{2}=x,\frac{b}{2}=y$

$$\Rightarrow a=2x, b=2y$$

Substituting these values of a and b in [2], the required equation of the locus is

$$\frac{3}{2x}-\frac{2}{2y}=1$$
$$3y-2x=2xy$$

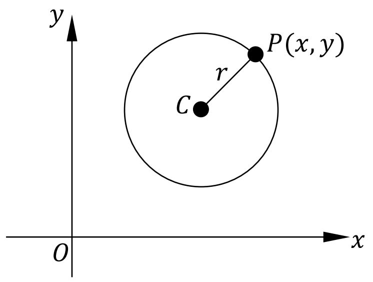
Chapter 15 The Circle

Introduction

A circle is the locus of a point which moves in such a way that its distance from a fixed point is always constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

The equation of a circle when the centre and radius are given

Let $C(h, k)$ be the centre and r be the radius of the circle. Let $P(x, y)$ be any point on the circle.



$$\overline{CP} = r$$
$$\sqrt{(x-h)^2 + (y-k)^2} = r$$
$$(x-h)^2 + (y-k)^2 = r^2$$

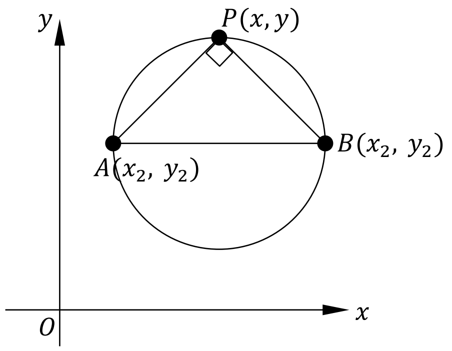
is the required equation of the circle.

Note:

If the centre of the circle is at the origin i.e. $(h, k) = (0, 0)$, then the equation of the circle is $x^2 + y^2 = r^2$

The equation of a circle if the endpoints of a diameter are given

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end points of a diameter. Let $P(x, y)$ be any point on the circle.



The angle in a semicircle is a right angle, thus PA is perpendicular to PB.

$$\Rightarrow (\text{slope of } PA)(\text{slope of } PB) = -1$$
$$\left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_2}\right) = -1$$
$$(y-y_1)(y-y_2) = -(x-x_1)(x-x_2)$$
$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$$

is the required equation of the circle.

Note: Alternatively, the equation of this circle can be obtained using the Pythagoras theorem.

$$\overline{PA}^2 + \overline{PB}^2 = \overline{AB}^2$$

The general equation of the circle

The general equation of the circle is in the form $x^2 + y^2 + 2gx + 2fy + c = 0$.

This equation can be written by completing the square as

$$x^2 + 2gx + g^2 + y^2 + 2fy + f^2 = g^2 + f^2 - c$$
$$(x+g)^2 + (y+f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$$
$$[x - (-g)]^2 + [y - (-f)]^2 = g^2 + f^2 - c$$

This is of the form

$$(x-h)^2 + (y-k)^2 = r^2$$

$\therefore$ The considered equation represents a circle with centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$

$\therefore$ The general equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$.

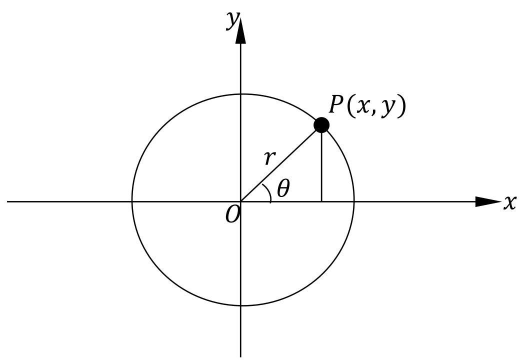
Note:

The general second degree equation $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents a circle if

1. $a = b$ i.e. coefficient of x^2 = coefficient of y^2
2. $h = 0$ i.e. no xy term.

Parametric form

Consider a circle with radius r and centre at the origin. Let $P(x, y)$ be any point on the circle. Assume that OP makes an angle θ with the positive direction of x-axis and PM is the perpendicular to the x-axis.



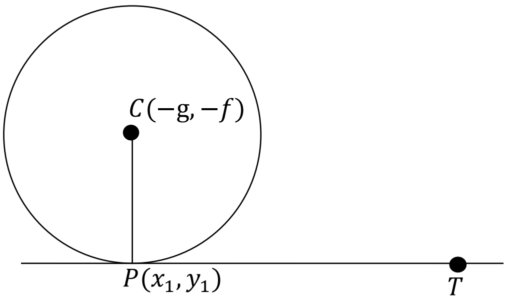
From the figure

$$\frac{x}{r} = \cos\theta, \frac{y}{r} = \sin\theta$$

In figure (a), the straight line AB does not touch or intersect the circle.
In figure (b), the straight line AB intersects the circle in two points and is called a secant.
In figure (c), the straight line AB touches the circle at exactly one point, and it is called a tangent.

A tangent to a circle is a straight line which intersects (touches) the circle in exactly one point.

Equation of the tangent to a circle at a point (x_1, y_1)



Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$
Let $P(x_1, y_1)$ be a given point on it

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$$

Let PT be the tangent at P
The centre of the circle is $C(-g, -f)$
Slope of the $CP = \frac{y_1+f}{x_1+g}$
Since CP is perpendicular to PT, slope of $PT = -\left(\frac{x_1+g}{y_1+f}\right)$
Equation of the tangent PT is

$$y - y_1 = m(x - x_1)$$
$$y - y_1 = -\left(\frac{x_1 + g}{y_1 + f}\right)(x - x_1)$$
$$(y - y_1)(y_1 + f) = -(x - x_1)(x_1 + g)$$
$$yy_1 - y_1^2 + fy - fy_1 = -[xx_1 - x_1^2 + gx - gx_1]$$
$$xx_1 + yy_1 + fy + gx = x_1^2 + y_1^2 + gx_1 + fy_1$$

Add $gx_1 + fy_1 + c$ on both sides

$$xx_1 + yy_1 + gx + gx_1 + fy + fy_1 + c$$
$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ is the required equation of the tangent at (x_1, y_1)

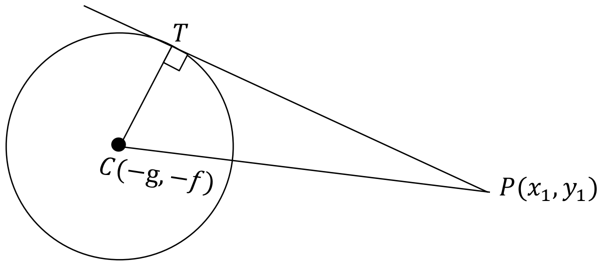
Note:
1. The equation of the tangent at (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.
2. To get the equation of the tangent at (x_1, y_1), replace x^2 as xx_1, y^2 as yy_1, x as $\frac{x+x_1}{2}$ and y as $\frac{y+y_1}{2}$ in the equation of the circle.
3. The gradient of the tangent to the circle can be obtained by differentiating the equation with respect to x i.e.

$$x^2 + y^2 + 2gx + 2fy + c = 0$$
$$2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$
$$(x + g) + (y + f)\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{x + g}{y + f}$$

So that the gradient of the circle at the point (x_1, y_1) is given by

$$\left(\frac{dy}{dx}\right)_{x=x_1} = -\frac{x_1 + g}{y_1 + f}$$

Length of the tangent to the circle from a point (x_1, y_1)



Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$
Let PT be the tangent to the circle from $P(x_1, y_1)$ outside it.
We know that the coordinate of the centre is $(-g, -f)$and radius, $r = CT = \sqrt{g^2 + f^2 - c}$
From the right-angled triangle PTC,

$$PT^2 = PC^2 - CT^2$$
$$= (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c)$$
$$= x_1^2 + 2gx_1 + g^2 + y_1^2 + 2fy_1 + f^2 - g^2 - f^2 + c$$
$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

$\therefore PT = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$, which is the length of the tangent from the point (x_1, y_1).

Note:
1. If the point P is on the circle, then $PT^2 = 0$ (PT is zero)
2. If the point P is outside the circle, then $PT^2 > 0$ (PT is real)
3. If the point P is inside the circle, then $PT^2 < 0$ (PT is imaginary).

The condition for the line $y = mx + c$ to be a tangent to the circle $x^2 + y^2 = a^2$

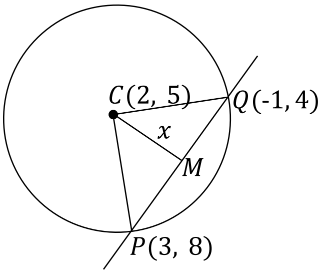
Substituting y from the equation of the line to the equation of the circle.

$$x^2 + (mx + c)^2 = a^2$$
$$x^2 + m^2x^2 + 2mcx + c^2 = a^2$$
$$(1 + m^2)x^2 + 2mcx + c^2 - a^2 = 0$$

For a tangent, $b^2 - 4ac = 0$

$$(2mc)^2 - 4(1 + m^2)(c^2 - a^2) = 0$$
$$4m^2c^2 - 4[c^2 - a^2 + m^2c^2 - m^2a^2] = 0$$
$$m^2c^2 - c^2 + a^2 - m^2c^2 + m^2a^2 = 0$$
$$a^2 + m^2a^2 = c^2$$
$$a^2(1 + m^2) = c^2$$

(c)

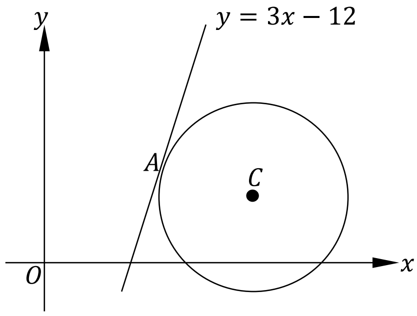


$$|PQ| = \sqrt{(8-4)^2 + (-1-3)^2} = \sqrt{32} = 4\sqrt{2}$$
$$|PM| = 2\sqrt{2}$$

By Pythagoras theorem,

$$\left(2\sqrt{2}\right)^2 + x^2 = \left(\sqrt{10}\right)^2$$
$$8 + x^2 = 10$$
$$x^2 = 2$$
$$x = \sqrt{2}$$

Example 24



The figure above shows a circle whose centre is at $C(8, k)$, where k is a constant. The straight line with equation $y = 3x - 12$ is a tangent to the circle at the point $A(5, 3)$.

(a) Find an equation of the normal to the circle at A

(b) Determine an equation of the circle.

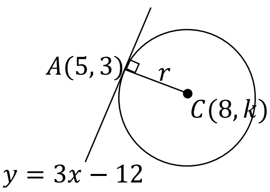
<u>Solution</u>

(a) As the tangent has gradient 3, the normal must have a gradient $-\frac{1}{3}$

$A(5, 3)$; $\quad y - y_0 = m(x - x_0)$

$$y - 3 = -\frac{1}{3}(x - 5)$$
$$3y - 9 = -x + 5$$
$$x + 3y = 14$$

(b)



The normal must pass through the centre $C(8, k)$

$$\Rightarrow 8 + 3k = 14$$
$$3k = 6$$
$$k = 2$$
$$\therefore A(5, 3) \text{ and } C(8, 2)$$
$$|AC| = \sqrt{(2-3)^2 + (8-5)^2} = \sqrt{1+9} = \sqrt{10}$$

The equation of the circle is given by

$$(x-8)^2 + (y-2)^2 = r^2$$
$$(x-8)^2 + (y-2)^2 = \left(\sqrt{10}\right)^2$$
$$(x-8)^2 + (y-2)^2 = 10$$

FAMILY OF CIRCLES

A collection of circles is called a family or a system of circles. Sometimes there exist more than one circle satisfying given conditions. The collection of all such circles is called a family of circles, satisfying the given conditions.

Concentric circles:

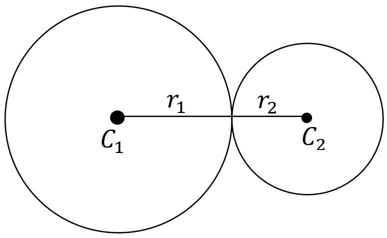
Two or more circles having the same centre are called concentric circles

Circles touching each other:

Two circles may touch each other either internally or externally. Let C_1, C_2 be the centres of the circle and r_1, r_2 be the radii and P the point of contact.

Case 1: The two circles touch externally

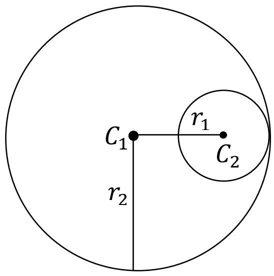
The distance between their centres is equal to the sum of their radii.



$$\overline{C_1C_2} = r_1 + r_2$$

Case 2: The two circles touch internally

The distance between their centres is equal to the difference of their radii.



$$\overline{C_1C_2} = \overline{C_1P} - \overline{C_2P} = r_1 - r_2$$

Orthogonal circles

Two circles are said to be orthogonal if the tangent at their point of intersection are at right angles.

$$3y - 3 = -4x + 12$$
$$3y + 4x - 15 = 0$$

To get point P, we need to solve the equations of the two lines simultaneously

$$3x - 4y + 20 = 0 \quad \ldots [1]$$
$$3y + 4x - 15 = 0 \quad \ldots [2]$$

4[1] −3[2];

$$-25y + 125 = 0$$
$$y = 5$$

From [1]; $3x - 4(5) + 20 = 0$

$$3x = 0$$
$$x = 0$$

$\therefore$ The point P is $(0, 5)$

Example 31

Prove that the circles whose equations are $x^2 + y^2 - 4y - 5 = 0$, $x^2 + y^2 - 8x + 2y + 1 = 0$ cut orthogonally and find the equation of the common chord.

Solution

C_1; $x^2 + y^2 - 4y - 5 = 0$

$x^2 + y^2 - 4y = 5$

$x^2 + (y-2)^2 - 4 = 5$

$x^2 + (y-2)^2 = 9$

Centre $(0, 2)$; radius $= 3$

C_2; $x^2 + y^2 - 8x + 2y + 1 = 0$

$x^2 - 8x + y^2 + 2y + 1 = 0$

$(x-4)^2 - 16 + (y+1)^2 = 0$

$(x-4)^2 + (y+1)^2 = 16$

Centre $(4, -1)$; radius $= 4$ units

Distance between centres,

$$d = \sqrt{(4-0)^2 + (-1-2)^2} = \sqrt{16+9} = 5$$

Now, $r_1^2 + r_2^2 = 3^2 + 4^2 = 9 + 16 = 25 = d^2$

$\Rightarrow$ The two circles are orthogonal

Now, to find the equation of the common chord, we need to subtract the equations of the two circles and eliminate the coefficients of the square terms.

$$x^2 + y^2 - 4y - 5 = 0 \quad \ldots [1]$$
$$x^2 + y^2 - 8x + 2y + 1 = 0 \quad \ldots [2]$$

Subtracting [1] – [2];

$$8x - 6y - 6 = 0$$
$$4x - 3y = 3$$

Example 32

The circles C_1 and C_2 have respective equations $x^2 + y^2 - 6x = 16$ and $x^2 + y^2 - 18x + 16y = 80$.

(a) By solving these equations simultaneously, show that C_1 and C_2 touch at a point P and determine its coordinates.

(b) Determine further whether C_1 and C_2 touch internally or externally.

Solution

(a) Solving simultaneously

$$x^2 + y^2 - 6x = 16$$
$$x^2 + y^2 - 18x + 16y = 80$$

Subtract;

$$12x - 16y = -64$$
$$3x - 4y = -16$$
$$3x = 4y - 16$$
$$x = \frac{4y - 16}{3}$$
$$x^2 + y^2 - 6x = 16$$
$$\left(\frac{4y-16}{3}\right)^2 + y^2 - 2(4y - 16) = 16$$
$$16y^2 - 128y + 256 + 9y^2 - 72y + 288 = 144$$
$$25y^2 - 200y + 400 = 0$$
$$y^2 - 8y + 16 = 0$$
$$(y-4)^2 = 0$$
$$y = 4$$

$y = 4$ is a repeated root, indeed the circles touch

$$\Rightarrow x = \frac{4(4) - 16}{3} = 0$$

The circles touch at $(0, 4)$

(b) Firstly, we need the circle particulars

C_1; $x^2 + y^2 - 6x = 16$

$x^2 - 6x + y^2 = 16$

$(x-3)^2 - 9 + y^2 = 16$

$(x-3)^2 + y^2 = 25$

Centre $(3, 0)$; radius 5

C_2; $x^2 + y^2 - 18x + 16y = 80$

$x^2 - 18x + y^2 + 16y = 80$

$(x-9)^2 - 81 + (y+8)^2 - 64 = 80$

$(x-9)^2 + (y+8)^2 = 225$

Centre $(9, -8)$, radius 15

Touching internally requires $d = 15 - 5 = 10$ and touching externally requires $d = 15 + 5 = 20$

Distance between the centres is given by

$$d = \sqrt{(3-9)^2 + (0+8)^2} = \sqrt{36+64} = \sqrt{100} = 10$$

$\therefore$ The circles touch internally

Equation of a circle passing through three given points

The form of the equation of the circle involves three parameters i.e. h, r, k and g$, f, c$ which require for their determination three conditions, and these three conditions may be that the circle shall pass through three given points.

Assume, then, that a circle is to pass through the points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$. Since each one of these points is on the curve, their coordinates must satisfy the equation of the curve. Hence

$$(x_1 - h)^2 + (y_1 - k)^2 = r^2$$
$$(x_2 - h)^2 + (y_2 - k)^2 = r^2$$
$$(x_3 - h)^2 + (y_3 - k)^2 = r^2$$

From these three equations, we can find the values of h, k, r and substituting them in the original equation, we shall have the equation of the desired circle.

[Ans: $x^2 + y^2 = 6x + 10y - 16 = 0$, $x^2 + y^2 + 12x - 14y - 40 = 0$; $3x - 4y = 4$, 10]

6. The circles $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 4x - 6y - 3 = 0$ intersect at the points A and B. Find (a) the equation of AB (b) the equation of the circle which passes through A, B and the point $(1, 2)$
[Ans: (a) $2x - 2y - 1 = 0$ (b) $x^2 + y^2 - 2y - 1 = 0$]
7. Given that the circles $x^2 + y^2 - 3y = 0$ and $x^2 + y^2 + 5x - 8y + 5 = 0$ intersect at P and Q, find the equation of the circle which passes through P, Q and (a) the point $(1, 1)$ (b) the origin (c) touches the x-axis.
[Ans: (a) $x^2 + y^2 + x - 4y + 1 = 0$, (b) $x^2 + y^2 - 3y = 0$ (c) $x^2 + y^2 + 4x - 7y + 4 = 0$]
8. The circle $x^2 + y^2 + 3x - 5y - 4 = 0$ and the straight line $y = 2x + 5$ intersect at the points A and B. Find the equation of the circle which passes through A, B and (a) the point $(3, 1)$, (b) the origin (c) has its centre on the y-axis.
[Ans: (a) $x^2 + y^2 + x - 4y - 9 = 0$ (b) $5x^2 + 5y^2 + 23x - 29y = 0$ (c) $2x^2 + 2y^2 - 7y - 23 = 0$]
9. Write down the perpendicular distance from the point (a, a) to the line $4x - 3y + 4 = 0$. The circle, with centre (a, a) and radius a, touches the line $4x - 3y + 4 = 0$ at the point P. Find a, and the equation of the normal to the circles at P. Show that P is the point $\left(\frac{1}{5}, \frac{8}{5}\right)$. Show that the equation of the circle which has centre P and which passes through the origin is $5(x^2 + y^2) - 2x - 16y = 0$.
[Ans: $\frac{1}{5}|a + 4|$; $a = 1$, $3x + 4y = 7$]
10. Find the centre and radius of each of the circles C_1 and C_2 whose equations are $x^2 + y^2 - 16y + 32 = 0$ and $x^2 + y^2 - 18x + 2y + 32 = 0$ respectively and show that the circles touch externally. Find the coordinates of their point of contact and show that the common tangent at that point passes through the origin. The other tangents from the origin, one to each circle, are drawn. Find, correct to the nearest degree, the angle between these tangents.
[Ans: $(0, 8)$, $4\sqrt{2}$; $(9, -1)$, $5\sqrt{2}$; $(4, 4)$; $167°$]
11. The circles whose equations are $x^2 + y^2 - x + 6y + 7 = 0$ and $x^2 + y^2 + 2x + 2y - 2 = 0$ intersect at the points A and B. Find (i) the equation of the line AB (ii) the coordinates of A and B. Show that the two given circles intersect at right angles and obtain the equation of the circle which passes through A and B and which also passes through the centres of the two circles.
[Ans: (i) $3x - 4y = 9$ (ii) $(-1, -3)$, $\left(\frac{23}{25}, -\frac{39}{25}\right)$, $2x^2 + 2y^2 + x + 8y + 5 = 0$]
12. Find the centre and the radius of the circle C which passes through the points $(4, 2)$, $(2, 4)$ and $(2, 6)$. If the line $y = mx$ is a tangent to C, obtain the quadratic equation satisfied by m. Hence or otherwise find the equations of the tangents to C which pass through the origin O. Find also (i) the angle between the two tangents, (ii) the equation of the circle which is the reflection of C in the line $y = 3x$.
[Ans: $(5, 5)$, $\sqrt{10}$; $3m^2 - 10m + 3 = 0$; $y = \frac{1}{3}x$, $y = 3x$; (i) $53.13°$ (ii) $x^2 + y^2 + 2x - 14y + 40 = 0$]
13. Show that the circles $x^2 + y^2 - 2x - 2y - 2 = 0$ and $x^2 - y^2 - 8x - 10y + 32 = 0$ touch externally and find the coordinates of the point of contact
[Ans: (11/5, 13/5)]
14. Find the centre and radius of the circle passing through the points $(3, 8)$, $(9, 6)$ and $(13, -2)$.
[Ans: $(3, -2)$; 10]
15. Determine the value of k so that $x^2 + y^2 - 8x + 10y + k = 0$ is the equation of a circle of radius 7.
[Ans: $k = -8$]
16. Find the equation of the circle with centre at $(1, 3)$ and tangent to the line $5x - 12y - 8 = 0$.
[Ans: $(x - 1)^2 + (y - 3)^2 = 9$]
17. Find an equation of the circle that contains the point $(3, 1)$ and passes through the points of intersection of the two circles $x^2 + y^2 - x - y - 2 = 0$ and $x^2 + y^2 + 4x - 4y - 8 = 0$.
[Ans: $3x^2 + 3y^2 - 13x + 3y + 6 = 0$]
18. Show that the circles $x^2 + y^2 - 2ax + c^2 = 0$ and $x^2 + y^2 - 2by - c^2 = 0$ are orthogonal.
19. Find the equation to the circle which passes through the points $(-2, 2)$, $(2, 4)$, $(5, -5)$. Show that the circle touches the circle $2x^2 + 2y^2 - 17x + 16y + 65 = 0$ at the point $(5, -5)$.
[Ans: $x^2 + y^2 - 4x + 2y - 20 = 0$]
20. Show that the circles $x^2 + y^2 + 4x - 2y - 11 = 0$ and $x^2 + y^2 - 4x - 8y + 11 = 0$ intersect at right angles and find the length of their common chord.
[Ans: $24/5$]
21. Prove that the circles $x^2 + y^2 + 2x - 8y + 8 = 0$; $x^2 + y^2 + 10x - 2y + 22 = 0$ touch one another. Find
(a) the point of contact
(b) the equation of the common tangent at this point
(c) the area of the triangle enclosed by this common tangent, the line of centres and the y-axis.
[Ans: (a) (-3.4, 2.2) (b) $4x + 3y + 7 = 0$ (c) 289/24]
22. Find the equation to the circle passing through the point $(-2, -6)$ and through the points of intersection of the circles $x^2 + y^2 - 3x + 4y - 2 = 0$ and $x^2 + y^2 + 5x - 3y = 8$.
[Ans: $x^2 + y^2 - 11x + 11y + 4 = 0$]

Chapter 16 Conic Sections

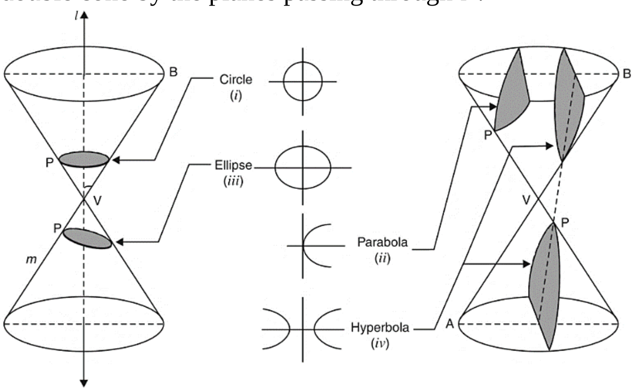
Introduction

Today, we find applications of the theory of conic sections in the orbits of planets and artificial satellites. The theory also applied to the lenses of telescopes, microscopes and other optical instruments, weather prediction, communication by satellites, geological surveying and the construction of buildings and bridges. Conics can also occur in the study of atomic structure, the long-range guidance systems for ships and aircrafts, the location of hidden gun emplacements and the detection of approaching enemy ships and aircrafts. The surfaces of revolution formed by the conic sections, such as paraboloid, ellipsoid, and hyperboloid find application in the sciences dealing with light, sound and radio waves.

It is helpful to visualize the conic sections formed by the intersections of a plane and a right circular cone. The cone is thought of as extending indefinitely on both sides of the vertex, that is a double right circular cone of infinite extent in both directions.

Conics as sections of a plane and a right circular cone

Take a point P on a cone and consider the sections of the double cone by the planes passing through P.



When the cutting plane is at right angles to the axis of the cone, the curve of intersection is a **circle**.

When the cutting plane is somewhat inclined to this direction and intersects only one nappe, the curve of intersection is an **ellipse**.

As the inclination increases, the ellipse gets more and more elongated till finally, when the cutting plane is parallel to the generator AB, which is diametrically opposite to P, the ellipse is infinitely long. This is then really a **parabola**.

When the cutting plane is still more inclined, it intersects both nappes, the curve of intersection is a **hyperbola**. A hyperbola consists of two branches.

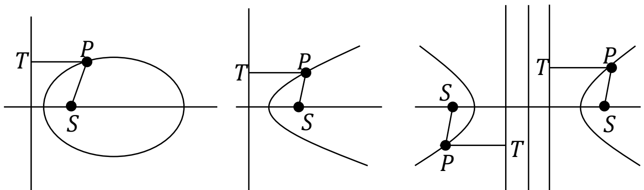
When P is the vertex V of the cone, the section is a pair of straight lines, which become coincident when the cutting plane touches the cone along a generator like AB. (Two parallel lines cannot be obtained as a plane section of a cone).

Finally, the locus is a single point if the plane contains the vertex and does not intersect either nappe of the cone.

Note: The circle, ellipse, parabola, and hyperbola are the main conic sections. Two straight lines, intersecting or coincident and a single point are called **degenerate** cases.

Definition: A conic section or conic, is the locus of a point which moves so that its distance from a fixed point is in a constant ratio to its distance from a fixed straight line.

The fixed point is called a **focus**, the fixed straight line is called a directrix, and the constant ratio is called the **eccentricity** which is denoted by e.



$0 < \frac{|PS|}{|PT|} < 1$ $\quad \frac{|PS|}{|PT|} = 1$ $\quad \frac{|PS|}{|PT|} > 1$

From the figure above $\frac{|PS|}{|PT|} = \text{constant} = e$

If $e = 1$, the curve is a parabola

If $e < 1$, the curve is an ellipse

If $e > 1$, the curve is a hyperbola.

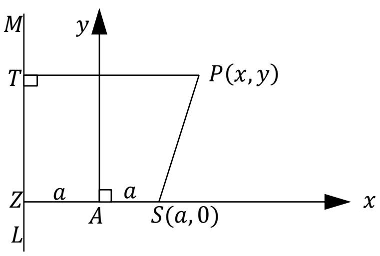
The Parabola

Standard equation of a parabola

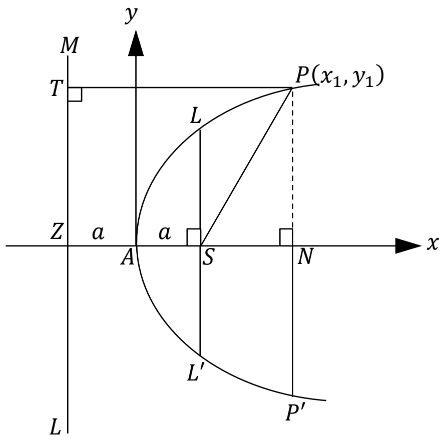
Let S be the fixed point, called the focus, and LM the fixed line, called the directrix. Draw SZ perpendicular to LM. Bisect SZ at A and let $SZ = 2a$ so that $|SA| = |AZ| = a$.

Take A as the origin, AS the positive x-axis and AY the positive y-axis perpendicular to the x-axis AS or AX.

Then the coordinates of the focus S are $(a, 0)$, and the equation of the directrix LM, being parallel to the y-axis, is $x = -a$ or $x + a = 0$.



axis or $y = 0$ is the axis of the parabola $y^2 = 4ax$. Note that the axis of the parabola passes through the focus and perpendicular to the directrix.



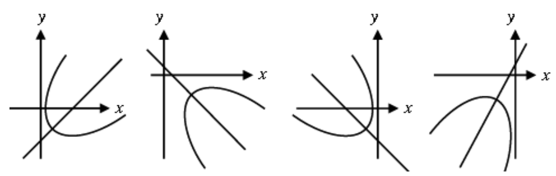
Vertex: The point of intersection of the parabola and its axis is called its vertex. Here, the vertex is $A(0, 0)$.
Focal distance: The focal distance is the distance between a point on the parabola and its focus.
Focal chord: A chord which passes through the focus of the parabola is called the focal chord of the parabola.
Latus rectum: The double ordinate passing through the focus of a parabola is called the Latus rectum. In the figure, LSL' is the latus rectum. Another name for the latus rectum is the principal diameter.

End points of latus rectum and length of latus rectum
To find the end points, solve the equation of latus rectum $x = a$ and $y^2 = 4ax$.

$$y^2 = 4a(a)$$
$$y^2 = 4a^2$$
$$y = \pm 2a$$

If L and L' are the end points of latus rectum, then L is $(a, 2a)$ and L' is $(a, -2a)$. The length of latus rectum $= LL' = 4a$.

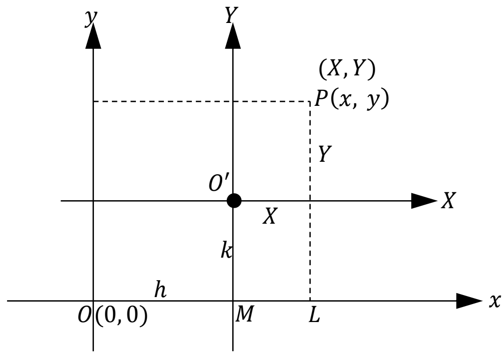
Note: So far we have discussed four standard types of parabolas. There are plenty of parabolas which cannot be classified under these standard types. For example, consider the following parabola.



For the above parabolas, the axes are neither parallel to x-axis nor parallel to y-axis. In such cases the equation of the parabolas include xy term, which is beyond the scope of this book, even though we will find the equation of the parabolas which are not in standard form. Note that for the standard types the axis is either parallel to the x-axis or parallel to y-axis.

All the parabolas discussed so far have vertex at origin. In general, the vertex need not be at the origin for any parabola. Hence, we need the concept of shifting the origin or translation of the axes.

The process of shifting the origin or translation of axes
Consider the xoy system. Draw a line parallel to x-axis (say X-axis) and draw a line parallel to y-axis (say Y-axis). Let $P(x, y)$ be the same point with respect to XOY system.
Let the co-ordinates O' with respect to xoy system be (h, k).



The co-ordinate of P with respect to xoy system:

$$OL = OM + ML = h + X$$
$$\text{i.e. } x = X + h$$

Similarly, $y = Y + k$
$\therefore$ The new co-ordinates of P with respect to XOY system

$$X = x - h$$
$$Y = y - k$$

General form of the standard equation of a parabola, which is open rightward (i.e. the vertex other than origin)
Consider a parabola with vertex V whose co-ordinates with respect to XOY system is $(0, 0)$ and with respect to xoy systems is (h, k).
Since it is open rightward, the equation of the parabola w.r.t XOY system is $Y^2 = 4aX$
By shifting the origin $X = x - h$ abd $Y = y - k$, the equation of the parabola with respect to old xoy system is

$$(y - k)^2 = 4a(x - h)$$

This is the general form of the standard equation of the parabola, which is open rightward. Similarly, the other general forms are

$$(y - k)^2 = -4a(x - h) \text{ (open leftwards)}$$
$$(x - h)^2 = 4a(y - k) \text{ (open upwards)}$$
$$(x - h)^2 = -4a(y - k) \text{ (open downwards)}$$

Note: To find the general form, replace x by $x - h$ and y by $y - k$ if the vertex is (h, k)

Example 1

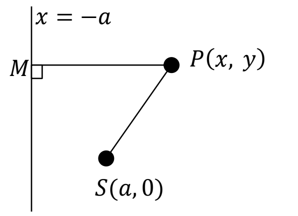
Find the equation of the following parabola with indicated focus and directrix.

(a) $(a, 0)$; $x = -a \quad a > 0$

(b) $(2, -3)$; $y - 2 = 0$

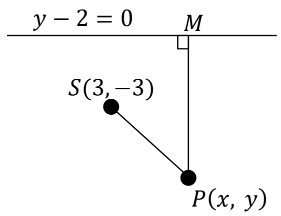
Solution

(a) Let $P(x, y)$ be any point on the parabola. If PM is drawn perpendicular to the directrix.



$$\frac{\overline{SP}}{\overline{PM}} = e = 1$$
$$\overline{SP}^2 = \overline{PM}^2$$
$$(x - a)^2 + (y - 0)^2 = \left(\frac{x + a}{\sqrt{1^2}}\right)^2$$
$$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$$
$$y^2 = 4ax$$

(b) Let $P(x, y)$ be any point on the parabola. If PM is drawn perpendicular to the directrix



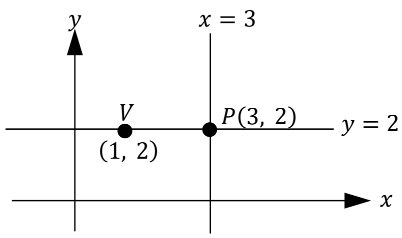
$$\frac{\overline{SP}}{\overline{PM}} = e = 1$$
$$\overline{SP}^2 = \overline{PM}^2$$
$$(x - 2)^2 + (y + 3)^2 = (y - 2)^2$$
$$x^2 - 4x + 4 + y^2 + 6y + 9 = y^2 - 4y + 4$$
$$x^2 - 4x + 10y + 9 = 0$$

Example 2

Find the equation of the parabola whose vertex is $(1, 2)$ and the equation of the latus rectum is $x = 3$.

Solution

From the given data, the parabola is open rightward.
The equation is of the form $(y - k)^2 = 4a(x - h)$
Here, the vertex $V(h, k)$ is $(1, 2)$.



Draw a perpendicular from V to the latus rectum. It passes through the focus. $\therefore F$ is $(3, 2)$
Again $VF = a = 2$
The required equation is

$$(y - 2)^2 = 4(2)(x - 1)$$
$$(y - 2)^2 = 8(x - 1)$$

Example 3

Find the equation of the parabola if the curve is open rightward, vertex is $(2, 1)$ and passing through point $(6, 5)$.

Solution

Since it is open rightward, the equation of the parabola is of the form $(y - k)^2 = 4a(x - h)$
The vertex $V(h, k)$ is $(2, 1)$

$$(y - 1)^2 = 4a(x - 2)$$

But it passes through $(6, 5)$

$$4^2 = 4a(6 - 2)$$
$$\Rightarrow a = 1$$

$\therefore$ The required equation is $(y - 1)^2 = 4(x - 2)$

Example 4

Find the equation of the parabola if the curve is open upward, vertex is $(-1, -2)$ and the length of the latus rectum is 4.

Solution

Since it is open upward, the equation is of the form

$$(x - h)^2 = 4a(y - k)$$

Length of the latus rectum = $4a = 4$ and this gives $a = 1$
The vertex $V(h, k)$ is $(-1, -2)$
Thus the required equation becomes

$$(x + 1)^2 = 4(y + 2)$$

Example 5

Find the equation of the parabola if the curve is open leftward, vertex is $(2, 0)$ and the distance between the latus rectum and directrix is 2.

Solution

Since it is open leftward, the equation is of the form

$$(y - k)^2 = -4a(x - h)$$

The vertex $V(h, k)$ is $(2, 0)$
The distance between latus rectum and directrix $= 2a = 2$ giving $a = 1$ and the equation of the parabola is

$$(y - 0)^2 = -4(1)(x - 2)$$
$$y^2 = -4(x - 2)$$

Example 6

Find the axis, vertex, focus, directrix, equation of the latus rectum, length of the latus rectum for the following parabolas and hence draw their graphs

(a) $y^2 = 4x$
(b) $x^2 = -4y$
(c) $(y + 2)^2 = -8(x + 1)$

The points of intersection of a straight line and parabola

The coordinates of the points of intersection of the straight line $y = mx + c$ and the parabola $y^2 = 4ax$ are the values of x which simultaneously satisfy both equations.

Substituting $y = mx + c$ in the equation of the parabola,

$$(mx + c)^2 = 4ax$$
$$m^2x^2 + 2(mc - 2a)x + c^2 = 0$$

The quadratic equation has real, equal or imaginary roots according as

$$[2(mc - 2a)]^2 - 4m^2c^2$$

is positive, zero or negative; i.e. according as c is less than, equal to, or greater than a/m.

The three possibilities can be illustrated as was done for the circle.

When $c < a/m$, the line intersects the parabola in two real points.

When $c > a/m$, it does not meet the parabola at all, or rather, it meets the curve in two imaginary points. If $c = a/m$, the line touches the parabola.

Substituting $c = \frac{a}{m}$ in the equation $y = mx + c$ to the line, we find that the line

$$y = mx + \frac{a}{m},$$

touches the parabola $y^2 = 4ax$ for all values of m

Parametric equations of a parabola

The point (x_1, y_1) lies on the parabola $y^2 = 4ax$ only if the relation $y_1^2 = 4ax_1$ between its two coordinates is satisfied. It is often convenient to be able to write down the coordinates of a point which always lies on the parabola. Such a point is one with coordinates $(at^2, 2at)$ for it is clear that if

$$x = at^2, y = 2at$$

then $y^2 = (2at)^2 = 4a(at^2) = 4ax$ for all values of t.

Equation of a tangent to a parabola at $P(at^2, 2at)$

$$x = at^2, y = 2at$$

Gradient at any point is

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dt} = 2a \times \frac{1}{2at} = \frac{1}{t}$$

Alternatively,

$$y^2 = 4ax$$
$$2y\frac{dy}{dx} = 4a$$
$$\frac{dy}{dx} = \frac{2a}{y}$$

At point $(at^2, 2at)$, $\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

Equation of a tangent becomes

$$\frac{y - 2at}{x - at^2} = \frac{1}{t}$$
$$ty - 2at^2 = x - at^2$$
$$x - ty + at^2 = 0$$

Equation of normal to the parabola at $P(at^2, 2at)$

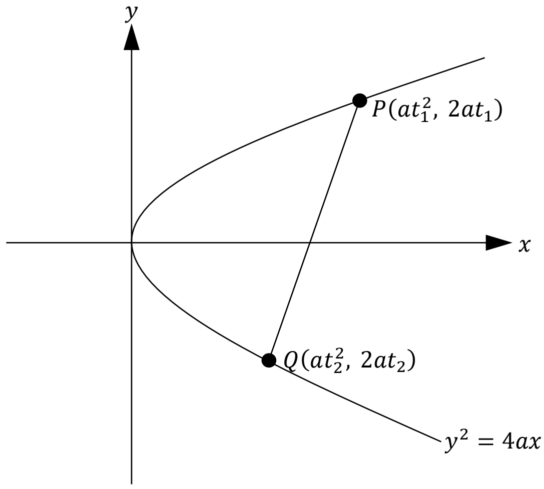
Gradient of tangent $= \frac{1}{t}$

Gradient of normal $= -t$

Equation of normal is thus given by

$$\frac{y - 2at}{x - at^2} = -t$$
$$y - 2at = -tx + at^3$$
$$y + tx = 2at + at^3$$

Equation of the chord joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$



Gradient of chord $= \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}$

Equation of chord is given by

$$\frac{y - 2at_1}{x - at_1^2} = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}$$
$$\frac{y - 2at_1}{x - at_1^2} = \frac{2(t_2 - t_1)}{(t_2 - t_1)(t_2 + t_1)}$$
$$\frac{y - 2at_1}{x - at_1^2} = \frac{2}{t_2 + t_1}$$
$$(t_1 + t_2)y - 2at_1(t_1 + t_2) = 2x - 2at_1^2$$
$$(t_1 + t_2)y - 2at_1^2 - 2at_1t_2 = 2x - 2at_1^2$$
$$2x - (t_1 + t_2)y + 2at_1t_2 = 0$$

Note: Many problems on the parabola are best solved by using the parametric equations. The above equations to the chord, tangent and normal are very useful in such work. The student should either remember them or (preferably) be able to derive them quickly.

Point of intersection of two tangents to the parabola

Consider tangents at $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ intersecting at a point T

Equation of a tangent at P is

$$x - py + ap^2 = 0 \text{ ... (i)}$$

Equation of a tangent at Q is

$$x - qy + aq^2 = 0 \text{ ... (ii)}$$

At the point of intersection, (i) and (ii) are satisfied simultaneously

(i) − (ii);

$$y(p-q)+ap^2-aq^2=0$$
$$y(p-q)=a(p^2-q^2)$$
$$y(p-q)=a(p-q)(p+q)$$
$$y=a(p+q)$$

Substituting for y in (i);

$$x-p[a(p+q)]+ap^2=0$$
$$x-ap^2-apq+ap^2=0$$
$$x=apq$$

The point of intersection, T is $[apq, a(p+q)]$

Since the chord passes through S, then the gradient of PS is equal to the gradient of SQ;

$$\frac{2ap}{ap^2-a}=\frac{2aq}{aq^2-a}$$
$$\frac{p}{p^2-1}=\frac{q}{q^2-1}$$
$$p(q^2-1)=q(p^2-1)$$
$$pq^2-qp^2=p-q$$
$$pq(q-p)=p-q$$

Since $p\neq q$, divide by $(q-p)$

$$\Rightarrow pq=-1$$

But the tangents at P and Q have gradients $\frac{1}{p}$ and $\frac{1}{q}$ respectively

The product of the gradients $=\frac{1}{pq}=-1$

Hence the tangents intersect at right angles

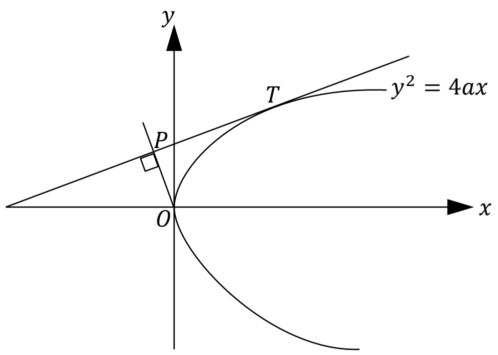
The coordinates of T become $[-a, a(p+q)]$

Hence T lies on the line $x=-a$ for all values of p and q

Example 12

The point $T(at^2, 2at)$ lies on the parabola with equation $y^2=4ax$. A straight line passing through the origin, intersects at right angles the tangent to the parabola at T, at the point P. Show that as t varies, the cartesian locus of P is $x^3+xy^2+ay^2=0$.

Solution



$$y^2=4ax$$
$$2y\frac{dy}{dx}=4a$$
$$\frac{dy}{dx}=\frac{2a}{y}$$
$$\left.\frac{dy}{dx}\right|_{y=2at}=\frac{1}{t}$$

Equation of tangent is

$$y-2at=\frac{1}{t}(x-at^2)$$

Equation of line OP is

$$y=-tx$$

Solving simultaneously;

$$-tx-2at=\frac{1}{t}(x-at^2)$$
$$-t^2x-2at^2=x-at^2$$
$$-at^2=x+t^2x$$
$$-at^2=x(1+t^2)$$
$$x=\frac{-at^2}{1+t^2}$$
$$y=\frac{at^3}{1+t^2}$$

i.e. $P\left(\frac{-at^2}{1+t^2}, \frac{at^3}{1+t^2}\right)$

Eliminate the parameter t;

$$X=\frac{-at^2}{1+t^2}\,;\,Y=\frac{at^3}{1+t^2}$$

Dividing

$$\frac{Y}{X}=-t$$
$$t=-\frac{Y}{X}$$

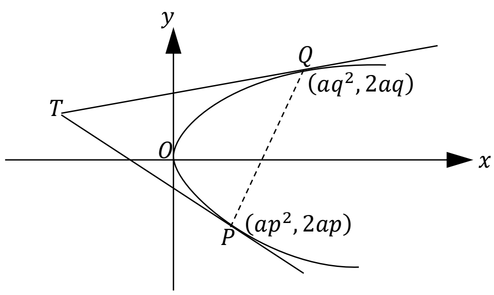
Substitute into either equation;

$$X=-\frac{a\left(\frac{-Y}{X}\right)^2}{1+\left(\frac{-Y}{X}\right)^2}$$
$$X=-\frac{\frac{aY^2}{X^2}}{1+\frac{Y^2}{X^2}}$$
$$X=-\frac{aY^2}{X^2+Y^2}$$
$$X^3+XY^2=-aY^2$$
$$X^3+XY^2+aY^2=0$$

Example 13

Find the equation of the tangent to the parabola $y^2=4ax$ at the point $P(ap^2, 2ap)$. If the tangents at P and Q with parameters p and q respectively intersect at T, find the locus of T, given that PQ is of constant length l.

Solution



Parametric equations of the parabola $x=at^2$, $y=2at$

Equation of QR is

$$\frac{y-0}{x+a}=\frac{2q}{q^2+1}$$

$$y=\frac{2q(x+a)}{q^2+1} \dots \text{(ii)}$$

To find the point of intersection R, we have to solve (i) and (ii) simultaneously i.e.

$$\frac{2}{p}x=\frac{2q(x+a)}{q^2+1}$$

$$x(q^2+1)=pq(x+a)$$

But $pq=-1$, thus

$$q^2+1=-\frac{x+a}{x}$$

$$q^2=-1-\frac{x+a}{x}$$

$$q^2=\frac{-2x-a}{x}$$

From (i); $p=\frac{2x}{y}$

$$pq=-1$$

$$p^2q^2=1$$

$$\frac{4x^2}{y^2}\left(\frac{-2x-a}{x}\right)=1$$

$$4x(-2x-a)=y^2$$

$$-8x^2-4ax=y^2$$

$$y^2+8x^2+4ax=0$$

Self-Evaluation exercise

1. Prove that the normal to the parabola $y^2=4ax$ at its points of intersection with the straight line $2x-3y+4a=0$ meet on the parabola
2. If the chord joining the points $(at_1^2,2at_1)$, $(at_2^2,2at_2)$ on the parabola $y^2=4ax$ passes through the focus $(a,0)$, find t_2 in terms of t_1. PQ is a focal chord and PL, QM are perpendicular to the axis of the parabola. Prove that $PL.QM$ is constant.
3. P is the point $(at_1^2,2at_1)$ and Q the point $(at_2^2,2at_2)$ on the parabola $y^2=4ax$. The tangent at P and Q intersect at R. Show that the area of the triangle PQR is $\frac{1}{2}a^2(t_1-t_2)^3$.
4. P is the point $(at^2,2at)$ on the parabola $y^2=4ax$. If PN is perpendicular from P to the x-axis and M is the point where the normal at P meets the x-axis, prove that the distance MN is independent of t.
5. P is a point on a parabola whose focus is S. D is the foot of the perpendicular from P to the directrix. Show that the tangent to the parabola at P bisects the angle SPD.
6. The normal to the parabola $y^2=4ax$ at the point $P(at^2,2at)$ meets the axis of the parabola at G and GP is produced beyond P, to Q so that $GP=PQ$. Show that the equation to the locus of Q is $y^2=16(x+2a)$
7. Show that the equation of the tangent to the curve $y^2=4ax$ at the point $(at^2,2at)$ is $ty=x+at^2$ and the equation of the tangent to the curve $x^2=4by$ at the point $(2bp,bp^2)$ is $y=px-bp^2$. The curves $y^2=32x$ and $x^2=4y$ intersect at the origin and at A. Find the equation of the common tangent to these curves and the coordinates of the points of contact B and C between the tangent and the curves. Calculate the area of the triangle ABC.

   [Ans: $2x+y+4=0$, $(2,-8)$, $(-4,4)$; 108]
8. Find the coordinates of the point of intersection R of the tangents to the parabola $y^2=4ax$ at the points $P(at_1^2,2at_1)$, $Q(at_2^2,2at_2)$. If the tangents at P, Q are inclined to one another at an angle of $45°$, show that the locus of R is the curve $y^2=x^2+6ax+a^2$

   [Ans: $[at_1t_2,a(t_1+t_2)]$]
9. P and Q are two points on the parabola $y^2=4ax$ whose coordinates are $(at_1^2,2at_1)$, $(at_2^2,2at_2)$. O is the origin of coordinates and OP is perpendicular to OQ. Show that $t_1t_2+4=0$ and that the tangents to the curve at P and Q meet on the line $x+4a=0$.
10. P is the point $(at^2,2at)$ on the parabola $y^2=4ax$. N is the foot of the perpendicular drawn from the origin to the tangent at P. Show that, as P varies, the locus of N is the curve $x(x^2+y^2)+ay^2=0$
11. The normal a point $P(at^2,2at)$ to the parabola $y^2=4ax$ meets the parabola again at point $R(aT^2,2aT)$. Prove that $T=-t-\frac{2}{t}$. Prove also that, if the normal at $Q(at_1^2,2at_1)$ passes through R,then $t_1=\frac{2}{t}$
12. The normal at the point $P(at^2,2at)$ to the parabola $y^2=4ax$ meets the curve again at the point $Q(at'^2,2at')$. Find t' in terms of t and hence, or otherwise, prove that the lines joining the origin to P and Q are at right angles if $t^2=2$.
13. Find the equation of the normal to the parabola $y^2=4ax$ at the point $(at^2,2at)$. The straight line $4x-9y+8a=0$ meets the parabola at the points P and Q; the normal to the parabola at the points P and Q meets at R. Find the coordinates of R, and verify that it lies on the parabola.

   [Ans: $\left(\frac{81}{4}a,-9aa\right)$]
14. Prove that the equation of the tangent to the parabola $y^2=4ax$ at the point $P(ap^2,2ap)$ on the curve is $py=x+ap^2$. Find the coordinates of the point of intersection, T, of the tangents at P and $Q(aq^2,2aq)$, simplifying your answers where possible. Given that S is the point $(a,0)$, verify that $SP.SQ=ST^2$

   [Ans: $[apq,a(p+q)]$]
15. Prove that the chord joining the points $P(ap^2,2ap)$and $Q(aq^2,2aq)$ on the parabola $y^2=4ax$ has the equation

$$(p+q)y=2r+2apq$$

The normal at the point (x_1, y_1) is the line through this point at right angles to the tangent. Its slope is therefore $\frac{a^2y_1}{b^2x_1}$ and its equation is

$$\frac{y-y_1}{x-x_1} = \frac{a^2y_1}{b^2x_1}$$

This can be written in the more symmetrical form

$$\frac{x-x_1}{x_1/a^2} = \frac{y-y_1}{y_1/b^2}$$

Example 7

Find the equations to the tangent and normal to the ellipse $5x^2 + 3y^2 = 137$ at the point $(5, 2)$

<u>Solution</u>

The equation of the ellipse can be written in the form

$$\frac{x^2}{(137/5)} + \frac{y^2}{(137/3)} = 1$$

so that $a^2 = 137/5$ and $b^2 = 137/3$

The equation of the tangent at (x_1, y_1) is given by

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

$$\frac{5x}{(137/5)} + \frac{2y}{(137/3)} = 1$$

$$25x + 6y = 137$$

The normal at (x_1, y_1) is given by

$$\frac{x-5}{5/(137/5)} = \frac{y-2}{2/(137/3)}$$

$$6x - 30 = 25y - 50$$

$$6x - 25y + 20 = 0$$

The points of intersection of a straight line and ellipse

The coordinates of the points of intersection of the straight line $y = mx + c$ and the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are the values of x and y which simultaneously satisfy both equations. Writing $y = mx + c$ in the equation to the ellipse,

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$$b^2x^2 + a^2(mx+c)^2 = a^2b^2$$

$$b^2x^2 + a^2m^2x^2 + 2a^2mcx + a^2c^2 = a^2b^2$$

$$(a^2m^2 + b^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0$$

This quadratic equation has real, equal or imaginary roots according as

$$(2a^2mc)^2 - 4(a^2m^2 + b^2)a^2(c^2 - b^2)$$

is positive, zero or negative i.e. according as c^2 is less, equal to or greater than $a^2m^2 + b^2$.

Again, the three possibilities can be illustrated as was done for the circle.

When $c^2 < a^2m^2 + b^2$, the line intersects the ellipse in two real points.

When $c^2 > a^2m^2 + b^2$, the line intersects the ellipse only in imaginary points.

If $c^2 = a^2m^2 + b^2$, the line is a tangent to the ellipse.

Writing $c = \sqrt{a^2m^2 + b^2}$ in the equation $y = mx + c$ to the line, we find that the line

$$y = mx + \sqrt{(a^2m^2 + b^2)}$$

always touches the ellipse.

Further, since the radical sign on the right-hand side may have either positive or negative signs attached to it, we see that there are two tangents to the ellipse having the same m. In other words, there are two tangents parallel to any given direction.

The parametric equations to an ellipse

When dealing with an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, working is generally made easier by using a parameter, but the question arises of what parameter to use. Now an equation in the form

$$(\quad)^2 + (\quad)^2 = 1$$

suggests the identity

$$\cos^2\theta + \sin^2\theta = 1$$

Thus

$$\frac{x}{a} = \cos\theta \Rightarrow x = a\cos\theta$$

$$\frac{y}{b} = \sin\theta \Rightarrow y = b\sin\theta$$

We therefore take a general point on the ellipse $(\boldsymbol{a\cos\theta}, \boldsymbol{b\sin\theta})$, θ is called the **eccentric angle** of the point.

Example 8

An ellipse has parametric equations

$$x = 4\cos\theta,\ y = \sqrt{7}\sin\theta$$

a) Find the coordinates of foci

b) Sketch the ellipse

<u>Solution</u>

(a) $x = 4\cos\theta \Rightarrow x^2 = 16\cos^2\theta$

$$\frac{x^2}{16} = \cos^2\theta$$

$y = \sqrt{7}\sin\theta,\ y^2 = 7\sin^2\theta$

$$\frac{y^2}{7} = \sin^2\theta$$

$$\therefore \frac{x^2}{16} + \frac{y^2}{7} = 1$$

Eccentricity relation: $b^2 = a^2(1 - e^2)$

$$7 = 16(1 - e^2)$$

$$\frac{7}{16} = 1 - e^2$$

$$e^2 = \frac{9}{16}$$

$$e = \frac{3}{4}$$

Foci at $(\pm ae, 0)$

Here $\left(\pm 4 \times \frac{3}{4}, 0\right)$

$$\therefore (\pm 3, 0)$$

i.e. $M(-\cos t, \sin t)$

$$X = -\cos t, Y = \sin t$$
$$\Rightarrow X^2 + Y^2 = 1$$

Example 13

Find the condition that the line $y = mx + c$ should touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

<u>Solution</u>

The equation of any tangent to the ellipse may be written

$$bx\cos\theta + ay\sin\theta - ab = 0$$

Let this equation represent the same tangent as the given line which we shall write as

$$mx - y + c = 0$$

Comparing coefficients,

$$\frac{b\cos\theta}{m} = \frac{a\sin\theta}{-1} = \frac{-ab}{c}$$
$$\therefore \cos\theta = -\frac{am}{c}, \sin\theta = \frac{b}{c}$$

But $\cos^2\theta + \sin^2\theta = 1$,

$$\therefore \frac{a^2m^2}{c^2} + \frac{b^2}{c^2} = 1$$

Therefore $y = mx + c$ touches the ellipse if

$$c^2 = a^2m^2 + b^2$$

Example 14

Find the equation of the chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ joining the points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$

<u>Solution</u>

$$\text{Gradient of } PQ = \frac{b\sin\theta - b\sin\phi}{a\cos\theta - a\cos\phi}$$
$$= \frac{2b\cos\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\theta-\phi)}{-2a\sin\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\theta-\phi)}$$
$$= -\frac{b\cos\frac{1}{2}(\theta+\phi)}{a\sin\frac{1}{2}(\theta+\phi)}$$

The equation of the chord PQ is

$$\frac{y - b\sin\theta}{x - a\cos\theta} = -\frac{b\cos\frac{1}{2}(\theta+\phi)}{a\sin\frac{1}{2}(\theta+\phi)}$$

$$ay\sin\frac{1}{2}(\theta+\phi) - ab\sin\theta\sin\frac{1}{2}(\theta+\phi)$$
$$= -bx\cos\frac{1}{2}(\theta+\phi) + ab\cos\theta\cos\frac{1}{2}(\theta+\phi)$$
$$bx\cos\frac{1}{2}(\theta+\phi) + ay\sin\frac{1}{2}(\theta+\phi)$$
$$= ab\left[\cos\theta\cos\frac{1}{2}(\theta+\phi) + \sin\theta\sin\frac{1}{2}(\theta+\phi)\right]$$
$$= ab\cos\left[\theta - \frac{1}{2}(\theta+\phi)\right]$$
$$= ab\cos\frac{1}{2}(\theta-\phi)$$

Hence the equation of the chord PQ may be written:

$$\frac{x}{a}\cos\frac{1}{2}(\theta+\phi) + \frac{y}{b}\sin\frac{1}{2}(\theta+\phi) = \cos\frac{1}{2}(\theta-\phi)$$

or

$$bx\cos\frac{1}{2}(\theta+\phi) + ay\sin\frac{1}{2}(\theta+\phi) = ab\cos\frac{1}{2}(\theta-\phi)$$

Example 15

The point $P(5\cos\theta, 4\sin\theta)$ lies on an ellipse E with Cartesian equation $16x^2 + 25y^2 = 400$

(a) Find the coordinates of the foci of E

(b) Show that an equation of the normal to the ellipse at P is $4y\cos\theta - 5x\sin\theta + 9\sin\theta\cos\theta = 0$

The normal to the ellipse intersects the coordinate axes at the points A and B, and the point M is the midpoint of AB.

(c) Show that the locus of M, as θ varies, is the ellipse with equation $100x^2 + 64y^2 = 81$

<u>Solution</u>

(a) $16x^2 + 25y^2 = 400$

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$
$$b^2 = a^2(1 - e^2)$$
$$16 = 25(1 - e^2)$$
$$\frac{16}{25} = 1 - e^2$$
$$e^2 = \frac{9}{25}$$
$$e = \frac{3}{5}$$

Foci at $(\pm ae, 0)$

$$\therefore (\pm 3, 0)$$

(b) Differentiate w.r.t x

$$32x + 50y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{16x}{25y}$$
$$\left.\frac{dy}{dx}\right|_p = \frac{-80\cos\theta}{100\sin\theta} = -\frac{4\cos\theta}{5\sin\theta}$$

Normal gradient is $\frac{5\sin\theta}{4\cos\theta}$

Thus,

$$y - 4\sin\theta = \frac{5\sin\theta}{4\cos\theta}(x - 5\cos\theta)$$
$$4y\cos\theta - 16\sin\theta\cos\theta = 5x\sin\theta - 25\sin\theta\cos\theta$$
$$4y\cos\theta - 5x\sin\theta + 9\sin\theta\cos\theta = 0$$

(c) When $x = 0$, $y = -\frac{9\sin\theta}{4}$

When $y = 0$, $x = \frac{9\cos\theta}{5}$

$$\therefore M\left(\frac{9\cos\theta}{10}, \frac{-9\sin\theta}{8}\right)$$
$$x = \frac{9\cos\theta}{10} \Rightarrow x^2 = \frac{81}{100}\cos^2\theta \Rightarrow \cos^2\theta = \frac{100}{81}x^2$$

When $y = \frac{4}{5}b$, $x^2 = 3b^2 - 3\left(\frac{4}{5}b\right)^2$

$$x^2 = 3b^2 - \frac{48b^2}{25} = \frac{27b^2}{25}$$

$$x = \pm\frac{3\sqrt{3}}{5}b$$

Ignore the negative value of x

$$\Rightarrow \left(\frac{3\sqrt{3}}{5}b, \frac{4}{5}b\right)$$

The perpendicular bisector of AB meets the ellipse at the two points $(0, -b)$ and $\left(\frac{3\sqrt{3}}{5}b, \frac{4}{5}b\right)$

Self-Evaluation exercise

1. Find in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the equations of the ellipses with
   (a) eccentricity $\frac{1}{2}$, foci $(\pm 2, 0)$ (b) eccentricity $\frac{3}{5}$, foci $(\pm 9, 0)$
   [Ans: (a) $\frac{x^2}{16} + \frac{y^2}{12} = 1$ (b) $\frac{x^2}{225} + \frac{y^2}{144} = 1$]
2. Use the Locus definition of the ellipse to find the equation of the ellipse with eccentricity $\frac{2}{3}$, focus $(2, 1)$ and directrix $x = -\frac{1}{2}$
   [Ans: $5x^2 + 9y^2 - 40x - 18y + 44 = 0$]
3. P is the point $(5\cos\theta, 4\sin\theta)$ on the curve $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and S' are the points with coordinates $(-3, 0)$ and $(3, 0)$, show that the value of $PS + PS'$ is independent of θ.
4. Show that the equation to the chord joining the two points whose eccentric angles are ϕ, ϕ' on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is
   $\frac{x}{a}\cos\frac{1}{2}(\phi + \phi') + \frac{y}{b}\sin\frac{1}{2}(\phi + \phi') = \cos\frac{1}{2}(\phi - \phi')$
   Deduce the equation to the tangent at the point ϕ
5. Show that if the tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ intersect at the point R, then the coordinates of R are $\left(\frac{a\cos\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}, \frac{b\sin\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}\right)$. If P and Q move on the ellipse in such a way that $\phi = \theta + \frac{1}{2}\pi$, find the equation of the locus of R. [Ans: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$]
6. The line $y = mx + c$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a > b > 0)$. Show that $c^2 = a^2m^2 + b^2$. The perpendicular distances from the points $\left(\sqrt{a^2 - b^2}, 0\right)$ and $\left(-\sqrt{a^2 - b^2}, 0\right)$, to any tangent to the ellipse are p_1, p_2. Show that $p_1p_2 = b^2$.
7. Prove that the equation of the tangent at the point (x_1, y_1) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$. The tangent at the point $\left(2\cos\theta, \sqrt{3}\sin\theta\right)$ on the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ passes through the point $P(2, 1)$. Show that $\sqrt{3}\cos\theta + \sin\theta = \sqrt{3}$.
8. Prove that the equation of the normal at $(\alpha\cos\phi, \beta\sin\phi)$ to the ellipse $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$ is $\alpha x\sec\phi - \beta y\operatorname{cosec}\phi = \alpha^2 - \beta^2$. P is the point $(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. M and N are the feet of the perpendiculars from P to the axes. Find the equation of MN. Prove that, for variable θ, MN is always normal to a fixed concentric ellipse and find the equation to this ellipse.
   [Ans: $bx\sec\theta + ay\operatorname{cosec}\theta = ab$; $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{ab}{a^2-b^2}\right)^2$]
9. Show that the tangents to the ellipse $x^2 + 2y^2 = 18$ at the points $(0, -3)$, $\left(-\frac{72}{17}, -\frac{3}{17}\right)$ intersect on the normal at the point $(4, 1)$.
10. The equation to a chord of the ellipse $x^2 + 4y^2 = 260$ is $x + 6y = 50$. Find the coordinates of its middle point.
    [Ans: $(5, \frac{15}{2})$]
11. Show that the x coordinates of any points of intersection of the line $y = mx + c$ and the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ are given by the solutions of the quadratic equation $(4 + 9m^2)x^2 + 18mcx + (9c^2 - 36) = 0$.
    If the line $y = mx + c$ is a tangent to the ellipse, prove that $c^2 = 4 + 9m^2$. The line $y = mx + c$ passes through the point $(2, 3)$. Write down a second equation connecting m and c, ane hence prove that m must satisfy the equation $5m^2 + 12m - 5 = 0$
    Prove that the two tangents drawn from the point $(2, 3)$ to the ellipse are perpendicular to each other.
    [Ans: $2m + c = 3$]
12. Show that the tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at points whose eccentric angles differ by 90° meet on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$.
13. Show that the equation of the tangent to the parabola $y^2 = 4ax$ at $P(ap^2, 2ap)$ is $x - py + ap^2 = 0$.
    Show also that the equation of the tangent to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ at $Q(a\cos\theta, b\sin\theta)$ is $bx\cos\theta + ay\sin\theta - ab = 0$. Hence or otherwise, show that the tangent to the parabola at P is also a tangent to the ellipse $4x^2 + y^2 = 4a^2$ if p satisfies the equation $1 + 4p^2 = p^4$.
    Deduce that there are exactly two such common tangents. Determine the points of contact between the tangents and the ellipse $4x^2 + y^2 = 4a^2$.

6

9. The tangents at the points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ on the rectangular hyperbola $xy = c^2$ intersect at the point R. Given that R lies on the rectangular hyperbola $xy = \frac{1}{2}c^2$, find the equation of the locus of the midpoint M of PQ as p and q vary.

[Ans: $xy = c^2$]

10. Show that the tangent at the point P, with parameter t, on the curve $x = ct, y = \frac{c}{t}$ has equation $x + t^2y = 2ct$. This tangent meets the x-axis in a point Q and the line through P parallel to the x-axis cuts the y-axis in a point R. Show that, for any position of P on the curve, QR is a tangent to the curve with parametric equations $x = ct$, $y = \frac{c}{2t}$

11. Prove that the equation of the normal to the rectangular hyperbola $xy = c^2$ at the point $P\left(ct, \frac{c}{t}\right)$ is $ty - t^3x = c(1 - t^4)$. The normal at P and the normal at the point $Q\left(ct, \frac{c}{t}\right)$, where $t > 1$, intersect at the point N. Show that $OPNQ$ is a rhombus, where O is the origin. Hence, or otherwise, find the coordinates of N. If the tangents to the hyperbola at P and Q intersect at T, prove that the product of the lengths of OT and ON is independent of T, prove that the product of the lengths of OT and ON is independent of t

[Ans: $\left(ct + \frac{c}{t}, ct + \frac{c}{t}\right)$]

12. Show that the equation to the chord two joining two points (x_1, y_1), (x_2, y_2) on the rectangular hyperbola $xy = c^2$ is

$$\frac{x}{x_1 + x_2} + \frac{y}{y_1 + y_2} = 1$$

13. The tangent at P to the rectangular hyperbola $xy = c^2$ meets the lines $x - y = 0$ and $x + y = 0$ at A and B, and Δ denotes the area of the triangle OAB where O is the origin. The normal at P meets the x-axis at C and the y-axis at D. If Δ_1 denotes the area of the triangle ODC show that $\Delta^2\Delta_1 = 8c^6$.

14. (a) Find the equation of the tangent to the curve $xy = c^2$ at the point $\left(ct_1, \frac{c}{t_1}\right)$
    (b) Find the equation of the normal to the curve $xy = c^2$ at the point $\left(ct_2, \frac{c}{t_2}\right)$
    (c) If the tangent of (a) meets the normal of (b) on the y-axis show that $2t_2 = t_1(1 - t_2^4)$

15. Find the equation of the chord joining the point $\left(ct_1, \frac{c}{t_1}\right)$ to the point $\left(ct_2, \frac{c}{t_2}\right)$ on the hyperbola $xy = c^2$.

16. By letting $t_1 = t_2 = t$ use your answer to part (a) to obtain the tangent to the curve $xy = c^2$ at the point $\left(ct, \frac{c}{t}\right)$

17. Given that $y = mx + c$ is a tangent to $xy = d^2$ prove that $m = -\frac{c^2}{4d^2}$

18. Show that the tangent at the point P, with parameter t, on the curve $x = ct$, $y = \frac{c}{t}$ has the equation $x + t^2y = 2ct$.

19. This tangent meets the x-axis in a point Q and the line through P parallel to the x-axis cuts the y-axis in a point R. Show that, for any position of P on the curve, QR is a tangent to the curve with parametric equations $x = ct$, $y = \frac{c}{2t}$

20. Show that the equation to the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(a\sec\theta, b\tan\theta)$ is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$. Find also the equation of the normal.

21. The tangent at the point $P\left(ct, \frac{c}{t}\right)$, where $t > 0$, on the rectangular $xy = c^2$ meets the x-axis at A and the y-axis at B. The normal at P to the rectangular hyperbola meets the line $y = x$ at C and the line $y = -x$ at D.
    (a) Show that P is the mid-point of both AB and CD
    (b) Prove that the points A, B, C and D form the vertices of a square. The normal at P meets the hyperbola again at the point Q and the midpoint of PQ is M
    (c) Prove that, as t varies, the point M lies on the curve
    $$c^2(x^2 - y^2)^2 + 4x^3y^3 = 0$$

22. The point $P\left(ap, \frac{a}{p}\right)$ lies on the rectangular hyperbola H, with Cartesian equation $xy = a^2$ where a is a positive constant and p is a parameter.
    (a) Show that the equation of a tangent at the point P is given by
    $$x + p^2y = 2ap$$
    The point $Q\left(aq, \frac{a}{q}\right)$ also lies on H, where q is a parameter, so that $q \neq p$. The tangent at P and at Q intersect at the point R.
    (b) Find simplified expressions for the coordinates of R.

    The values of p and q are such so that $p = 3q$
    (c) Find a Cartesian locus of R as p varies

[Ans: (b) $R\left(\frac{2apq}{p+q}, \frac{2a}{p+q}\right)$ (c) $xy = \frac{3}{4}a^2$]

23. The general point $P\left(cp, \frac{c}{p}\right)$, $p \neq 0$, where p is a parameter, lies on the rectangular hyperbola, with Cartesian equation $xy = c^2$.
    (a) Show that the equation of the tangent to the hyperbola at P is given by $yp^2 + x = 2cp$

    Another point $Q\left(cq, \frac{c}{q}\right), p \neq \pm q$ also lies on the hyperbola. The tangents to the hyperbola at P and Q meet at the point R.

(b) Show that the coordinates of R are given by

$$\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$$

(c) Given that PQ is perpendicular to OR, show that

$$p^2q^2 = 1$$

24. The general point $P\left(5t, \frac{5}{t}\right)$ where t is a parameter lies on the hyperbola with cartesian equation $xy = 25$.

(a) Show that an equation of the normal to the hyperbola at the point P is $y = t^2x + \frac{5}{t} - 5t^3$

The normal to the hyperbola at P meets the hyperbola again at the point Q.

(b) Show that the coordinates of Q are given by

$$\left(-\frac{5}{t^3}, -5t^3\right)$$

(c) Show that the Cartesian form of the locus of the midpoint of PQ, as t varies is given by

$$4xy + 25\left(\frac{y}{x} - \frac{x}{y}\right)^2 = 0$$

25. The general point $P\left(\frac{p}{2}, \frac{1}{2p}\right)$ where p is a parameter, lies on a rectangular hyperbola, with Cartesian equation $4xy = 1$

The normal to the hyperbola meets the hyperbola again at the point Q.

Show that the Cartesian form of the locus of the midpoint of PQ, as p varies, is

$$(y^2 - x^2)^2 + 16x^3y^3 = 0$$

26. Two distinct points $P\left(2p, \frac{2}{p}\right)$ and $Q\left(2q, \frac{2}{q}\right)$, lie on the hyperbola with Cartesian equation $xy = 4$.

The tangents to the hyperbola at the points P and Q, meet at the point R.

(a) Show that the coordinates of the point R are given by $x = \frac{4pq}{p+q}$, $y = \frac{4}{p+q}$

(b) Given that the point R traces the rectangular hyperbola $xy = 3$, find the two possible relationships between p and q, in the form $p = f(q)$

[Ans: $p = 3q$, $p = \frac{1}{3}q$]

27. A hyperbola H and a line L have Cartesian equations $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $y = mx + c$ respectively where a,b, m and c are non-zero constants

(a) Show that the x coordinates of the points of intersection between L and H satisfy the equation

$$(a^2m^2 - b^2)x^2 + (2a^2mc)x + a^2(b^2 + c^2) = 0$$

(b) Given the line is a tangent to the hyperbola, show that $a^2m^2 = b^2 + c^2$

(c) Find the equations of the two tangents to the hyperbola with Cartesian equation $\frac{x^2}{25} - \frac{y^2}{16} = 1$ that pass through the point $(1, 4)$ and for each tangent the coordinates of their point of tangency.

[Ans: $y = x + 3$, $\left(-\frac{25}{3}, -\frac{16}{3}\right)$; $y = -\frac{4}{3}x + \frac{16}{3}$, $\left(\frac{25}{4}, -3\right)$]

28. A hyperbola H has Cartesian equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where a and b are positive constants.

The straight line T_1 is the tangent to H at the point $(a\cos h\theta, b\sin h\theta)$. T_1 meets the x-axis at the point P.

The straight line T_2 is a tangent to the hyperbola at the point $(a, 0)$. T_1 and T_2 meet each other at the point Q.

Given further that M is the midpoint of PQ, show that as θ varies, the locus of M traces the curve with equation

$$x(4y^2 + b^2) = ab^2$$

SUMMARY

Cartesian form:	**Parabola**	**Ellipse**	**Hyperbola**
Equation of chord joining (x_1, y_1) and (x_2, y_2)	$y - y_1 = \frac{4a}{y_1 + y_2}(x - x_1)$	$y - y_1 = -\frac{b^2(x_1 + x_2)}{a^2(y_1 + y_2)}(x - x_1)$	$y - y_1 = \frac{b^2(x_1 + x_2)}{a^2(y_1 + y_2)}(x - x_1)$
Equation of tangent at (x_1, y_1)	$yy_1 = 2a(x + x_1)$	$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$	$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$
Equation of normal at (x_1, y_1)	$xy_1 + 2ay = x_1y_1 + 2ay_1$	$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$	$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$
Parametric form:			
Equation of chord	Chord joining the points t_1 and t_2 is $y(t_1 + t_2) = 2x + 2at_1t_2$	Chord joining the points θ_1 and θ_2 is $\frac{x}{a}\cos\frac{(\theta_1+\theta_2)}{2} + \frac{y}{b}\sin\frac{(\theta_1+\theta_2)}{2} = \cos\frac{(\theta_1-\theta_2)}{2}$	Chord joining the points θ_1 and θ_2 is $\frac{x}{a}\cos\frac{(\theta_1-\theta_2)}{2} + \frac{y}{b}\sin\frac{(\theta_1+\theta_2)}{2} = \cos\frac{(\theta_1+\theta_2)}{2}$
Equation of tangent	at t is $yt = x + at^2$	at θ is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$	at θ is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$
Equation of normal	at t is $tx + y = 2at + at^3$	$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$	$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$

Chapter 17 Examination Questions

SECTION A

1. The equation of an ellipse is $4x^2 + 25y^2 + 8x - 100y + 4 = 0$. Detemine the;
   (a) coordinates of the centre of the ellipse.
   (b) eccentricity of the ellipse
   [2024, No. 3]
2. If a line $y = mx + c$ is a tangent to the curve $4x^2 + 3y^2 = 12$, show that $c^2 = 4 + 3m^2$.
   [2023, No. 2]
3. A line L passes through the point of intersection of the lines $x - 3y - 4 = 0$ and $y + 3x - 2 = 0$. If L is perpendicular to the line $4y + 3x = 0$, determine the equation of the line L.
   [2022, No. 7]
4. (a) Show that the curve whose parametric equations are $x = 9\cos\theta$ and $y = 12\sin\theta$ represents an ellipse
   (b) Determine the eccentricity of the ellipse
   [2020, No. 7]
5. A point P moves such that its distances from two points $A(-2, 0)$ and $B(8, 6)$ are in the ratio $AP{:}PB = 3 : 2$. Show that the locus of P is a circle.
   [2018, No. 5: Ans: $x^2 + y^2 - 32x - 21.6y + 176.8 = 0$]
6. The equation of a curve is given by $y^2 - 6y + 20x + 49 = 0$.
   (a) Show that the curve is a parabola
   (b) Find the coordinates of its vertex
   [2017, No. 7: Ans: (b) $(-2, 3)$]
7. Find the angle between the lines $2x - y = 3$ and $11x + 2y = 13$.
   [2016, No. 2: Ans: $36.87°$]
8. Find the equation of a line through the point $(5, 3)$ and perpendicular to the line $2x - y + 4 = 0$.
   [2015, No. 2: Ans: $2y + x = 11$]
9. A focal chord PQ, to the parabola $y^2 = 4x$, has a gradient $m = 1$. Find the coordinates of the midpoint of PQ.
   [2014, No. 2: Ans: $(3, 2)$]
10. Given that $r = 3\cos\theta$ is an equation of a circle, find its Cartesian form.
    [2013, No. 3: Ans: $x^2 + y^2 - 3x = 0$]
11. The line $y = mx + c$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ when $c = \pm\sqrt{a^2m^2 + b^2}$. Find the equations of the tangents to the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$ from the point $(0, \sqrt{5})$.
    [2012, No. 6: Ans: $y = \pm x + \sqrt{5}$]
12. Find the equation of a line through the point $(2, 3)$ and perpendicular to the line $x + 2y + 5 = 0$.
    [2011, No. 2: Ans: $y = 2x - 1$]
13. The points A and B lie on the positive sides of the x-axis and y-axis respectively. If the length AB is 5 units and angle OAB is θ, where O is the origin, find the equation of the line AB. (Leave θ in your answer)
    [2010, No. 2: Ans: $y = -x\tan\theta + 5\sin\theta$]
14. Given the points $O(0, 0)$ and $P(4, 2)$, A is the locus of the points such that $OA : AP = 1{:}2$. Q is the mid-point of AP. Find the locus of Q in its simplest form.
    [2008, No. 5: Ans: $3X^2 + 3Y^2 - 8X - 4Y = 0$]
15. Find the locus of the point $P(x, y)$ which moves such that its distance from the point $S(-3, 0)$ is equal to its distance from a fixed line $x = 3$.
    [2007, No. 7: Ans: $y^2 = -12x$]
16. Prove that $y = -3x + 6$ is a tangent to the rectangular hyperbola whose parametric co-ordinates are of the form $\left(\sqrt{3}t, \frac{\sqrt{3}}{t}\right)$
    [2006, No. 4]
17. Sketch the parabola $y^2 = 12(x - 4)$. State the focus and equation of the directrix.
    [2005, No. 7: Ans: $(7, 0)$, $x = 1$]
18. A is the point $(1, 3)$ and B is the point $(4, 6)$. P is a variable point which moves in such a way that $(\overline{AP})^2 + (\overline{PB})^2 = 34$. Show that the locus of P describes a circle. Find the centre and radius of the circle.
    [2004, No. 5: Ans: $\left(\frac{5}{2}, \frac{9}{2}\right)$, $r = \frac{5\sqrt{2}}{2}$ units]
19. The points $A(2, 1)$, $P(\alpha, \beta)$ and point $B(1, 2)$ lie in the same plane. PA meets the x-axis at the point $(h, 0)$ and PB meets the y-axis at the point $(0, k)$. Find h and k in terms of α and β.
    [2003, No. 7: Ans: $h = \frac{2\beta-\alpha}{\beta-1}$, $k = \frac{2\alpha-\beta}{\alpha-1\alpha}$]
20. The points $R(2, 0)$ and $P(3, 0)$ lie on the x-axis and $Q(0, -y)$ lies on the y-axis. The perpendicular from the origin meets PQ at $S(X, -Y)$. Determine the locus of S in terms of X and Y.
    [2002, No. 4: Ans: $2X^2 + 3Y^2 - 6X = 0$]
21. Find the locus of the point which is equidistant from the line $x = 2$ and the circle $x^2 + y^2 = 1$. Illustrate this with a sketch.
    [2001, No. 7: Ans: $y^2 + 6x - 9 = 0$]
22. Show that the line $x - 2y + 10 = 0$ is a tangent to the ellipse $\frac{x^2}{64} + \frac{y^2}{9} = 1$

13. (a) Find the equation of the tangent to the parabola $y^2 = \frac{x}{8}$ at the point $\left(t^2, \frac{t}{4}\right)$
(b) If the tangents to the parabola in (a) above at the points $P\left(p^2, \frac{p}{4}\right)$ and $Q\left(q^2, \frac{q}{4}\right)$ meet on the line $y = 2$
(i) show that $p + q = 16$,
(ii) deduce that the mid-point of PQ lies on the line $y = 2$
[2011, No. 13: Ans: (a) $x - 8ty + t^2 = 0$]

14. (a) (i) Find the co-ordinates of the points where the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ cuts the axes
(ii) Express the given equation in a(i) above, in its polar form.
(b) If the line $y = mx + c$ is tangent to the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, show that $c^2 = 4m^2 + 9$
[2010, No. 13: Ans: (a) (i) $(0, -3)$, $(0, 3)$ (ii) $(3\cos\theta, 2\sin\theta)$]

15. (a) Find the equation of the tangent and normal to the ellipse
(b) If the tangent in (a) cuts the y-axis at a point A and the x-axis at a point B, and the normal cuts the x-axis at point C, find the co-ordinates of points A, B and C.
[2009, No. 15: Ans: (a) $\frac{x}{2}\cos\theta + y\sin\theta = 1$; $\frac{2}{3}x\sec\theta - \frac{1}{3}y\,\text{cosec}\,\theta = 1$ (b) $A(0, \text{cosec}\,\theta)$, $B(2\sec\theta, 0)$, $C(1.5\cos\theta, 0)$]

16. A circle cuts the y-axis at two points A and B. It touches the x-axis at a distance of 4 units from the origin and distance AB is 6 units. A is the point $(0, 1)$. Find the:
(a) equation of the circle
(b) equations of the tangents to the circle at A and B
[2008, No. 9]

17. (i) Show that the equation of the tangent to the hyperbola $(a\sec\theta, b\tan\theta)$ is
$$bx - ay\sin\theta - ab\cos\theta = 0$$
(ii) Find the equations of the tangents to $\frac{x^2}{4} - \frac{y^2}{9} = 1$, at the points where $\theta = 45°$ and where $\theta = -135°$
(iii) Find the asymptotes
[2007, No. 11: Ans: (ii) $y = \left(\frac{3}{2}\sqrt{2}\right)x - 3$, $y = \frac{3\sqrt{2}}{2}x + 3$]

18. (a) Form the equation of a circle that passes through the points $A(-1, 4)$, $B(2, 5)$ and $C(0, 1)$.
(b) The line $x + y = c$ is a tangent to the circle $x^2 + y^2 - 4y + 2 = 0$. Find the coordinates of the points of contact of the tangent for each value of c
[2006, No. 13: Ans: (a) $x^2 + y^2 - 2x - 6y + 5 = 0$ (b) $c = 0, 4$; $(-1, 1)$; $(1, 3)$]

19. (a) Find the equation of a circle which passes through the points $(5, 7)$, $(1, 3)$ and $(2, 2)$.
(b) (i) If $x = 0$ and $y = 0$ are tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, show that $c^2 = g^2 = f^2$
(ii) Given that the line $3x - 4y + 6 = 0$ is also a tangent to the circle in (b) (i) above, determine the equation of the circle lying in the first quadrant.
[2005, No. 12: Ans: (a) $x^2 + y^2 - 7x - 9y + 24 = 0$ (b)(ii) $x^2 + y^2 - 2x - 2y + 1 = 0$]

20. (a) Show that the line $5y = 4x + 25$ is a tangent to the ellipse
$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$
(b) Find the equation of the normal to the ellipse at the point of contact
(c) Determine the eccentricity of the ellipse.
[2004, No. 13: Ans: (b) $y = -\frac{5}{4}x - \frac{16}{5}$ (c) $e = \pm\frac{4}{5}$]

21. (a) A conic section is given by $x = 4\cos\theta$; $y = 3\sin\theta$. Show that the conic section is an ellipse and determine its eccentricity.
(b) Given that the line $y = mx + c$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, show that $c^2 = a^2m^2 + b^2$. Hence determine the equations of the tangents at the point $(-3, 3)$ to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$
[2003, No. 14: Ans: (a) $e = \frac{\sqrt{7}}{4}$ (b) $y = 3$, $y = \frac{18}{7}x + \frac{75}{7}$]

22. P is a variable point given by the parametric equations
$$x = \frac{1}{2}\left(t + \frac{1}{t}\right);\ y = \frac{b}{2}\left(t - \frac{1}{t}\right)$$
Show that the locus of P is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
State the asymptotes. Determine the coordinates of the points where the tangent from P meets the asymptotes.
[2002, No. 10: Ans: $y = \pm\frac{b}{a}x$; (at, bt), $\left(\frac{a}{t}, -\frac{b}{t}\right)$]

23. (a) (i) Find the equation of the chord through the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ of the parabola $y^2 = 4ax$
(ii) Show that the chord cuts the directrix when
$$y = \frac{2a(t_2t_1 - 1)}{t_1 + t_2}$$
(b) Find the equation of the normal to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ and determine its point of intersection with the directrix.
[2001, No. 12: Ans: (b) $y + tx - at^3 - 2at = 0$, $(-a, at(t^2 + 3))$]

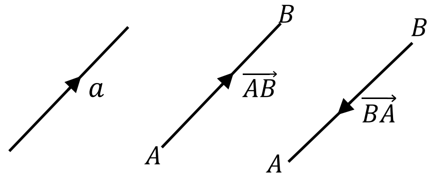
24. (a) A point P is twice as far from the line $x + y = 5$ as from the point $(3, 0)$. Find the locus of P
(b) A point Q is given parametrically by $x = 2t$, $y = \frac{2}{t} + 1$.
Determine the cartesian equation of Q and sketch it.

Vectors

Chapter 18 Vectors

Vector quantities
Vector quantities are those that have both magnitude and direction. Examples of vectors include velocity, acceleration, displacement.

Vector representation
A vector can be represented by a section of a straight line, whose length represents the magnitude of the vector and whose direction, indicated by an arrow, represents the direction of the vector.
Such vectors can be denoted by a small letter say a.
Alternatively, we can represent a vector by the magnitude and direction of a line joining A to B. When we denote the vector by $\overrightarrow{AB}$ or AB, the vector in the opposite direction i.e. from B to A is written $\overrightarrow{BA}$ or BA



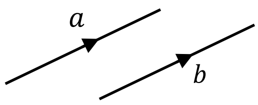
Modulus of a vector
The modulus of a vector a is its magnitude and it is written $|a|$ i.e. $|a|$ is the length of the line representing a.
The modulus of a vector $a = \begin{pmatrix} x \\ y \end{pmatrix} = xi + yj$ is given by

$$|a| = \sqrt{x^2 + y^2}$$

The modulus of a vector $a = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = xi + yj + zk$ is

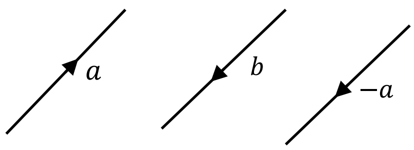
$$|a| = \sqrt{x^2 + y^2 + z^2}$$

Equal vectors
Two vectors with the same magnitude and same direction are equal



i.e. $a = b$ if $|a| = |b|$ and the directions of a and b are the same.

Negative vectors



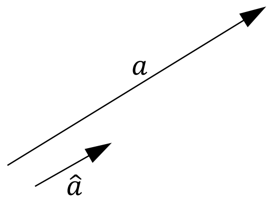
If two vectors, a and b, have the same magnitude but opposite directions, we say that

$$b = -a$$

i.e. $-a$ is a vector of magnitude $|a|$ and in the direction opposite to that of a.
We also say that a and b are equal and opposite vectors.

Parallel vectors
Two vectors a and b are parallel if one is a scalar multiple of the other i.e. $a = \lambda b$ where λ is a scalar.

Unit vectors
The unit vector, as the name implies, is a vector having a magnitude (length) of one unit.

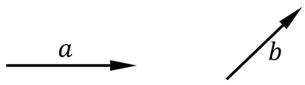


We use the notation $\hat{a}$. Then, $\hat{a}$ is a vector of length one unit in the same direction as a.
For example, if we had a vector r of length 4 units, to find the corresponding unit vector, $\hat{r}$, we would need to divide the vector r by four, resulting in a vector parallel to r but of unit length.
We then have the definition that

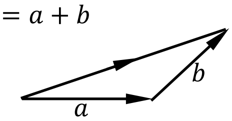
$$\hat{a} = \frac{a}{|a|}$$

Addition of vectors
The vector sum of two vectors a and b is given by the unique vector c (also known as the resultant) by using the Triangle Law of Addition or the Parallelogram law of addition.

Triangle law of addition
Vector b is translated so that its tail coincides with the head of vector a.



$c = a + b$



Then, the vectors sum of a and b is the vector c which closes the triangle.

$$25x^2 = 9$$
$$x = \pm\frac{3}{5}$$

Substituting for x, we have

$$y = \pm\frac{4}{5}$$

Therefore, both $v = \frac{3}{5}i + \frac{4}{5}j$ and $v = -\left(\frac{3}{5}i + \frac{4}{5}j\right)$ are perpendicular to u

Example 9

Find a vector perpendicular to both $a = 2i + j - k$ and $b = i + 3j + k$

Solution

Let $c = xi + yj + zk$ be perpendicular to both a and b. Then we have

$$a \cdot c = 0 \text{ and } b \cdot c = 0$$

From $a \cdot c = 0$, we obtain

$$(2i + j - k\,) \cdot (xi + yj + zk) = 0$$
$$2x + y - z = 0 \ldots (1)$$

From $b \cdot c = 0$, we obtain

$$(i + 3j + k\,) \cdot (xi + yj + zk) = 0$$
$$x + 3y + z = 0 \ \ldots (2)$$

In order to solve for three unknowns, we need one more equation. We note that if c is perpendicular to a and b, then so too will be the unit vector $\hat{c}$. So, without any loss in generality, we can assume that c is a unit vector. This will provide a third equation

$$|c| = 1$$
$$x^2 + y^2 + z^2 = 1 \ldots\ldots (3)$$

We can now solve for x, y and z

$(1) + (2)$; $\quad 3x + 4y = 0 \Rightarrow x = -\frac{4}{3}y$

$2 \times (1) - (2)$; $\ 5y + 3z = 0 \Rightarrow z = -\frac{5}{3}y$

Substituting for x and z into (3) gives;

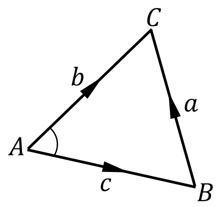
$$\left(-\frac{4}{3}y\right)^2 + y^2 + \left(-\frac{5}{3}y\right)^2 = 1$$
$$16y^2 + 9y^2 + 25y^2 = 1$$
$$50y^2 = 9$$
$$y = \pm\frac{3}{5\sqrt{2}}$$
$$y = \pm\frac{3\sqrt{2}}{10}$$
$$x - -\frac{4}{3} \times \pm\frac{3\sqrt{2}}{10} = \mp\frac{2\sqrt{2}}{5}$$
$$z = -\frac{5}{3} \times \pm\frac{3\sqrt{2}}{10} = \mp\frac{\sqrt{2}}{2}$$

$\therefore \mp\frac{2\sqrt{2}}{5}i \pm \frac{3\sqrt{2}}{10}j \mp \frac{\sqrt{2}}{2}k$ or $\mp\frac{1}{10}\left(4\sqrt{2}i - 3\sqrt{2}j + 5\sqrt{2}k\right)$ are two vectors perpendicular to a and b. Of course, any multiple of this vector will also be perpendicular to a and b.

Example 10

Prove the cosine rule in the form $a^2 = b^2 + c^2 - 2bc\cos A$ using the scalar product

Solution



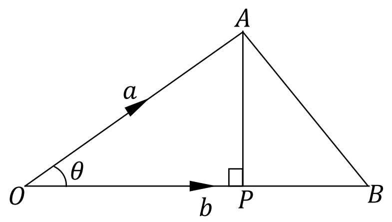
In $\triangle ABC$, let $a = \overrightarrow{BC}$, $b = \overrightarrow{AC}$ and $c = \overrightarrow{AB}$, then

$$a^2 = a \cdot a = (b - c) \cdot (b - c)$$
$$= b \cdot (b - c) - c \cdot (b - c)$$
$$= b \cdot b - b \cdot c - c \cdot b + c \cdot c$$
$$= b^2 + c^2 - 2b \cdot c$$
$$\therefore a^2 = b^2 + c^2 - 2bc\cos A$$

Example 11

The points A and B have position vectors a and b with respect to an origin O. Show that the area of triangle OAB is given by $\frac{1}{2}\sqrt{a^2b^2 - (a \cdot b)^2}$.

Solution



$$OP = OA\cos\theta$$
$$OP = |a|\cos\theta$$

Since $a \cdot b = |a||b|\cos\theta \Rightarrow \cos\theta = \frac{a \cdot b}{|a||b|}$

$$OP = |a| \times \frac{a \cdot b}{|a||b|} = \frac{a \cdot b}{|b|}$$

Using Pythagoras' theorem in ΔOAP, $OP^2 + AP^2 = OA^2$

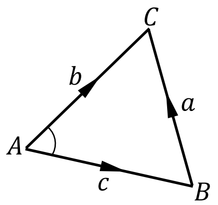
$$AP^2 = OA^2 - OP^2 = a^2 - \frac{(a \cdot b)^2}{b^2}$$

Hence the area of $\Delta OAB = \frac{1}{2} \times OB \times AP$

$$= \frac{1}{2}\sqrt{b^2 \times \left(a^2 - \frac{(a \cdot b)^2}{b^2}\right)}$$
$$= \frac{1}{2}\sqrt{a^2b^2 - (a \cdot b)^2}$$

To show that three points are vertices of a triangle

Consider three points A, B and C as vertices of a triangle ABC as shown below



$$\Rightarrow AB + BC = AC$$
$$\text{or } AB + BC + CA = 0$$

Example 12

Show that the points $A(4, 5, 2)$, $B(1, 7, 3)$ and $C(2, 4, 5)$ are vertices of a triangle.

<u>Solution</u>

Let $OA = \begin{pmatrix}4\\5\\2\end{pmatrix}$, $OB = \begin{pmatrix}1\\7\\3\end{pmatrix}$ and $OC = \begin{pmatrix}2\\4\\5\end{pmatrix}$

ABC is a triangle if $AB + BC = AC$

$$AB = OB - OA = \begin{pmatrix}1\\7\\3\end{pmatrix} - \begin{pmatrix}4\\5\\2\end{pmatrix} = \begin{pmatrix}-3\\2\\1\end{pmatrix}$$
$$BC = OC - OB = \begin{pmatrix}2\\4\\5\end{pmatrix} - \begin{pmatrix}1\\7\\3\end{pmatrix} = \begin{pmatrix}1\\-3\\2\end{pmatrix}$$
$$AC = OC - OA = \begin{pmatrix}2\\4\\5\end{pmatrix} - \begin{pmatrix}4\\5\\2\end{pmatrix} = \begin{pmatrix}-2\\-1\\3\end{pmatrix}$$

Now,

$$AB + BC = \begin{pmatrix}-3\\2\\1\end{pmatrix} + \begin{pmatrix}1\\-3\\2\end{pmatrix} = \begin{pmatrix}-2\\-1\\3\end{pmatrix}$$
$$= AC$$

$\therefore$ The points A, B and C are vertices of a triangle

Example 13

Show that the vectors $2i - j + k$, $i - j - 5k$, $3i - 4j - 4k$ form the vertices of a right-angled triangle.

<u>Solution</u>

Let $A = 2i - j + k, B = i - 3j - 5k, C = 3i - 4j - 4k$

We know that two vectors are perpendicular to each other, i.e. have an angle of $90°$ between them, if their scalar product is zero.

$$AB = OB - OA = (i - 3j - 5k) - (2i - j + k)$$
$$AB = -i - 2j - 6k$$
$$BC = OC - OB = (3i - 4j - 4k) - (i - 3j - 5k)$$
$$BC = 2i - j + k$$
$$CA = OC - OA = (2i - j + k) - (3i - 4j - 4k)$$
$$CA = -i + 3j + 5k$$

Now,

$$BC \cdot CA = (2i - j + k) \cdot (-i + 3j + 5k)$$
$$= (2)(-1) + (-1)(3) + (1)(5)$$
$$= 0$$

Since $BC \cdot CA = 0$, BC is perpendicular to CA hence ΔABC is a right-angled triangle.

Alternatively;

Considering ΔABC as a right-angled triangle, by Pythagoras theorem

$$|AB|^2 = |BC|^2 + |CA|^2$$
$$|AB|^2 = (-1)^2 + (-2)^2 + (-6)^2 = 41$$
$$|BC|^2 = 2^2 + (-1)^2 + 1^2 = 6$$
$$|CA|^2 = (-1)^2 + 3^2 + 5^2 = 35$$
$$|BC|^2 + |CA|^2 = 6 + 35 = 41$$

Thus, $|AB|^2 = |BC|^2 + |CA|^2$

So, ABC is a right-angled triangle

VECTOR PRODUCT/CROSS PRODUCT

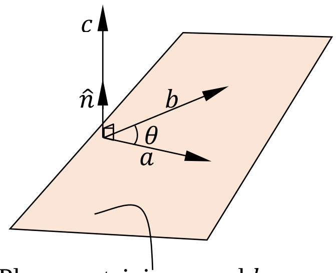
Unlike the scalar product of two vectors, which results in a scalar value, the vector product or as often, called the cross product, produces a vector.

We define the vector product as follows

The vector product (or cross product) of two vectors, a and b, produces a third vector, c, where

$$c = a \times b = |a||b| \sin\theta\, \hat{n}$$

and θ is the angle between a and b and $\hat{n}$ is a unit vector perpendicular to both a and b i.e. to the plane of $a \times b$. This means that the vectors a, b and $\hat{n}$ (in that order) form a right-handed system.



Plane containing a and b

We now consider some properties of the vector product.

1. Direction of $a \times b$
   The resulting vector, $c = a \times b$ is a vector that is parallel to the unit vector $\hat{n}$ (unless $a \times b = 0$)
2. Magnitude of $a \times b$
   The magnitude of $a \times b$ is given by
   $$|a \times b| = |a||b||\sin\theta||\hat{n}|$$
   But, $|\hat{n}| = 1$ and $0 \leq \theta \leq \pi \Rightarrow \sin\theta \geq 0$, thus
   $$|a \times b| = |a||b| \sin\theta$$
3. From these two properties, we can also conclude that
   If $a \times b = 0$, then either
   (a) $a = 0$ or $b = 0$ or both a and b are zero
   (b) $\sin\theta = 0 \Rightarrow \theta = 0 \text{ or } \pi$
4. $a \times b = -b \times a$
5. $a \times (b + c) = (a \times b) + (a \times c)$
6. $(a + b) \times c = (a \times c) + (b \times c)$

7. $a \times (kb) = (ka) \times b = k(a \times b)$, k a scalar
8. $a \times a = 0$
9. $a \cdot (a \times b) = 0$
10. $b \cdot (a \times b) = 0$

VECTOR FORM OF THE VECTOR PRODUCT

1. Component form

The vector product is only defined when both vectors are three dimensional.

The vector product of $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is given by

$$a \times b = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

This is known as the component form of the cross product. The result is a third vector that is at right angles to the two original vectors. This can be verified by making use of the dot product i.e.

$$a \cdot (a \times b) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$
$$= a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1)$$
$$= a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_2a_1b_3 + a_3a_1b_2 - a_3a_2b_1$$
$$= 0$$

2. Determinant form

When vectors are given in base vector notation, a more convenient method of finding the vector/cross product relies on a determinant representation. Given two vectors $a = a_1i + a_2j + a_3k$ and $b = b_1i + b_2j + b_3k$, the vector product $a \times b$ is defined as

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

Example 14

Find the vector product $\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix}$

Solution

$$\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \times -2 - 1 \times 4 \\ 1 \times -1 - (-2) \times 2 \\ 2 \times 4 - (-1) \times 4 \end{pmatrix} = \begin{pmatrix} -12 \\ 3 \\ 12 \end{pmatrix}$$

Example 15

Find $a \times b$ if $a = 2i + k$ and $b = 3i - 4j + 2k$. Hence, find $|a \times b|$

Solution

Using the determinant form of the cross product we have:

$$a \times b = \begin{vmatrix} i & j & k \\ 2 & 0 & 1 \\ 3 & -4 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -4 & 2 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} j + \begin{vmatrix} 2 & 0 \\ 3 & -4 \end{vmatrix} k$$
$$= \big(0 - (-4)\big)i - (4 - 3)j + (-8 - 0)k$$
$$= 4i - j - 8k$$

Therefore, $|a \times b| = \sqrt{(4)^2 + (-1)^2 + (-8)^2} = \sqrt{81} = 9$

Example 16

Find the angle between vectors a and b if $a = 2i - j + k$ and $b = 3i - 4j + 2k$

Solution

Let us first determine $a \times b$

$$a \times b = \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 3 & -4 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -4 & 2 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} j + \begin{vmatrix} 2 & -1 \\ 3 & -4 \end{vmatrix} k$$
$$= 2i - j - 5k$$

Next, $|a \times b| = \sqrt{4 + 1 + 25} = \sqrt{30}$

From $a \times b = |a||b| \sin\theta\, \hat{n}$, we have that

$$|a \times b| = \big||a||b| \sin\theta\, \hat{n}\big| = |a||b| \sin\theta$$

where θ is the angle between a and b.

$|a| = \sqrt{4 + 1 + 1} = \sqrt{6}$ and $|b| = \sqrt{9 + 16 + 4} = \sqrt{29}$

$$\sqrt{30} = \sqrt{6} \times \sqrt{29} \sin\theta$$
$$\sin\theta = \frac{\sqrt{30}}{\sqrt{6} \times \sqrt{29}}$$
$$\theta = 24.53°$$

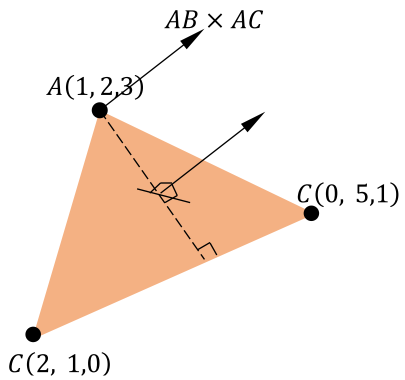
Of course, it would have been easier to do the above example using the scalar product.

Example 17

Find a unit vector that is perpendicular to the plane containing the points $A(1, 2, 3)$, $B(2, 1, 0)$ and $C(0, 5, 1)$

Solution

Let's draw a diagram of the situation described so that the triangle ABC lies on the planes containing the points A, B and C.



Then, the vector, perpendicular to the plane containing the points A, B and C will be parallel to the vector produced by the cross product $AB \times AC$.

$$AB = OB - OA = \begin{pmatrix}0\\10\\1\end{pmatrix} - \begin{pmatrix}1\\6\\3\end{pmatrix} = \begin{pmatrix}-1\\4\\-2\end{pmatrix}$$
$$AC = OC - OA = \begin{pmatrix}5\\8\\3\end{pmatrix} - \begin{pmatrix}1\\6\\3\end{pmatrix} = \begin{pmatrix}4\\2\\0\end{pmatrix}$$

Next, we calculate the vector product:

$$AB \times AC = \begin{pmatrix}-1\\4\\-2\end{pmatrix}\begin{pmatrix}4\\2\\0\end{pmatrix} = \begin{pmatrix}4(0)-(-2)(2)\\(-2)(4)-(-1)(0)\\(-1)(2)-(4)(4)\end{pmatrix} = \begin{pmatrix}4\\-8\\-18\end{pmatrix}$$

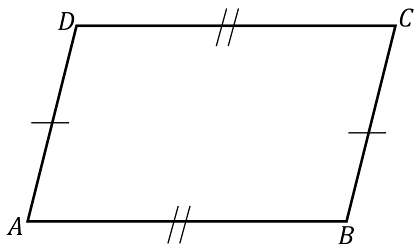
Area of triangle $ABC = \frac{1}{2}|AB \times AC|$

$$= \frac{1}{2}\sqrt{4^2 + (-8)^2 + (-18)^2} = \sqrt{101}\text{ unit}^2$$

Example 21

Show that the quadrilateral with vertices $A(4, 1, 0)$, $B(7, 6, 2)$, $C(5, 5, 4)$ and $D(2, 0, 2)$ is a parallelogram. Hence find its area.

Solution



If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

If $AB \cong DC$ and $AD \cong BC$, then $ABCD$ is a parallelogram.

$$AD = OD - OA = \begin{pmatrix}2\\0\\2\end{pmatrix} - \begin{pmatrix}4\\1\\0\end{pmatrix} = \begin{pmatrix}-2\\-1\\2\end{pmatrix}$$
$$BC = OC - OB = \begin{pmatrix}5\\5\\4\end{pmatrix} - \begin{pmatrix}7\\6\\2\end{pmatrix} = \begin{pmatrix}-2\\-1\\2\end{pmatrix}$$
$$\Rightarrow AD = DC$$

Now,

$$AB = OB - OA = \begin{pmatrix}7\\6\\2\end{pmatrix} - \begin{pmatrix}4\\1\\0\end{pmatrix} = \begin{pmatrix}3\\5\\2\end{pmatrix}$$
$$DC = OC - OD = \begin{pmatrix}5\\5\\4\end{pmatrix} - \begin{pmatrix}2\\0\\2\end{pmatrix} = \begin{pmatrix}3\\5\\2\end{pmatrix}$$
$$\Rightarrow AB = DC$$

Thus the quadrilateral $ABCD$ is a parallelogram

Area $= |AB \times AC|$

$$AC = OC - OA = \begin{pmatrix}5\\5\\4\end{pmatrix} - \begin{pmatrix}4\\1\\0\end{pmatrix} = \begin{pmatrix}1\\4\\4\end{pmatrix}$$
$$AB \times AC = \begin{pmatrix}3\\5\\2\end{pmatrix}\begin{pmatrix}1\\4\\4\end{pmatrix} = \begin{pmatrix}5(4)-2(4)\\-[3(4)-2(1)]\\3(4)-5(1)\end{pmatrix} = \begin{pmatrix}12\\-10\\7\end{pmatrix}$$

Area $= |AB \times AC| = \sqrt{12^2 + (-10)^2 + 7^2} = \sqrt{293}$ unit^2

Example 22

Prove that $|a \times b| = |a|^2|b|^2 - (a \cdot b)^2$

Solution

$$|a \times b| = |a||b|\sin\theta$$
$$|a \times b|^2 = |a|^2|b|^2\sin^2\theta$$
$$|a \times b|^2 = |a|^2|b|^2(1 - \cos^2\theta)$$
$$|a \times b|^2 = |a|^2|b|^2 - |a|^2|b|^2\cos^2\theta$$
$$|a \times b|^2 = |a|^2|b|^2 - (|a||b|\cos\theta)^2$$

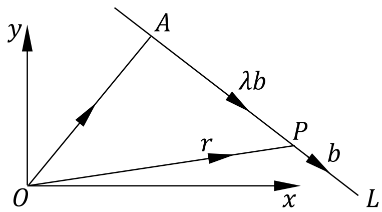
But $a \cdot b = |a||b|\cos\theta$

$$\therefore |a \times b|^2 = |a|^2|b|^2 - (a \cdot b)^2$$

THE EQUATION OF A STRAIGHT LINE

A straight line is located uniquely in space if either it passes through a known fixed point and has a known direction, or it passes through two known fixed points.

Vector equation of a line in 2D



The vector equation of a line L in the direction of vector b, passing through the point A with position vector a is given by

$$r = a + \lambda b$$

where λ is a scalar parameter

Proof:

Let the point $P(x, y)$ be any point on the line L, then the vector AP is parallel to the vector b

$$r = OP$$
$$= OA + AP$$
$$\therefore r = a + \lambda b$$

We can now derive two other forms for equations of a line. We start by letting the coordinates of A be (a_1, a_2), the coordinates of P be (x, y) and the vector $b = \begin{pmatrix}b_1\\b_2\end{pmatrix}$

For $r = a + \lambda b$, we have

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}a_1\\a_2\end{pmatrix} + \lambda\begin{pmatrix}b_1\\b_2\end{pmatrix} = \begin{pmatrix}a_1 + \lambda b_1\\a_2 + \lambda b_2\end{pmatrix}$$

This provides us with the **parametric form** for the equations of a straight line:

$$x = a_1 + \lambda b_1$$
$$y = a_2 + \lambda b_2$$

Next, from the parametric form, we have

$$x = a_1 + \lambda b_1 \Rightarrow \lambda = \frac{x - a_1}{b_1} \quad \text{... [1]}$$
$$y = a_2 + \lambda b_2 \Rightarrow \lambda = \frac{y - a_2}{b_2} \quad \text{.... [2]}$$

LINES IN THREE DIMENSIONS

In three dimensional work, always try to visualize situations very clearly. Because diagrams are never very satisfactory, it is useful to use the corner of a table with an imagined vertical line for axes; then pencils become lines and books or sheets of paper become planes.

It is tempting to generalize from a two-dimensional line like $x+y=8$ and think that the Cartesian equation of a three-dimensional line will have the form $x+y+z=8$. This is not correct – as we will see later **this represents a plane, not a line**.

Cartesian equation of a line in three dimensions

Consider the cartesian form of any straight line L passing through the point $P(x_1, y_1, z_1)$:

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

From this equation, we obtain the parametric form of the straight line

$$\frac{x-x_1}{a}=\lambda \Rightarrow x=x_1+\lambda a$$

$$\frac{y-y_1}{b}=\lambda \Rightarrow y=y_1+\lambda b$$

$$\frac{z-z_1}{c}=\lambda \Rightarrow z=z_1+\lambda c$$

which then leads to the vector form of the straight line:

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+\lambda\begin{pmatrix}a\\b\\c\end{pmatrix}$$

Example 26

Find the Cartesian equations of the line that is parallel to the vector $2i+3j+4k$ and which passes through the point A, position vector $3i-j+2k$

<u>Solution</u>

The vector equation of the line is

$$r=3i-j+2k+\lambda(2i+3j+4k)$$

Thus

$$x=3+2\lambda \Rightarrow \lambda=\frac{x-3}{2}$$

$$y=-1+3\lambda \Rightarrow \lambda=\frac{y+1}{3}$$

$$z=2+4\lambda \Rightarrow \lambda=\frac{z-2}{4}$$

The cartesian equations are therefore

$$\frac{x-3}{2}=\frac{y+1}{3}=\frac{z-2}{4}(=\lambda)$$

Example 27

Find the Cartesian form of the straight line passing through the point $(4,6,3)$ and having direction vector $3i-2j+k$.

<u>Solution</u>

The vector equation of the line is given by

$$r=(4i+6j+3k)+\lambda(3i-2j+k)$$

From the vector equation we obtain the parametric form of the line

$$x=4+3\lambda,\ y=6-2\lambda \text{ and } z=3+\lambda$$

From these equations we have;

$$\lambda=\frac{x-4}{3},\qquad \lambda=\frac{y-6}{-2},\ \ \lambda=\frac{z-3}{1}$$

Then, eliminating λ, we have

$$\frac{x-4}{3}=\frac{y-6}{-2}=\frac{z-3}{1}$$

or

$$\frac{x-4}{3}=\frac{y-6}{-2}=z-3$$

Showing that a point lies on a given line

Showing that a point lies on a line is as simple as finding the value of the parameter λ (or whatever letter it is) that gives your point's coordinates as the output.

Example 28

Show that the point $(3,6,-2)$ lies on the line given by the equation $r=\begin{pmatrix}-5\\2\\6\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-2\end{pmatrix}$.

<u>Solution</u>

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-5+2\lambda\\2+\lambda\\6-2\lambda\end{pmatrix}$$

Solve the x-direction equation

$$3=-5+2\lambda$$

$$\lambda=4$$

Then check the other two directions by putting the λ-value.

$$y=2+\lambda=2+4=6$$

$$z=6-2\lambda=6-2(4)=-2$$

Therefore, the point $(3,6,-2)$ lies on the line.

Example 29

Show that the points whose position vectors are $(-2i+3j+5k)$, $(i+2j+3k)$ and $(7i-k)$ are collinear.

<u>Solution</u>

Let A, B, C be the three given points whose position vectors are a, b, c respectively. Then

$a=-2i+3j+5k$, $b=i+2j+3k$, $c=7i-k$

The vector equation of AB is given by

$$r=a+\lambda(b-a)$$

$$r=(-2i+3j+5k)+\lambda[(i+2j+3k)-(-2i+3j+5k)]$$

$$r=(-2i+3j+5k)+\lambda(3i-j-2k)\ \ \ldots[1]$$

The three points A, B, C will be collinear if C lies on it, i.e. if C satisfies [1]

$$7i-k=(-2i+3j+5k)+\lambda(3i-j-2k)$$

$$7i-k=(-2+3\lambda)i+(3-\lambda)j+(5-2\lambda)k$$

Now,

$$-2+3\lambda=7 \Rightarrow \lambda=3$$

$$3-\lambda=0 \Rightarrow \lambda=3$$

$$5-2\lambda=-1 \Rightarrow \lambda=3$$

PLANES

A plane is a surface such that if two points are taken in it, straight line joining them lies wholly in the surface.

General equation of the plane

The locus of the equation will be a plane, if every point of the line joining any two points on the locus lies on the locus. Let $P(x_1, y_1, z_1)$ and $Q(x_2.y_2, z_2)$ be any two points on the locus represented by

$$ax + by + cz + d = 0 \quad \ldots\ldots [1]$$

Since P and Q lie on [1], then

$$ax_1 + by_1 + cz_1 + d = 0 \quad \ldots. [2]$$
$$ax_2 + by_2 + cz_2 + d = 0 \quad \ldots. [3]$$

Multiplying [3] by k and adding to [2], we get

$$a(x_1 + kx_2) + b(y_1 + ky_2) + c(z_1 + kz_2) + d(1 + k) = 0$$
$$a\left(\frac{x_1 + kx_2}{1 + k}\right) + b\left(\frac{y_1 + ky_2}{1 + k}\right) + c\left(\frac{z_1 + kz_2}{1 + k}\right) + d = 0$$

This shows that the point $\left(\frac{x_1+kx_2}{1+k}, \frac{y_1+ky_2}{1+k}, \frac{z_1+kz_2}{1+k}\right)$ is also on the locus. But for different values f k, these are the general coordinates of any point on the line PQ. Thus, every point on the straight line joining any arbitrary points on the locus lies on the locus. Therefore, by definition of a plane, the locus of [1] is a plane.

Notes:

1. Number of constants in $ax + by + cz + d = 0$. The equation $ax + by + cz + d = 0$ can be written as $\frac{a}{d}x + \frac{b}{d}y + \frac{c}{d}z + 1 = 0$ or $a_1x + b_1y + c_1z + 1 = 0$, where $a_1 = \frac{a}{d}$, $b_1 = \frac{b}{d}$, $c_1 = \frac{c}{d}$. This shows that although the given equation contains four constants a, b, c, d yet in reality it contains only three independent constants. Hence, a plane can be uniquely determined if three independent conditions are given.
2. The plane passing through the origin is $ax + by + cz = 0$ as in this case $d = 0$

One point form of the equation of the plane

Let (x_1, y_1, z_1) be a given point.

Let the plane containing this point be

$$ax + by + cz + d = 0 \quad \ldots [1]$$

Since the point (x_1, y_1, z_1) lies on it, therefore,

$$ax_1 + by_1 + cz_1 + d = 0 \quad \ldots [2]$$

Subtracting [2] from [1], we get

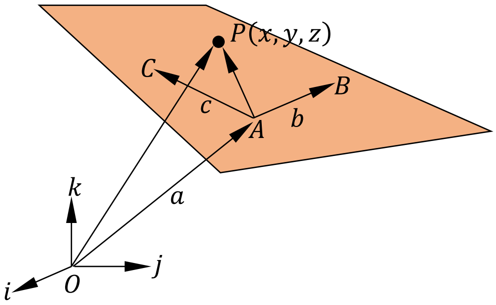
$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

which is the required equation of the plane passing through a given point (x_1, y_1, z_1).

Any three points not a straight line determine a plane. While this characterisation of a plane is quite simple. It is not convenient for beginning the study of planes. Instead it is more convenient to find the equation of a plane perpendicular to a line.

Vector equation of a plane

Let $P(x, y, z)$, whose position vector is $r = OP$ be any point on the plane relative to some origin O.



Consider three points A, B and C on this plane where $OA = a$, $AB = b$ and $AC = c$ i.e. the plane contains the vectors b and c, where $b \neq c \neq 0$ and the vectors a, b and c are non-coplanar.

Now, as AP, b and c are coplanar, then we can express AP in terms of b and c.

$$AP = \lambda b + \mu c$$

for some real λ and μ

Then, $r = OP = OA + AP = a + \lambda b + \mu c$

Thus, the vector equation of a plane is given by

$$\boldsymbol{r = a + \lambda b + \mu c}$$

This means that to find the vector form of the equation of a plane, we need to know

(i) the position vector of a point A in the plane, and
(ii) two non-parallel vectors in the plane

Example 40

Find the vector equation of the plane containing the vectors $\begin{pmatrix}2\\1\\1\end{pmatrix}$ and $\begin{pmatrix}3\\0\\-1\end{pmatrix}$ which also includes the point $(1, 2, 0)$

<u>**Solution**</u>

Let $b = \begin{pmatrix}2\\1\\1\end{pmatrix}$ and $c = \begin{pmatrix}3\\0\\-1\end{pmatrix}$ be two vectors on the plane.

Then, as the point $(1, 2, 0)$ lies on the plane, we let $a = \begin{pmatrix}1\\2\\0\end{pmatrix}$ be the position of this point.

Using the vector form of the equation of a plane i.e.

$$r = a + \lambda b + \mu c$$
$$r = \begin{pmatrix}1\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\1\end{pmatrix} + \mu\begin{pmatrix}3\\0\\-1\end{pmatrix}$$

Cartesian equation of a plane

In the same way we are able to produce a Cartesian equation for a line, we now derive the Cartesian equation of a plane. Using the example above to obtain the parametric equations and use them to derive the Cartesian equation of the plane.

Example 42

Find the vector and Cartesian equation of the pane through the point $(1, 2, 3)$ and perpendicular to the line with direction ratios $(2, 3, -4)$.

Solution

The position vector of the point $(1, 2, 3)$ is

$$a = i + 2j + 3k$$

Normal vector n perpendicular to the plane is

$$n = 2i + 3j - 4k$$

Therefore, the vector equation of the plane passing through the point A with position vector n is given by

$$(r - a) \cdot n = 0$$

$$[r - (i + 2j + 3k)] \cdot (2i + 3j - 4k) = 0$$

Cartesian form

Let $r = xi + yj + zk$, then

$$[(xi + yj + zk) - (i + 2j + 3k)] \cdot (2i + 3j - 4k) = 0$$

$$[(x - 1)i + (y - 2)j + (2 - 3)k] \cdot (2i + 3j - 4k) = 0$$

$$2(x - 1) + 3(y - 2) - 4(z - 3) = 0$$

$$2x + 3y - 4z + 4 = 0$$

Example 43

Find the cartesian equation of the plane containing the point $A(3, 1, 1)$ and with the normal vector given by $n = 3i - 2j + 4k$

Solution

Using the normal vector, $n = 3i - 2j + 4k$ and a vector on the plane passing through the point $A(3, 1, 2)$ i.e. the vector

$$AP = (x - 3)i + (y - 1)j + (z - 1)k$$

where $P(x, y, z)$ is an arbitrary point on the plane, we have

$$n \cdot AP = 0$$

$$(3i - 2j + 4k) \cdot [(x - 3)i + (y - 1)j + (z - 1)k] = 0$$

$$3(x - 3) + (-2)(y - 1) + 4(z - 1) = 0$$

$$3x - 9 - 2y + 2 + 4z - 4 = 0$$

$$3x - 2y + 4z = 11$$

Alternatively, using $r \cdot n = n \cdot a$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

$$3x - 2y + 4z = 11$$

Example 44

The foot of the perpendicular drawn from the origin to a plane is $(4, -2, -5)$. Find the equation of the plane.

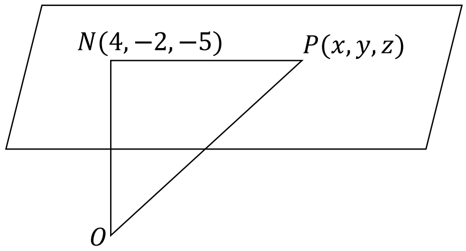
Solution

Let O be the origin and $N(4, -2, -5)$ on the plane be the foot of the perpendicular from O to the given plane. Let $P(x, y, z)$ be any point on the plane.

Then, direction ratios of NP and ON are given as follows;

$$NP = OP - ON = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 4 \\ -2 \\ -5 \end{pmatrix} = \begin{pmatrix} x - 4 \\ y + 2 \\ z + 5 \end{pmatrix}$$

$$ON = \begin{pmatrix} 4 \\ -2 \\ -5 \end{pmatrix}$$



Since $ON \perp NP$, therefore,

$$\begin{pmatrix} x - 4 \\ y + 2 \\ z + 5 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -2 \\ -5 \end{pmatrix} = 0$$

$$4(x - 4) - 2(y + 2) - 5(z + 5) = 0$$

$$4x - 16 - 2y - 4 - 5z - 25 = 0$$

$$4x - 2y - 5z - 45 = 0$$

Example 45

Find a normal vector to the plane $x + 2y + 3z - 6 = 0$.

Solution

The equation of the plane can be rewritten as:

$$x + 2y + 3z = 6$$

$$\text{or } (xi + yj + zk) \cdot (i + 2j + 3k) = 6$$

$$\text{or } r \cdot n = 6$$

$$\text{when } n = i + 2j + 3k$$

Thus, normal unit vector to the given plane is

$$\frac{n}{|n|} = \frac{i + 2j + 3k}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{1}{\sqrt{14}}(i + 2j + 3k)$$

Example 46

Find the vector equation of a plane which is at a distance of 6 units from the origin and has $2, -1, 2$ as the direction ratios of a normal to it. Also, find the coordinates of the of the foot of the normal drawn from the origin.

Solution

Let n be a vector normal to the plane. Then the direction vectors of n are $2, -1, 2$

$$n = 2i - j + 2k$$

$$|n| = \sqrt{2^2 + (-1)^2 + 2^2} = 3$$

$$\hat{n} = \frac{n}{|n|} = \frac{2}{3}i - \frac{1}{3}j + \frac{2}{3}k$$

Since the plane is at a distance of 6 units from the origin, its equation is

$$r \cdot \hat{n} = 6$$

$$r \cdot \left(\frac{2}{3}i - \frac{1}{3}j + \frac{2}{3}k\right) = 6$$

The position vector of the foot of the normal drawn from the origin is

$$p\hat{n} = 6\left(\frac{2}{3}i - \frac{1}{3}j + \frac{2}{3}k\right) = 4i - 2j + 4k$$

So the coordinates of the foot of the normal are $(4, -2, 4)$

Alternative method

The equation of the plane passing through the point (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

The equation of the plane passing through the point $(1, 0, -1)$ is

$$A(x-1) + B(y) + C(z+1) = 0 \quad \ldots [1]$$

Since it passes through $(3, 1, 4)$ and $(2, -2, 0)$, therefore

$$A(3-1) + B(1) + C(4+1) = 0$$

$$2A + B + 5C = 0 \ldots. [2]$$

and $A(2-1) + B(-2) + C(0+1) = 0$

$$A - 2B + C = 0 \quad \ldots. [3]$$

$[2] - 2 \times [3]$;

$$5B + 3C = 0$$

$$\Rightarrow B = -\frac{3C}{5}$$

Substituting for B in [2];

$$2A - \frac{3C}{5} + 5C = 0$$

$$10A - 3C + 25C = 0$$

$$10A + 22C = 0$$

$$A = -\frac{11C}{5}$$

Substituting for B and A in [1];

$$-\frac{11C}{5}(x-1) - \frac{3C}{5}(y) + C(z+1) = 0$$

$$-11x + 11 - 3y + 5z + 5 = 0$$

$$-11x - 3y + 5z + 16 = 0$$

Example 49

Find the equation of the plane containing the vectors $b = 3i - j + 2k$ and $c = 2i + 2j + k$ and passing through the point $A(2, 1, 6)$.

<u>Solution</u>

The cross product $b \times c$ represents a vector that is perpendicular to the plane containing the vectors b and c.

$$n = b \times c = \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 2 & 2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} i - \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} j + \begin{vmatrix} 3 & -1 \\ 2 & 2 \end{vmatrix} k$$

$$= 5i + j + 8k$$

Let $P(x, y, z)$ be any point on the plane. As P lies on the plane, the vector AP must also be perpendicular to the vector n i.e. $n \cdot AP = 0$.

$$AP = OP - OA = (x-2)i + (y-1)j + (z-6)k$$

$$(5i + j + 8k) \cdot [(x-2)i + (y-1)j + (z-6)k] = 0$$

$$5(x-2) + y - 1 + 8(z-6) = 0$$

$$5x - y - 8z + 39 = 0$$

We can check this result by use of the parametric form of the plane

From the vector, $r = \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, we obtain the parametric equations:

$$x = 2 + 3\lambda + 2\mu \ \ldots.. [1]$$

$$y = 1 - \lambda + 2\mu \ \ldots. [2]$$

$$z = 6 + 2\lambda + \mu \quad \ldots. [3]$$

$[1] - [2]$;

$$x - y = 1 + 4\lambda \ \ldots.. [4]$$

$[2] - 2 \times [3]$;

$$y - 2z = -11 - 5\lambda \ \ldots. [5]$$

From [4]; $\lambda = \frac{x-y-1}{4}$

Substituting in [5] we obtain;

$$y - 2z = -11 - 5\left(\frac{x-y-1}{4}\right)$$

$$4y - 8z = -44 - 5x + 5y + 5$$

$$5x - y - 8z = -39$$

$$5x - y - 8z + 39 = 0$$

As expected, we produce the same equation

Example 50

Find the equation of a plane through three given points $(-2, 6, -6)$, $(-3, 10, -9)$ and $(-5, 0, -6)$.

<u>Solution</u>

Equation of the plane passing through the given points $(-2, 6, -6)$, $(-3, 10, -9)$ and $(-5, 0, -6)$ is

$$\begin{vmatrix} x - (-2) & y - 6 & z - (-6) \\ -3 - (-2) & 10 - 6 & -9 - (-6) \\ -5 - (-2) & 0 - 6 & -6 - (-6) \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x + 2 & y - 6 & z + 6 \\ -1 & 4 & -3 \\ -3 & -6 & 0 \end{vmatrix} = 0$$

$$(x+2)(0-18) - (y-6)(0-9) + (z+6)(6+12) = 0$$

$$-18x - 36 + 9y - 54 + 18z + 108 = 0$$

$$-18x + 9y + 18z + 18 = 0$$

$$-2x + y + 2z + 2 = 0$$

$$2x - y - 2z - 2 = 0$$

Alternatively;

The equation of the plane passing through the point $(-2, 6, -6)$ is

$$A(x+2) + B(y-6) + C(z+6) = 0 \ \ldots [1]$$

Since it passes through $(-3, 10, -9)$ and $(-5, 0, -6)$, then

$$A(-3+2) + B(10-6) + C(-9+6) = 0$$

$$-A + 4B - 3C = 0 \ \ldots [2]$$

and

$$A(-5+2) + B(0-6) + C(-6+6) = 0$$

$$-3A - 6B = 0 \ldots [3]$$

From [3]; $\quad A = -2B$

Substituting for A in [2];

$$2B + 4B - 3C = 0$$

$$C = 2B$$

Substituting for A and C in [1];

$$-2B(x+2) + B(y-6) + 2B(z+6) = 0$$

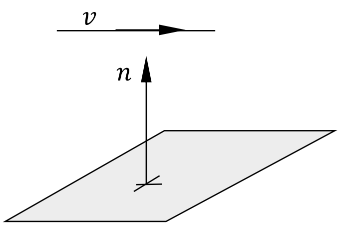
$$-2x - 4 + y - 6 + 2z + 12 = 0$$

$$-2x + y + 2z + 2 = 0$$

$$2x - y - 2z - 2 = 0$$

Solution

We need to prove that the normal, n to the plane is perpendicular to the direction vector, b of the line.



Rewriting $x - 2y + 2z = 11$ in the normal vector form, we have $r \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 11$.

From this equation, $n = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

From vector equation of line, direction vector is $b = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$

$$b \cdot n = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 4 - 6 + 2 = 0$$

The vectors are perpendicular and so the line and plane are parallel.

Example 55

Find the vector equation of the line passing through $(1, 2, 3)$ and parallel to the planes $r \cdot (i - j + 2k) = 5$ and $r \cdot (3i + j + k) = 6$.

Solution

Let the direction of the line be $d = d_1 i + d_2 j + bd_3 k$

Equation of the line passing through $(1, 2,3)$ and having the direction d is

$$r = i + 2j + 3k + \lambda d$$

$$r = i + 2j + 3k + \lambda(d_1 i + d_2 j + d_3 k) \ \ldots [1]$$

Line [1] and plane $r \cdot (i - j + 2k) = 5$ are parallel

⇒ Normal of the plane is perpendicular to line [1]

$$(d_1 i + d_2 j + d_3 k) \cdot (i - j + 2k) = 0$$

$$d_1 - d_2 + 2d_3 = 0 \ \ldots [2]$$

Again line [1] and plane $r \cdot (3i + j + k) = 6$ are parallel,

⇒ Normal of the plane is perpendicular to line [1]

$$(d_1 i + d_2 j + d_3 k) \cdot (3i + j + k) = 0$$

$$3d_1 + d_2 + d_3 = 0 \ \ldots [3]$$

Adding [2] and [3] gives;

$$4d_1 + 3d_3 = 0$$

$$d_1 = -\frac{3d_3}{4}$$

Substituting for d_3 in [3];

$$3d_1 + d_2 - \frac{4d_1}{3} = 0$$

$$5d_1 + 3d_2 = 0$$

$$d_1 = -\frac{3d_2}{5}$$

$$\therefore d_1 = -\frac{3d_2}{5} = -\frac{3d_3}{4}$$

$$\frac{d_1}{3} = \frac{d_2}{-5} = \frac{d_3}{-4}$$

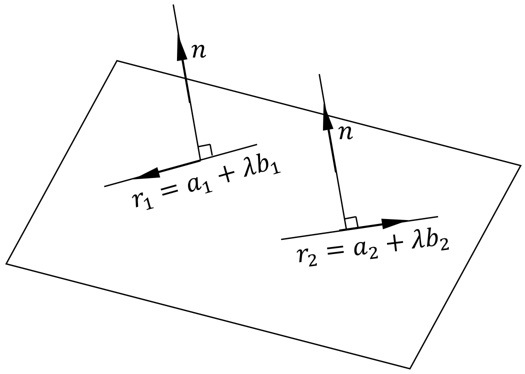
$3, -5, -4$ are the direction ratios of the line.

Hence the equation of the required line is

$$r = i + 2j + 3k + \lambda(3i - 5j - 4k)$$

Equation of a plane containing two nonconcurrent lines

The equation of a plane containing any two nonconcurrent lines can be obtained using three approaches as shall be discussed as follows:



Consider a plane containing two lines L_1 and L_2 with equations $r_1 = a_1 + \lambda b_1$ and $r_2 = a_2 + \mu b_2$ respectively.

Method 1:

We can choose one position vector (a_1 or a_2) of a point from any of the two lines and then use the two direction vectors of the lines (b_1 and b_2). The vector equation of the plane will be in the form

$$r = a_1 + \lambda b_1 + \mu b_2$$

$$\text{or } r = a_2 + \lambda b_1 + \mu b_2$$

We can then use this vector equation to obtain parametric equations and hence the cartesian equation.

Method 2:

Since we have two vectors on the plane i.e. the direction vectors b_1 and b_2, we can use the dot/scalar product to obtain the normal to the plane i.e.

$$n \cdot b_1 = 0 \ \text{ and } n \cdot b_2 = 0$$

Once we have the normal to the plane, we can use the normal form to obtain the equation of the plane i.e.

$$r \cdot n = n \cdot a_1 \ \text{ or } r \cdot n = n \cdot a_2$$

Method 3:

We can use the cross product to obtain the normal to the plane i.e. the normal, n to the plane will be given by

$$n = b_1 \times b_2$$

Once we get the normal, we can use $r \cdot n = n \cdot a_1$ or $r \cdot n = n \cdot a_2$ to obtain the equation of the plane.

$$x = 2 + 5s$$
$$y = 1 + s$$
$$z = -s$$

If this line lies on the plane, then the parametric equations must satisfy the Cartesian equation of the plane. Substituting into equation $x - 3y + 2z = -1$, we get

$$L.H.S = (2 + 5s) - 3(1 + s) - 2s$$
$$= 2 + 5s - 3 - 3s - 2s$$
$$= -1$$
$$= R.H.S$$

Therefore, the line lies in the plane

Note: An alternative approach to this is to show that the line is parallel to the plane and the distance between the line and the plane is zero.

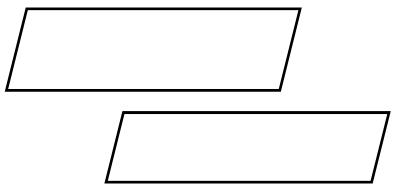
Parallel and perpendicular planes

If two planes Π_1 and Π_2 have normal vectors, $n_1 = a_1 i + b_1 j + c_1 k$ and $n_2 = a_2 i + b_2 j + c_2 k$ respectively, then the two planes, Π_1 and Π_2 are

1. **parallel** if and only if their normal vectors are parallel i.e. iff $n_1 = \lambda \times n_2$ where λ is a scalar
   i.e. $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \lambda$
2. perpendicular if and only if their normal vectors are perpendicular i.e. iff $n_1 \cdot n_2 = 0$
   $$(a_1 i + b_1 j + c_1 k) \cdot (a_2 i + b_2 j + c_2 k) = 0$$
   $$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$
   Hence, the planes $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$ are perpendicular if $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

Taking this one step further, this result also means that we can use the normal to find the angle between two planes.

Equation of a plane parallel to a given plane



Let the given plane be

$$ax + by + cz + d = 0 \text{ ... [1]}$$

Let $\quad a_1 x + b_1 y + c_1 z + d_1 = 0 \quad \text{... [2]}$

[1] and [2] are parallel if

$$\frac{a_1}{a} = \frac{b_1}{b} = \frac{c_1}{c} = \lambda$$
$$\therefore a_1 = \lambda a, b_1 = \lambda b, c_1 = \lambda c$$

Substituting in [2], we get

$$a\lambda x + b\lambda y + c\lambda z + d_1 = 0$$
$$ax + by + cz + \frac{d_1}{\lambda} = 0$$

Suppose $k = \frac{d_1}{\lambda}$,

$$ax + by + cz + k = 0$$

Hence, the equation of any plane parallel to the given plane $ax + by + cz + d = 0$ is

$$\boldsymbol{ax + by + cz + k = 0}$$

Example 58

Find the equation of the plane through $(2, 3,4\ )$ parallel to the plane $x + 2y + 4z = 5$.

<u>Solution</u>

$$x + 2y + 4z = 5 \quad \text{... [1]}$$

Any plane parallel to [1] is

$$x + 2y + 4z = d \quad \text{... [2]}$$

If [2] passes through $(2, 3, 4)$, then

$$2 + 6 + 16 = d$$
$$d = 24$$

Substituting in [2], we get

$$x + 2y + 4z = 24$$

This is the required equation

Example 59

Find the equation of the plane perpendicular to the plane $2x + 3y - 5z - 6 = 0$ and passes through the points $P(2, -1, -1)$ and $Q(1, 2, 3)$.

<u>Solution</u>

Method I:

Let the required plane be $ax + by + cz + d = 0$. Since this plane is perpendicular to the plane $2x + 3y - 5z - 6 = 0$, then

$$2a + 3b - 5c = 0 \quad \text{... [1]}$$

Since P and Q also lie on the plane $ax + by + cz + d = o$,

$$2a - b - c + d = 0 \quad \text{... [2]}$$
$$a + 2b + 3c + d = 0 \quad \text{... [3]}$$

Subtracting [3] from [2] gives;

$$a - 3b - 4c = 0 \quad \text{... [4]}$$

Adding [1] and [4] gives;

$$3a = 9c$$
$$a = 3c$$

Substituting for a in [4];

$$3c - 3b - 4c = 0$$
$$b = -\frac{c}{3}$$

Substituting for a and b in [2] gives;

$$2(3c) + \frac{c}{3} - c + d = 0$$
$$d = -\frac{16c}{3}$$

Finally substituting these values in $ax + by + cz + d = 0$,

$$3cx - \frac{c}{3}y + cz - \frac{16c}{3} = 0$$
$$9x - y + 3z - 16 = 0$$

Method II:

Any plane passing through $(2, -1, -1)$ is

$$a(x - 2) + b(y + 1) + c(z + 1) = 0 \quad \text{... [1]}$$

This passes through $(1, 2, 3)$ if

$$a(1-2) + b(2+1) + c(3+1) = 0$$

$$-a + 3b + 4c = 0 \ \dots [2]$$

The plane [1] is perpendicular to the plane $2x + 3y - 5z - 6 = 0$ if the normals to the planes are at right angles to each other. Thus

$$2a + 3b - 5c = 0 \ \dots [3]$$

Subtracting [3] – [2] gives;

$$3a - 9c = 0$$

$$a = 3c$$

Substituting for a in [2];

$$-3c + 3b + 4c = 0$$

$$b = -\frac{c}{3}$$

Substituting for a and b in [1];

$$3c(x-2) - \frac{c}{3}(y+1) + c(z+1) = 0$$

$$9x - 18 - y - 1 + 3z + 3 = 0$$

$$9x - y + 3z - 16 = 0$$

Example 60

Find the equation(s) of the plane(s) parallel to the plane $3x - 6y + 2z + 14$ and

(a) 6 units from the origin,
(b) 4 units from $A(2, 1, -3)$,
(c) 5 units from the above plane

<u>Solution</u>

The equation of any plane parallel to the given plane is of the form

$$3x - 6y + 2z + k = 0 \ \dots [1]$$

(a) Since [1] is 6 units from the origin

$$\frac{|3(0) - 6(0) + 2(0) + k|}{\sqrt{3^2 + (-6)^2 + 2^2}} = 6$$

$$|k| = 42$$

$$k = \pm 42$$

Hence the equations of the planes are $3x - 6y + 2z \pm 42 = 0$

(b) Since [1] is 4 units from $A(2, 1, -3)$, therefore

$$\frac{|3(2) - 6(1) + 2(-3) + k|}{\sqrt{3^2 + (-6)^2 + 2^2}} = 4$$

$$|k - 6| = 28$$

$$k - 6 = \pm 28$$

$$k = 34 \text{ or } k = -22$$

The required equations of the planes are $3x - 6y + 2z + 34 = 0$ and $3x - 6y + 2z - 22 = 0$.

(c) Take a point $(0, 0, -7)$ on $3x - 6y + 2z + 14 = 0$

Since the perpendicular distance of $(0, 0, -7)$ from $3x - 6y + 2z + k = 0$ is equal to 5, then

$$\frac{|3(0) - 6(0) + 2(-7) + k|}{\sqrt{3^2 + (-6)^2 + 2^2}} = 5$$

$$|k - 14| = 35$$

$$k - 14 = \pm 35$$

$$k = 49 \text{ or } -21$$

Hence, the required equations of the planes are

$$3x - 6y + 2z + 49 = 0 \text{ and } 3x - 6y + 2z - 21 = 0$$

Example 61

Find the equation of the plane which passes through $P(2, 3, 4)$ and is perpendicular to the planes $2x - y + 2z - 8 = 0$ and $x + 2y - 3z + 7 = 0$.

<u>Solution</u>

Any plane passing through $P(2, 3, 4)$ is given by

$$a(x-2) + b(y-3) + c(z-4) = 0 \quad \dots [1]$$

where $ai + bj + ck$ is the normal vector to the plane

The plane [1] is perpendicular to the given planes if

$$2a - b + 2c = 0 \quad \dots. [2]$$

$$a + 2b - 3c = 0 \quad \dots. [3]$$

Subtracting [2] $-2 \times$[3] gives;

$$-5b + 8c = 0$$

$$\Rightarrow b = \frac{8c}{5}$$

Substituting for b in [2];

$$2a - \frac{8c}{5} + 2c = 0$$

$$10a + 2c = 0$$

$$a = -\frac{c}{5}$$

Substituting for a and b in [1];

$$-\frac{c}{5}(x-2) + \frac{8c}{5}(y-3) + c(z-4) = 0$$

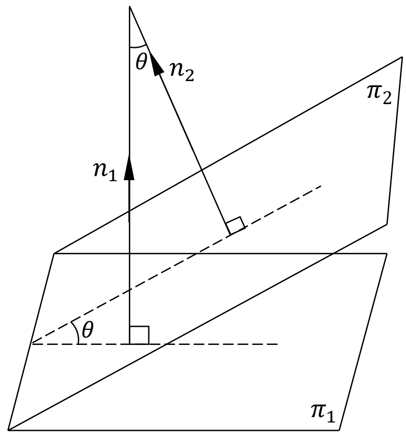
$$-x + 2 + 8y - 24 + 5z - 20 = 0$$

$$-x + 8y + 5z - 42 = 0$$

$$x - 8y - 5z + 42 = 0$$

Angle between planes

The angle between two planes is defined as the angle between their normals.



If two planes Π_1 and Π_2 have normal vectors, $n_1 = a_1 i + b_1 j + c_1 k$ and $n_2 = a_2 i + b_2 j + c_2 k$ respectively, and intersect at an acute angle θ (or $180° - \theta$ depending on their direction), the acute can be found from

$$= \frac{|(1)(1) + (2)(1) + (-1)(0) - 3|}{\sqrt{1^2 + 2^2 + (-1)^2}} = 0$$

This shows that the point lies in the plane.

Example 67

Show that the points $i - j + 3k$ and $3(i + j + k)$ are equidistant from the plane $r \cdot (5i + 2j - 7k) + 9 = 0$.

Solution

The given plane can be represented in Cartesian form as

$$5x + 2y - 7z + 9 = 0$$

Let $a = i - j + 3k$ and $b = 3(i + j + k)$

Let A and B be the points whose position vectors are a and b

$$A = (1, -1, 3),\ B = (3, 3, 3)$$

Now, the distance of A from the plane

$$= \frac{|5(1) + 2(-1) + (-7)(3) + 9|}{\sqrt{5^2 + 2^2 + (-7)^2}} = \frac{9}{\sqrt{78}}$$

and the distance of A from the plane

$$= \frac{|5(3) + 2(3) + (-7)(3) + 9|}{\sqrt{5^2 + 2^2 + (-7)^2}} = \frac{9}{\sqrt{78}}$$

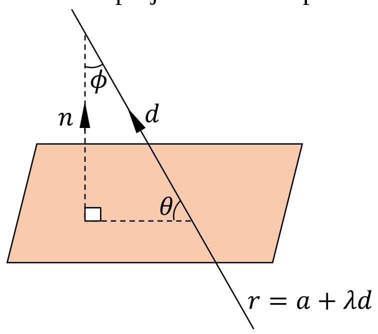
∴ The points A and B are equidistant from the plane.

Intersection of a line and a plane

In the previous examples, we considered the case of a line and a plane being parallel, and the case of a line lying in a plane. If neither of these happens, then the line and plane must intersect in a point.

The angle between a line and a plane

The angle between a line and a plane is defined as the angle between the line and its projection on the plane.



To find the angle between a line and a plane, we look at the vectors n (perpendicular to the plane) and d (in the direction of the line)

We can find angle ϕ from the formula

$$\cos\phi = \frac{d \cdot n}{|b||n|}$$

then subtract from 90° to find θ

Alternatively, we can use the fact that

$$\cos\phi = \cos(90° - \theta) = \sin\theta \text{ to write directly}$$

$$\sin\theta = \frac{d \cdot n}{|d||n|}$$

Example 68

Find the position vector of the point where the line $r = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}$ meets the plane $r \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 12$

Solution

The position vector of the point of intersection will satisfy both the equation of the line and that of the plane.

If r_1 is the position vector of the point of intersection, then

$$r_1 = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix} \text{ and } r_1 \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 12$$

Thus

$$\begin{pmatrix} 5 + \lambda \\ 3 - 4\lambda \\ -1 + 2\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 12$$

$$10 + 2\lambda + 3 - 4\lambda - 3 + 6\lambda = 12$$

$$\lambda = \frac{1}{2}$$

The required position vector is $\begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{11}{2} \\ 1 \\ 0 \end{pmatrix}$

Example 69

Find the point of intersection of the line $\frac{x}{2} = \frac{y+6}{2} = 3z - 1$ and the plane $3x + y - z = 9$. Find also the angle between the line and the plane.

Solution

Introducing the parameter λ, we have the parametric equations

$$x = 2\lambda,\ y = 2\lambda - 6 \text{ and } \tfrac{z+1}{3}$$

Substituting each of these values into the equation of the plane, we obtain

$$6\lambda + (2\lambda - 6) - \frac{\lambda + 1}{3} = 9$$

$$18\lambda + 6\lambda - 18 - \lambda - 1 = 27$$

$$23\lambda = 46$$

$$\lambda = 2$$

Substituting for $\lambda = 2$, we get

$$x = 4,\ y = -2 \text{ and } z = 1$$

∴ The point of intersection is $(4, -2, 1)$

Now writing the equation of the plane as $r \cdot \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = 9$ and the equation of the line as $r = \begin{pmatrix} 0 \\ -6 \\ \frac{1}{3} \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ \frac{1}{3} \end{pmatrix}$, we have

$n = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$ and $d = \begin{pmatrix} 2 \\ 2 \\ \frac{1}{3} \end{pmatrix}$

$$d \cdot n = 6 + 2 - \tfrac{1}{3} = \tfrac{23}{3}$$

$$|d| = \sqrt{2^2 + 2^2 + \left(\tfrac{1}{3}\right)^2} = \sqrt{\tfrac{73}{9}}$$

$$x = -\frac{8}{5}(z+1) + 2z + 3$$
$$x = \frac{2z+7}{5}$$

Collecting this information together

$$\frac{5x-7}{2} = \frac{5y+8}{-8} = z$$
$$\frac{x-7/5}{2} = \frac{y+8/5}{-8} = \frac{z}{5}$$

which is a set of equations of the line passing through the point $(7/5, -8/5, 0)$ having direction $(2, -8, 5)$. In vector equation, this would be expressed as

$$r = \begin{pmatrix} 7/5 \\ -8/5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -8 \\ 5 \end{pmatrix}$$

There are a couple of useful checks which can be used here. We can test whether $(7/5, -8/5, 0)$ really does lie on both planes by substituting into both equations.

Example 73

Find the equation of the line of intersection of the planes $x + 3y + z = 5$ and $2x - y - z = 1$. Find also the angle between the two planes.

<u>Solution</u>

$$x + 3y + z = 5 \quad \ldots.[1]$$
$$2x - y - z = 1 \quad \ldots.[2]$$

Eliminate z and hence write x in terms of y

Adding [1] and [2];

$$3x + 2y = 6 \Rightarrow x = \frac{6-2y}{3}$$

Now we eliminate y and write x in terms of z

Adding [1] to $3 \times [2]$;

$$7x - 2z = 8 \Rightarrow x = \frac{2z+8}{7}$$

Putting these together into a single equation, we have the line

$$x = \frac{6-2y}{3} = \frac{2z+8}{7}$$

Note: Having found this line, it is worth choosing a simple-valued point on the line, such as $(2, 0, 3)$ and checking that it lies on both planes – which in this case it does.

To find the angle between the planes, we find the angle between their normal vectors.

Rewriting the equations as $r \cdot (i + 3j + k) = 5$ and $r \cdot (2i - j - k) = 1$, we can calculate

$$(i + 3j + k) \cdot (2i - j - k) = -2$$
$$|i + 3j + k| = \sqrt{11}, |2i - j - k| = \sqrt{6}$$
$$\cos\theta = \frac{-2}{\sqrt{66}}$$
$$\theta = 104.3°$$

If the acute angle was required, it would be $(180° - 104.3°) = 75.7°$

Example 74

Find the equation of the plane through the intersection of the planes $3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$ and the point $(2, 2, 1)$.

<u>Solution</u>

Given planes are:

$$3x - y + 2z - 4 = 0 \text{ and } x + y + z - 2 = 0$$

Any plane through their intersection is

$$3x - y + 2z - 4 + \lambda(x + y + z - 2) = 0 \ldots [1]$$

Since point $(2, 2, 1)$ lies on it,

$$3(2) - 2 + 2(1) - 4 + \lambda(2 + 2 + 1 - 2) = 0$$
$$2 + 3\lambda = 0$$
$$\lambda = -\frac{2}{3}$$

Substituting for λ in [1] gives;

$$3x - y + 2z - 4 - \frac{2}{3}(x + y + z - 2) = 0$$
$$9x - 3y + 6z - 12 - 2x - 2y - 2z + 4 = 0$$
$$7x - 5y + 4z - 8 = 0$$

Example 75

Find the equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$.

<u>Solution</u>

The given planes are

$$x + y + z = 1 \quad \ldots [1]$$
$$2x + 3y + 4z = 5 \quad \ldots [2]$$
$$x - y + z = 0 \quad \ldots [3]$$

Any plane through the intersection of [1] and [2] is given by

$$x + y + z - 1 + \lambda(2x + 3y + 4z - 5) = 0$$
$$(1 + 2\lambda)x + (1 + 3\lambda)y + (1 + 4\lambda)z - 1 - 5\lambda = 0 \ldots [4]$$

The direction ratios of the normal of [3] are $1, -1, 1$

Also, the direction ratios of the normal of [4] are $1 + 2\lambda$, $1 + 3\lambda$, $1 + 4\lambda$

Two planes are perpendicular if their normal are perpendicular i.e. $n_1 \cdot n_2 = 0$

$$\Rightarrow 1 + 2\lambda - 1 - 3\lambda + 1 + 4\lambda = 0$$
$$\lambda = -\frac{1}{3}$$

Now the equation of the required planes is given by

$$\left(1 - \frac{2}{3}\right)x + (1 - 1)y + \left(1 - \frac{4}{3}\right)z - 1 + \frac{5}{3} = 0$$
$$x - z + 2 = 0$$

Intersection of three planes

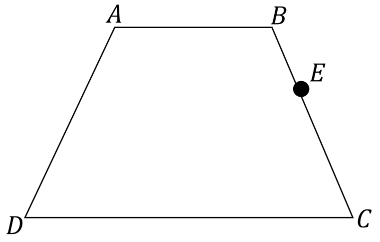
We shall consider these systematically. Let us call the three planes π_1, π_2 and π_3, and let n_1, n_2, n_3 be vectors normal to these planes respectively.

Case 1

The three planes are coincident. Then every point on each plane lies on all three planes is a plane, namely $\pi_1 (= \pi_2 = \pi_3)$

Self-Evaluation exercise

1. Prove that if $\overrightarrow{AO}+\overrightarrow{OB}=\overrightarrow{BO}+\overrightarrow{OC}$, then A, B, C are collinear.
2. If M is the midpoint of $\overline{AB}$ and N is the midpoint of $\overline{CD}$, show that $2MN = AC + BD$
3. If the vectors $a = 2u - 3v$ and $b = 5u + 4v$, find the scalars m and n such that $c = ma + nb$ where $c = 12u + 7v$ and u and v are non-parallel vectors.
[Ans: $m = \frac{13}{23}, n = \frac{50}{23}$]
4. Consider the parallelogram $ABCD$ where the point P is such that $AP : PD = 1 : 2$ and BD intersects CP at Q where $DQ : QB = 1 : 3$. Find the scalar m if $CP = mCQ$
[Ans: $m = \frac{4}{3}$]
5. Find the values(s) of x for which the vectors $xi + j - k$ and $xi - 2xj - k$ are perpendicular.
[Ans: 1]
6. P, Q and R are three points in space with coordinates $(2,-1,4)$, $(3,1,2)$ and $(-1,2,5)$ respectively. Find angle Q in the triangle PQR
[Ans: $105.2°$]
7. Find the values of x and y if $u = xi + 2yj - 8k$ is perpendicular to both $v = 2i - j + k$ and $w = 3i + 2j - 4k$
[Ans: $x = -\frac{16}{7}, y = -\frac{44}{7}$]
8. Find the unit vector that is perpendicular to both $a = 3i + 6j - k$ and $b = 3i + 2j - 4k$
[Ans: $\pm\frac{1}{\sqrt{11}}(-i + j + 3k)$]
9. Use vector methods to prove the Pythagoras' theorem
10. Use vector methods to prove that the diagonals of a rhombus bisect each other at right angles.
11. Show that if a, b and c are non-zero vectors such that $a \cdot b = a \cdot c$, then either $b = c$ or a is perpendicular to $(b - c)$
12. Vectors a, b and c are such that $a \cdot c = 3$ and $b \cdot c = 4$. Given that the vector $d = a + \lambda b$ is perpendicular to c, find the value of λ
[Ans: $-\frac{3}{4}$]
13. In the trapezium shown below, $BE{:}BC = 1{:}3$



14. Show that $3AC \cdot DE = 2(4m^2 - n^2)$ where $|AB| = m$, $|DC| = 2$ and $|DA| = n$
15. Find the equation of the line that passes through the point $A(2,7)$ and is perpendicular to the line with equation $r = -i - 3j + \lambda(3i - 4j)$
[Ans: $r = 2i + 7j + t(4i + 3j)$]
16. Find the Cartesian equation of the line passing through the points $A(5,2,6)$ and $B(-2,4,2)$.
[Ans: $\frac{x-5}{-7} = \frac{y-2}{2} = \frac{z-6}{-4}$]
17. Show that the lines $\frac{x-1}{2} = 2 - y = 5 - z$ and $\frac{4-x}{4} = \frac{5+z}{2}$ are parallel.
18. Find the coordinates of the point where the line $r = \begin{pmatrix}-2\\5\\3\end{pmatrix} + t\begin{pmatrix}-1\\2\\1\end{pmatrix}$ intersects x-y plane
[Ans: $(1,-1,0)$]
19. The line $\frac{x-3}{4} = y + 2 = \frac{4-z}{5}$ passes through the point $(a,1,b)$. Find the values of a and b.
[Ans: $a = 15, b = -11$]
20. Find the acute angle between the following lines
(a) $r = \begin{pmatrix}0\\2\\3\end{pmatrix} + s\begin{pmatrix}3\\4\\5\end{pmatrix}$ and $r = \begin{pmatrix}-2\\5\\3\end{pmatrix} + t\begin{pmatrix}-1\\2\\1\end{pmatrix}$
(b) $r = \begin{pmatrix}2\\1\\4\end{pmatrix} + s\begin{pmatrix}-2\\0\\1\end{pmatrix}$ and $r = \begin{pmatrix}1\\1\\1\end{pmatrix} + s\begin{pmatrix}1\\1\\3\end{pmatrix}$
(c) $\frac{x-3}{-1} = \frac{2-y}{3} = \frac{z-4}{2}$ and $\frac{x-1}{2} = \frac{y-2}{-2} = z - 2$
[Ans: (a) $54.74°$ (b) $82.25°$ (c) $57.69°$]
21. Find the point of intersection of the lines
(a) $\frac{x-5}{-2} = y - 10 = \frac{z-9}{12}$ and $x = 4, \frac{y-9}{-2} = \frac{z+9}{6}$
(b) $\frac{2x-1}{3} = \frac{y+5}{3} = \frac{z-1}{-2}$ and $\frac{2-x}{4} = \frac{y+3}{2} = \frac{4-2z}{1}$
[Ans: (a) $(4, 10.5, 15)$ (b) do not intersect]
22. Show that the lines $\frac{x-1}{-3} = y - 2 = \frac{7-z}{11}$ and $\frac{x-2}{3} = \frac{y+1}{8} = \frac{z-4}{-7}$ are skew.
23. Find the equation of the line passing through the origin and the point of intersection of the lines with equations $x - 2 = \frac{y-1}{4}, z = 3$ and $\frac{x-6}{2} = y - 10 = z - 4$
[Ans: $\frac{x}{4} = \frac{y}{9} = \frac{z}{3}$]
24. Find the value(s) of k, such that the lines $\frac{x-2}{k} = \frac{y}{2} = \frac{3-z}{3}$ and $\frac{x}{k-1} = \frac{y+2}{3} = \frac{z}{4}$ are perpendicular.
[Ans: 3 or -2]
25. Find a direction vector of the line that is perpendicular to both $\frac{x+1}{3} = \frac{y+1}{8} = \frac{z+1}{12}$ and $\frac{1-2x}{-4} = \frac{3y+1}{9} = \frac{z}{6}$
[Ans: $12i + 6j - 7k$ (or any multiple thereof)]
26. Are the lines $\frac{x-1}{5} = \frac{y+2}{4} = \frac{4-z}{3}$ and $\frac{x+2}{3} = \frac{y+7}{2} = \frac{2-z}{3}$ parallel? Find the point of intersection of these lines. What do you conclude?
[Ans: Not parallel. Do not intersect. Lines are skew]

67. Find the equation of the plane passing through the line of intersection of the planes $x + 2y + 3z - 4 = 0$ and $3z - y = 0$ and perpendicular to the plane $3x + 4y - 2z + 6 = 0$

[Ans: $2x + 3y + 9z = 8$]

68. Find the vector equation of the line passing through the point $(-1, 2, 1)$ and parallel to the line $r = 2i + 3j - k + \lambda(i - 2j + k)$. Also, find the distance between them.

[Ans: $r = -i + 2j + k + \mu(i - 2j + k)$; $\sqrt{\frac{83}{6}}$ units]

69. Find the equation of the plane passing through the points $A(2, 1, -3)$, $B(-3, -2, 1)$ and $C(2, 4, -1)$.

[Ans: $18x - 10y + 15z + 19 = 0$]

70. Find the Cartesian of a line which passes through the point $(-4, 2, -3)$ and is parallel to the line $\frac{-x-2}{4} = \frac{y+3}{-2} = \frac{2z-6}{3}$

[Ans: $\frac{x+4}{-8} = \frac{y-2}{-4} = \frac{z+3}{3}$]

71. Find the angle between the two straight lines $r = 3i - 2j + 4k + \lambda(-2i + j + 2k)$ and $r = i + 3j - 2k + \mu(3i - 2j + 6k)$.

[Ans: 79°]

72. Find the coordinates of the foot of the perpendicular from the point $A(1, 2, 1)$ to the line joining $B(1, 4, 6)$ and $C(5, 4, 4)$. Also, find the length of the perpendicular.

[Ans: $(3, 4, 5)$; $2\sqrt{6}$ units]

73. Show that the lines $r_1 = i + 2j + 3k + \lambda(2i + 3j + 4k)$ and $r_2 = 2i + 3j + 4k + \mu(3i + 4j + 5k)$ intersect.

74. The position vectors of two points A and B are $3i + j + 2k$ and $i - 2j - 4k$ respectively. Find the equation of a plane through B and perpendicular to AB. Calculate the distance of the plane so obtained from the point $(-1, 1, 1)$.

[Ans: $r \cdot (2i + 3j + 6k) + 28 = 0$; 5 units]

75. Find the angle between the planes whose equations are $r \cdot (-i + 2j + 2k) = 3$ and $r \cdot (3i - 2j + 6k) + 7 = 0$

[Ans: 76.2°]

76. Find the equation of the plane through the point $(3, 4, -1)$, which is parallel to the plane $r \cdot (2i - 3j + 5k) + 7 = 0$

[Ans: $2x - 3y + 5z + 11 = 0$]

77. Find the equation of the plane passing through the points $P(1, -1, 2)$ and $Q(2, -2, 2)$ and perpendicular to the plane $6x - 2y + 2z = 9$.

[Ans: $x + y - 2z + 4 = 0$]

78. Find the vector equation of the plane passing through the line of intersection of the planes $r \cdot (2i - 7j + 4k) = 3$ and $r \cdot (3i - 5j + 4k) + 11 = 0$ and passing through the points $(-2, 1, 3)$.

[Ans: $r \cdot (15i - 47j + 28k) = 7$]

79. Find the equation of the plane passing through the line of intersection of the planes $2x - y = 0$ and $3z - y = 0$ and perpendicular to the plane $4x + 5y - 3z = 8$.

[Ans: $28x - 17y + 9z = 0$]

80. Show that the plane whose vector equation is $r \cdot (i + 2j - k) = 1$ and the line whose vector equation is $r \cdot (i + j + k) + \lambda(2i + j + 4k)$ are parallel. Also, find the distance between them.

[Ans: $\frac{1}{6}\sqrt{6}$ units]

81. Find the foot of perpendicular from the point $(2, 3, -8)$ to the line $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$. Also find the perpendicular distance from the given point to the line.

[Ans: $(2, 6, -2)$; $3\sqrt{5}$]

82. Find the length and the foot of perpendicular from the point $\left(1, \frac{3}{2}, 2\right)$ to the plane $2x - 2y + 4z + 5 = 0$.

[Ans: $\sqrt{6}$]

83. Find the equation of the plane through the points $(2, 1, -1)$ and $(-1, 3, 4)$, and perpendicular to the plane $x - 2y + 4z = 10$.

[Ans: $18x + 17y + 4z = 49$]

84. Find the equation of the plane which is perpendicular to the plane $5x + 3y + 6z + 8 = 0$ and which contains the line of intersection of the planes $x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$.

[Ans: $51x + 15y - 50z + 173 = 0$]

85. Find the equation of the plane through the intersection of the planes $r \cdot (i + 3j) - 6 = 0$ and $r \cdot (3i - j - 4k) = 0$ whose perpendicular distance from origin is unity.

[Ans: $4x + 2y - 4z - 6 = 0$ and $-2x + 4y + 4z - 6 = 0$]

86. Show that the points $(i - j + 3k)$ and $3(i + j + k)$ are equidistant from the plane $r \cdot (5i + 2j - 7k) + 9 = 0$ and lies on opposite side of it.

87. $AB = 3i - j + k$ and $CD = -3i + 2j + 4k$ are two vectors. The position vectors of the points A and C are $6i + 7j + 4k$ and $-9j + 2k$, respectively. Find the position vectors of point P on the line AB and a point Q on the line CD such that PQ is perpendicular to both AB and CD.

[Ans: $3i + 8j + 3k$, $-3i - 7j + 6k$]

Chapter 19 Examination Questions

SECTION A

1. The point $C(a, 4, 5)$ divides the line joining points $A(1, 2, 3)$ and $B(6, 7, 8)$ in the ratio $\lambda : 3$. Using vectors, find the values a and λ.

2. [2024, No. 7]

3. Find the angle between the line $r = \begin{pmatrix}2\\0\\5\end{pmatrix} + \lambda\begin{pmatrix}3\\12\\4\end{pmatrix}$ and the plane $-x + 2y + 2z - 66 = 0$.

[2023, No. 4]

4. The position vectors of points P and Q are given by $OP = i - 2j + k$ and $OQ = 3i - 4j + 6k$ respectively. Point R divides the line $\overline{PQ}$ in the ratio $2 : -3$. Determine the coordinates of the point R.

[2022, No. 3]

5. A plane is perpendicular to the vector $r = (i + 3j - 2k)$ and contains the point $P(-2, 0, 4)$. Determine the equation of the plane.

[2020, No. 4]

6. Given the plane $4x + 3y - 3z - 4 = 0$;
   (a) show that the point $A(1, 1, 1)$ lies on the plane
   (b) find the perpendicular distance from the plane to the point $B(1, 5, 1)$

[2019, No. 4]

7. Determine the angle between the line $\frac{x+4}{8} = \frac{y-2}{2} = \frac{z+1}{-4}$ and the plane $4x + 3y - 3z + 1 = 0$

[2018, No. 2: Ans: $69.33°$]

8. The vertices of a triangle are $P(4, 3)$, $Q(6, 4)$ and $R(5, 8)$. Find the angle RPQ using vectors.

[2017, No. 5: Ans: $52.13°$]

9. Three points $A(2, -1, 0)$, $B(-2, 5, -4)$ and C are on a straight line such that $3AB = 2AC$. Find the coordinates of C.

[2016, No. 8: Ans: $C(-4, 8, -6)$]

10. Given that $D(7, 1, 2)$, $E(3, -1, 4)$ and $F(4, -2, 5)$ are points on a plane, show that $\boldsymbol{ED}$ is perpendicular to $\boldsymbol{EF}$.

[2015, No. 5]

11. Find the equation of a line through $S(1, 0, 2)$ and $T(3, 2, 1)$ in the form $r = a + \lambda b$.
    Hence, deduce the cartesian equation of the line.

[2014, No. 5: Ans: $\frac{x-1}{2} = \frac{y}{2} = \frac{z-2}{-1} = \lambda$]

12. The position vector of point A is $2i + 3j + k$, of B is $5j + 4k$ and of C is $i + 2j + 12k$. Show that ABC is a triangle.

[2013, No. 4]

13. A line passes through the points $A(4, 6, 3)$ and $B(1, 3, 3)$.
    (a) Find the vector equation of the line
    (b) Show that the point $C(2, 4, 3)$ lies on the line in (a) above.

[2012, No. 4: Ans: (a) $\begin{pmatrix}4\\6\\3\end{pmatrix} + \lambda\begin{pmatrix}-3\\-3\\0\end{pmatrix}$]

14. Show that the points A, B and C with position vectors $3i + 3j + k$, $8i + 7j + 4k$ and $11i + 4j + 5k$ respectively, are vertices of a triangle.

[2011, No. 7]

15. Given the points $A(-3, 3, 4)$, $B(5, 7, 2)$ and $C(1, 1, 4)$, find the vector equation of a line which joins the mid-points of AB and BC.

[2010, No. 5: Ans: $r = \begin{pmatrix}1\\5\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\0\end{pmatrix}$]

16. Find the equation of a line through the point $(1, 3, -2)$ and perpendicular to the plane whose equation is $4x + 3y - 2z - 16 = 0$.

[2009, No. 7: Ans: $r = \begin{pmatrix}1\\3\\-2\end{pmatrix} + \lambda\begin{pmatrix}4\\3\\-2\end{pmatrix}$]

17. The points $P(2, 3)$, $Q(-11, 8)$ and $R(-4, -5)$ are vertices of a parallelogram $PQRS$ which has PR as a diagonal. Find the coordinates of the vertex S.

[2009, No. 5: Ans: $S(9, -10)$]

18. Given vectors $a = i - 3j + 3k$ and $b =$-$i - 3j + 2k$, find the:
    (i) acute angle between the vectors a and b
    (ii) equation of the plane containing a and b

[2008, No. 4: Ans: (i) $30.86°$ (ii) $-3x + 5y + 6z = 0$]

19. A point P has co-ordinates $(1, -2, 3)$ and a certain plane has the equation $x + 2y + 2z = 8$. The line through P parallel to the line $\frac{x}{3} = \frac{y+1}{-1} = z + 1$ meets the plane at Q. Find the co-ordinates of Q.

[2007, No. 5: Ans: $\left(6, -\frac{11}{3}, \frac{14}{3}\right)$]

20. Find the point of intersection of the plane $11x - 3y + 7z = 8$ and the line
    $r = \begin{pmatrix}-3\\1\\5\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\-2\end{pmatrix}$, where λ is a scalar

[2006, No. 5: Ans: $(-4, -1, -7)$]

21. Given that the vectors $ai - 2j + k$ and $2ai + aj - 4k$ are perpendicular, find the values of a

[2005, No. 4: Ans: $-1, 2$]

16. The position vectors of points A and B are $OA = 2i - 4j - k$ and $OB = 5i - 2j + 3k$ respectively. The line AB produced to meet the plane $2x + 6y - 3z = -5$ at a point C. Find the:
    (a) co-ordinates of C
    (b) angle between AB and the plane.
    [2008, No. 12: Ans: (a) $(8, 0, 7)$ (b) $9.16°$]
17. Given that the position vectors of A, B and C are $OA = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$, $OB = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and $OC = \begin{pmatrix} 7 \\ 10 \\ -7 \end{pmatrix}$
    (i) Prove that A, B and C are collinear
    (ii) Find the acute angle between OA and OB
    (iii) If $OABD$ is a parallelogram, find the position vectors of E and F such that E divides DA in the ratio $1:2$ and F divides it externally in the ratio $1:2$.
    [2007, No. 15: Ans: (ii) $106.1°$ (iii) $OE = \begin{pmatrix} \frac{5}{3} \\ 2 \\ -\frac{4}{3} \end{pmatrix}$; $OF = \begin{pmatrix} 3 \\ 10 \\ -8 \end{pmatrix}$]
18. (a) Given the vectors $a = 3i - 2j + k$ and $b = i - 2j + 2k$, find
    (i) the acute angle between the vectors
    (ii) vector c such that it is perpendicular to both vectors a and b
    (b) Given that $OA = a$ and $OB = b$, point R is on $\overline{OB}$ such that $\overline{OR} = \overline{OA}$ are both produced they meet at point Q. Find:
    (i) OR and OP in terms of a and b
    (ii) OQ in terms of a
    [2006, No. 11: Ans: (a) (i) $36.7°$ (ii) $2i + 5j + 4k$ (b) (i) $\frac{1}{5}(2a + 3b)$ (ii) $\frac{8}{5}a$]
19. (i) Determine the coordinates of the point of intersection of the line $\frac{x+1}{2} = \frac{y-3}{5} = \frac{z+2}{-1}$ and the plane $x + y + z = 12$
    (ii) Find the angle between the line $\frac{x+1}{2} = \frac{y-3}{5} = \frac{z+2}{-1}$ and the plane $x + y + z = 12$
    [2005, No. 10: Ans: (i) $(3, 13, -4)$ (ii) $39.25°$]
20. (a) Find the equation of the line through $A(2, 2, 5)$ and $B(1, 2, 3)$
    (b) If the line in (a) above meets the line $\frac{x-1}{1} = \frac{y-2}{0} = \frac{z-1}{3}$ at P, find the:
    (i) coordinates of P,
    (ii) angle between the two lines
    [2004, No. 11: Ans: (b) (i) $P(3, 2, 7)$ (ii) $8.1°$ or $171.9°$]
21. (a) in a triangle ABC, the altitudes from B and C meet the altitudes from B and C meet the opposite sides at E and F respectively. BE and CF intersect at O. Taking O as the origin, use the dot product to prove that AO is perpendicular to BC.
    (b) Prove that $\angle ABC = 90°$ given that A is $(0, 5, -3)$, $B(2, 3, -4)$ and $C(1, -1, 2)$. Find the coordinates of D, if $ABCD$ is a rectangle.
    [2003, No. 11: Ans: (b) $D(-1, 1, 3)$]
22. (a) Find the equation of the perpendicular line from point $A = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ onto the line $r = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$. What is the distance from A to r?
    (b) Find the angle contained between line OR and the x-y plane, where $OR = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$
    [2002, No. 11: Ans: (a) 1.795 units (b) $41.81°$]
23. (a) Find the Cartesian equation of the plane through $A(0, 3, , -4)$, $B(2, -1, 2)$ and $C(7, 4, -1)$. Show that $Q(10, 13, -10)$ lies in the same plane.
    (b) Express the equation of the plane in (a) in the scalar product form
    (c) Find the area of triangle ABC in (a)
    [2001, No. 14: Ans: (b) $r \cdot \begin{pmatrix} 3 \\ -6 \\ -5 \end{pmatrix} = 2$ (c) 25.1 sq. units]
24. (a) Show that the equation of the plane through points A with the position vector $-2i + 4k$ perpendicular to the vector $i + 3j - 2k$ is $x + 3y - 2z + 10 = 0$
    (b) (i) Show that the vector $2i - 5j + 3.5k$ is perpendicular to the line $r = 2i - j + \lambda(4i + 3j + 2k)$
    (ii) Calculate the angle between the vector $3i - 2j + k$ and the line in (b)(i) above.
    [2000, No. 12: Ans: (b) (ii) $66.6°$]
25. (a) Find in Cartesian form the equation of the line passing through the points $A(1, 2, 5)$, $B(1, 0, 4)$ and $C(5, 2, 1)$
    (b) Find the angle between the line $\frac{x+4}{8} = \frac{y-2}{2} = \frac{z-1}{-4}$ and the plane $4x + 3y - 3z + 1 = 0$
    [1999, No. 16: Ans: (a) A, B and C are non-collinear (b) $69.3°$]
26. The vector equations of lines P and Q are given as $r_p = t(4i + 3j)$ and $r_q = 2i + 12j + 5(i - j)$.
    (a) Use the dot product to find the angle between P and Q
    (b) If the lines P and Q meet at M, find the coordinates of M. Find also the equation of the line through M perpendicular to the line Q.
    [Nov 1998, No. 12: Ans: (a) $8.13°$ (b) N/A]

Trigonometry

Chapter 20 Introduction to Trigonometry

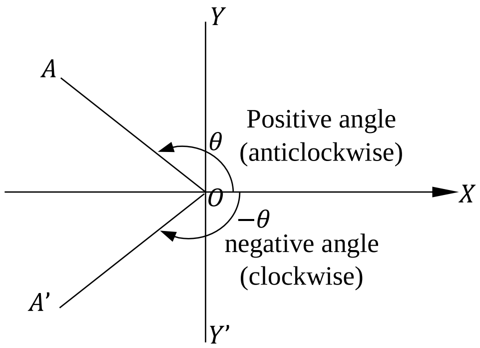
Trigonometry

Trigonometry is the branch of mathematics that deals with the measurement of sides and angles of triangles and their relationship with each other. There are many applications where the knowledge of trigonometry is used such as astronomy, navigation (on the oceans, in aircraft and in space), electronics seismology, medical imaging and many other physical sciences.

Angles

An angle is defined as the amount of rotation of a revolving line from the initial position to the terminal position. Counter-clockwise rotations will be called positive and the clockwise will be called negative.

Consider a rotating ray OA with its endpoint at the origin O.



The rotating ay OA is often called the terminal side of the angle and the positive half of the x-axis (OX) is called the initial side.

The positive angle θ is XOA (counter-clockwise direction)

The negative angle θ is XOA' (clockwise direction)

Note:

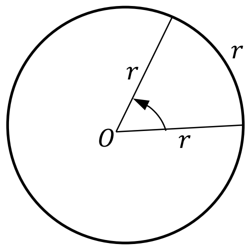
1. one complete rotation (counter-clockwise) = 360°
2. If there is no rotation, the measure of the angle is 0°

Measurement of angles

If a rotation from the initial position to the terminal position is $\left(\frac{1}{360}\right)^{th}$ of the revolution, the angle is said to have a measure of one degree and written as 1°. A degree is divided into minutes, and a minute is divided into seconds.

Radian measure

One radian, is the measure of an angle subtended at the centre O of a circle of radius r by an arc of length r.

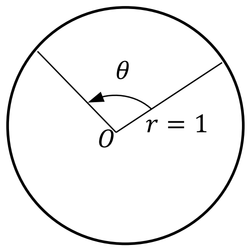


Note:

1. To express the measure of an angle as a real number, we use radian measure.
2. The word "radians" is optional and often omitted. Thus, if no unit is given for a rotation, it is understood to be in radians.

Relationship between Degrees and Radians

Since a circle of radius r has a circumference of $2\pi r$, a circle of radius 1 unit (which is referred to as a unit circle) has a circumference 2π. When θ is a complete rotation, P travels the circumference of a unit circle completely.



If θ is a complete rotation (counter-clockwise) then $\theta = 2\pi$ radian. On the other hand, we already know that one complete rotation (counter-clockwise) is 360°, consequently $360° = 2\pi$ radians or $180° = \pi$ radian. It follows that

$$1° = \frac{\pi}{180} \text{ radian and } \frac{180°}{\pi} = 1 \text{ radian}$$

Therefore $1° \approx 0.01746$ radian and $1 \text{ radian} = 180° \times \frac{7}{22} \approx 57°16'$.

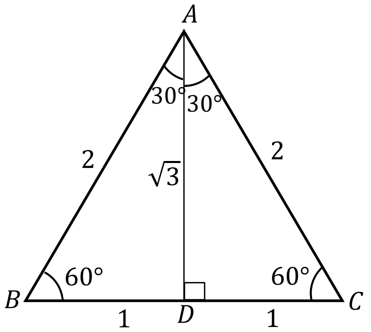
Conversions for some special angles

Degrees	30°	45°	60°	90°	180°	270°	360°
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{0.500} = 2.00$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{0.625}{0.500} = 1.25$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{0.500}{0.625} = 0.80$$

Trigonometric ratios of special angles $30°, 45°$ and $60°$

Consider an equilateral triangle ABC of side 2 units. AD bisects angle A and bisects the side BC.



Using the Pythagoras' theorem on triangle ABD;

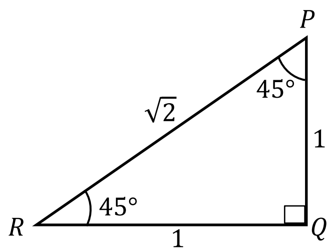
$$AD = \sqrt{2^2 - 1^2} = \sqrt{3}$$

Hence,

$$\sin 30° = \frac{BD}{AD} = \frac{1}{2}$$
$$\cos 30° = \frac{AD}{AB} = \frac{\sqrt{3}}{2}$$
$$\tan 30° = \frac{BD}{AD} = \frac{1}{\sqrt{3}}$$
$$\sin 60° = \frac{AD}{AB} = \frac{\sqrt{3}}{2}$$
$$\cos 60° = \frac{BD}{AB} = \frac{1}{2}$$
$$\tan 60° = \frac{AD}{BD} = \sqrt{3}$$

Consider an isosceles triangle PQR with $PQ = QR = 1$ unit.
By Pythagoras' theorem,

$$PR = \sqrt{1^2 + 1^2} = \sqrt{2}$$



Hence,

$$\sin 45° = \frac{1}{\sqrt{2}}$$
$$\cos 45° = \frac{1}{\sqrt{2}}$$
$$\tan 45° = 1$$

It can be observed that
$\sin 30° = \cos 60°$, $\sin 45° = \cos 45°$ and
$\sin 60° = \cos 30°$
In general,

$$\sin\theta = \cos(90° - \theta)$$
$$\cos\theta = \sin(90° - \theta)$$

For example, it may be checked by calculator that $\sin 25° = \cos 65°$, $\sin 42° = \cos 48°$ and so on.

The trigonometric ratios of $30°, 45°$ and $60°$ are summarized in the table below.

θ	$\sin\theta°$	$\cos\theta°$	$\tan\theta°$
30	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Example 4

Using surd forms, evaluate

$$\frac{3\tan 60° - 2\cos 30°}{\tan 30°}$$

Solution

$\tan 60° = \sqrt{3}$, $\cos 30° = \frac{\sqrt{3}}{2}$ and $\tan 30° = \frac{1}{\sqrt{3}}$

$$\frac{3\tan 60° - 2\cos 30°}{\tan 30°} = \frac{3(\sqrt{3}) - 2\left(\frac{\sqrt{3}}{2}\right)}{\frac{1}{\sqrt{3}}}$$
$$= \frac{3\sqrt{3} - \sqrt{3}}{\frac{1}{\sqrt{3}}}$$
$$= 2\sqrt{3} \times \frac{\sqrt{3}}{1} = 6$$

Evaluating trigonometric ratios of any angles

The easiest method of evaluating trigonometric functions of any angle is by using a calculator. The following values, correct to 4 decimal places, may be checked.

$$\sin 18° = 0.3090$$
$$\sin 241.63° = -0.8799$$
$$\cos 56° = 0.5592$$
$$\cos 331.78° = 0.8811$$
$$\tan 178° = -0.0349$$
$$\tan 296.42° = -2.0127$$

To evaluate, say, $\sin 42°23'$ using a calculator means finding $\sin 42\frac{23°}{60}$ since there are 60 minutes in 1 degree.

$$\frac{23}{60} = 0.3833, \text{ thus } 42°23' = 42.3833°$$

Thus $\sin 42°23' = \sin 42.3833° = 0.6741$, correct to 4 d.p

Similarly, $\cos 72°38' = \cos 72\frac{38°}{60} = 0.2985$, correct to 4 d.p

Chapter 21 Trigonometric identities & Equations

Trigonometric identities

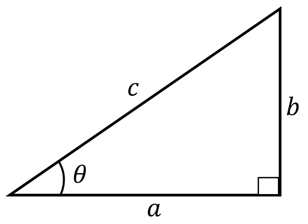
A trigonometric identity is a relationship that is true for all values of the unknown variable.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \cot\theta = \frac{\cos\theta}{\sin\theta}, \sec\theta = \frac{1}{\cos\theta}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \text{ and } \cot\theta = \frac{1}{\tan\theta}$$

are examples of trigonometric identities.

Applying Pythagoras' theorem to the right-angled triangle shown below



$$a^2 + b^2 = c^2 \ldots (1)$$

Dividing each term by c^2 gives:

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2}$$

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

$$\boldsymbol{\cos^2\theta + \sin^2\theta = 1}$$

Dividing each term of equation (1) by a^2 gives:

$$\frac{a^2}{a^2} + \frac{b^2}{a^2} = \frac{c^2}{a^2}$$

$$1 + \left(\frac{b}{a}\right)^2 = \left(\frac{c}{a}\right)^2$$

$$\boldsymbol{1 + \tan^2\theta = \sec^2\theta}$$

Dividing each term of equation (1) by b^2 gives:

$$\frac{a^2}{b^2} + \frac{b^2}{b^2} = \frac{c^2}{b^2}$$

$$\left(\frac{a}{b}\right)^2 + 1 = \left(\frac{c}{b}\right)^2$$

$$\boldsymbol{\cot^2\theta + 1 = \operatorname{cosec}^2\theta}$$

The above three are further examples of trigonometric identities.

Example 1

Prove the identity $\sin^2\theta\cot\theta\sec\theta = \sin\theta$

Solution

With trigonometric identities, it is necessary to start with the left-hand side (LHS) and attempt to make it equal to the right-hand side (RHS) or vice versa. It is often useful to

$$\text{LHS} = \sin^2\theta\cot\theta\sec\theta$$

$$= \sin^2\theta\left(\frac{\cos\theta}{\sin\theta}\right)\left(\frac{1}{\cos\theta}\right)$$

$$= \sin\theta = \text{RHS}$$

Example 2

Prove that

$$\frac{\tan x + \sec x}{\sec x\left(1 + \frac{\tan x}{\sec x}\right)} = 1$$

Solution

$$\text{LHS} = \frac{\tan x + \sec x}{\sec x\left(1 + \frac{\tan x}{\sec x}\right)}$$

$$= \frac{\frac{\sin x}{\cos x} + \frac{1}{\cos x}}{\left(\frac{1}{\cos x}\right)\left(1 + \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}}\right)}$$

$$= \frac{\frac{\sin x + 1}{\cos x}}{\left(\frac{1}{\cos x}\right)\left[1 + \left(\frac{\sin x}{\cos x}\right)\left(\frac{\cos x}{1}\right)\right]}$$

$$= \frac{\frac{\sin x + 1}{\cos x}}{\left(\frac{1}{\cos x}\right)[1 + \sin x]}$$

$$= \left(\frac{\sin x + 1}{\cos x}\right)\left(\frac{\cos x}{1 + \sin x}\right)$$

$$= 1 = \text{RHS}$$

Example 3

Prove that $\dfrac{1 + \cot\theta}{1 + \tan\theta} = \cot\theta$

Solution

$$\text{LHS} = \frac{1 + \cot\theta}{1 + \tan\theta}$$

$$= \frac{1 + \frac{\cos\theta}{\sin\theta}}{1 + \frac{\sin\theta}{\cos\theta}}$$

$$= \frac{\frac{\sin\theta + \cos\theta}{\sin\theta}}{\frac{\cos\theta + \sin\theta}{\cos\theta}}$$

$$= \left(\frac{\sin\theta + \cos\theta}{\sin\theta}\right)\left(\frac{\cos\theta}{\cos\theta + \sin\theta}\right)$$

$$= \frac{\cos\theta}{\sin\theta} = \cot\theta = \text{RHS}$$

$$\frac{2\sin\theta}{\cos\theta} = 1$$
$$2\tan\theta = 1$$
$$\tan\theta = \frac{1}{2}$$
$$\theta = \tan^{-1}\frac{1}{2} = 26.57°$$

Since tangent is positive in the first and third quadrants.

$$\theta = 26.57° \text{ and } 206.57°$$

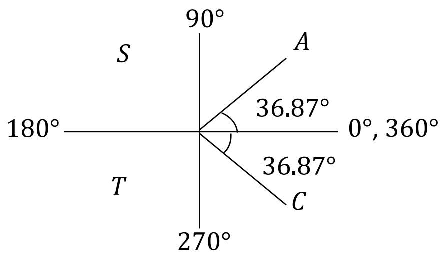
Example 17

Solve $4\sec t = 5$ for values of t between $0°$ and $360°$

Solution

$$\sec t = \frac{5}{4}$$
$$\frac{1}{\cos t} = \frac{5}{4}$$
$$\cos t = \frac{4}{5} = 0.8$$
$$t = \cos^{-1} 0.8 = 36.87°$$

Since cosine is positive in the first and fourth quadrants.



$$t = 36.87° \text{ or } 360° - 36.87°$$

Example 18

Solve $2 - 4\cos^2 A = 0$ for values of A in the range $0° < A < 360°$

Solution

$$2 - 4\cos^2 A = 0$$
$$\cos^2 A = \frac{2}{5} = 0.5$$
$$\cos A = \sqrt{0.5} = \pm 0.7071$$
$$A = \cos^{-1}(0.7071)$$

Cosine is positive in quadrants one and four and negative in quadrants wo and three. Thus in this case, there are four solutions, one in each quadrant.

The acute angle $\cos^{-1} 0.7071 = 45°$

Hence, $A = 45°, 135°, 225°$ or $315°$

Example 19

Solve the equation

$$8\sin^2\theta + 2\sin\theta - 1 = 0$$

for all values of θ between $0°$ and $360°$.

Solution

Factorising $8\sin^2\theta + 2\sin\theta - 1 = 0$

$$8\sin^2\theta + 4\sin\theta - 2\sin\theta - 1 = 0$$
$$4\sin\theta\,(2\sin\theta + 1) - (2\sin\theta + 1) = 0$$
$$(2\sin\theta + 1)(4\sin\theta - 1) = 0$$

Hence $4\sin\theta - 1 = 0$, from which $\sin\theta = \frac{1}{4} = 0.25$

or $2\sin\theta + 1 = 0$, from which $\sin\theta = -\frac{1}{2} = -0.5$

$\theta = \sin^{-1} 0.25 = 14.48°$ or $165.52°$, since sine is positive in the first and second quadrants, or

$\theta = \sin^{-1} 0.5 = 210°$ or $330°$, since sine is negative in the third and fourth quadrants.

Hence $\theta = 14.48°, 165.52°, 210°$ or $330°$

Example 20

Solve $6\cos^2\theta + 5\cos\theta - 6 = 0$ for values of θ from $0°$ to $360°$.

Solution

$$6\cos^2\theta + 5\cos\theta - 6 = 0$$
$$6\cos^2\theta + 9\cos\theta - 4\cos\theta - 6 = 0$$
$$3\cos\theta\,(2\cos\theta + 3) - 2(2\cos\theta + 3) = 0$$
$$(2\cos\theta + 3)(3\cos\theta - 2) = 0$$

Hence $3\cos\theta - 2 = 0$, from which $\cos\theta = \frac{2}{3} = 0.6667$

or $2\cos\theta + 3 = 0$, from which, $\cos\theta = -\frac{3}{2} = -1.5$

The minimum value of a cosine is -1, hence the latter expression has no solution and is thus neglected.

Hence,

$$\theta = \cos^{-1} 0.6667 = 48.18°$$

and since cosine is positive in the first and fourth quadrants,

$$\theta = 48.18° \text{ or } 311.82°$$

Example 21

Solve $5\cos^2 t + 3\sin t - 3 = 0$ for values of t from $0°$ to $360°$.

Solution

Since $\cos^2 t + \sin^2 t = 1$, $\cos^2 t = 1 - \sin^2 t$. Substituting for $\cos^2 t$ in the given equation gives:

$$5(1 - \sin^2 t) + 3\sin t - 3 = 0$$
$$5 - 5\sin^2 t + 3\sin t - 3 = 0$$
$$-5\sin^2 t + 3\sin t + 2 = 0$$
$$5\sin^2 t - 3\sin t - 2 = 0$$
$$5\sin^2 t - 5\sin t + 2\sin t - 2 = 0$$
$$5\sin t\,(\sin t - 1) + 2(\sin t - 1) = 0$$
$$(5\sin t + 2)(\sin t - 1) = 0$$

Hence,

$5\sin t + 2 = 0$ from which, $\sin t = -\frac{2}{5} = -0.4$

or $\sin t - 1 = 0$, from which, $\sin t = 1$

$$t = \sin^{-1} 0.4 = 23.58°$$

Since sine is negative in the third and fourth quadrants

$$t = 203.58° \text{ or } 336.42°$$

Also, $t = \sin^{-1} 1 = 90°$ and sine is positive in the first and fourth quadrants.

Hence, $t = 90°, 203.58°$ or $336.42°$

Example 22

Solve $18\sec^2 A - 3\tan A = 21$ for values of A between 0° and 360°.

Solution

$$1 + \tan^2 A = \sec^2 A$$

Substituting for $\sec^2 A$ n the given equation

$$18(1 + \tan^2 A) - 3\tan A = 21$$
$$18 + 18\tan^2 A - 3\tan A - 21 = 0$$
$$18\tan^2 A - 3\tan A - 3 = 0$$
$$6\tan^2 A - \tan A - 1 = 0$$
$$6\tan^2 A - 3\tan A + 2\tan A - 1 = 0$$
$$3\tan A(2\tan A - 1) + (2\tan A - 1) = 0$$
$$(2\tan A - 1)(3\tan A + 1) = 0$$

Hence $2\tan A - 1 = 0$, from which $\tan A = \frac{1}{2} = 0.5$

or $3\tan A + 1 = 0$, from which $\tan A = -\frac{1}{3} = -0.3333$

$A = \tan^{-1} 0.5 = 26.67°$ or $206.57°$, since tangent is positive in the first and third quadrants, or

$A = \tan^{-1}(0.3333) = 161.57°$ or $341.57°$, since tangent is negative in the second and fourth quadrants.

Hence, $A = 26.57°, 161.57°, 206.57°$ or $341.57°$

Example 23

Solve $3\,\mathrm{cosec}^2\theta - 5 = 4\cot\theta$ in the range $0° < \theta < 360°$.

Solution

$$\cot^2\theta + 1 = \mathrm{cosec}^2\theta$$

Substituting for $\mathrm{cosec}^2\theta$ in the given equation gives:

$$3(\cot^2\theta + 1) - 5 = 4\cot\theta$$
$$3\cot^2\theta + 3 - 5 = 4\cot\theta$$
$$3\cot^2\theta - 4\cot\theta - 2 = 0$$

Since the LHS does not factorise, the quadratic formula is used. Thus,

$$\cot\theta = \frac{-(-4) \pm \sqrt{[(-4)^2 - 4(3)(-2)]}}{2(3)}$$
$$= \frac{4 \pm \sqrt{40}}{6}$$
$$= 1.708 \text{ or} - 0.3874$$
$$\Rightarrow \tan\theta = \frac{1}{1.708} \text{ or} - \frac{1}{0.3874}$$

$\theta = \tan^{-1}\left(\frac{1}{1.708}\right) = 30.17°$ or $210.17°$ since tangent is positive in the first and third quadrants, or

$\theta = \tan^{-1}\left(\frac{1}{0.3874}\right) = 111.18°$ or $291.18°$ since tangent is negative in the second and fourth quadrants.

Hence, $\theta = 30.17°, 111.18°, 210.17°$ or $291.18°$

Example 24

Solve the equation $2\tan^2\theta = 11\sec\theta - 7$ for $0 \le \theta < 360°$

Solution

$$2\tan^2\theta = 11\sec\theta - 7$$
$$2(\sec^2\theta - 1) = 11\sec\theta - 7$$
$$2\sec^2\theta - 11\sec\theta + 5 = 0$$
$$2\sec^2\theta - \sec\theta - 10\sec\theta + 5 = 0$$
$$\sec\theta(2\sec\theta - 1) - 5(2\sec\theta - 1) = 0$$
$$(2\sec\theta - 1)(\sec\theta - 5) = 0$$
$$2\sec\theta - 1 = 0 \text{ or} \sec\theta - 5 = 0$$
$$\sec\theta = \frac{1}{2} \quad \text{or} \sec\theta = 5$$
$$\Rightarrow \cos\theta = 2 \quad \text{or } \cos\theta = \frac{1}{5} = 0.2$$
$$\cos^{-1} 2 = \text{undefined}$$
$$\cos^{-1} 0.2 = 78.5°$$
$$\therefore \theta = 78.5°, 281.5°$$

Example 25

Solve the equation $4\cot^2 x - 9\,\mathrm{cosec}\, x + 6 = 0$, for $0 \le x < 360°$.

Solution

$$4\cot^2 x - 9\,\mathrm{cosec}\, x + 6 = 0$$
$$4(\mathrm{cosc}^2 x - 1) - 9\,\mathrm{cosec}\, x + 6 = 0$$
$$4\,\mathrm{cosec}^2 x - 9\,\mathrm{cosec}\, x + 2 = 0$$
$$4\,\mathrm{cosec}^2 x - 8\,\mathrm{cosec}\, x - \mathrm{cosec}\, x + 2 = 0$$
$$4\,\mathrm{cosec}\, x(\mathrm{cosec}\, x - 2) - (\mathrm{cosec}\, x - 2) = 0$$
$$(\mathrm{cosec}\, x - 2)(4\,\mathrm{cosec}\, x - 1) = 0$$
$$\mathrm{cosec}\, x - 2 = 0 \text{ or} \,\mathrm{cosec}\, x - 1 = 0$$
$$\mathrm{cosec}\, x = 2 \quad \text{or} \quad \mathrm{cosec}\, x = \frac{1}{4}$$
$$\Rightarrow \sin x = \frac{1}{2} \quad \text{or} \quad \sin x = 4$$
$$\sin^{-1} 4 = \text{undefined}$$
$$\sin^{-1} 0.5 = 30°$$
$$\therefore x = 30°, 150°$$

Example 26

Solve $\sec^2 y + \tan y = 3$ for $0 \le x < 360°$

Solution

$$\sec^2 y + \tan y = 3$$
$$(1 + \tan^2 y) + \tan y = 3$$
$$\tan^2 y + \tan y - 2 = 0$$
$$(\tan y - 1)(\tan y + 2) = 0$$
$$\tan y - 1 = 0 \quad \text{or } \tan y + 2 = 0$$
$$\tan y = 1 \text{ or } \tan y = -2$$
$$\tan^{-1} 1 = 45°$$
$$y = 45°, 225°$$
$$\tan^{-1} 2 = 63.4°$$
$$y = 116.6°, 296.6°$$
$$\therefore y = 45°, 116.6°, 225°, 296.6°$$

Example 27

Solve $2\,\mathrm{cosec}^2\varphi + \cot^2\varphi = 11$ for $0 \le \varphi < 360°$

Chapter 22 Compound angles

Introduction

It is often useful to be able to express a trig ratio of an angle $A + B$ in terms of trig ratios of A and of B.

It is dangerously easy to think, for instance, that $\sin(A + B)$ is $\sin A + \sin B$. However, this is false as can be seen by considering

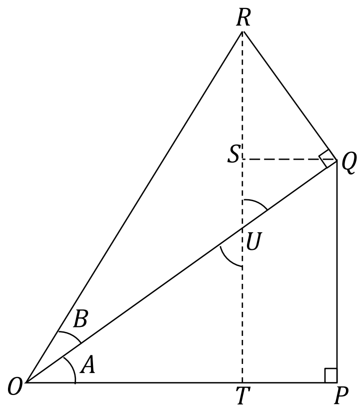
$$\sin(45° + 45°) = \sin 90° = 1$$

whereas $\sin 45° + \sin 45° = \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2} \neq 1$

Thus the sine function is not distributive and neither are the other trig functions.

The correct identity is $\sin(A + B) = \sin A \cos B + \cos A \sin B$.

This is proved geometrically when A and B are both acute, from the diagram below.



The right-angled triangles OPQ and OQR contain angles A and B as shown.

From the diagram, $\angle URQ = A$

$$\sin(A + B) = \frac{TR}{OR} = \frac{TS + SR}{OR} = \frac{PQ + SR}{OR}$$

$$= \frac{PQ}{OQ} \times \frac{OQ}{OR} + \frac{SR}{QR} \times \frac{QR}{OR}$$

$$\therefore \sin(A + B) = \sin A \cos B + \cos A \sin B$$

This identity is in fact valid for all angles and it can be adapted to give the full set of compound angle formulae.

To find an identity for the sine of a difference, we can use the identity just derived, substituting $-B$ for B

$$\sin(A - B) = \sin[A + (-B)]$$

$$= \sin A \cos(-B) + \cos A \sin(-B)$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

To develop an identity for $\cos(A + B)$, we recall the following

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

In this identity, we shall substitute $A + B$ for θ.

$$\cos(A + B) = \sin\left[\frac{\pi}{2} - (A + B)\right] = \sin\left[\left(\frac{\pi}{2} - A\right) - B\right]$$

We now use the identity of sine of a difference

$$\sin\left[\left(\frac{\pi}{2} - A\right) - B\right] = \sin\left(\frac{\pi}{2} - A\right)\cos B - \sin B \cos\left(\frac{\pi}{2} - A\right)$$

$$= \cos A \cos B - \sin B \sin A$$

$$\therefore \cos(A + B) = \cos A \cos B - \sin A \sin B$$

Now, let us consider $\cos(A - B)$. This is equal to $\cos[A + (-B)]$ and by cosine of a sum, we have the following

$$\cos(A - B) = \cos A \cos(-B) - \sin A \sin(-B)$$

$\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

An identity for the tangent of a sum can be derived using identities already established.

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)}$$

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

Divide both the Numerator and Denominator by $\cos A \cos B$

$$= \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}}$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Similarly, an identity for a tangent of a difference can be established.

It is given by

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Collecting these results we have:

$$\mathbf{\sin(A + B) \equiv \sin A \cos B + \cos A \sin B}$$
$$\mathbf{\sin(A - B) \equiv \sin A \cos B - \cos A \sin B}$$

$$\mathbf{\cos(A + B) \equiv \cos A \cos B - \sin A \sin B}$$
$$\mathbf{\cos(A - B) \equiv \cos A \cos B + \sin A \sin B}$$

$$\mathbf{\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}}$$
$$\mathbf{\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}}$$

Solving trigonometric equations

Example 12

Solve the equation

$$4\sin(x - 20°) = 5\cos x$$

for values of x between 0° and 90°

Solution

From the formula of $\sin(A - B)$

$$4\sin(x - 20°) = 4[\sin x\cos 20° - \cos x\sin 20°]$$
$$= 4[0.9397\sin x - 0.3420\cos x]$$
$$= 3.7588\sin x - 1.3680\cos x$$

Since $4\sin(x - 20°) = 5\cos x$

$$3.7588\sin x - 1.3680\cos x = 5\cos x$$
$$3.7588\sin x = 5\cos x + 1.3680\cos x$$
$$\frac{\sin x}{\cos x} = \frac{6.3680}{3.7588} = 1.6942$$
$$\tan x = 1.6942$$
$$x = \tan^{-1} 1.6942 = 59.45°$$

Example 13

$\sin(\theta - 45°) = \sin\theta,\quad 0 \le \theta < 360°$

Solution

$$\sin(\theta - 45) = \sin\theta$$
$$\sin\theta\sin 45 - \cos\theta\sin 45 = \sin\theta$$
$$\sin\theta \times \frac{\sqrt{2}}{2} - \cos\theta \times \frac{\sqrt{2}}{2} = \sin\theta$$
$$\sqrt{2}\sin\theta - \sqrt{2}\cos\theta = 2\sin\theta$$
$$\frac{\sqrt{2}\sin\theta}{\cos\theta} - \frac{\sqrt{2}\cos\theta}{\cos\theta} = \frac{2\sin\theta}{\cos\theta}$$
$$\sqrt{2}\tan\theta - \sqrt{2} = 2\tan\theta$$
$$\left(\sqrt{2} - 2\right)\tan\theta = \sqrt{2}$$
$$\tan\theta = \frac{\sqrt{2}}{\sqrt{2} - 2} = -2.4142$$
$$\tan^{-1}(-2.4142) = 67.5°$$
$$\therefore \theta = 112.5°, 292.5°$$

Example 14

$\cos(x - 30°) = \sin(x + 30°),\quad 0 \le \theta < 360°$

Solution

$$\cos(x - 30°) = \sin(x + 30°)$$
$$\cos x\cos 30 + \sin x\sin 30 = \sin x\cos 30 + \cos x\sin 30$$
$$\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x$$
$$\sqrt{3}\cos x + \sin x = \sqrt{3}\sin x + \cos x$$
$$\frac{\sqrt{3}\cos x}{\cos x} + \frac{\sin x}{\cos x} = \frac{\sqrt{3}\sin x}{\cos x} + \frac{\cos x}{\cos x}$$
$$\sqrt{3} + \tan x = \sqrt{3}\tan x + 1$$
$$\sqrt{3} - 1 = \left(\sqrt{3} - 1\right)\tan x$$
$$\tan x = 1$$
$$\tan^{-1} 1 = 45°$$
$$\therefore x = 45°, 225°$$

Example 15

$\cos(y - 30°) = \sin(y + 45°),\quad 0 < y < 360°$

Solution

$$\cos(y - 30) = \sin(y + 45)$$
$$\cos y\cos 30 + \sin y\sin 30 = \sin y\cos 45 + \cos y\sin 45$$
$$\frac{\sqrt{3}}{2}\cos y + \frac{1}{2}\sin y = \frac{\sqrt{2}}{2}\sin y + \frac{\sqrt{2}}{2}\cos y$$
$$\sqrt{3}\cos y + \sin y = \sqrt{2}\sin y + \sqrt{2}\cos y$$
$$\frac{\sqrt{3}\cos y}{\cos y} + \frac{\sin y}{\cos y} = \frac{\sqrt{2}\sin y}{\cos y} + \frac{\sqrt{2}\cos y}{\cos y}$$
$$\sqrt{3} + \tan y = \sqrt{2}\tan y + \sqrt{2}\tan y + \sqrt{2}$$
$$\sqrt{3} - \sqrt{2} = \left(\sqrt{2} - 1\right)\tan y$$
$$\tan y = \frac{\sqrt{3} - \sqrt{2}}{\sqrt{2} - 1} = 0.7673$$
$$\tan^{-1} 0.7673 = 37.5°$$
$$\therefore y = 37.5°, 217.5°$$

Example 16

$\sin(\varphi - 30°) = \cos(\varphi - 45°),\quad 0 \le \varphi < 360°$

Solution

$$\sin(\varphi - 30°) = \cos(\varphi - 45°)$$
$$\sin\varphi\cos 30 - \cos\varphi\sin 30 = \cos\varphi\cos 45 + \sin\varphi\sin 45$$
$$\frac{\sqrt{3}}{2}\sin\varphi - \frac{1}{2}\cos\varphi = \frac{\sqrt{2}}{2}\cos\varphi + \frac{\sqrt{2}}{2}\sin\varphi$$
$$\sqrt{3}\sin\varphi - \cos\varphi = \sqrt{2}\cos\varphi + \sqrt{2}\sin\varphi$$
$$\frac{\sqrt{3}\sin\varphi}{\cos\varphi} - \frac{\cos\varphi}{\cos\varphi} = \frac{\sqrt{2}\cos\varphi}{\cos\varphi} + \frac{\sqrt{2}\sin\varphi}{\cos\varphi}$$
$$\sqrt{3}\tan\varphi - 1 = \sqrt{2} + \sqrt{2}\tan\varphi$$
$$\sqrt{3}\tan\varphi - \sqrt{2}\tan\varphi = \sqrt{2} + 1$$
$$\tan\varphi = \frac{\sqrt{2} + 1}{\sqrt{3} - \sqrt{2}} = 7.5958$$
$$\tan^{-1} 7.5958 = 82.5°$$
$$\therefore \varphi = 82.5°, 262.5°$$

Example 17

$\cos(\alpha - 60°) = \cos(\alpha + 60°),\quad 0 \le \alpha < 360°$

Solution

$$\cos(\alpha - 60) = \cos(\alpha + 60)$$
$$\cos\alpha\cos 60 + \sin\alpha\sin 60 = \cos\alpha\cos 60 + \sin\alpha\sin 60$$
$$2\sin\alpha\sin 60 = 0$$
$$\sqrt{3}\sin\alpha = 0$$
$$\sin\alpha = 0$$
$$\sin^{1} 0 = 0°$$
$$\therefore \alpha = 0°, 180°$$

Self-Evaluation exercise

Prove the following identities

1. $\cot(A + B) \equiv \frac{\cot A\cot B - 1}{\cot A + \cot B}$
2. $(\sin A + \cos A)(\sin B + \cos B) \equiv \sin(A + B) + \cos(A - B)$

Using this identity for $\tan 2A$ again, but this time with $A = 2\theta$, gives

$$\tan 4\theta = \frac{2\tan 2\theta}{1-\tan^2 2\theta} = \frac{2\left(\frac{24}{7}\right)}{1-\left(\frac{24}{7}\right)^2} = -\frac{336}{527}$$

Example 2

Eliminate θ from the equations $x = \cos 2\theta$, $y = \sec\theta$

Solution

Using $\cos 2\theta = 2\cos^2\theta - 1$ gives

$x = 2\cos^2\theta - 1$ and $y = \frac{1}{\cos\theta}$

$$x = 2\left(\frac{1}{y}\right)^2 - 1$$

$$(x+1)y^2 = 2$$

Example 3

Prove that: $\dfrac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta$

Solution

$$\text{LHS} = \frac{1-\cos 2\theta}{\sin 2\theta} = \frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta}$$

$$= \frac{2\sin^2\theta}{2\sin\theta\cos\theta}$$

$$= \frac{\sin\theta}{\cos\theta} = \tan\theta = \text{RHS}$$

Example 4

Prove the following identities

(a) $\sec\theta\operatorname{cosec}\theta \equiv 2\operatorname{cosec} 2\theta$

Solution

$$\text{LHS} = \sec\theta\operatorname{cosec}\theta = \frac{1}{\cos\theta}\times\frac{1}{\sin\theta}$$

$$= \frac{1}{\sin\theta\cos\theta}$$

$$= \frac{2}{2\sin\theta\cos\theta}$$

$$= \frac{2}{\sin 2\theta}$$

$$= 2\operatorname{cosec} 2\theta = \text{RHS}$$

(b) $\tan\theta + \cot\theta \equiv 2\operatorname{cosec} 2\theta$

Solution

$$\text{LHS} = \tan\theta + \cot\theta = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}$$

$$= \frac{1}{\sin\theta\cos\theta}$$

$$= \frac{2}{2\sin\theta\cos\theta}$$

$$= \frac{2}{\sin 2\theta}$$

$$= 2\operatorname{cosec} 2\theta = \text{RHS}$$

(c) $\dfrac{1-\cos 2x}{\sin 2x} \equiv \tan x$

Solution

$$\text{LHS} = \frac{1-\cos 2x}{\sin 2x} = \frac{1-(1-2\sin^2 x)}{2\sin x\cos x}$$

$$= \frac{2\sin^2 x}{2\sin x\cos x}$$

$$= \frac{\sin x}{\cos x} = \tan x = \text{RHS}$$

(d) $\dfrac{\cos 2\theta}{\cos\theta - \sin\theta} \equiv \cos\theta + \sin\theta$

Solution

$$\text{LHS} = \frac{\cos 2\theta}{\cos\theta - \sin\theta} = \frac{\cos^2\theta - \sin^2\theta}{\cos\theta - \sin\theta}$$

$$= \frac{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}{\cos\theta - \sin\theta}$$

$$= \cos\theta + \sin\theta = \text{RHS}$$

(e) $\dfrac{\cos 2x}{\sin x} + \dfrac{\sin 2x}{x} \equiv \operatorname{cosec} x$

Solution

$$\text{LHS} = \frac{\cos 2x}{\sin x} + \frac{\sin 2x}{x} = \frac{1-2\sin^2 x}{\sin x} + \frac{2\sin x\cos x}{\cos x}$$

$$= \frac{1}{\sin x} - \frac{2\sin^2 x}{\sin x} + 2\sin x$$

$$= \operatorname{cosec} x - 2\sin x + 2\sin x$$

$$= \operatorname{cosec} x = \text{RHS}$$

Alternatively;

$$\text{LHS} = \frac{\cos 2x}{\sin x} + \frac{\sin 2x}{x} = \frac{\cos 2x\cos x + \sin x\sin 2x}{\sin x\cos x}$$

$$= \frac{\cos(2x - x)}{\sin x\cos x}$$

$$= \frac{\cos x}{\sin x\cos x}$$

$$= \operatorname{cosec} x = \text{RHS}$$

Example 5

Prove that $\sin 3A = 3\sin A - 4\sin^3 A$

Solution

$$\sin 3A = \sin(2A + A)$$

$$= \sin 2A\cos A + \cos 2A\sin A$$

$$= (2\sin A\cos A)\cos A + (1-2\sin^2 A)\sin A$$

$$= 2\sin A\cos^2 A + \sin A - 2\sin^3 A$$

$$= 2\sin A\,(1-\sin^2 A) + \sin A - 2\sin^3 A$$

$$= 3\sin A - 4\sin^3 A$$

Example 6

Solve the equation $\cos 2\theta + 3\sin\theta = 2$ for θ in the range $0° \le \theta \le 360°$.

Solution

Replacing the double angle term with the relationship $\cos 2\theta = 1 - 2\sin^2\theta$ gives:

$$1 - 2\sin^2\theta + 3\sin\theta = 2$$

Rearranging gives:

$$-2\sin^2\theta + 3\sin\theta - 1 = 0$$

Half angles

These can be derived from the double angle formula.

From the double angle formula

$$\cos 2A = 2\cos^2 A - 1$$

Dividing the angles by 2

$$\cos A = 2\cos^2\frac{A}{2} - 1$$

$$\cos^2\frac{A}{2} = \frac{1}{2}(1 + \cos A)$$

Similarly,

$$\cos A = 1 - 2\sin^2\frac{A}{2}$$

$$2\sin^2\frac{A}{2} = 1 - \cos A$$

$$\sin^2\frac{A}{2} = \frac{1}{2}(1 - \cos A)$$

The t-formulae

If $\tan\frac{x}{2}$, we can express $\sin x$, $\cos x$, $\tan x$ in terms of t which helps to solve trigonometric equations.

From $\cos 2x = \cos^2 x - \sin^2 x$

$$\cos x = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{1}$$

But

$$1 = \cos^2\frac{x}{2} + \sin^2\frac{x}{2}$$

$$\cos x = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}}$$

Dividing each term on RHS by $\cos^2\frac{x}{2}$

$$\cos x = \frac{1 - \tan^2\frac{x}{2}}{1 - \tan^2\frac{x}{2}} \quad \text{but } t = \tan\frac{x}{2}$$

$$\cos x = \frac{1 - t^2}{1 + t^2}$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\sin\frac{x}{2}\cos\frac{x}{2} \times \frac{\cos\frac{x}{2}}{\cos\frac{x}{2}}$$

$$\sin x = 2\tan\frac{x}{2}\cos^2\frac{x}{2}$$

$$\sin x = \frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}}$$

$$\sin x = \frac{2t}{1 + t^2}$$

$$\tan x = \frac{\tan\frac{x}{2} + \tan\frac{x}{2}}{1 - \tan^2\frac{x}{2}}$$

$$\tan x = \frac{2t}{1 - t^2}$$

The three formulae

$$\sin x = \frac{2t}{1 + t^2}, \quad \cos x = \frac{1 - t^2}{1 + t^2}, \quad \tan x = \frac{2t}{1 - t^2}$$

where $t = \tan\frac{1}{2}x$ are useful in the solution of a certain type of trigonometric equation. They also have other important applications.

Example 7

Use the t-formula to solve the following equations giving values of θ from 0° to 360° inclusive.

(a) $2\cos\theta + 3\sin\theta - 2 = 0$

(b) $7\cos\theta + \sin\theta - 5 = 0$

Solution

(a) Let $t = \tan\frac{\theta}{2}$

$$\cos\theta = \frac{1 - t^2}{1 + t^2} \qquad \sin\theta = \frac{2t}{1 + t^2}$$

$$2\cos\theta + 3\sin\theta - 2 = 0$$

$$2\left(\frac{1 - t^2}{1 + t^2}\right) + 3\left(\frac{2t}{1 + t^2}\right) - 2 = 0$$

$$2 - 2t^2 + 6t - 2(1 + t^2) = 0$$

$$2 - 2t^2 + 6t - 2 - 2t^2 = 0$$

$$-4t^2 + 6t = 0$$

$$-2t(2t - 3) = 0$$

Either $-2t = 0$ or $2t - 3 = 0$

$t = 0$ $\qquad$ $t = \frac{3}{2}$

$$\tan\frac{\theta}{2} = 0$$

$$\frac{\theta}{2} = \tan^{-1} 0$$

$$\frac{\theta}{2} = 0°, 180°$$

$$\theta = 0°, 360°$$

$$\tan\frac{\theta}{2} = \frac{3}{2}$$

$$\frac{\theta}{2} = \tan^{-1}\frac{3}{2}$$

$$\frac{\theta}{2} = 56.3°$$

$$\frac{\theta}{2} = 56.3°, 236.3°$$

$$\theta = 112.6°, 472.6°$$

$$\therefore \theta = 0°, 112.6°, 360°$$

(b)

$$7\cos\theta + \sin\theta - 5 = 0$$

$$7\left(\frac{1 - t^2}{1 + t^2}\right) + \frac{2t}{1 + t^2} - 5 = 0$$

$$7(1 - t^2) + 2t - 5(1 + t^2) = 0$$

$$7 - 7t^2 + 2t - 5 - 5t^2 = 0$$

$$-12t^2 + 2t + 2 = 0$$

$$6t^2 - t - 1 = 0$$

$$6t^2 - 3t + 2t - 1 = 0$$

$$3t(2t - 1) + (2t - 1) = 0$$

$$(2t - 1)(3t + 1) = 0$$

Either $2t - 1 = 0$ or $3t + 1 = 0$

$$= 1 - 2\sin A\,[\sin A + \sin(B - C)]$$

But $A = 180° - (B + C)$

$$\sin A = \sin[180° - (B + C)] = \sin(B + C)$$

Thus

L.H.S $= 1 - 2\sin A\,[\sin(B + C) + \sin(B - C)]$

$$= 1 - 2\sin A\,[2\sin B\cos C]$$

$$= 1 - 4\sin A\sin B\cos C = \text{R.H.S}$$

Example 3

If $A + B + C = \pi$, prove that $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2\sin A\sin B\cos C$

Solution

L.H.S $= \cos^2 A + \cos^2 B - \cos^2 C = (1 - \sin^2 A) + \cos^2 B - \cos^2 C$

$$= 1 + (\cos^2 B - \sin^2 A) - \cos^2 C$$

$$= 1 + \cos(A + B)\cos(A - B) - \cos^2 C$$

$$= 1 + \cos(\pi - C)\cos(A - B) - \cos^2 C$$

$$= 1 - \cos C\cos(A - B) - \cos^2 C$$

$$= 1 - \cos C\,[\cos(A - B) + \cos C]$$

$$= 1 - \cos C\,[\cos(A - B) - \cos(A + B)]$$

$$= 1 - \cos C\,[2\sin A\sin B]$$

$$= 1 - 2\sin A\sin B\cos C = \text{R.H.S}$$

Example 4

If A, B, C are the angles of a triangle, prove that

$$\cos A + \cos B + \cos C - 1 = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

Solution

Split the left-hand side into two pairs.

$$\cos A + \cos B = 2\cos\frac{A + B}{2}\cos\frac{A - B}{2}$$

$A + B + C = 180° \Rightarrow A + B = 180° - C$ hence

$$\frac{A + B}{2} = 90° - \frac{C}{2}$$

$$\therefore \cos\frac{A + B}{2} = \sin\frac{C}{2}$$

Seeing this factor $\sin\frac{C}{2}$ on the right-hand side, write

$$\cos C - 1 = -2\sin^2\frac{C}{2}$$

Therefore

$$\cos A + \cos B + \cos C - 1 = 2\sin\frac{C}{2}\cos\frac{A - B}{2} - 2\sin^2\frac{C}{2}$$

$$= 2\sin\frac{C}{2}\left(\cos\frac{A - B}{2} - \sin\frac{C}{2}\right)$$

$$= 2\sin\frac{C}{2}\left(\cos\frac{A - B}{2} - \cos\frac{A + B}{2}\right)$$

$$= -2\left(\cos\frac{A + B}{2} - \cos\frac{A - B}{2}\right)\sin\frac{C}{2}$$

$$= -2\left(-2\sin\frac{A}{2}\sin\frac{B}{2}\right)\sin\frac{C}{2}$$

$$\cos A + \cos B + \cos C - 1 = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

Example 5

Prove that $\tan(A + B + C)$

$$= \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan B\tan C - \tan C\tan A - \tan A\tan B}$$

Hence prove that if A, B, C are angles of a triangle, then

$$\tan A + \tan B + \tan C = \tan A\tan B\tan C$$

Solution

$$\tan[A + (B + C)] = \frac{\tan A + \tan(B + C)}{1 - \tan A\tan(B + C)}$$

But $\tan(B + C) = \frac{\tan B + \tan C}{1 - \tan B\tan C}$

$$\tan(A + B + C) = \frac{\tan A + \dfrac{\tan B + \tan C}{1 - \tan B\tan C}}{1 - \tan A \times \dfrac{\tan B + \tan C}{1 - \tan B\tan C}}$$

$$= \frac{\dfrac{\tan A - \tan A\tan B\tan C + \tan B + \tan C}{1 - \tan A\tan B}}{\dfrac{1 - \tan B\tan C - \tan A\tan B - \tan A\tan C}{1 - \tan B\tan C}}$$

$$= \frac{\tan A - \tan A\tan B\tan C + \tan B + \tan C}{1 - \tan B\tan C - \tan A\tan B - \tan A\tan C}$$

$$= \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan B\tan C - \tan C\tan A - \tan A\tan B}$$

Hence $\tan(A + B + C)$

$$= \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan B\tan C - \tan C\tan A - \tan A\tan B}$$

If A, B & C are angles of a triangle

$$A + B + C = 180°$$

$$\tan(A + B + C) = \tan 180°$$

$$\tan 180° = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan B\tan C - \tan C\tan A - \tan A\tan B}$$

$$0 = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan B\tan C - \tan C\tan A - \tan A\tan B}$$

$$0 = \tan A + \tan B + \tan C - \tan A\tan B\tan C$$

$$\tan A + \tan B + \tan C = \tan A\tan B\tan C$$

The form $a\cos\theta \pm b\sin\theta$

Using the compound angle formulas, the equation of the form $a\cos\theta \pm b\sin\theta + c = 0$ can be solved. Also the maximum and minimum values of functions involving functions $a\cos\theta \pm b\sin\theta + c$ or $\frac{1}{a\cos\theta \pm b\sin\theta + c}$ can be obtained.

Our problem is to express $a\sin\theta \pm b\cos\theta$ in the form $R\sin(\theta \pm \alpha)$ where a, b, R and α are positive constants.

Let

$$a\sin\theta + b\cos\theta \equiv R\sin(\theta + \alpha)$$

Using the compound angle formula, we can expand $R\sin(\theta + \alpha)$ as follows

$$R\sin(\theta + \alpha) = R(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$$
$$= R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$$

So

$$a\sin\theta + b\cos\theta = R\cos\alpha\sin\theta + R\sin\alpha\cos\theta$$

Equating the coefficients of $\sin\theta$ and $\cos\theta$ in this identity, we have

$$a = R\cos\alpha \;\; \ldots\text{(i)}$$
$$b = R\sin\alpha \ldots\text{(ii)}$$

Eqn (ii) ÷ (i);

$$\frac{b}{a} = \frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha$$
$$\Rightarrow \alpha = \tan^{-1}\frac{b}{a}$$

(α is a positive acute angle and a and b are positive)

Now we square each of equations (i) and (ii) and add them to find an expression for R

$$a^2 + b^2 = R^2\cos^2\alpha + R^2\sin^2\alpha$$
$$a^2 + b^2 = R^2(\cos^2\alpha + \sin^2\alpha)$$
$$a^2 + b^2 = R^2$$

So

$$R = \sqrt{a^2 + b^2}$$

(we only take the positive root)

then we have expressed $a\sin\theta + b\cos\theta$ in the form required.

$$a\sin\theta + b\cos\theta = R\sin(\theta + \alpha)$$

The minus case

Similarly, for the minus case, we equate $a\sin\theta - b\cos\theta$ with the expansion $R\sin(\theta - \alpha)$ as follows.

$$a\sin\theta - b\cos\theta = R\cos\alpha\sin\theta - R\sin\alpha\cos\theta$$

Once again, we will obtain (try it yourself)

$$\alpha = \tan^{-1}\frac{b}{a}$$

and

$$R = \sqrt{a^2 + b^2}$$

Our equation for the minus case is:

$$a\sin\theta - b\cos\theta = R\sin(\theta - \alpha)$$

Equations of the type $a\sin\theta \pm b\cos\theta = c$

To solve an equation in the form $a\sin\theta \pm b\cos\theta = c$, express the LHS in the form $R\sin(\theta \pm \alpha)$ and then solve

$$R\sin(\theta \pm \alpha) = c$$

Example 1

(a) Express $4\sin\theta + 3\cos\theta$ in the form $R\sin(\theta + \alpha)$

(b) Hence, solve the equation $4\sin\theta + 3\cos\theta = 2$

Solution

(a) Let $4\sin\theta + 3\cos\theta \equiv R\sin(\theta + \alpha)$

$$4\sin\theta + 3\cos\theta \equiv R\sin\theta\cos\alpha + R\sin\alpha\cos\theta$$

Thus

$$R\cos\alpha = 4$$
$$R\sin\alpha = 3$$
$$R = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$
$$\tan\alpha = \frac{3}{4}$$
$$\alpha = \tan^{-1}\left(\frac{3}{4}\right) = 36.87°$$
$$\therefore 4\sin\theta + 3\cos\theta \equiv 5\sin(\theta + 36.87°)$$

(b) Now,

$$5\sin(\theta + 36.87°) = 2$$
$$\sin(\theta + 36.87°) = 0.4$$
$$\sin^{-1}0.4 = 23.58°$$
$$\theta + 36.87° = 23.58°, 156.42°$$
$$\theta = 23.58° - 36.87° = -13.29° = 346.71°$$
$$\theta = 156.42° - 36.87° = 119.55°$$
$$\therefore \theta = 119.55°, 346.71°$$

Example 2

Solve the equation

$$\sin\theta - \sqrt{2}\cos\theta = 0.8 \text{ for } 0 \le \theta < 360°$$

Solution

Let $\sin\theta - \sqrt{2}\cos\theta \equiv R\sin(\theta - \alpha)$

$$\sin\theta - \sqrt{2}\cos\theta \equiv R\sin\theta\cos\alpha - R\sin\alpha\cos\theta$$
$$R\cos\alpha = 1$$
$$R\sin\alpha = \sqrt{2}$$
$$R = \sqrt{2+1} = \sqrt{3}$$
$$\tan\alpha = \sqrt{2} \Rightarrow \alpha = \tan^{-1}\sqrt{2} = 54.74°$$
$$\sin\theta - \sqrt{2}\cos\theta \equiv \sqrt{3}\sin(\theta - 54.74°)$$

Now,

$$\sqrt{3}\sin(\theta - 54.74°) = 0.8$$
$$\sin(\theta - 54.74°) = 0.4619$$
$$\sin^{-1}0.4619 = 27.51°$$
$$\theta - 54.74° = 27.51°, 152.49°$$
$$\theta = (27.51° + 54.74°), (152.49° + 54.74°)$$
$$\therefore \theta = 82.25°, 207.23°$$

Example 3

Solve $7\sin 3\theta - 6\cos 3\theta = 3.8$ for $0 \le \theta < 360°$

Solution

Firstly, express the LHS in the form $R\sin(3\theta - \alpha)$.

(r) $(\cos x + \sin x)(\operatorname{cosec} x - \sec x) \equiv 2\cot 2x$

(s) $\frac{2\sec^2\theta - \cos 2\theta - 1}{2\tan\theta + \sin 2\theta} \equiv \tan\theta$

(t) $4\operatorname{cosec}^2 2\theta - \sec^2\theta \equiv \operatorname{cosec}^2\theta$

(u) $2\cos^4\theta + \frac{1}{2}\sin^2 2\theta - 1 \equiv \cos 2\theta$

(v) $\frac{\cos 2x}{\sqrt{1+\sin 2x}} \equiv \cos x - \sin x$

(w) $\frac{\sqrt{2-2\cos x}}{\sin x} \equiv \sec\frac{x}{2}$

(x) $8\cos^4\left(\frac{1}{2}\theta\right) \equiv \cos 2\theta + 4\cos\theta + 3$

(y) $\sin^4 + \cos^4\theta \equiv \frac{1}{4}(3 + \cos 4\theta)$

(z) $\sqrt{1+\sin 2\theta} \equiv \sin\theta + \cos\theta$

2. Solve each of the following trigonometric equations.

(a) $2\sin 2\theta = \cot\theta, \quad 0 \le \theta \le \pi$

[Ans: $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$]

(b) $3\sin 2x = 2\cos x, \ 0 \le x \le 180°$

[Ans: $19.5°, 90°, 160.5°$]

(c) $\sin 4y = \sin 2y, \quad 0 \le y \le 180°$

[Ans: $0°, 30°, 90°, 150°$]

(d) $\sin\varphi + \frac{1}{4}\sec\varphi = 0, \quad 0 \le \varphi < \pi$

[Ans: $\varphi = \frac{7\pi}{12}, \frac{11\pi}{12}$]

(e) $\cos\theta - \sin 2\theta = 0, \ 0 \le \theta \le 360°$

[Ans: $30°, 90°, 150°, 270°$]

(f) $\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = 4, \quad 0 \le x < 360°$

[Ans: $15°, 75°, 195°, 255°$]

(g) $2\cos y = 2\tan y \sin y + \sec y, 0 \le y \le 2\pi$

[Ans: $y = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$]

(h) $2\cos\varphi + \operatorname{cosec}\varphi = 0, \ 0 \le \varphi < 2\pi$

[Ans: $\varphi = \frac{3\pi}{4}, \frac{7\pi}{4}$]

(i) $2\cos 2\theta = 1 + \cos\theta, 0 \le \theta < 360°$

[Ans: $0°, 138.6°, 221.4°$]

(j) $\cos 2x + 3\sin x = 2, \quad 0 \le x < 360°$

[Ans: $30°, 90°, 150°$]

(k) $2(1 - \cos 2\varphi) = \tan\varphi, 0 \le \varphi < 180°$

[Ans: $0°, 15°, 75°$]

(l) $3\cos 2\theta - 5\sin\theta = 4, \ 0 \le \theta < 360°$

[Ans: $199.5°, 210°, 330°, 340.5°$]

(m) $3\cos 2x = 1 - \sin x, \ 0 \le x < 360°$

[Ans: $41.8°, 138.2°, 210°, 330°$]

(n) $2\cos 2\varphi = 1 - 2\sin\varphi, \ 0 \le \varphi < 360°$

[Ans: $54°, 126°, 198°, 342°$]

3. Show that $\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta$

4. Express $\tan(A+B+C)$ in terms of $\tan A$, $\tan B$ and $\tan C$.

5. Prove that

$$\frac{\sin\alpha\sin\beta}{\cos\alpha + \cos\beta} = \frac{2\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}{1 - \tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}}$$

6. Prove the identity

$$(\sin 2\alpha - \sin 2\beta)\tan(\alpha+\beta) = 2(\sin^2\alpha - \sin^2\beta)$$

7. If $\tan\frac{1}{2}x = \operatorname{cosec} x - \sin x$, prove that $\tan^2\frac{1}{2}x = -2 \pm \sqrt{5}$.

8. If $\sin\theta + \sin 2\theta = a$ and $\cos\theta + \cos 2\theta = b$, prove that

$$(a^2+b^2)(a^2+b^2-3) = 2b$$

9. Show that $\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta$

10. Prove that $\cot(A+B) = \frac{\cot A\cot B - 1}{\cot A + \cot B}$

11. Show that

$$\sin(A+B+C) = \cos A\cos B\cos C\,(\tan A + \tan B + \tan C - \tan A\tan B\tan C)$$

and deduce that, if A, B, C are the angles of a triangle, then $\cot A\cot B + \cot B\cot C + \cot C\cot A = 1$

12. Show that $a\sin x + b\cos x = \sqrt{(a^2+b^2)}\sin(x+\alpha)$ where $\tan\alpha = \frac{b}{a}$

13. If $k\cos\theta = \cos(\theta - \alpha)$, show that $\tan\theta = k\operatorname{cosec}\alpha - \cot\alpha$.

14. Prove that

$$\frac{\sin(A-B)}{\cos A\cos B} + \frac{\sin(B-C)}{\cos B\cos C} + \frac{\sin(C-A)}{\cos C\cos A} = 0$$

15. If $\tan^2\alpha - 2\tan^2\beta = 1$, prove that $\cos 2\alpha + \sin^2\beta = 0$.

16. If $\sin 3\theta = p$ and $\sin^2\theta = \frac{3}{4} - q$, prove that $p^2 + 16q^3 = 12q^2$.

17. Prove that $2\cot\frac{1}{2}A + \tan A = \tan A\cot^2\frac{1}{2}A$

18. If $2\cos\theta = x + \frac{1}{x}$, show that $2\cos 3\theta = x^3 + \frac{1}{x^3}$.

19. If $\sec A - \tan A = x$, prove that $\tan\frac{1}{2}A = \frac{1-x}{1+x}$

20. Prove that

$$4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right) = \frac{\pi}{4}$$

21. Prove that

$$\cot^{-1}\left(\frac{1}{3}\right) = \cot^{-1}3 + \cos^{-1}\left(\frac{3}{5}\right)$$

22. Find x from the equation $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$

[Ans: 1/6]

23. Prove the following identities

(a) $\frac{\cos B + \cos C}{\sin B - \sin C} = \cot\frac{B-C}{2}$

(b) $\frac{\cos B - \cos C}{\sin B + \sin C} = -\tan\frac{B-C}{2}$

(c) $\frac{\sin B + \sin C}{\cos B + \cos C} = \tan\frac{B+C}{2}$

(d) $\frac{\sin B - \sin C}{\sin B + \sin C} = \cot\frac{B+C}{2}\tan\frac{B-C}{2}$

24. Prove the following identities. A, B, C are to be taken as the angles of a triangle.

(a) $\sin A + \sin(B-C) = 2\sin B\cos C$

(b) $\cos A - \cos(B-C) = -2\cos B\cos C$

(c) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(d) $\sin 2A + \sin 2 + \sin 2C = 4\sin A\sin B\sin C$

Chapter 22 Solutions of triangles

Introduction

To 'solve a triangle' means 'to find the values of unknown sides and angles'. If a triangle is right-angled, trigonometric ratios and the theorem of Pythagoras may be used for its solution. However, for a non-right-angled triangle, trigonometric ratios and Pythagoras' theorem cannot be used. Instead, two rules, called the sine rule and cosine rule are used.

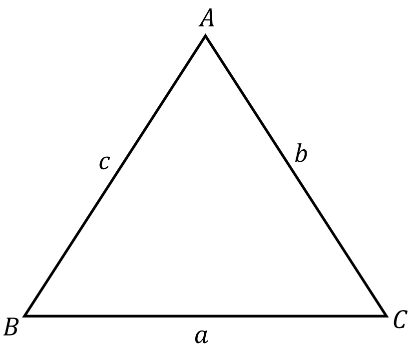
Note:

A triangle possesses six elements i.e. the three sides and the three angles. If any three elements (other than three angles.) are given, the remaining three elements can be found. This is called solving the triangle.

In solving the triangle, two geometrical facts are useful i.e.

1. In any triangle the sum of the angles is 180°
2. In any triangle, the largest side is opposite the greatest angle and the shortest side is opposite the smallest angle

The sine rule



With reference to triangle ABC above, the sine rule states:

$$\frac{\boldsymbol{a}}{\sin \boldsymbol{A}} = \frac{\boldsymbol{b}}{\sin \boldsymbol{B}} = \frac{\boldsymbol{c}}{\sin \boldsymbol{C}} = \boldsymbol{2R}$$

where R is the radius of the circumcircle of the triangle.

The rule may be used only when:

(i) 1 side and any 2 angles are initially given, or

(ii) 2 sides and an angle (not the included angle) are initially given

The cosine rule

With reference to triangle ABC, the cosine rule states:

$$\boldsymbol{a^2 = b^2 + c^2 - 2bc\cos A}$$

$$\textbf{or } \boldsymbol{b^2 = a^2 + c^2 - 2ac\cos B}$$

$$\textbf{or } \boldsymbol{c^2 = b^2 + a^2 - 2ab\cos C}$$

The rule may be used only when:

(i) 2 sides and the included angle are initially given, or

(ii) 3 sides are initially given

Area of any triangle

The area of any triangle such as ABC is given by:

(i) $A = \frac{1}{2} \times$ base $\times$ perpendicular height, or

(ii) $A = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A$, or

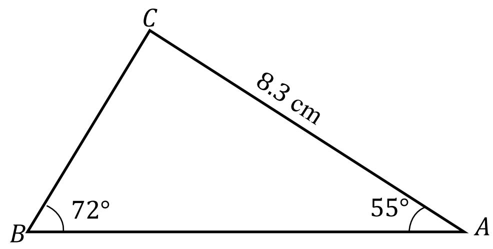
(iii) $A = \sqrt{s(s-a)(s-b)(s-c)}$

where $s = \frac{a+b+c}{2}$

The latter formula is called **Hero's formula**

Example 1

Find the length of the side BC in the given triangle



Solution

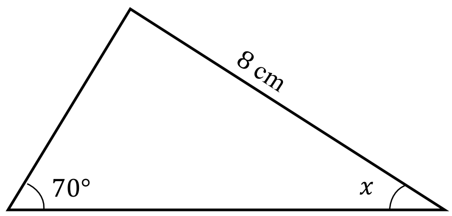
Using the sine rule;

$$\frac{BC}{\sin 55} = \frac{8.3}{\sin 72}$$

$$BC = \frac{8.3\sin 55}{\sin 72} = 7.15\ cm$$

Example 2

Find the angle x in the given triangle



Solution

Using the sine rule;

$$\frac{8}{\sin 70°} = \frac{6}{\sin x}$$

$$\sin x = \frac{6\sin 70°}{8} = 0.7048$$

$$x = \sin^{-1}(0.7048) = 44.81°$$

Chapter 23 Examination Questions

SECTION A

1. Show that
$$\frac{1-\cos 2x+2\sin x\cos^2 x}{1+\cos 2x}=\sin x+\tan^2 x$$
[2024, No. 6]
2. Express $2\sin\theta+3\cos\theta$ in the form $R\sin(\theta+\alpha)$
[2023, No. 7]
3. Solve $5\tan^2 A-5\tan A=2\sec^2 A$ for $0°\le A\le 360°$
[2022, No. 2]
4. Solve the equation: $\sin x+\sin 2x+\sin 3x=0$ for $0°\le x\le 180°$
[2020, No. 1]
5. Solve $2\cos 2\theta-5\cos\theta=4$ for $0°\le\theta\le 360°$
[2019, No. 2]
6. In triangle ABC, $a=7$ cm, $b=4$ cm and $c=5$ cm. Find the value of:
   (a) $\cos A$
   (b) $\sin A$
[2018, No. 1: Ans: (a) -0.2 (b) 0.9798]
7. Solve the equation $3\tan^2\theta+2\sec^2\theta=2(5-3\tan\theta)$ for $0°\le\theta\le 180°$.
[2017, No. 2: Ans: 38.66°, 116.57°]
8. Solve the equation $\sin 2\theta+\cos 2\theta\cos 4\theta=\cos 4\theta\cos 6\theta$ for $0\le\theta\le\frac{\pi}{4}$.
[2016, No. 6: Ans: 0, $\frac{3\pi}{16}$]
9. In a triangle ABC, all the angles are acute. Angle $ABC=50°$, $a=10$ cm and $b=9$ cm. Solve the triangle.
[2015, No. 5: Ans: 58.34°, 71.66°, $c=11.15$ cm]
10. Given that $\cos 2A-\cos 2B=-p$ and $\sin 2A-\sin 2B=q$, prove that $\sec\,(A+B)=\frac{1}{q}\sqrt{p^2+q^2}$
[2014, No. 3]
11. Solve $5\cos^2 3\theta=3(1+\sin 3\theta)$ for $0°\le\theta\le 90°$
[2013, No. 5: Ans: 7.859°, 52.141°]
12. (a) Prove that $\frac{2\tan\theta}{1+\tan^2\theta}=\sin 2\theta$
   (b) Solve $\sin 2\theta=\cos\theta$ for $0°\le\theta\le 90°$
[2012, No. 2: Ans: 30°, 90°]
13. Show that: $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)=\tan^{-1}\left(\frac{7}{9}\right)$
[2011, No. 5]
14. Express $\sin x+\cos x$ in the form $R\cos(x-\alpha)$. Hence, find the greatest value of $\sin x+\cos x-1$
[2010, No. 7: Ans: $\sqrt{2}\cos(x-45°)$; 0.4142]
15. Given that $\sin(\theta-45°)=3\cos(\theta+45°)$, show that $\tan\theta=1$. Hence find θ for $0°\le\theta\le 360°$
[2009, No. 2: Ans: 45°, 225°]
16. Without using tables or calculators, show that
$$\tan 15°=2-\sqrt{3}$$
[2008, No. 2]
17. Show that $\frac{\sin\theta-2\sin 2\theta+\sin 3\theta}{\sin\theta+2\sin 2\theta+\sin 3\theta}=-\tan^2\frac{\theta}{2}$
[2007, No. 4]
18. Prove that $\tan\left(\frac{\pi}{4}+\theta\right)-\tan\left(\frac{\pi}{4}-\theta\right)=2\tan 2\theta$
[2006, No. 1]
19. Solve the equation $2\sin^2\theta+3\cos\theta=0$, $0°\le\theta\le 360°$.
[2005, No. 6: Ans: 120°, 240°]
20. Solve $\cos\theta+\sin 2\theta=0$ for $0°\le\theta\le 360°$
[2004, No. 3: Ans: 90°, 210°, 270°, 330°]
21. Solve the equation $\cos 2\theta+\cos 3\theta+\cos\theta=0$; $0°\le\theta\le 180°$.
[2003, No. 3: Ans: 45°, 120°, 135°]
22. Solve the equation $2\cos\theta-\operatorname{cosec}\theta=0$; $0°<\theta<270°$
[2002, No. 1: Ans: 45°, 225°]
23. Given that $\sin 2\theta=\cos 3\theta$, find the value of $\sin\theta$, $0\le\theta\le\pi$.
[2001, No. 3: Ans: 0.309]
24. Solve $\cos\theta+\sqrt{3}\sin\theta=2$
[2000, No. 2: Ans: $2\pi n+\frac{\pi}{3}$]
25. Solve $\cos(\theta+35°)=\sin(\theta+25°)$ for $0°\le\theta\le 360°$.
[1999, No. 3: Ans: 15°, 195°]
26. Solve $\cos\theta+\sqrt{3}\sin\theta=2$ for $0°\le\theta\le 360°$
[Nov, 1998, No. 1: Ans: 60°]
27. Show that $\cos 4\theta=\frac{\tan^4\theta-6\tan^2\theta+1}{\tan^4\theta+2\tan^2\theta+1}$
[Mar 1998, No. 3]
28. Solve the equation $4\cos x-2\cos 2x=3$ for $0°\le x\le\pi$
[1997, No. 1: Ans: $\frac{\pi}{3}$]

Calculus

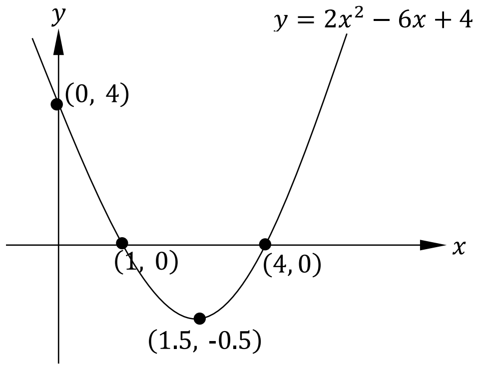
We now investigate for the nature of the turning point of the curve by using the values on the immediate left and right of the turning point.

	L	1.5	R
Sign of $\frac{dy}{dx}$	$-$	0	$+$

We now come to realize that $(1.5, -0.5)$ is a minimum turning point

Alternatively; using the second derivative method, $\frac{d^2y}{dx^2} = 4$ which is greater than 0

Implying that the curve has a maximum turning point.



Example 21

Sketch the curve $y = 4x - x^2$

<u>Solution</u>

Intercepts

when $y = 0, 4x - x^2 = 0$

$$x(4 - x) = 0$$

Either $x = 0$ or $x = 4$,

$\Rightarrow (0, 0)$ and $(4, 0)$ are the x −intercepts

When $x = 0, y = 0$

$(0, 0)$ is the y-intercept

Turning point

$$y = 4x - x^2$$

$$\frac{dy}{dx} = 4 - 2x$$

$$4 - 2x = 0 \Rightarrow x = 2$$

when $x = 2, y = 4(2) - 2^2 = 4$

$(2, 4)$ is a turning point

We now investigate for the nature of the turning point of the curve

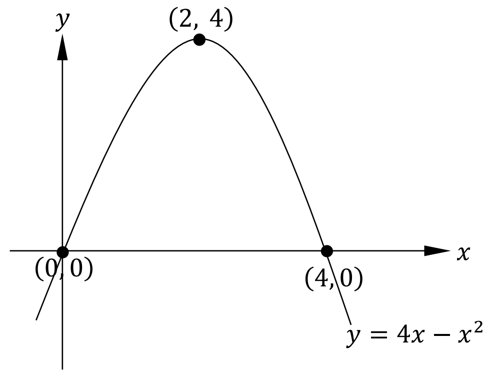
	L	2	R
Sign of $\frac{dy}{dx}$	$+$	0	$-$

We observe that $(2, 4)$ is a maximum turning point

Alternatively; if we would wish to investigate the nature of the turning point using the second derivative, we find out that $\frac{d^2y}{dx^2} = -2$ which is less than 0 $(\frac{d^2y}{dx^2} < 0)$

Hence, the curve has a maximum turning point

We can now sketch the curve



Example 22

Sketch the curve $y = x^3 - x^2 - 5x + 6$

<u>Solution</u>

Intercepts

when $x = 0, y = 0$

$(0, 6)$ is the y-intercept

When $y = 0$,

$$x^3 - x^2 - 5x + 6 = 0$$

Inspection approach is used to find the first factor i.e. $(x - 2)$, then the other factor is obtained by long division.

$$\begin{array}{r l}
 & x^2 + x - 3 \\
x - 2\,) & x^3 - x^2 - 5x + 6 \\
 & -\ \ x^3 - 2x^2 \\
 & \quad -x^2 - 5x + 6 \\
 & \quad - -x^2 - 2x \\
 & \qquad 3x + 6 \\
 & \qquad -\ \ 3x + 6 \\
 & \qquad\ \ -\ \ -
\end{array}$$

$$x^3 - x^2 - 5x + 6 = (x - 2)(x^2 + x^2 - 3) = 0$$

$$x^2 + x - 3 = 0$$

$$x = \frac{-1 \pm \sqrt{1 + 12}}{2} = \frac{-1 \pm 3.6}{2}$$

$$x = 1.3 \text{ or } -2.3$$

Hence the x-intercepts are $(2, 0)$, $(1.3, 0)$ and $(-2.3, 0)$

Turning points

$$y = x^3 - x^2 - 5x + 6$$

$$\frac{dy}{dx} = 3x^2 - 2x - 5$$

At turning point, $\frac{dy}{dx} = 0$

$$3x^2 - 2x - 5 = 0$$

$$(3x - 5)(x + 1) = 0$$

$$\frac{d^2A}{dx^2} = 4 + 216x^{-3}$$

$$\Rightarrow 4 + 216 \times 3^{-3} = +\text{ve} \ \therefore \text{a min}$$

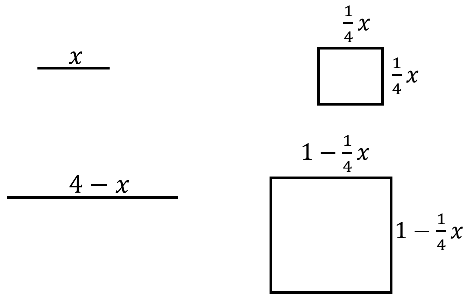
Minimum surface area is when the cuboid is a cube with all sides equal to x.

Example 3

A piece of wire, length $4\,m$, is cut into 2 pieces (not necessarily equal), and each piece is bent into square. How should this be done to have:

(a) the smallest total area from both squares?
(b) the largest total area from both squares?

Solution



Total area is:

$$A = \left(\frac{1}{4}x\right)^2 + \left(1 - \frac{1}{4}x\right)^2$$

$$= \frac{1}{16}x^2 + 1 - \frac{1}{2}x + \frac{1}{16}x^2$$

$$= \frac{1}{8}x^2 - \frac{1}{2}x + 1$$

$$\frac{dy}{dx} = \frac{1}{4}x - \frac{1}{2}$$

For max/min, $\frac{dy}{dx} = 0$

$$\frac{1}{4}x - \frac{1}{2}x = 0$$

$$x = 2$$

Now, $\frac{d^2y}{dx^2} = \frac{1}{4} > 0 \therefore$ a maximum

(a) smallest area is therefore when $x = 2$ (i.e. when wire is cut in half)

$$A = \frac{1}{8}(2^2) - \frac{1}{2}(2) + 1 = \frac{1}{2}m^2$$

(b) biggest area must be when $x = 0$,

$$\Rightarrow A = 1\,m^2$$

Example 4

A rectangle has perimeter 28 m. What is this maximum area?

Solution

Let x and y metres be the sides of the rectangle

Perimeter $= 2x + 2y = 14$

$$y = 14 - x$$

$$A = xy = x(14 - x) = 14x - x^2$$

$$\frac{dA}{dx} = 14 - 2x$$

When A is a maximum, $\frac{dA}{dx} = 0$

$$14 - 2x = 0$$

$$x = 7$$

$$\frac{d^2A}{dx^2} = -2 < 0 \Rightarrow \text{a max}$$

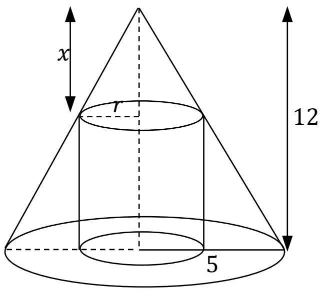
When $x = 7$, $y = 14 - 7 = 7$

So the maximum area is $7^2 = 49$ m

Example 5

A hollow cone of radius 5 cm and height 12 cm, is placed on a table. What is the largest cylinder that can be filled underneath it?

Solution



Volume of cone $= \frac{1}{3}\pi r^2 h$, volume of cylinder $= \pi r^2 h$

Consider the cone split into two cones:

Ratio of radius/height of large cone to small cone

$$5 : 12 = r : x$$

$$x = \frac{12}{5}r$$

$\therefore$ Height of cylinder $= 12 - \frac{12}{5}r$

Volume of cylinder $= \pi r^2\left(12 - \frac{12}{5}r\right)$

$$= 12\pi r^2 - \frac{12}{5}\pi r^3$$

$$\frac{dV}{dx} = 24\pi r - \frac{36}{5}\pi r^2$$

Min or max: $24\pi r - \frac{36}{5}\pi r^2 = 0$

$$120\pi r - 36\pi r^2 = 0$$

$$\pi r(10 - 3r) = 0$$

$$r = 0 \text{ or } r = \frac{10}{3}$$

$r = 0$ means no cylinder – reject solution

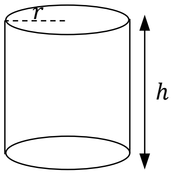
Maximum Volume $= \pi\left(\frac{10}{3}\right)^2\left(12 - \frac{12}{5} \times \frac{10}{3}\right)$

$$= \frac{400}{9}\pi$$

Example 6

A cylinder has height h and radius r. The volume of the cylinder is $250\ cm^3$. Find the optimum value of r to ensure the surface area is a minimum.

Solution



We need to find a formula for the volume and surface area in terms of h & r. Then eliminate one of the variables to give a function that can be differentiated.

Surface area: $A = 2\pi r^2 + 2\pi rh \quad ...(1)$

Volume of cylinder: $V = \pi r^2 h \quad ....(2)$

Eliminate h to give V in terms of r:

$$\text{From (2)}, h = \frac{V}{\pi r^2}$$

Surface area: $A = 2\pi r^2 + 2\pi r \times \frac{V}{\pi r^2}$

$$= 2\pi r^2 + 2Vr^{-1}$$

$$\frac{dA}{dr} = 4\pi r - 2Vr^{-2}$$

For max/min: $\frac{dA}{dr} = 0$

$$4\pi r - 2Vr^{-2} = 0$$

$$4\pi r^3 - 2V = 0$$

$$r^3 = \frac{2V}{4\pi}$$

$$r = \sqrt[3]{\frac{V}{2\pi}} = \sqrt[3]{\frac{250}{2\pi}} = 3.414$$

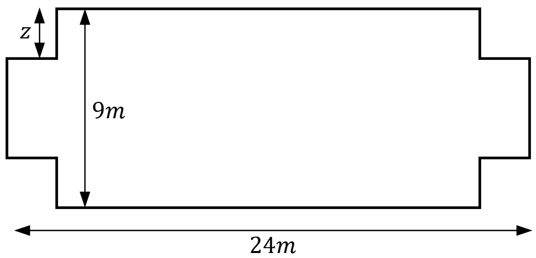
To determine if this is max or min, find the second derivative

$$\frac{d^2A}{dr^2} = 4\pi + 4Vr^{-3} > 0 \ \text{ since } r > 0$$

Example 7

A piece of cardboard $9m \times 24m$ is cut out to make a box. What is the value of z for the optimum volume?

<u>Solution</u>



Area of the box $= (24 - 2z)(9 - 2z)$

$$= z(216 - 48z - 18z + 4z^2)$$

$$= z(216 - 66z + 4z^2)$$

$$= 4z^3 - 66z^2 + 216z$$

$$\frac{dV}{dz} = 12z^2 - 132z + 216$$

For max/min, $\frac{dV}{dz} = 0$

$$12z^2 - 132z + 216 = 0$$

$$z^2 - 11z + 18 = 0$$

$$z^2 - 2z - 9z + 18 = 0$$

$$z(z - 2) - 9(z - 2) = 0$$

$$(z - 2)(z - 9) = 0$$

$$z = 2 \text{ or } z = 9$$

When $z = 9$, $9 - 2z \Rightarrow (9 - 18) = -9$

Hence $z = 9$ is invalid

$$\frac{d^2V}{dz^2} = 24z - 132$$

If $z = 2$, $\frac{d^2V}{dz^2} = 48 - 132 = -84$, hence a maximum volume

Example 8

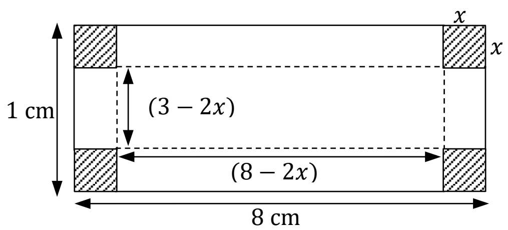
The lengths of the sides of a rectangular sheet of metal are 8 cm and 3 cm. A square of side x cm is cut from each corner of the sheet and the remaining piece is folded to make an open box.

(a) Show that the volume V of the box is given by

$$V = 4x^3 - 2x^2 + 24x \text{ cm}^3$$

(b) Find the value of x for which the volume of the box is a maximum. Calculate the maximum volume.

<u>Solution</u>



(a) The volume of the box is given by

$$V = (8 - 2x)(3 - 2x)x$$

$$= (24 - 16x - 6x + 4x^2)x$$

$$= 4x^3 - 22x^2 + 24x \text{ cm}^3$$

(b) Differentiating with respect to x

$$\frac{dV}{dx} = 12x^2 - 44x + 24$$

For a maximum (or minimum) value of V, $\frac{dV}{dx} = 0$

$$12x^2 - 44x + 24 = 0$$

$$3x^2 - 11x + 6 = 0$$

$$3x^2 - 9x - 2x + 6 = 0$$

$$(3x - 2)(x - 3) = 0$$

$$x = \frac{2}{3} \text{ or } 3$$

Clearly, x cannot be 3 cm since the width of the sheet initially is only 3 cm. So $x = \frac{2}{3}$

Differentiating again gives;

$$\frac{d^2V}{dx^2} = 24x - 44$$

$$\text{when } x = \frac{2}{3}, \frac{d^2V}{dx^2} < 0 \text{ i.e. } V \text{ is a maximum}$$

The maximum volume is obtained from

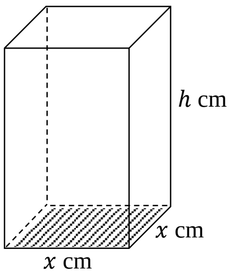
$$V = 4\left(\frac{2}{3}\right)^3 - 22\left(\frac{2}{3}\right)^2 + 24\left(\frac{2}{3}\right) = \frac{200}{27} = 7.41 \text{ cm}^3$$

Example 9

A box is to be constructed in such a way that it must have a fixed volume of 800 cm^3 and a square base. If the box is to be open ended at one end, find the dimensions of the box that will require the least amount of material.

Solution

Let the square base have side lengths x cm and let the height be h cm.



Therefore the volume of the box is $x^2h\text{ cm}^3$

As the volume is 800 cm^3, we have

$$x^2h = 800 \quad \ldots.[1]$$

Let the surface area of the box by $S\text{ cm}^2$,

$$S = x^2 + 4xh \quad \ldots.[2]$$

We wish to minimise S, therefore we need to find the critical point(s) of S. However, we must first obtain an expression for S in terms of x (exclusively)

From [1], $h = \frac{800}{x^2}$

Substituting in [2];

$$S(x) = x^2 + 4x\left(\frac{800}{x^2}\right) = x^2 + \frac{3200}{x}$$

$$S'(x) = 2x - \frac{3200}{x^2}$$

For stationary points, we need to solve $S'(x) = 0$

$$2x - \frac{3200}{x^2} = 0$$

$$x^3 = 1600$$

$$x = \sqrt[3]{1600} = 11.70$$

Next, we check the nature of the stationary point.

$$S''(x) = 2 + 6400x^{-3}$$

$$S''(11.7) = 2 + 6400(11.7)^{-3} > 0 \;\therefore \text{a minimum}$$

$$h = \frac{800}{11.7^2} = 5.85$$

Therefore there is a local minimum at $x = 11.70$ and the dimensions of material required is least when $x = 11.70$ cm and $h = 5.85$ cm

Example 10

Determine the height and radius of a closed cylinder of volume 200 cm^3 which has the least surface area.

Solution

Let the cylinder have radius r and perpendicular height h.

Volume of cylinder,

$$V = \pi r^2h = 200$$

Surface area of cylinder,

$$A = 2\pi rh + 2\pi r^2$$

Least surface area means minimum surface area and a formula for the surface area in terms of one variable is required.

$$h = \frac{200}{\pi r^2}$$

Hence surface area,

$$A = 2\pi r\left(\frac{200}{\pi r^2}\right) + 2\pi r^2$$

$$= \frac{400}{r} + 2\pi r^2$$

$$= 400r^{-1} + 2\pi r^2$$

$$\frac{dA}{dr} = -\frac{400}{r^2} + 4\pi r$$

For minimum, $\frac{dA}{dr} = 0$

$$-\frac{400}{r^2} + 4\pi r = 0$$

$$4\pi r = \frac{400}{r^2}$$

$$r^3 = \frac{400}{4\pi}$$

$$r = \sqrt[3]{\frac{100}{\pi}} = 3.169\text{ cm}$$

$$\frac{d^2A}{dr^2} = \frac{800}{r^3} + 4\pi$$

When $r = 3.169$ cm, $\frac{d^2A}{dr^2}$ is positive, giving a minimum value.

When $r = 3.169$,

$$h = \frac{200}{\pi(3.169)^2} = 6.339\text{ cm}$$

Hence for the least surface area, a cylinder of volume 200 cm^3 has a radius of 3.169 cm and height of 6.339 cm.

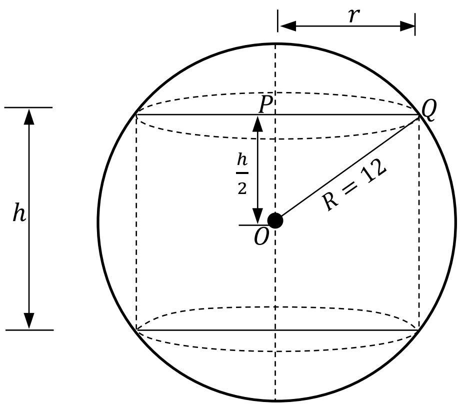
Example 11

Find the diameter and height of a cylinder of maximum volume which can be cut from a sphere of radius 12 cm.

Solution

A cylinder of radius r and height h is shown enclosed in a sphere of radius $R = 12$ cm below.

Volume of cylinder, $V = \pi r^2h$ …… (1)



$$\frac{dV}{dx} = 3x^2$$
$$\delta V \approx 3x^2 \delta x$$

When $\delta x = 0.01x$, $\delta V = 3x^2 \times 0.01x = 0.03x^3$

Percentage change in volume $= \frac{\delta V}{V} \times 100$

$$= \frac{0.03x^3}{x^3} \times 100 = 3\%$$

IMPLICIT DIFFERENTIATION

Most of the expressions we have dealt with so far have been expressed in the form $y = f(x)$. For example, $y = x^3 - 2x$, $y = \ln(x - e^x)$ i.e. y has been expressed **explicitly** in terms of x so that for any one given value of x we obtain a unique value of y by substituting the x-value into the given equation.

Expressions such as $x^2y + y - 2 = 0$, $\sin(xy) = 1$, $e^x = x + y$ are called **implicit equations** because these equations define y implicitly as a function of x.

To differentiate y^3 with respect to x, with the assumption that y is a function of x, we use the chain rule as follows:

$$\frac{d}{dx}(y^3) = \frac{d}{dy}(y^3) \cdot \frac{dy}{dx} = 3y^2 \frac{dy}{dx}$$

To differentiate xy^2 with respect to x, with the assumption that y is a function of x, we use the product rule as follows:

$$\frac{d}{dx}(xy^2) = \frac{d}{dx}(x) \times y^2 + x \times \frac{d}{dx}(y^2)$$
$$= 1 \times y^2 + x \times \left[\frac{d}{dx}(y^2) \cdot \frac{dy}{dx}\right]$$
$$= y^2 + x\left[2y\frac{dy}{dx}\right]$$
$$\therefore \frac{dy}{dx} = y^2 + 2xy\frac{dy}{dx}$$

In general, if we have a term of the form $x^m y^n$, then we use the product rule and obtain

$$\frac{d}{dx}(x^m y^n) = x^m \frac{d}{dx}(y^n) + y^n \frac{d}{dx}(x^m)$$
$$= nx^m\, y^{n-1}\frac{dy}{dx} + mx^{m-1}y^n$$

Example 1

Find the gradient of the curve $2x^2 + y^3 - y = 2$ at the point $(1, 1)$.

<u>Solution</u>

We start by differentiating both sides of the equation w.r.t x

$$\frac{d}{dx}(2x^2 + y^3 - y) = \frac{d}{dx}(2)$$

Then, we differentiate each term in the expression w.r.t x

$$\frac{d}{dx}(2x^2) + \frac{d}{dx}(y^3) - \frac{d}{dx}(y) = 0$$

Use the chain rule

$$4x + \frac{d}{dy}(y^3) \cdot \frac{dy}{dx} - \frac{d}{dy}(y) \cdot \frac{dy}{dx} = 0$$
$$4x + 3y^2\frac{dy}{dx} - \frac{dy}{dx} = 0$$

Then group the $\frac{dy}{dx}$ terms and factorise:

$$4x + (3y^2 - 1)\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{4x}{3y^2 - 1}$$

To find the gradient of the curve at the point $(1, 1)$, we substitute the values $x = 1$ and $y = 1$ into the equation of the derivative

$$\frac{dy}{dx} = -\frac{4}{3 - 1} = -2$$

Example 2

Find $\frac{dy}{dx}$ when $x^3 + 8xy + y^3 = 64$

<u>Solution</u>

Differentiating on both sides with respect to x

$$3x^2 + 8\left[x\frac{dy}{dx} + y\,(1)\right] + 3y^2\frac{dy}{dx} = 0$$
$$3x^2 + 8y + 8x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$
$$(3x^2 + 8y) + (8x + 3y^2)\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{(3x^2 + 8y)}{(8x + 3y^2)}$$

Example 3

Find the equation of the tangent to the curve defined by $x^2y - y = x^2 - 4$ at the point where it crosses the positive x-axis.

<u>Solution</u>

We first determine the gradient function;

$$\frac{d}{dx}(x^2y - y) = \frac{d}{dx}(x^2 - 4)$$
$$2xy + x^2\frac{dy}{dx} - \frac{dy}{dx} = 2x$$
$$(x^2 - 1)\frac{dy}{dx} = 2x - 2xy$$
$$\frac{dy}{dx} = \frac{2x(1 - y)}{(x^2 - 1)}$$

At the point where the curve crosses the x-axis, we have $y = 0$, so substituting $y = 0$ into the equation of the curve we have:

$$x^2(0) - (0) = x^2 - 4$$
$$x^2 = 4$$
$$x = \pm 2$$

As we are only interested in the positive x-axis, we choose $x = 2$.

So the gradient of the tangent at the point $(2, 0)$ is given by

$$\frac{dy}{dx} = \frac{2 \times 2(1 - 0)}{2^2 - 1} = \frac{4}{3}$$

Now, the equation of the tangent is given by

RATES OF CHANGE

We have seen that $\frac{dy}{dx}$ measures the rate of change of a quantity y with respect to another quantity x. In the same way, we have that

$\frac{dA}{dr}$ measures the rate of change of A w.r.t r

$\frac{dV}{dt}$ measures the rate of change of V w.r.t t

$\frac{dP}{dV}$ measures the rate of change of P w.r.t V

For example, if $A\ m^2$ measures the area of a circle of radius $r\ m$, then $\frac{dA}{dr}$ measures the rate of change of the area A with respect to its radius r. Then as $A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$

We note that a rate of change statement needs to have two quantities specified:

1. what quantity is changing, and
2. what it is changing with respect to

Example 1

Find the rate of change of the volume of a sphere with respect to its radius.

Solution

Volume V of a sphere of radius r is given by

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$

Example 2

Let A be the surface area of a spherical balloon. What is the rate of increase in the surface area of the balloon when the radius r is 6 cm, and the radius is increasing at 0.08 cm/sec?

Solution

We want to find $\frac{dA}{dt}$ and we know that $A = 4\pi r^2$

$$\frac{dA}{dr} = 8\pi r$$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

$$\frac{dA}{dt} = 8\pi r \times \frac{dr}{dt}$$

When $r = 6$ and $\frac{dr}{dt} = 0.08$

$$\frac{dA}{dt} = 8\pi \times 6 \times 0.08 = 3.84\pi\ \text{cm}^2/\text{sec}$$

Example 3

The radius of a circular oil patch is increasing at a rate of 1.2 cm per minute. Find the rate at which the surface area of the patch is increasing when the radius is 25 cm.

Solution

We need to find $\frac{dA}{dt}$ when $r = 25$ given $\frac{dr}{dt} = 1.2$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

Area of circular patch, $A = \pi r^2$

$$\frac{dA}{dr} = 2\pi r$$

$$\Rightarrow \frac{dA}{dt} = 2\pi r \times \frac{dr}{dt}$$

With $r = 25$ and $\frac{dr}{dt} = 1.2$ we have

$$\frac{dA}{dt} = 2\pi(25) \times 1.2 = 60\pi = 188.5\ \text{cm}^2\text{min}^{-1}$$

Example 4

The volume of a cube is increasing at $24\ \text{cm}^3\text{s}^{-1}$. At what rate are the side lengths increasing when the volume is $1000\ \text{cm}^3$?

Solution

Let the $V\ \text{cm}^3$ denote the volume of the cube of side length x cm.

We want $\frac{dx}{dt}$ when $V = 1000$, given $\frac{dV}{dt} = 24$

$$V = x^3$$

$$\frac{dV}{dx} = 3x^2$$

$$\frac{dx}{dt} = \frac{dx}{dV} \times \frac{dV}{dt}$$

$$\frac{dx}{dt} = \frac{1}{3x^2} \times \frac{dV}{dt}$$

When $V = 1000$, $x^3 = 1000 \Rightarrow x = 10$

$$\therefore \frac{dx}{dt} = \frac{1}{3(10)^2} \times 24 = 0.08\ \text{cms}^{-1}$$

Example 5

A container in the shape of an inverted right circular cone of base radius 10 cm and height 50 cm has water poured into it at a rate of $5\ \text{cm}^3\text{min}^{-1}$. Find the rate at which the level of the water is rising when it reaches a height of 10 cm.

Solution

Let the water level at time t min have a height h cm with a corresponding radius r cm and volume V cm^3

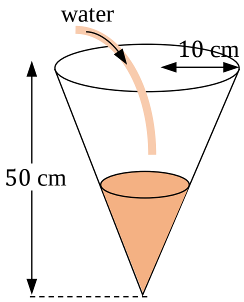


Figure A

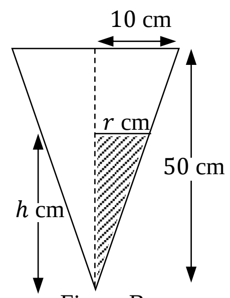


Figure B

We want $\frac{dh}{dt}$ when $h = 10$ given $\frac{dV}{dt} = 5$

Before we can find $\frac{dV}{dh}$, we need an expression for V in terms of h and make use of figure B (similar triangles)

Determine the maximum volume of the box if 6 m^2 of metal are used in its construction.

[Ans: 0.8 m^2]

23. Determine the coordinates of the maximum and minimum values of the graph $y = \frac{x^3}{3} - \frac{x^2}{2} - 6x + \frac{5}{3}$ and distinguish between them. Sketch the graph.

[Ans: (a) max $(-2, 9)$; min $\left(3, -11\frac{5}{6}\right)$]

24. Show that the curve $y = \frac{2}{3}(t-1)^3 + 2t(t-2)$ has a maximum value of $\frac{2}{3}$ and a minimum value of -2.

25. A closed cylindrical container has a surface area of 400 cm^2. Determine the dimensions for the maximum volume.

[Ans: $r = 4.607$ cm; $h = 9.212$ cm]

26. Calculate the height of a cylinder of maximum volume that can be cut from a cone of height 20 cm and base radius 80 cm.

[Ans: 6.67 cm]

27. Find the height and radius of a closed cylinder of volume 125 cm^3 which has the least surface area.

[Ans: $r = 2.71$ cm; $h = 5.42$ cm]

28. The radius of a circular oil patch is increasing at a rate of 1.2 cm per minute. Find the rate at which the surface area of the patch is increasing when radius is 25 cm.

[Ans: 188.5 $cm^2 min^{-1}$]

29. A right circular cylinder of radius r cm and height h cm is to have a fixed volume of 30 cm^3.
    (a) Show that the surface area, $A\,cm^2$ of such a cylinder is given by
$$A = 2\pi r\left(r + \frac{30}{\pi r^2}\right)$$
    (b) Determine the value of r that will yield the minimum surface area.

[Ans: $\sqrt[3]{\frac{15}{\pi}}$]

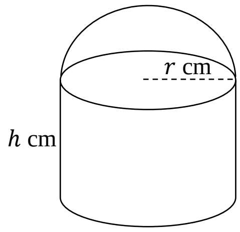
30. A right-circular cone of radius r cm contains a sphere of radius 12 cm.
    (a) If the height of the cone is h cm, express h in terms of r.
    (b) If V cm^3 denotes the volume of the cone, find an expression for V in terms of r.
    (c) Find the dimensions of the cone with the smallest volume.

[Ans: (a) $h = \frac{24r^2}{r^2-144}$ (b) $\frac{8\pi r^4}{r^2-144}$ (c) $r = 12\sqrt{2}$, $h = 48$]

31. A piece of wire 30 cm long is cut into 2 pieces. One of the pieces is bent into a square while the other is bent into a circle. Find the ratio of the side length of the square to the radius of the circle which provides the smallest area sum.

[Ans: 2 ∶ 1]

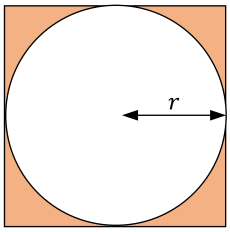
32. A closed tin is to be constructed as shown in the diagram. It is made up of a cylinder of height h cm and radius base r cm which is surmounted by a hemispherical cap.



    (a) Find an expression in terms of r and h for
        (i) its volume, V cm^3
        (ii) its surface area, A cm^2
    (b) Given that $V = \pi k^3$, $k > 0$, show that its surface area is given by
$$A = \frac{2\pi k^3}{r} + \frac{5\pi r^2}{3}$$
    (c) Find the ratio $r : h$ for A to be minimum

[Ans: (a) (i) $\pi r^2 h + \frac{2}{3}\pi r^3$ (ii) $3\pi r^2 + 2\pi rh$ (c) 1 ∶ 1]

33. A closed cylindrical tin can be designed. Its volume is to be 1000 cm^3. Its curved surface is to be shaped from a rectangular piece of tin sheet. Each of its circular ends (of radius r cm) is to be cut from a square of tin sheet, as shown, with the shaded sections being wasted.



    (a) If h cm is the height of the can, express h in terms of r.
    Hence show that the area, A cm^2, of tin sheet needed for a can is given by
$$A = \frac{2000}{r} + 8r^2$$
    (b) Find the radius and height of the can for which the area of tin sheet needed is a minimum.

[Ans: (a) $h = \frac{1000}{\pi r^2}$ (b) $r = 5$ cm, height $= 12.7$ cm]

Chapter 25 Differentiation of exponential and logarithmic functions

Differentiation of e^x

Recall from Exponential & Log functions that the value of e is chosen such that the gradient function of $y = e^x$ is the same as the original function and when $x = 0$, the gradient of $y = e^x$ is 1.

Hence:

$$y = e^x \qquad \frac{dy}{dx} = e^x$$

$$y = e^{kx} \qquad \frac{dy}{dx} = ke^{kx}$$

$$y = e^{f(x)} \qquad \frac{dy}{dx} = f'(x)e^{f(x)}$$

Example 1

Differentiate $y = 5e^{3x} + 2e^{-4x}$

<u>Solution</u>

$$\frac{dy}{dx} = 5 \times 3e^{5x} + 2 \times (-4)e^{4x}$$

$$\frac{dy}{dx} = 15e^{3x} - 8e^{-4x}$$

Example 2

Differentiate $y = e^{x^3}$

<u>Solution</u>

$$u = x^3 \qquad \frac{du}{dx} = 3x^2$$

$$y = e^u \qquad \frac{dy}{du} = e^u$$

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$$\frac{dy}{dx} = 3x^2 \times e^u = 3x^2e^u$$

$$\frac{dy}{dx} = 3x^2e^{x^3}$$

Example 3

Differentiate $y = e^{(x-1)^2}$

<u>Solution</u>

$$u = (x - 1\,)^2 \qquad \frac{du}{dx} = 2(x-1)^1(1) = 2(x-1)$$

$$y = e^u \qquad \frac{dy}{du} = e^u$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = e^u \times 2(x-1)$$

$$\frac{dy}{dx} = 2(x-1)e^{(x-1)^2}$$

Differentiation of $\ln x$

Recall that $\ln x$ is the reciprocal function of e^x and that $y = e^x$ is a reflection of $y = \ln x$ in the line $y = x$

$$y = \ln x \qquad \frac{dy}{dx} = \frac{1}{x}$$

$$y = \ln f(x) \qquad \frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

Note that if:

$$y = \ln kx \qquad \Rightarrow y = \ln k + \ln x$$

$$\frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x}$$

This can be shown thus:

Recall that $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

If $\quad y = \ln x \quad$ then $\; x = e^y$

Differentiate w.r.t y

$$\frac{dx}{dy} = e^y$$

Hence $\frac{dx}{dy} = x$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{x}$$

Example 4

Differentiate $y = \ln x^2$

<u>Solution</u>

$$u = x^2 \qquad \frac{du}{dx} = 2x$$

$$y = \ln u \qquad \frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 2x \times \frac{1}{u} = \frac{2x}{x^2} = \frac{2}{x}$$

Example 5

Differentiate $y = \ln\left(x^2\sqrt{2x^3+3}\right)$

<u>Solution</u>

$$y = \ln\left(x^2\sqrt{2x^3+3}\right)$$

$$y = \ln x^2 + \ln\sqrt{2x^3+3}$$

$$y = 2\ln x + \frac{1}{2}\ln(2x^3+3)$$

$$\frac{dy}{dx} = \frac{2}{x} + \frac{1}{2}\left(\frac{6x^2}{2x^3+3}\right)$$

$$\frac{dy}{dx} = \frac{2(2x^3+3) + 3x^3}{x(2x^3+3)} = \frac{7x^3+3}{x(2x^3+3)}$$

Example 6

Differentiate $y = e^{x\ln 2}$

$$\frac{1}{y}\frac{dy}{dx} = \frac{x}{x^2+2} + \frac{1}{2(x-1)} - \frac{1}{x+3}$$
$$\frac{dy}{dx} = y\left[\frac{x}{x^2+2} + \frac{1}{2(x-1)} - \frac{1}{x+3}\right]$$
$$\frac{dy}{dx} = y\left[\frac{x(x-1)(x+3)+(x^2+2)(x+3)-2(x^2+2)(x-1)}{2(x^2+2)(x-1)(x+3)}\right]$$
$$\frac{dy}{dx} = y\left[\frac{7x^2-5x+10}{2(x^2+2)(x-1)(x+3)}\right]$$
$$\frac{dy}{dx} = \frac{(1-x)\sqrt{x^2+2}}{(x+3)\sqrt{x-1}}\left[\frac{7x^2-5x+10}{2(x^2+2)(x-1)(x+3)}\right]$$
$$\frac{dy}{dx} = \frac{-(7x^2-5x+10)}{2(x+3)^2\sqrt{(x-1)(x^2+2)}}$$

Example 14

Given that $x = \ln(\sec 3y)$. Find an expression for $\frac{dy}{dx}$ in terms of x.

Solution

$$x = \ln(\sec 3y)$$
$$x = \log_e \sec 3y$$
$$\sec 3y = e^x$$
$$3\sec 3y \tan 3y \; dy = e^x \, dx$$
$$\frac{dy}{dx} = \frac{1}{3\tan 3y}$$
$$\frac{dy}{dx} = \frac{1}{3\sqrt{\sec^2 3y - 1}}$$
$$\frac{dy}{dx} = \frac{1}{3\sqrt{e^{2x}-1}}$$

Example 15

A curve has equation $y = x^{-x}$. Show that

$$y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 - \frac{y^2}{x}$$

Solution

$$y = x^{-x}$$

Taking logs on both sides

$$\ln y = \ln x^{-x}$$
$$\ln y = -x\ln x$$
$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(-x\ln x)$$
$$\frac{1}{y}\frac{dy}{dx} = -1\times\ln x - x\times\frac{1}{x}$$
$$\frac{1}{y}\frac{dy}{dx} = -\ln x - 1$$
$$\frac{dy}{dx} = -y(1+\ln x)$$

Differentiate w.r.t x again

$$\frac{d^2y}{dx^2} = -1\frac{dy}{dx}(1+\ln x) - y\left(0+\frac{1}{x}\right)$$
$$\frac{d^2y}{dx^2} = -\frac{dy}{dx}(1+\ln x) - \frac{y}{x}$$

From the expression of $\frac{dy}{dx}$, $1+\ln x = -\frac{1}{y}\frac{dy}{dx}$

Thus

$$\frac{d^2y}{dx^2} = -\frac{dy}{dx}\left(-\frac{1}{y}\frac{dy}{dx}\right) - \frac{y}{x}$$
$$\frac{d^2y}{dx^2} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x}$$
$$y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 - \frac{y^2}{x}$$

Example 16

A curve has equation $y = x - 2\ln(x^2+4)$.

(a) Show clearly that

$$\frac{d^2y}{dx^2} = \frac{4(x^2-4)}{(x^2+4)^2}$$

The curve has a single stationary point.

(b) Find its exact coordinates and determine its nature.

Solution

(a) $y = x - 2\ln(x^2+4)$

$$\frac{dy}{dx} = 1 - \frac{2}{x^2+4}\times 2x$$
$$\frac{dy}{dx} = 1 - \frac{4x}{x^2+4}$$
$$\frac{d^2y}{dx^2} = -\frac{(x^2+4)\times 4 - 4x(2x)}{(x^2+4)^2}$$
$$\frac{d^2y}{dx^2} = -\frac{4x^2+16-8x^2}{(x^2+4)^2}$$
$$\frac{d^2y}{dx^2} = \frac{4x^2-16}{(x^2+4)^2}$$
$$\frac{d^2y}{dx^2} = \frac{4(x^2-4)}{(x^2+4)^2}$$

(b) $\frac{dy}{dx} = 0$

$$1 - \frac{4x}{x^2+4} = 0$$
$$\frac{x^2+4-4x}{x^2+4} = 0$$
$$x^2-4x+4 = 0$$
$$(x-2)^2 = 0$$
$$x = 2$$
$$y = 2 - \ln 8$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=2} = \frac{4(2^2-4)}{(2^2+4)^2} = 0$$

$\therefore$ $(2, 2-6\ln 2)$ is a stationary point of inflection

Example 17

Given that $y = 3\cos(\ln x) + 2\sin(\ln x)$, show clearly that

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y$$

Solution

$$y = 3\cos(\ln x) + 2\sin(\ln x)$$
$$\frac{dy}{dx} = -3\sin(\ln x)\times\frac{1}{x} + 2\cos(\ln x)\times\frac{1}{x}$$
$$x\frac{dy}{dx} = -3\sin(\ln x) + 2\cos(\ln x)$$

$$\cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$$
$$\sin\frac{\theta}{2} \approx \frac{\theta}{2}$$
$$\cos\theta \approx 1 - \frac{2\theta^2}{4} = 1 - \frac{\theta^2}{2}$$

Derivative of $\sin x$ and $\cos x$

1. $\sin x$

Let $y = \sin x$
$$y + \delta y = \sin(x + \delta x)$$
$$\delta y = \sin(x + \delta x) - y = \sin(x + \delta x) - \sin x$$
$$\delta y = 2\cos\left(\frac{x + \delta x + x}{2}\right)\sin\left(\frac{x + \delta x - x}{2}\right)$$
$$= 2\cos\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)$$
Dividing by δx;
$$\frac{\delta y}{\delta x} = \frac{\cos\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\frac{1}{2}\delta x}$$
The limit as $\delta x \to 0$,

$\frac{\delta y}{\delta x} \to \frac{dy}{dx}$, $\sin\left(\frac{\delta x}{2}\right) \approx \frac{\delta x}{2}$

Therefore $\cos\left(x + \frac{\delta x}{2}\right) \approx \cos x$
$$\frac{dy}{dx} = \cos x$$
$$\therefore \frac{d}{dx}(\sin x) = \cos x$$

2. $\cos x$

Let $y = \cos x$

$y + \delta y = \cos(x + \delta x)$

$\delta y = \cos(x + \delta x) - \cos x$
$$= -2\sin\left(\frac{x + \delta x + x}{2}\right)\sin\left(\frac{x + \delta x - x}{2}\right)$$
$$\frac{\delta y}{\delta x} = \frac{-\sin\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}$$
The limit as $\delta x \to 0$; $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$, $\sin\left(\frac{\delta x}{2}\right) \approx \frac{\delta x}{2}$
$$\therefore \frac{dy}{dx} = \frac{-\sin x \frac{\delta x}{2}}{\frac{\delta x}{2}} = -\sin x$$
$$\therefore \frac{d}{dx}(\cos x) = -\sin x$$

Defining other Trigonometric functions

This depends on 3 ideas:

- Definitions of $\tan x$, $\cot x$, $\sec x$ & $\operatorname{cosec} x$ in terms of $\sin x$ and $\cos x$
- The differential of $\sin x$ and $\cos x$
- Product and quotient rules of differentiation

3. $\tan x$
$$y = \tan x$$
$$y = \frac{\sin x}{\cos x}$$
Using the quotient rule;
$$\frac{dy}{dx} = \frac{\cos x \times \cos x - \sin x \times -\sin x}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

4. $\cot x$
$$y = \cot x = \frac{1}{\tan x} = (\tan x)^{-1}$$
Let $u = \tan x$, $\frac{du}{dx} = \sec^2 x$

$y = u^{-1}$, $\frac{dy}{du} = -u^{-2} = -\frac{1}{u^2} = -\frac{1}{\tan^2 x}$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{1}{\tan^2 x} \times \sec^2 x$$
$$\frac{dy}{dx} = -\frac{\cos^2 x}{\sin^2 x} \times \frac{1}{\cos^2 x} = -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

5. $\sec x$
$$y = \sec x = \frac{1}{\cos x} = (\cos x)^{-1}$$
Let $u = \cos x$, $\frac{du}{dx} = -\sin x$
$$y = u^{-1} \Rightarrow \frac{dy}{du} = -u^{-2} = -\frac{1}{u^2} = -\frac{1}{\cos^2 x}$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{1}{\cos^2 x} \times -\sin x$$
$$\frac{dy}{dx} = \frac{\sin x}{\cos x} \times \frac{1}{\cos x} = \tan x \sec x$$

6. $\operatorname{cosec} x$
$$y = \operatorname{cosec} x = \frac{1}{\sin x} = (\sin x)^{-1}$$
Let $u = \sin x \Rightarrow \frac{du}{dx} = \cos x$
$$y = u^{-1} \Rightarrow \frac{dy}{du} = -u^{-2}$$
Use the chain rule: $\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$
$$\frac{dy}{dx} = \cos x \times (-u^{-2})$$
$$\frac{dy}{dx} = -\frac{\cos x}{\sin^2 x} = -\frac{1}{\tan x \sin x} = -\cot x \operatorname{cosec} x$$
Or use the quotient rule:
$$u = 1;\ \frac{du}{dx} = 0$$
$$v = \sin x;\ \frac{dv}{dx} = \cos x$$
$$\frac{dy}{dx} = \frac{\sin x \times 0 - 1 \times \cos x}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\cot x \operatorname{cosec} x$$

Example 4

Prove by first principles, and using small angle approximations for $\sin x$ and $\cos x$, that

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

Solution

$$f'(x) = \lim_{\delta x \to 0}\left[\frac{f(x+\delta x) - f(x)}{\delta x}\right]$$

$$\frac{d}{dx}(\tan x) = \lim_{\delta x \to 0}\left[\frac{\tan(x+\delta x) - \tan x}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{\frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{\sin(x+\delta x)\cos x - \cos(x+\delta x)\sin x}{\delta x \cos x \cos(x+\delta x)}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{\sin[(x+\delta x) - x]}{\delta x \cos x \cos(x+\delta x)}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{\sin \delta x}{\delta x \cos x \cos(x+\delta x)}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{1}{\cos x \cos(x+\delta x)} \times \frac{\sin \delta x}{\delta x}\right]$$

As $\delta x \to 0$, $\frac{\sin \delta x}{\delta x} \to 1$ and $\cos(x+\delta x) \to \cos x$

$$f'(x) = \frac{1}{\cos^2 x} = \sec^2 x$$

Example 5

Prove by first principles, and by using small angle approximations for $\sin x$ and $\cos x$, that

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

Solution

$$f'(x) = \lim_{\delta x \to 0}\left[\frac{f(x+\delta x) - f(x)}{\delta x}\right]$$

$$\frac{d}{dx}(\sec x) = \lim_{\delta x \to 0}\left[\frac{\sec(x+\delta x) - \sec x}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{\frac{1}{\cos(x+\delta x)} - \frac{1}{\cos x}}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{\cos x - \cos(x+\delta x)}{\delta x \cos x \cos(x+\delta x)}\right]$$

Using the trigonometric identity

$$\cos A - \cos B \equiv -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos x - \cos(x+\delta x) = -2\sin\left(\frac{x+x+\delta x}{2}\right)\sin\left(\frac{x-x-\delta x}{2}\right)$$

$$= -2\sin\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{-\delta x}{2}\right)$$

$$\frac{d}{dx}(\sec x) = \lim_{\delta x \to 0}\left[\frac{-2\sin\left(x+\frac{\delta x}{2}\right)\sin\left(\frac{-\delta x}{2}\right)}{\delta x \cos x \cos(x+\delta x)}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{\sin\left(x+\frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)\cos x \cos(x+\delta x)}\right]$$

As $\delta x \to 0$, $\frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}} \to 1$

$$\frac{d}{dx}(\sec x) = \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \times \frac{1}{\cos x} = \tan x \sec x$$

Summary

Function $f(x)$	**Differential $\frac{dy}{dx} = f'(x)$**
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec} x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$

Example 6

Differentiate the following

(a) $y = x^3 \sin x$

Let $u = x^3 \Rightarrow \frac{du}{dx} = 3x^2$

$v = \sin x \Rightarrow \frac{dv}{dx} = \cos x$

$$\frac{dy}{dx} = x^3 \times \cos x + \sin x \times 3x^2$$

$$\frac{dy}{dx} = x^2(x\cos x + 3\sin x)$$

(b) $y = \frac{1}{x}\cos x$

$y = \frac{\cos x}{x}$

Let $u = \cos x \Rightarrow \frac{du}{dx} = -\sin x$

$v = x \Rightarrow \frac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{x \times (-\sin x) - \cos x}{x^2}$$

$$\frac{dy}{dx} = \frac{-x\sin x - \cos x}{x^2} = -\frac{x\sin x + \cos x}{x^2}$$

(c) $y = \cos^4 x$

$$u = \cos x \Rightarrow \frac{du}{dx} = -\sin x$$

$$y = u^4 \Rightarrow \frac{dy}{du} = 4u^3$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = 4u^3 \times (-\sin x)$$

$$\frac{dy}{dx} = 4\cos^3 x(-\sin x) = -4\cos^3 x \sin x$$

(d) $y = \ln \sec x$

$u = \sec x$, $y = \ln u$

$\frac{du}{dx} = \sec x \tan x \quad \frac{du}{dx} = \frac{1}{u}$

$$\frac{dy}{dx} = \sec x \tan x \times \frac{1}{\sec x} = \tan x$$

2. Given that $y = \frac{\sin x - \cos x}{\sin x + \cos x}$; show that $\frac{dy}{dx} = 1 + y^2$. Prove that $\frac{d^2}{dx^2}$ is zero only when $y = 0$.
3. Given that $f(x) = \sin^4 x - \cos^4 x$, prove that $f'(x) = 2\sin 2x$.
4. Given that $x = \theta - \sin\theta$, $y = 1 - \cos\theta$, show that $\frac{dy}{dx} = \cot\frac{\theta}{2}$ and that $\frac{dy^2}{dx^2} + \frac{1}{y^2} = 0$.
5. If $y = \frac{x\cos x + \sin x}{x^2}$, find $\frac{dy}{dx}$ and simplify your answer as much as possible.

   [Ans: $\frac{(x^2+2)\sin x}{x^2}$]
6. A curve has the equation $y = x\sin 2x$. Find the gradient of the curve at $x = \frac{\pi}{3}$.

   [Ans: $\frac{\sqrt{3}}{2} - \frac{\pi}{2}$]
7. Differentiate the following functions w.r.t x
   (a) $\sec x + \tan x$

   [Ans: $\sec x\,(\sec x + \tan x)$]

   (b) $\sec x \tan x$

   [Ans: $\sec x\,(\sec^2 x + \tan^2 x)$]

   (c) $\cos x \tan x$

   [Ans: $-\cos x - \cot x \operatorname{cosec} x$]

   (d) $\sec^2 x + \tan x$

   [Ans: $\sec^2 x\,(1 + 2\tan x)$]

   (e) $\sec^2 x + \tan^2 x$

   [Ans: $4\sec^2 x \tan x$]

   (f) $(\cos x + \sin x)(\sec x + \tan x)$

   [Ans: $(\sec x + \tan x)(1 - \sin x + \cos x + \tan x)$]

   (g) $\frac{\sec x}{\sin x + \cos x}$

   [Ans: $\frac{\sec x(\sin x \tan x + 2\sin x - \cos x)}{(\sin x + \cos x)^2}$]

   (h) $\frac{\sec x}{1 + \sec x}$

   [Ans: $\frac{\sec x \tan x}{(1+\sec x)^2}$]

   (i) $\frac{1 - \tan x}{1 + \tan x}$

   [Ans: $-\frac{2\sec^2 x}{(1+\tan x)^2}$]

   (j) $\frac{\sin^2 x}{\cos x(\cos x + \sin x)}$

   [Ans: $\frac{2\sin x\cos x + \sin^2 x}{\cos^2 x(\cos x + \sin x)^2}$]

   (k) $\sin^3 x \tan 2x$

   [Ans: $3\sin^2 x\cos x \tan 2x + 2\sin^3 x \sec^2 2x$]

   (l) $(1 + \sin^2 x)(1 - \sin^2 x)$

   [Ans: $-4\sin^3 x \cos x$]

   (m) $\left(\frac{\cos x}{1 + \sin x}\right)^3$

   [Ans: $\frac{-3\cos^2 x}{(1+\sin x)^3}$]

   (n) $(1 - \cos^4 x)(1 + \cos^4 x)$

   [Ans: $8\cos^7 x \sin x$]

   (o) $\frac{\sin x}{2 + \sin^2 x}$

   [Ans: $\frac{4\sin x\cos x}{(2+\sin^2 x)^2}$]

   (p) $\sec^3(\tan^2 3x)$

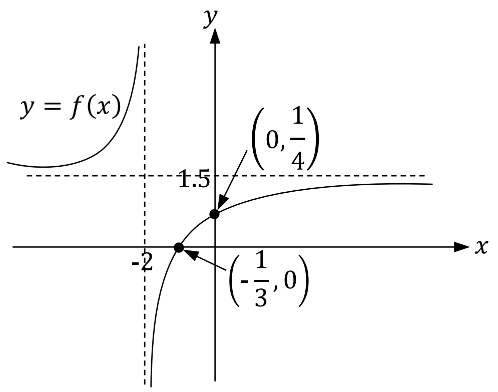
   [Ans: $18\sec^3(\tan^2 3x)\tan(\tan^2 3x)\tan 3x \sec^2 3x$]

   (q) $\sqrt{\frac{1+\cos x}{1-\cos x}}$

   [Ans: $-\frac{1}{1-\cos x}$]
8. If $y = \sin x$, show that $\frac{d^2y}{dx^2} = -y$.
9. If $y = \tan\theta$, show that $\frac{d^2y}{dx^2} = 2y(1 + y^2)$
10. Show that $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$
11. Show that $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{x\sqrt{(x^2-1)}}$
12. If $y = \tan x + \frac{1}{3}\tan^3 x$, prove that $\frac{dy}{dx} = (1 + \tan^2 x)^2$
13. If $y = \frac{\cos\theta}{\theta}$, find $\frac{dy}{d\theta}$ and $\frac{d^2y}{d\theta}$, prove that
$$\theta^2\frac{d^2y}{d\theta^2} + 4\theta\frac{dy}{d\theta} + (\theta^2 + 2)y = 0$$
14. If $y = \sqrt{(4 + 3\sin x)}$ prove that
$$2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y^2 = 4$$
15. Find $\frac{d\theta}{dt}$ when
    (a) $\theta = \sin t \sin 3t$
    (b) $\theta = t^2 \sin^{-1} t$

    [Ans: (a) $3\sin t\cos 3t + \cos t \sin 3t$

    (b) $\frac{2t\sqrt{1-t^2}\sin^{-1} t + t^2}{\sqrt{(1-t^2)}}$]
16. Show that, if P and Q are constants and $y = P\cos(\ln t) + Q\sin(\ln t)$, then
$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + y = 0$$



Example 2

Sketch the graph of $y = 2 + \frac{x}{x^2-1}$

Solution

Intercepts:

y-intercept:

When $x = 0$, $y = 2$

$$\Rightarrow (0, 2) \text{ is the } y\text{-intercept}$$

x-intercepts:

When $y = 0$, $2 + \frac{x}{x^2-1} = 0$

$$2x^2 - 2 + x = 0$$
$$2x^2 + x - 2 = 0$$
$$x = \frac{-1 \pm \sqrt{17}}{4}$$
$$x = 0.781 \text{ or } -1.281$$

The curve cuts the x-axis at $(0.781, 0)$ and $(-1.281, 0)$

Asymptotes:

Vertical asymptotes:

These will correspond to values of x for which the denominator is zero i.e.

$$x^2 - 1 = 0$$
$$(x-1)(x+1) = 0$$
$$x = 1 \text{ or } x = -1$$

The vertical asymptotes are $x = 1$ and $x = -1$

Horizontal asymptote:

We look at the behavior of y as $|x| \longrightarrow \infty$ or $x \to \pm\infty$

$$y = 2 + \frac{x}{x^2-1}$$

As $x \to \pm\infty$, $y \to 2 + 0$

$y = 2$ is the horizontal asymptote

Stationary points:

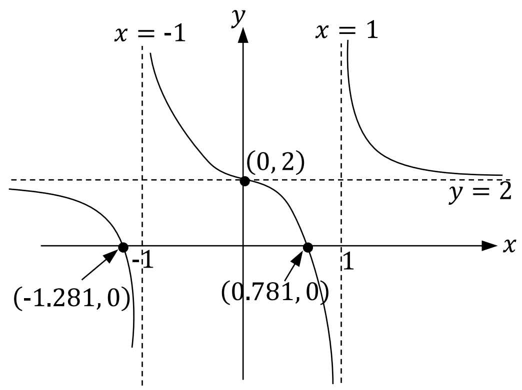
We need to solve $\frac{dy}{dx} = 0$

$$y = 2 + \frac{x}{x^2-1}$$
$$\frac{dy}{dx} = \frac{(1)(x^2-1) - (x)(2x)}{(x^2-1)^2} = \frac{-(x^2+1)}{(x^2-1)}$$
$$\frac{-(x^2+1)}{(x^2-1)} = 0$$
$$-(x^2+1) = 0$$

for which there are no real solutions.

Therefore, there are no stationary points on this curve

After all this, we can now sketch the curve $y = 2 + \frac{x}{x^2-1}$



Example 3

Sketch the curve $y = \dfrac{2x-3}{x^2+2x-3}$

Solution

Intercepts:

When $x = 0$, $y = 1$

The curve cuts the y-axis at $(0, 1)$.

When $y = 0$, $2x - 3 = 0$

$$x = \frac{3}{2}$$

The curve cuts the y-axis at $\left(\frac{3}{2}, 0\right)$

Asymptotes:

Vertical asymptotes, denominator = 0

$$x^2 + 2x - 3 = 0$$
$$x^2 + 3x - x - 3 = 0$$
$$x(x+3) - (x+3) = 0$$
$$(x+3)(x-1) = 0$$

$x = 1$ and $x = -3$ are vertical asymptotes

Horizontal asymptote:

$$y = \frac{2x-3}{x^2+2x-3}$$

As $x \to \pm\infty$, $y \to 0$

$y = 0$ (x-axis) is the horizontal asymptote

Critical values: $x = -3, 1, \frac{3}{2}$

	$x < -3$	$-3 < x < 1$	$1 < x < \frac{3}{2}$	$x > \frac{3}{2}$
$x + 3$	−	+	+	+
$x - 1$	−	−	+	+
$2x - 3$	−	−	−	+
y	−	+	−	+

Rearranging the equation as a quadratic in x:

$$y(x^2 + 2x - 3) = 2x - 3$$
$$yx^2 + (2y-2)x - 3y + 3 = 0$$

For real values of x,

$$(2y-2)^2 - 4y(-3y+3) \geq 0$$
$$4(y-1)^2 + 12y(y-1) \geq 0$$
$$4(y-1)\{(y-1) + 3y\} \geq 0$$

Example 7

A curve C has equation

$$y = \frac{x^2+3}{x-1}$$

Sketch the graph of C

Solution

This is an improper fraction which can be simplified by long division.

$$\begin{array}{r|l} & x+1 \\ \hline x-1 & x^2+3 \\ & -\ \ x^2-x \\ \hline & \quad x+3 \\ & \quad -x-1 \\ \hline & \qquad 4 \end{array}$$

$$y = \frac{x^2+3}{x-1} = x+1+\frac{4}{x-1}$$

Turning points:

$$\frac{dy}{dx} = 1 - \frac{4}{(x-1)^2}$$

For turning points, $\frac{dy}{dx} = 0$

$$1 - \frac{4}{(x-1)^2} = 0$$
$$(x-1)^2 = 4$$
$$x-1 = \pm 2$$
$$x = 3 \text{ or } x = -1$$

When $x = 3$, $y = 3 + 1 + \frac{4}{3-1} = 6 \Rightarrow (3, 6)$

When $x = -1$, $y = -1 + 2 + \frac{4}{-1-2} = -2 \Rightarrow (-1, -2)$

Determining the nature of the turning points;

x	L	3	R	L	-1	R
$\frac{dy}{dx}$	$-$ \	0 _	$+$ /	$+$ /	0 —	$-$ \
		min			max	

$(3, 6)$ is a minimum and $(-1, -2)$ is a maximum

Alternatively; determining the nature of the roots using the second derivative.

$$\frac{d^2y}{dx^2} = \frac{8}{(x-1)^3}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=3} = 1 > 0 \Rightarrow \text{minimum}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = -1 < 0 \Rightarrow \text{maximum}$$

Asymptotes:

$x = 1$ is a vertical asymptote (denominator = 0)

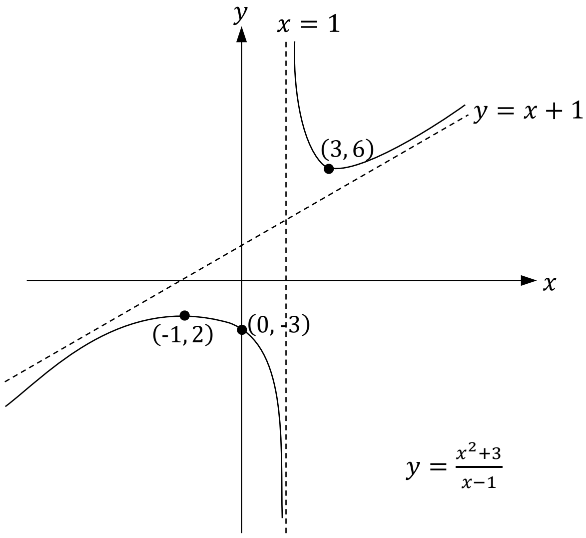
As $x \to \pm\infty$, $y \to x + 1$

$y = x + 1$ is the slanting asymptote

Intercepts:

When $x = 0$, $y = -3 \Rightarrow (0, -3)$

When $y = 0$, no solutions i.e. no x-intercepts



Example 8

A curve C has equation

$$y = \frac{x^2-2x-8}{x-6}$$

Sketch the graph of C

Solution

$$\begin{array}{r|l} & x+4 \\ \hline x-6 & x^2-2x-8 \\ & -\ \ x^2-6x \\ \hline & \quad 4x-8 \\ & \ -\ 4x-24 \\ \hline & \qquad 16 \end{array}$$

Rewrite $y = \frac{x^2-2x-8}{x-6}$ as $x + 4 + \frac{16}{x-6}$

Intercepts:

When $x = 0$, $y = \frac{4}{3}$ $\Rightarrow$ $\left(0, \frac{4}{3}\right)$

When $y = 0$, $x^2 - 2x - 8 = 0$

$$(x-4)(x+2) = 0$$
$$x = 4 \text{ or } x = -2$$
$$\Rightarrow (4, 0) \text{ and } (-2, 0)$$

Asymptotes:

Vertical asymptote, $x = 6$ (denominator = 0)

Oblique/slanting asymptote:

As $x \to \pm\infty$, $y \to x + 4$

$\Rightarrow y = x + 4$ is the slanting asymptote

Turning points:

$$\frac{dy}{dx} = 1 - \frac{16}{(x-6)^2}$$
$$1 - \frac{16}{(x-6)^2} = 0$$
$$(x-6)^2 = 16$$
$$x - 6 = \pm 4$$
$$x = 10 \text{ or } x = 2$$

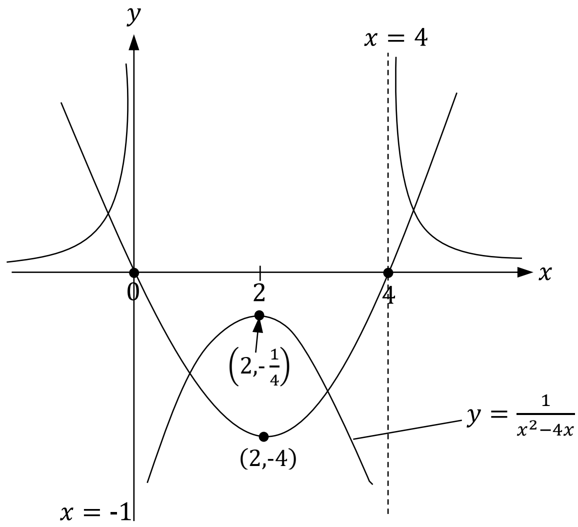
(a) For $y = \frac{1}{f(x)}$

$y \to \pm\infty$, $x = 4$, $x = -1$ are vertical asymptotes

(b) When $f(x) \to \infty$, $\frac{1}{f(x)} \to 0$

$y = 0$ is a horizontal asymptote

(c) Turning point $\left(\frac{3}{2}, \frac{4}{25}\right)$ is a minimum turning point



Example 12

Sketch on the same axes the graphs of

(a) $y = x^2 - 4x$

(b) $y = \frac{1}{x^2-4x}$

The sketch must include

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any stationary points.
- The equations of any asymptotes.

<u>**Solution**</u>

Intercepts:

When $x = 0$, $y = 0$

The curve cuts the y-axis at $(0, 0)$

When $y = 0$, $x^2 - 4x = 0$

$$x(x-4) = 0$$

$$x = 0 \text{ or } x = 4$$

The curve cuts the x-axis at $(0, 0)$ and $(4, 0)$

Turning points:

$$y = x^2 - 4x$$

$$\frac{dy}{dx} = 2x - 4$$

$$2x - 4 = 0$$

$$x = 2$$

When $x = 2$, $y = 2^2 - 4(2) = -4$

$\Rightarrow (2, -4)$ is a turning point

$\frac{d^2y}{dx^2} = 2 > 0 \Rightarrow$ minimum turning point

Now

For $y = \frac{1}{x^2-4x}$

As $y \to \pm\infty$, $x^2 - 4x \to 0$,

$x = 0$ and $x = 4$ are vertical asymptotes.

As $y \to \infty$, $\frac{1}{x^2-4x} \to 0$

$y = 0$ is a horizontal asymptote

Inversing the turning point;

$\left(2, -\frac{1}{4}\right)$ is a maximum turning point

Critical values: $x = 0$, $x = 4$

	$x < 0$	$0 < x < 4$	$x > 4$
$x(x-4)$	$+$	$-$	$+$

Self-Evaluative exercise

1. Find the stationary points of the function given by
$$f(x) = \frac{(x+1)(x+4)}{x}$$
Sketch the graph of $f(x)$ and find the range of $f(x)$.
[Ans: The range for $f(x)$ is $f(x) \le 1$ or $f(x) \ge 9$]

2. Find the stationary points of the function
$$f(x) = \frac{4x-5}{(x-1)(x+1)}$$
and determine the nature of each point.
Sketch the graph of $f(x)$ and give the equations of the asymptotes.
Give the range of $f(x)$
[Ans: $f(x) \le 1$ or $f(x) \ge 4$]

3. Sketch on the same axes the graphs of
(c) $y = (x+1)(2x-3)$
(d) $y = \frac{1}{(x+1)(2x-3)}$

4. Given that $f(x) \equiv \frac{3x+2}{(2x-1)(x+3)}$, express $f(x)$ in partial fractions.
Sketch the $y = f(x)$, showing the asymptotes and the points of intersection of the curve with the axes.
Evaluate $\int_1^5 f(x)\,dx$ and shade on your sketch the region whose area is equal to this integral.
[Ans: 1.792]

5. Show that $f(x) = \frac{x(x-5)}{(x-3)(x+2)}$ has no turning points.
Sketch the curve $y = f(x)$
If $g(x) = \frac{1}{f(x)}$, sketch the graph of $y = g(x)$ on the same axes. Show the asymptotes and where $f(x)$ and $g(x)$ intersect.

Chapter 28 Maclaurin's Expansion

Introduction

One of the simplest kinds of function to deal with, in either algebra or calculus, is a polynomial (i.e. an expression of the form $a + bx + cx^2 + dx^3 + \cdots$. Polynomials are easy to substitute numerical values into, and they are easy to differentiate. One useful application of Maclaurin's series is to approximate, to a polynomial, functions which are not already in polynomial form.

Some mathematical functions may be represented as power series, containing terms in ascending powers of the variable. For example,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Using a series, called Maclaurin's series, mixed functions containing, say, algebraic, trigonometric and exponential functions, may be expressed solely as algebraic functions, and the differentiation and integration can often be more readily performed.

To expand a function using Maclaurin's theorem, some knowledge of differentiation is needed. Given a general function $f(x)$, then $f'(x)$ is the first derivative, $f''(x)$ is the second derivative, and so on. Also, $f(0)$ means he value of the function when $x = 0$, $f'(0)$ means the value of the first derivative when $x = 0$ and so on.

Derivation of Maclaurin's theorem

Let the power series for $f(x)$ be

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots \text{ (i)}$$

where $a_0, a_1, a_2, \ldots$ are constants.

When $x = 0$, $f(0) = a_0$

Differentiating (i) w.r.t x gives;

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_2x^3 + 5ax^4 + \cdots \text{ (ii)}$$

When $x = 0$, $f'(0) = a_1$

Differentiating (ii) w.r.t x gives;

$$f''(x) = 2a_2 + (3)(2)a_3x + (4)(3)a_4x^2 + (5)(4)a_5x^3 + \ldots \text{ (iii)}$$

When $x = 0$, $f''(0) = 2a_2 = 2!\,a_2$ i.e. $a_2 = \frac{f''(0)}{2!}$

Differentiating (iii) w.r.t. x gives:

$$f'''(x) = (3)(2)a_3 + (4)(3)(2)a_4x + (5)(4)(3)a_5x^2 + \ldots \text{ (iv)}$$

When $x = 0$, $f'''(0) = (3)(2)a_3 = 3!\,a_3$ i.e. $a_3 = \frac{f'''(0)}{3!}$

Continuing the same procedure gives $a_4 = \frac{f^{iv}(0)}{4!}$, $a_5 = \frac{f^{v}(0)}{5!}$, and so on.

Substituting for $a_0, a_1, a_2, \ldots$ (i) gives:

$$\boldsymbol{f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^r(0)}{r!}x^r + \ldots}$$

This result is called Maclaurin's theorem and the series obtained is known as the Maclaurin's series for $f(x)$.

It is possible to find a Maclaurin's series for any function $f(x)$ whose derivatives $f'(0)$, $f''(0)$, $f'''(0)$, … can be determined. The series obtained may converge to the sum $f(x)$ for all values of x. However, for many functions, Maclaurin's theorem holds only within a restricted range of values $f(x)$.

Conditions of Maclaurin's series

Maclaurin's series may be used to represent any function, say $f(x)$, as power series provided that at $x = 0$ the following conditions are met.

1. $f(0) \neq \infty$
   For example, for the function $f(x) = \cos x$, $f(0) = \cos 0 = 1$, thus $\cos x$ meets the condition. However, if $f(x) = \ln x$, $f(0) = \ln 0 = -\infty$, thus $\ln x$ does not meet this condition.
2. $f'(0), f''(0), f'''(0), \ldots \neq \infty$
   For example, for the function $f(x) = \cos x$, $f'(0) = -\sin 0 = 0$, $f''(0) = -\cos 0 = -1$, and so on; thus $\cos x$ meets this condition. However, if $f(x) = \ln x$, $f'(0) = \frac{1}{0} = \infty$, thus $\ln x$ does not meet this condition.
3. The resultant Maclaurin's series must be convergent.
   In general, this means that the values of the terms, or groups of terms, must get progressively smaller and the sum of the terms must reach a limiting value.

Example 1

Determine the first four terms of the power series for $\cos x$. Hence or otherwise produce a power series for $\cos^2 2x$ as far as the term in x^6.

Solution

The values of $f(0)$, $f'(0)$, $f''(0)$, … in Maclaurin's series are obtained as follows:

$f(x) = \cos x$ $\qquad f(0) = \cos 0 = 1$

$f'(x) = -\sin x$ $\qquad f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x$ $\qquad f''(0) = -\cos 0 = -1$

$f'''(x) = \sin x$ $\qquad f'''(0) = \sin 0 = 0$

$f^{iv}(x) = \cos x$ $\qquad f^{iv}(0) = \cos 0 = 1$

$f^{v}(x) = -\sin x$ $\qquad f^{v}(0) = -\sin 0 = 0$

Chapter 29 Integration I

Introduction

The process of integration reverses the process of differentiation. In differentiation, if $f(x) = 2x^2$, then $f'(x) = 4x$. Thus the integral of $4x$ is $2x^2$ i.e. integration is the process of moving from $f'(x)$ to $f(x)$. By similar reasoning, the integral of $2t$ is t^2.

Integration is a process of summation or adding parts together and an elongated S, shown as $\int$, is used to replace the words 'the integral of'. Hence, from above, $\int 4x = 2x^2$ and $\int 2t = 2t^2$

In differentiation, the differential coefficient $\frac{dy}{dx}$ indicates that a function of x is being differentiated with respect to x, the dx indicating that it is 'with respect to x'. In integration the variable of integration is shown by adding d(the variable) after the function the function to be integrated.

Thus $\int 4x\,dx$ means 'the integral of $4x$ with respect to x' and $\int 2t\,dt$ means 'the integral of $2t$ with respect to t'

As stated above, the differential coefficient of $2x^2$ is $4x$, hence $\int 4x\,dx = 2x^2$. However, the differential coefficient of $2x^2 + 7$ is also $4x$. Hence $\int 4x\,dx$ is also equal to $2x^2 + 7$. To allow for possible presence of a constant, whenever the process of integration is performed, a constant 'c' is added to the result.

Thus $\displaystyle\int 4x\,dx = 2x^2 + c$ and $\displaystyle\int 2t\,dt = t^2 + c$

'c' is called the **arbitrary constant of integration**

The general solution of integrals of the form ax^n

The general solution of integrals of the form $\int ax^n\,dx$, where a and n are constants is given by

$$\boldsymbol{\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c}$$

This is true when n is fractional, zero, or a positive or negative integer, with the exception of $n = -1$.

Using this rule gives:

(i) $\displaystyle\int 3x^4\,dx = \frac{3x^{4+1}}{4+1} + c = \frac{3}{5}x^5 + c$

(ii) $\displaystyle\int \frac{2}{x^2}\,dx = \int 2x^{-2}\,dx = \frac{2x^{-2+1}}{-2+1} + c$

$$= \frac{2x^{-1}}{-1} + c = -\frac{2}{x} + c$$

(iii) $\displaystyle\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}\sqrt{x^3} + c$$

Each of these results may be checked by differentiation

Notes:

(a) The integral of a constant k is $kx + c$. For example

$$\int 8\,dx = 8x + c$$

(b) When a sum of several items is integrated, the result is the sum of the separate terms. For example

$$\int (3x + 2x^2 - 5)\,dx$$

$$= \int 3x\,dx + \int 2x^2\,dx - \int 5\,dx$$

$$= \frac{3x^2}{2} + \frac{2x^3}{3} - 5x + c$$

Standard integrals

Since integration is the reverse process of differentiation the standard integrals listed below may be deduced and readily checked by differentiation.

(i) $\displaystyle\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c \quad n \neq -1$

(ii) $\displaystyle\int \cos ax\,dx = \frac{1}{a}\sin ax + c$

(iii) $\displaystyle\int \sin ax\,dx = -\frac{1}{a}\cos ax + c$

(iv) $\displaystyle\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + c$

(v) $\displaystyle\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + c$

(vi) $\displaystyle\int \operatorname{cosec} ax \cot ax\,dx = -\frac{1}{a}\operatorname{cosec} ax + c$

(vii) $\displaystyle\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + c$

(viii) $\displaystyle\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$

(ix) $\displaystyle\int \frac{1}{x}\,dx = \ln x + c$

Example 1

Determine (a) $\displaystyle\int 5x^2\,dx$ (b) $\displaystyle\int 2t^3\,dt$

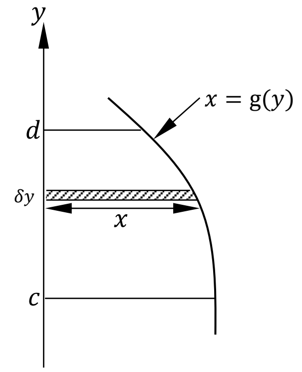
<u>Solution</u>

(a) $\displaystyle\int 5x^2\,dx = \frac{5x^{2+1}}{2+1} + c = \frac{5x^3}{3} + c$

$$\text{Area} = \int_a^b [f_1(x) - f_2(x)]dx$$

(v) Area between a curve and the y-axis

If $x = g(y) \geq 0$ for $c \leq y \leq d$, area of elemental strip $\approx x\,\delta y = g(y)\delta y$

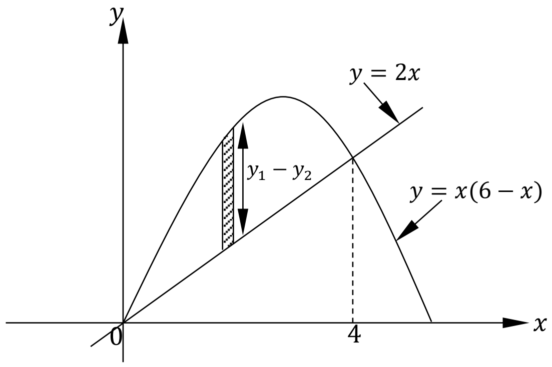


$$\text{Area} = \int_c^d x\,dy\,dy = \int_c^d g(y)\,dy$$

Example 1

Calculate the area of the finite region bounded by the curve $y = x(6-x)$ and the straight line $y = 2x$.

Solution



The two curves intersect when $2x = x(6-x)$

$$x = 0 \text{ or } 4$$

Area of the shaded element $\approx (y_1 - y_2)\delta x$

where $y_1 = x(6-x)$ and $y_2 = 2x$

$$\text{Area of the finite region} = \int_0^4 \{(6x - x^2) - 2x\}\,dx$$

$$= \int_0^4 (4x - x^2)\,dx$$

$$= \left[2x^2 - \frac{x^3}{3}\right]_0^4$$

$$= 32 - \frac{64}{3} = \frac{32}{3} \text{ square units}$$

Example 2

Determine the area enclosed between the curves between the curves $y = x^2 + 1$ and the line $y = 7 - x$.

Solution

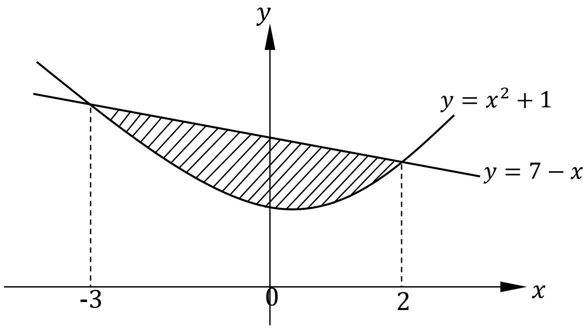
At the points of intersection the curves are equal. Thus, equating the y values of each curve gives:

$$x^2 + 1 = 7 - x$$

$$x^2 + x - 6 = 0$$

$$(x-2)(x+3) = 0$$

$$x = 2 \text{ and } x = -3$$



$$\text{Shaded area} = \int_{-3}^{2} (7-x)\,dx - \int_{-3}^{2} (x^2+1)\,dx$$

$$= \int_{-3}^{2} [(7-x) - (x^2+1)]\,dx$$

$$= \int_{-3}^{2} (6 - x - x^2)\,dx$$

$$= \left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-3}^{2}$$

$$= \left(12 - 2 - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + 9\right)$$

$$= 20\frac{5}{6} \text{ square units}$$

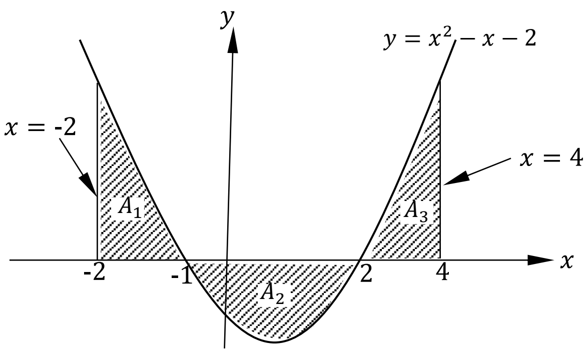
Example 3

Find the area between the curves $y = x^2 - x - 2$, x-axis and the lines $x = -2$ and $x = 4$.

Solution

$$y = x^2 - x - 2 = (x+1)(x-2)$$

This curve intersects x-axis at $x = -1$ and $x = 2$.



Required area $= A_1 + A_2 + A_3$

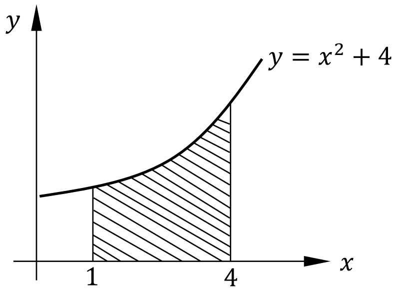
The part A_2 lies below the x-axis

$$\Rightarrow A_2 = \int_{-1}^{2} y\,dx$$

Example 1

The curve $y = x^2 + 4$ is rotated one about the x-axis between the limits $x = 1$ and $x = 4$. Determine the volume of solid revolution produced.

Solution



Resolving the shaded area about the axis produces a solid of revolution given by:

$$\text{Volume} = \int_1^4 \pi y^2\,dx = \int_1^4 \pi(x^2+4)^2\,dx$$

$$= \int_1^4 \pi(x^4 + 8x^2 + 16)\,dx$$

$$= \pi\left[\frac{x^5}{5} + \frac{8x^3}{3} + 16x\right]_1^4$$

$$= \pi\left[\left(\frac{4^5}{5} + \frac{8(4)^3}{3} + 16(4)\right) - (0.2 + 2.67 + 16)\right]$$

$$= 420.6\pi \text{ cubic units}$$

Volume between two solids

The volume of the solid generated by the revolution about the x-axis of the area bounded by the curves $x = f(x)$, $y = \mathrm{g}(x)$ and the ordinates $x = a$, $x = b$ is

$$V = \int_a^b \pi(y_1^2 - y_2^2)\,dx$$

where y_1 is the y of the upper curve and y_2 is the y of the lower curve.

Example 2

Find the volume of the solid of revolution obtained by rotating the area included between the curves $y^2 = x^3$ and $x^2 = y^3$ about the x-axis.

Solution

The curve $y^2 = x^3$ is symmetrical about x-axis and the curve $x^2 = y^3$ is symmetrical about y-axis.

Eliminating y from the given curves, we get

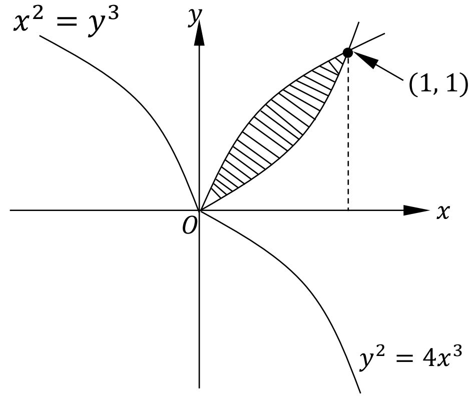
$$x^2 = \left(x^{\frac{3}{2}}\right)^3$$

$$x^2 - x^{\frac{9}{2}} = 0$$

$$x^2\left(1 - x^{\frac{5}{2}}\right) = 0$$

$$x = 0 \text{ and } x = 1$$

Therefore, the points of intersection of the curves are $(0,0)$, $(1,1)$.



$$\text{Required volume} = \pi\int_0^1 (y_1^2 - y_2^2)\,dx$$

(where y_1 is the y of upper curve $x^2 = y^3$ and y_2 is the y of the lower curve $y^2 = x^3$)

$$V = \pi\int_0^1 \left(x^{\frac{4}{3}} - x^3\right)dx$$

$$= \pi\left[\frac{3}{7}x^{\frac{7}{3}} - \frac{x^4}{4}\right]_0^1$$

$$= \pi\left[\frac{3}{7} - \frac{1}{4}\right]$$

$$= \frac{5}{28}\pi$$

Example 3

Determine the area enclosed by the two curves $y = x^2$ and $y^2 = 8x$. If this area is rotated $360°$ about the x-axis, determine the volume of the solid of revolution produced.

Solution

At the points of intersection, the coordinates of the curves are equal. Since $y = x^2$, then $y^2 = x^4$. Hence equating the y^2 values at the pints of intersection.

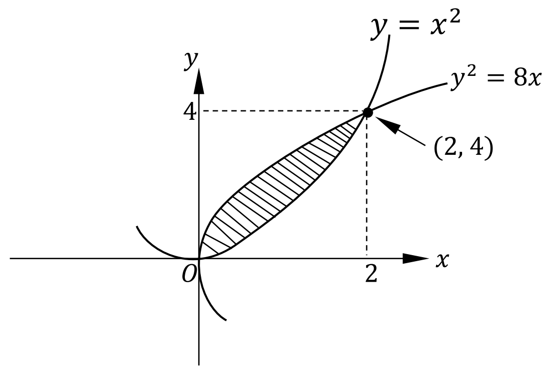
$$x^4 = 8x$$

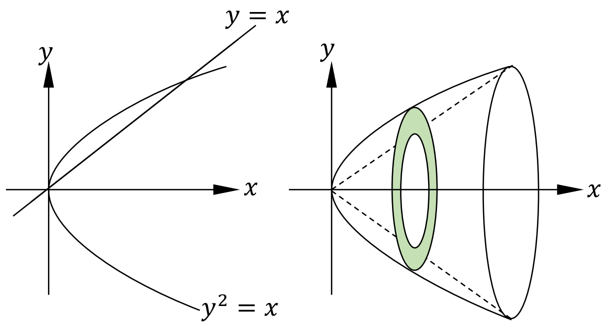
$$x^4 - 8x = 0$$

$$x(x^3 - 8) = 0$$

Hence, at the points of intersection, $x = 0$ and $x = 2$.

When $x = 0$, $y = 0$ and when $x = 2$, $y = 4$. The points of intersection of the curves $y = x^2$ and $y^2 = 8x$ are therefore $(0,0)$ and $(2,4)$.





The solid of revolution of similar shapes is the difference between the solids of rotation of the two separate curves or lines. The limits are found from the intersection of the straight line and the curve. The intersection points are easy found to be $(0, 0)$ and $(1, 1)$

$$V = \int_a^b \pi(y_1^2 - y_2^2)dx$$

In this case: $y_1^2 = x$ and $y_2 = x$

$$V = \int_0^1 \pi(x - x^2)\,dx$$

$$= \pi\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \pi\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= \frac{\pi}{6} \text{ cubic units}$$

Example 7

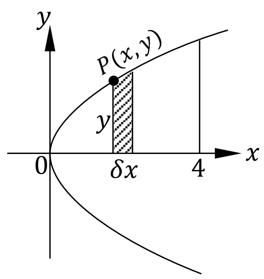
The region bounded by the curve $y^2 = x$ and the line $x = 4$ is revolved around the line indicated.

(a) the x-axis (b) the y-axis (c) $y = 2$

Find the volume of revolution of the solid in each case

<u>Solution</u>

(a)

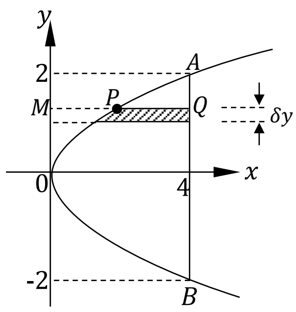


Element volume $\approx \pi y^2 \delta x$

$$\text{Volume} = \int_0^4 \pi y^2 dx = \int_0^4 \pi x\,dx$$

$$= \pi\left[\frac{x^2}{2}\right]_0^4 = 8\pi \text{ cubic units}$$

(b) Rotation about the y-axis



Element of volume is the ring of radii $PM = x$ and $QM = 4$, thickness δy

Area of the cross-section at $P(x, y)$

= Area of the ring of internal radius PM and the external radius QM, $x_1 = QM = 4$, $x_2 = PM = x$

Solve the equations $y^2 = x$ and $x = 4$

$$\therefore y = \pm 2$$

This gives $A(4, 2)$ and $B(4, -2)$

Then

$$\text{Area} = \pi(x_1^2 - x_2^2) = \pi(16 - x^2) = \pi(16 - y^4)$$

$$V = \int_{-2}^2 \pi x^2\,dy$$

Using the symmetry of the curve,

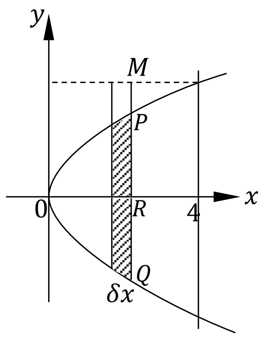
$$V = 2\int_0^2 \pi(16 - y^4)dy$$

$$= 2\pi\left[16y - \frac{y^5}{5}\right]_0^2$$

$$= 2\pi\left(32 - \frac{32}{5}\right)$$

$$= \frac{256\pi}{5} \text{ cubic units}$$

(c) Rotation about the line $y = 2$



Element of volume is a ring of radii QM (external) and PM (internal), thickness δx at $P(x, y)$ with $Q(x, -y)$.

$$QM = 2 - (-y) = 2 + y, \;\; PM = 2 - y$$

Area of cross-sectional $= \pi(QM^2 - PM^2)$

$$= \pi[(2 + y)^2 - (2 - y)^2]$$

$$= 8\pi y$$

$$\delta V = 8\pi y \delta x$$

Chapter 30 Integration by Inspection

Introduction

This covers two forms of integration which involve a function combined with its differential, either as a product or a quotient. These include:

Integrals of the form $\int \frac{kf'(x)}{f(x)}dx$

Integrals of the form $\int kf'(x)[f(x)]^n\,dx$

Integrals f the form $\int kf'(x)e^{f(x)}\,dx$

Integration of these types is often called 'integration by inspection' or 'integration by recognition', because once proficient in using this method, you should be able to just write down the answer by 'inspecting' the function.

It is derived from reversing the 'function of a function' rule for differentiation i.e. the chain rule.

The key to using this method is recognising that one part of the integral is the differential (or scalar multiple) of the other part.

There are several methods of integrating fractions and products, depending on the form of the original function and recognition of this form will save a good deal of calculations. A common alternative to this method is "integration by substitution".

Quotients

Integrals of the form $\int \frac{kf'(x)}{f(x)}dx$ are basically fractions with a function in the denominator and a multiple of its differential in the numerator, assuming that the function is rational.

E.g $\int \frac{4x}{x^2+1}dx \Rightarrow \frac{2\times\text{differential of the denominator}}{\text{a function with a differential of } 2x}$

$\int \frac{4\sin x}{\cos x+1}dx \Rightarrow \frac{-4\times\text{differential of the denominator}}{\text{a function with a differential of } -\sin x}$

Remember, using the chain rule that

$$\text{if } y=\ln x \quad \text{then} \quad \frac{dy}{dx}=\frac{1}{x}$$

$$\text{and if } y=\ln f(x) \quad \text{then} \quad \frac{dy}{dx}=\frac{1}{f(x)}\times f'(x)$$

Reversing the differential by integrating we get:

$$\int \frac{kf'(x)}{f(x)}dx = k\ln|f(x)|+c$$

Note that he modulus sign indicates that you cannot take natural log of a negative number.

Following our method, our first guess should therefore, be

(guess) = ln|denominator|

Example 1

Find $\int \frac{x^2}{1+x^2}dx$

Solution

Guess: $\ln|1+x^3|$

Test: $\frac{d}{dx}\ln|1+x^3| = \frac{1}{1+x^3}\times 3x^2 = \frac{3x^2}{1+x^3}$

Reverse: $\int \frac{3x^2}{1+x^3}dx = \ln|1+x^3|+c$

Adapt: $\int \frac{x^2}{1+x^3}dx = \frac{1}{3}\ln|1+x^3|+c$

Note: Adjustment has to be a number only.

Example 2

Find $\int \frac{\cos x-\sin x}{\sin x+\cos x}dx$

Solution

Guess: $\ln|\sin x+\cos x|$

Test: $\frac{d}{dx}[\ln|\sin x+\cos x|] = \frac{1}{\sin x+\cos x}\times(\cos x-\sin x)$

Reverse: $\int \frac{\cos x-\sin x}{\sin x+\cos x}dx = \ln|\sin x+\cos x|+c$

Adapt: Not required because the numerator is the exact of the denominator.

Example 3

Find $\int \frac{2x}{x^2+9}dx$

Solution

of the form $\int \frac{f'(x)}{f(x)}dx$

$$\therefore \int \frac{2x}{x^2+9}dx = \ln|x^2+9|+c$$
$$= \ln(x^2+9)+c$$

Note: for all real values of x, $(x^2+9)>0$, hence modulus sign not required.

Products

Integrals of the form $\int kf'(x)[f(x)]^n\,dx$ and $\int k\,f'(x)e^{f(x)}\,dx$ involves a function raised to a power or e raised to the power of the function, multiplied by a multiple of the differential of $f(x)$. Note that many of these examples can also be solved by other methods like substitution.

$\int x(x^2+1)^2\,dx \qquad f(x)=x^2+1 \;\Rightarrow\; f'(x)=2x$

$\int x^2(3x^3+1)^4\,dx \qquad f(x)=3x^3+1 \;\Rightarrow\; f'(x)=9x^2$

Chapter 31 Integration by Substitution

Introduction

Also known as integration by change of variable. This is the nearest to the chain rule that integration can get. It is used to perform integrations that cannot be done by other methods, and is also an alternative method to some other methods. It is worth checking if integration can be done by inspection, which may be simpler.

Substitution is often used to define some standard integrals. The objective is to substitute some inner part of the function by a second variable u, and charge all the instances of x to be in terms of u, including dx.

The basic argument for integration by substitution is:

$$\text{If} \quad y = \int f(x)\,dx$$

$$\frac{dy}{dx} = f(x)$$

For the chain rule, if u is a function of x

$$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$$

$$\frac{dy}{du} = f(x)\frac{dx}{du}$$

$$\int \frac{dy}{du}\,du = \int f(x)\frac{dx}{du}\,du$$

$$y = \int f(x)\frac{dx}{du}\,du$$

$$\therefore \int f(x)\,dx = \int f(x)\frac{dx}{du}\,du$$

Required knowledge

$$\int (ax+b)^2 = \frac{1}{a(n+1)}(ax+b)^{n+1} + c$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$$

$$\int e^{(ax+b)}\,dx = \frac{1}{a}e^{(ax+b)} + c$$

Example 1

Use substitution to find $\int (5x-3)^3\,dx$

<u>Solution</u>

Let $u = 5x - 3$

$$\frac{du}{dx} = 5 \quad \Rightarrow \frac{dx}{du} = \frac{1}{5}$$

$$\int f(x)\,dx = \int f(x)\frac{dx}{du}\,du$$

Substituting;

$$\int (5x-3)^3\,dx = \int (u)^3\frac{1}{5}du = \frac{1}{5}\int u^3 du$$

$$= \frac{1}{5} \times \frac{1}{4}u^4 + c = \frac{1}{20}(5x-3)^4 + c$$

Example 2

Use substitution to find: $\displaystyle\int \frac{1}{4x+2}\,dx$

<u>Solution</u>

Let $u = 4x + 2$

$$\frac{du}{dx} = 4 \Rightarrow \frac{dx}{du} = \frac{1}{4}$$

Substituting

$$\int \frac{1}{4x+2}\,dx = \int \frac{1}{u}\cdot\frac{1}{4}du = \frac{1}{4}\int \frac{1}{u}du$$

$$= \frac{1}{4}\ln u + c$$

$$= \frac{1}{4}\ln(4x+2) + c$$

This is a standard result:

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$$

Example 3

Use substitution to find:

$$\int \frac{1}{x+\sqrt{x}}\,dx$$

<u>Solution</u>

Let $u = \sqrt{x} \Rightarrow \quad u = x^{\frac{1}{2}}$

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \quad \Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}} \quad \Rightarrow dx = 2\sqrt{x}\,du$$

but we still have x involved, so substitute for x, thus

$$dx = 2u\,du$$

Substituting into the original:

$$\int \frac{1}{x+\sqrt{x}}\,dx = \int \frac{1}{u^2+u}2u\,du = \int \frac{2}{(u+1)}du$$

$$= 2\ln|u+1| + c$$

$$= 2\ln\left|\sqrt{x}+1\right| + c$$

Example 4

Use substitution to find:

$$\int 3x\sqrt{1+x^2}\,dx$$

<u>Solution</u>

Let $\quad u = x^2$

$$\frac{du}{dx} = 2x \qquad \Rightarrow \qquad dx = \frac{du}{2x}$$

Substituting:

$$\int 3x\sqrt{1+x^2}\,dx = 3\int x(1+u)^{\frac{1}{2}}\,du = \frac{3}{2}\int (1+u)^{\frac{1}{2}}\,du$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{1+\tan^2 u}\sec^2 u\,du$$
$$= \int_0^{\frac{\pi}{4}} du$$
$$= [u]_0^{\frac{\pi}{4}}$$
$$= \frac{\pi}{4}$$

Example 13

Find $\int_0^2 x(2x-1)^6\,dx$

<u>Solution</u>

Let $u = 2x - 1 \quad \Rightarrow \quad x = \frac{1}{2}(u+1)$

$\frac{du}{dx} = 2 \quad \Rightarrow \quad dx = \frac{1}{2}du$

Limits:

x	u
1	-1
2	3

Substituting:

$$\int_0^2 x(2x-1)^6\,dx = \int_{-1}^3 \frac{1}{2}(u+1)(u^6)\frac{1}{2}du$$
$$= \frac{1}{4}\int_{-1}^3 u^7 + u^6\,du$$
$$= \frac{1}{4}\left[\frac{1}{8}u^8 + \frac{1}{7}u^7\right]_{-1}^3$$
$$= \frac{1}{4}\left[\frac{1}{8}(3)^8 + \frac{1}{7}(3)^7\right] - \frac{1}{4}\left[\frac{1}{8}(-1)^8 + \frac{1}{7}(-1)^7\right]$$
$$= 283.14$$

Example 14

Evaluate $\int_{-1}^2 x^2\sqrt{(x^3+1)}dx$

<u>Solution</u>

Let $u = x^3 + 1$

$\frac{du}{dx} = 3x^2 \qquad dx = \frac{1}{3x^2}du$

Limits:

x	u
-1	0
2	9

Substituting:

$$\int_{-1}^2 x^2\sqrt{(x^3+1)}dx = \int_0^9 x^2 u^{\frac{1}{2}}\frac{1}{3x^2}du$$
$$= \frac{1}{3}\int_0^9 u^{\frac{1}{2}}\,du$$
$$= \frac{1}{3}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^9$$
$$= \frac{1}{3}\left[\frac{2}{3}(9)^{\frac{3}{2}}\right] - 0$$
$$= 6$$

Reverse substitution

This is where we have to recognize the substitution by ourselves, by recognizing the reverse chain rule.

Example 15

Use substitution to find: $\int \frac{1}{x}\ln x\,dx$

<u>Solution</u>

Let: $x = e^u$

$$\frac{dx}{du} = e^u \qquad \therefore \quad dx = e^u\,du$$

Substituting

$$\int \frac{1}{x}\ln x\,dx = \int \frac{1}{e^u}\ln e^u \times e^u\,du = \int u\,du$$
$$\int u\,du = \frac{u^2}{2} + c$$
$$x = e^u \quad \therefore \quad u = \ln x$$
$$\int \frac{1}{x}\ln x\,dx = \frac{(\ln x)^2}{2} + c$$

Example 16

Find $\int \frac{6x}{\sqrt{1+x^2}}dx$

<u>Solution</u>

Let $u = 1 + x^2$

$\frac{du}{dx} = 2x \quad \Rightarrow \quad dx = \frac{du}{2x}$

$$\int \frac{6x}{\sqrt{1+x^2}}dx = \int \frac{6x}{\sqrt{u}} \times \frac{du}{2x}$$
$$= \int \frac{3}{\sqrt{u}}du$$
$$= \int 3u^{-\frac{1}{2}}\,du$$
$$= 6u^{\frac{1}{2}} + c$$
$$= 6(1+x^2)^{\frac{1}{2}} + c$$
$$= 6\sqrt{1+x^2} + c$$

Example 17

Find: $\int x^2\sqrt{1+x^3}\,dx$

<u>Solution</u>

Let $u = 1 + x^3$

$$\frac{du}{dx} = 3x^2 \quad \Rightarrow \quad dx = \frac{du}{3x^2}$$
$$\int x^2\sqrt{1+x^3}\,dx = \int x^2\sqrt{u} \times \frac{du}{3x^2}$$
$$= \frac{1}{3}\int u^{\frac{1}{2}}\,du$$
$$= \frac{1}{3}\left[\frac{2}{3}u^{\frac{3}{2}}\right] + c$$
$$= \frac{2}{9}\left[(1+x^3)^{\frac{3}{2}}\right] + c$$

Example 18

Find $\int \frac{7x}{(1+2x^2)^3}dx$

Solution

Let $u = 1 + 2x^2$

$\frac{du}{dx} = 4x \quad \Rightarrow \quad dx = \frac{du}{4x}$

$$\int \frac{7x}{(1+2x^2)^3}dx = \int \frac{7x}{u^3}\cdot\frac{du}{4x}$$
$$= \int \frac{7}{4u^3}du$$
$$= \int \frac{7}{4}u^{-3}\,du$$
$$= \frac{7}{4}\int u^{-3}\,du$$
$$= \frac{7}{4}\left[\frac{1}{2}u^{-2}\right] + c$$
$$= -\frac{7}{8}u^{-2} + c$$
$$= -\frac{7}{8u^2} + c$$
$$= -\frac{7}{8}(1+2x^2)^{-2} + c$$
$$= -\frac{7}{8(1+2x^2)^2} + c$$

Example 19

Evaluate $\int_1^2 \frac{x^3}{\sqrt{4-x^2}}dx$

Solution

Let $u^2 = 4 - x^2$, then $2u\frac{du}{dx} = -2x$

$$x\frac{dx}{du} = -u$$

When $x = 1$, $u = \sqrt{3}$; when $x = 2$, $u = 0$

$$\int_1^2 \frac{x^3}{\sqrt{4-x^2}}dx = \int_{\sqrt{3}}^0 \frac{x^2}{\sqrt{4-x^2}}\left(x\frac{dx}{du}\right)du$$
$$= \int_{\sqrt{3}}^0 \frac{(4-u^2)}{u}(u\,du)$$
$$= \left[\frac{u^3}{3} - 4u\right]_{\sqrt{3}}^0$$
$$= -\frac{3\sqrt{3}}{3} + 4\sqrt{3}$$
$$= 3\sqrt{3}$$

Example 20

Using the substitution $u^2 = 3 - x$, or otherwise, find

$$\int (x+1)\sqrt{3-x}\,dx$$

Solution

Let $u^2 = 3 - x$, then $2u\frac{du}{dx} = -1$

$$dx = -2u\,du$$

$$\int (x+1)\sqrt{3-x}\,dx = \int (4-u^2)(u)(-2u)du$$
$$= -2\int (4u^2 - u^4)\,du$$
$$= -2\left(\frac{4}{3}u^3 - \frac{1}{5}u^5\right)$$
$$= -\frac{2}{15}u^3(20 - 3u^2)$$
$$= -\frac{2}{15}(3-x)^{\frac{3}{2}}(20 - 9 + 3x)$$
$$= -\frac{2}{15}(3-x)^{\frac{3}{2}}(11 + 3x) + c$$

Example 21

Using the substitution $t = \tan\frac{\theta}{2}$, or otherwise, evaluate

$$\int_0^{\frac{\pi}{2}} \frac{6}{1 + \sin\theta + 3\cos\theta}$$

Solution

Let $= \tan\frac{\theta}{2}$, then $\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2} = \frac{1}{2}(1+t^2)$

$$\frac{d\theta}{dt} = \frac{2}{(1+t^2)}$$

When $\theta = 0$, $t = 0$; when $\theta = \frac{\pi}{2}$, $t = 1$

$$\int_0^1 \frac{6}{1+\sin\theta+3\cos\theta}\frac{d\theta}{dt}\;dt$$
$$= \int_0^1 \left(\frac{6}{1 + \frac{2t}{1+t^2} + 3\frac{(1-t^2)}{1+t^2}}\right)\frac{2}{1+t^2}dt$$
$$= \int_0^1 \frac{12}{1 + t^2 + 2t + 3 - 3t^2}dt$$
$$= \int_0^1 \frac{6}{2 + t - t^2}dt$$

By partial fractions,

$$\frac{6}{2+t-t^2} = \frac{6}{(2-t)(1+t)} = \frac{2}{2-t} + \frac{2}{1+t}$$
$$\int_0^1 \frac{6}{2+t-t^2}dt = 2\int\left(\frac{1}{2-t} + \frac{1}{1+t}\right)dt$$
$$= 2[-\ln|2-t| + \ln|1+t|]_0^1$$
$$= 2(-\ln 1 + \ln 2) - 2(-\ln 2 + \ln 1)$$
$$= 4\ln 2 = 2.77$$

Example 22

By using the substitution $u = e^x$, or otherwise, evaluate

$$\int_0^1 \frac{e^{3x}}{1+e^{2x}}dx$$

Solution

$$u = e^x \quad \Rightarrow \quad \frac{du}{dx} = e^x = u$$
$$\int_0^1 \frac{e^{3x}}{1+e^{2x}}dx = \int_1^e \frac{u^2}{1+u^2}du$$

$$= \frac{1+t^2}{7(1+t^2) - 3(2t) + 6(1-t^2)}$$
$$= \frac{1+t^2}{7+7t^2-6t+6-6t^2}$$
$$= \frac{1+t^2}{t^2-6t+13}$$
$$\int \frac{dx}{7-3\sin x + 6\cos x} = \int \frac{1+t^2}{t^2-6t+13} \times \frac{2dt}{1+t^2}$$
$$= \int \frac{2\,dt}{t^2-6t+13}$$
$$= \int \frac{2dt}{(t-3)^2+2^2}$$
$$= 2\left[\frac{1}{2}\tan^{-1}\left(\frac{t-3}{2}\right)\right] + c$$
$$\int \frac{dx}{7-3\sin x+6\cos x} = \tan^{-1}\left(\frac{\tan\frac{x}{2} - 3}{2}\right) + c$$

Example 29

Evaluate $\displaystyle\int_0^{\pi/2} \frac{4}{3+5\cos\theta}\,d\theta$

Solution

Let $\tan\frac{\theta}{2} = t$

$$\frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta = dt$$
$$\left(1+\tan^2\frac{\theta}{2}\right)d\theta = 2dt$$
$$d\theta = \frac{2}{1+t^2}dt$$
$$3+5\cos\theta = 3+5\left(\frac{1-t^2}{1+t^2}\right) = \frac{3(1+t^2)+5(1-t^2)}{1+t^2}$$
$$= \frac{8-2t^2}{1+t^2}$$

Changing limits;

$\theta = \frac{\pi}{2}, \Rightarrow t = 1$

$\theta = 0 \ \Rightarrow t = 0$

$$\int_0^{\pi/2} \frac{4}{3+5\cos\theta}\,d\theta = \int_0^1 \frac{4(1+t^2)}{8-2t^2} \times \frac{2}{1+t^2}dt$$
$$= \int_0^1 \frac{4}{4-t^2}dt$$
$$= \int_0^1 \frac{4}{(2+t)(2-t)}dt$$
$$= \int_0^1 \left\{\frac{1}{2+t} + \frac{1}{2-t}\right\}dt$$
$$= [\ln(2+t) - \ln(2-t)]_0^1$$
$$= (\ln 3 - \ln 1) - (\ln 2 - \ln 2)$$
$$= \ln 3$$

The change of variable $t = \tan x$

An integrand containing $\sin x$ and $\cos x$, particularly even powers of these, may often be expressed as a function of $\tan x$ and $\sec x$. In such a case, the change of variable $t = \tan x$ is worth trying.

Example 30

Find $\displaystyle\int \frac{1}{1+\sin^2 x}dx$

Solution

Dividing both the numerator and denominator by $\cos^2 x$

$$\int \frac{1}{1+\sin^2 x}dx = \int \frac{\sec^2 x}{\sec^2 x + \tan^2 x}dx$$

Let $t = \tan x \Rightarrow \frac{dt}{dx} = \sec^2 x = 1+\tan^2 x = 1+t^2$

$$dx = \frac{1}{1+t^2}dt$$
$$\int \frac{1}{1+\sin^2 x}dx = \int \frac{\sec^2 x}{1+2\tan^2 x} \times \frac{1}{1+t^2}dt$$
$$= \int \frac{1+t^2}{1+2t^2} \times \frac{1}{1+t^2}dt$$
$$= \int \frac{1}{1+2t^2}dt$$
$$= \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}) + c$$
$$= \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x) + c$$

Self-Evaluation exercise

1. Carry out the following integrations by substitution

(a) $\int 4x(2x-1)^4\,dx$

[Ans: $\frac{1}{6}(2x-1)^6 + \frac{1}{5}(2x-1)^5 + c$]

(b) $\int \frac{2x}{2x+1}dx$

[Ans: $\frac{1}{2}(2x+1) - \frac{1}{2}\ln|2x+1| + c$]

(c) $\int x(4-x)^{-\frac{1}{2}}dx$

[Ans: $-(4-x^2)^{\frac{1}{2}} + c$]

(d) $\int \frac{8x}{\sqrt{4x-1}}dx$

[Ans: $\frac{1}{3}(4x-1)^{\frac{3}{2}} + (4x-1)^{\frac{1}{2}} + c$]

(e) $\int \frac{2x^2}{\sqrt{2x^3+1}}dx$

[Ans: $\frac{2}{3}(2x^3+1)^{\frac{1}{2}} + c$]

(f) $\int \frac{4-3x}{x+2}dx$

[Ans: $10\ln|x+2| - 3(x+2) + c$]

(g) $\int \frac{4x^2}{2x-1}dx$

[Ans: $\frac{1}{4}(2x-1)^2 + (2x-1) + \frac{1}{2}\ln|2x-1| + c$]

(h) $\int \sec^4 x\,dx$

[Ans: $\tan x + \frac{1}{3}\tan^3 x + c$]

Chapter 32 Integration of Trigonometric Functions

Introduction

Integrating trig functions is mainly a matter of recognizing the standard derivative and reversing it to find the standard integral. You need a very good working knowledge of the trig identities and be able to use the chain rule.

Integrals of $\sin x$, $\cos x$ and $\sec^2 x$

From the standard derivative of the basic trig functions, the integral can be found by reversing the process. Thus:

$$\frac{d}{dx}(\sin x) = \cos x \quad \Rightarrow \quad \int \cos x\, dx = \sin x + c$$

$$\frac{d}{dx}(\cos x) = -\sin x \quad \Rightarrow \quad \int \sin x\, dx = -\cos x + c$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \Rightarrow \quad \int \sec^2 x\, dx = \tan x + c$$

Using reverse differentiation

In a similar manner the following be found:

$y = f(x)$	$\int f(x)\, dx$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
$\sin kx$	$-\frac{1}{k}\cos kx + c$
$\cos kx$	$\frac{1}{k}\sin kx + c$
$\sec^2 kx$	$\frac{1}{k}\tan kx + c$
$\sec x \tan x$	$\sec x + c$
$\operatorname{cosec} x \cot x$	$-\operatorname{cosec} x + c$
$\operatorname{cosec}^2 x$	$-\cot x + c$
$\cot x$	$\ln\lvert\sin x\rvert$

Integrals of $\tan x$ and $\cot x$

To find the integrals, recognize the standard integral type:

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$$

Now

$$\tan x = \frac{\sin x}{\cos x}$$

$$\begin{aligned}\int \tan x\, dx &= \int \frac{\sin x}{\cos x} dx \\ &= -\int \frac{-\sin x}{\cos x} dx \\ &= -\ln|\cos x| + c \\ &= \ln|(\cos x)^{-1}| + c \\ &= \ln\left|\frac{1}{\cos x}\right| + c \\ &= \ln|\sec x| + c\end{aligned}$$

$$\int \tan x\, dx = -\ln|\cos x| + c = \ln|\sec x| + c$$

For the general case;

$$\int \tan ax\, dx = \frac{1}{a}\ln|\sec ax| + c$$

Similarly:

$$\begin{aligned}\int \cot x\, dx &= \int \frac{\cos x}{\sin x} dx \\ &= \ln|\sin x| + c\end{aligned}$$

$$\int \cot ax\, dx = \frac{1}{a}\ln|\sin x| + c$$

Recognising the opposite of the chain rule

Reversing the derivatives (found using the chain rule), the following can be derived;

$$\frac{d}{dx}\sin(ax+b) = a\cos(ax+b)$$

$$\Rightarrow \int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + c$$

$$\frac{d}{dx}\cos(ax+b) = -a\sin(ax+b)$$

$$\Rightarrow \int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b) + c$$

$$\frac{d}{dx}\tan(ax+b) = a\sec^2(ax+b)$$

$$\Rightarrow \int \sec^2(ax+b)\, dx = \frac{1}{a}\tan(ax+b) + c$$

Integrals of type: $\cos A \cos B$, $\sin A \cos B$ & $\sin A \sin B$

This type of problem covers the most questions. Use the addition (compound angle) trig identities.

Example 1

Find $\int \sin 3x \cos 4x\, dx$

Solution

Use formula: $2\sin A\cos B = \sin(A-B) + \sin(A+B)$

Let $A = 3x,\ B = 4x$

$$\begin{aligned}2\sin 3x\cos 4x &= \sin(3x-4x) + \sin(3x+4x) \\ &= \sin(-x) + \sin 7x\end{aligned}$$

$$\sin 3x \cos 4x = \frac{1}{2}(-\sin x + \sin 7x)$$

$$\begin{aligned}\int \sin 3x\cos 4x\, dx &= \frac{1}{2}\int -\sin x + \sin 7x\, dx \\ &= \frac{1}{2}\left(\cos x - \frac{1}{7}\cos 7x\right) + c \\ &= \frac{1}{2}\cos x - \frac{1}{14}\cos 7x + c\end{aligned}$$

Example 2

Find $\int \sin 4x \cos 4x\, dx$

Solution

Example 9

Find: $\int \sin^2(2x+3)\,dx$

Solution

$$\sin^2 A = \frac{1}{2}(1-\cos 2A)$$

$$\int \sin^2(2x+3)\,dx = \frac{1}{2}\int (1-\cos 2(2x+3))\,dx$$

$$= \frac{1}{2}\int (1-\cos(4x+6))dx$$

Recall: $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+c$

$$\int \sin^2(2x+3)\,dx = \frac{1}{2}\left[x-\frac{1}{4}\cos(4x+6)\right]+c$$

Integrals of the type: $\cos^n A \sin A$, $\sin^n A \cos A$

This is another example of applying the reverse of the differentiation and the chain rule.

From the chain rule, the derivative required is

$$\frac{d}{dx}(\sin^n x) = n\sin^{n-1}x\cos x$$

In reverse

$$\int \sin^n x\cos x\,dx = \frac{1}{n+1}\sin^{n+1}x + c$$

Similarly;

$$\int \cos^n x\sin x\,dx = -\frac{1}{n+1}\cos^{n+1}x + c$$

For example;

$$\int \sin^4 x\cos x\,dx = \frac{1}{5}\sin^5 x + c$$

$$\int \cos^7 x\sin x\,dx = \frac{1}{8}\cos^8 x + c$$

Three ways of integrating $\sin x\cos x$:

$$\int \sin x\cos x = \frac{1}{2}\sin^2 x + c \qquad (\sin^n x\cos x)$$

$$= -\frac{1}{2}\cos^2 x + c \qquad (\cos^n x\sin x)$$

$$= \int \frac{1}{2}\sin 2x\,dx$$

$$= -\frac{1}{4}\cos 2x + c$$

Example 10

Evaluate $\int_0^{\frac{\pi}{2}} \sin^2 x\cos^3 x\,dx$

Solution

$$\int_0^{\frac{\pi}{2}} \sin^2 x\cos^3 x\,dx = \int_0^{\frac{\pi}{2}} \sin^2 x\cos^2 x\cos x\,dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2 x)(1-\sin^2 x)(\cos x)dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 x\cos x - \sin^4 x\cos x\,dx$$

$$= \left[\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5}\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{\left(\sin\frac{\pi}{2}\right)^3}{3} - \frac{\left(\sin\frac{\pi}{2}\right)^5}{5}\right] - [0-0]$$

$$= \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$$

Example 11

Find $\int \sin^2 t\cos^4 t\,dt$

Solution

$$\sin^2 t\cos^4 t = \sin^2 t\,(\cos^2 t)^2$$

$$= \left(\frac{1-\cos 2t}{2}\right)\left(\frac{1+\cos 2t}{2}\right)^2$$

$$= \frac{1}{8}(1-\cos 2t)(1+2\cos 2t+\cos^2 2t)$$

$$= \frac{1}{8}(1+2\cos 2t+\cos^2 2t-\cos 2t-2\cos^2 2t - \cos^3 2t)$$

$$= \frac{1}{8}(1+\cos 2t-\cos^2 2t-\cos^3 2t)$$

$$= \frac{1}{8}\left[1+\cos 2t-\left(\frac{1+\cos 4t}{2}\right)-\cos 2t\,(1-\sin^2 2t)\right]$$

$$= \frac{1}{8}\left[\frac{1}{2}-\frac{\cos 4t}{2}+\cos 2t\sin^2 2t\right]$$

$$\int \sin^2 t\cos^4 t\,dt = \frac{1}{8}\int \frac{1}{2}-\frac{\cos 4t}{2}+\cos 2t\sin^2 2t\,dt$$

$$= \frac{1}{8}\left(\frac{t}{2}-\frac{\sin 4t}{8}+\frac{\sin^3 2t}{6}\right)+c$$

Integrating odd powers of $\sin x$ & $\cos x$

This technique is entirely different – change all but one of the sin/cos functions to the opposite by using the Pythagoras identity.

$$\cos^2 x + \sin^2 x = 1$$

Hence:

$$\sin^2 x = 1-\cos^2 x$$
$$\cos^2 x = 1-\sin^2 x$$

Example 12

Find: $\int \sin^3 x\,dx$

Solution

$$\int \sin^3 x\,dx = \int \sin x\sin^2 x\,dx$$

$$= \int \sin x\,(1-\cos^2 x)\,dx$$

$$= \int (\sin x - \cos^2 x\sin x)\,dx$$

Recognise standard type $\int \cos^n x\sin x\,dx$

$$\int \sin^3 x\,dx = -\cos x + \frac{1}{3}\cos^3 x + c$$

Chapter 33 Integration by Partial Fractions

Introduction

Sometimes expressions which at first sight look impossible to integrate using standard techniques may in fact be integrated by first expressing them as simpler partial fractions and then using earlier learned techniques. As explained in Chapter 5, the algebraic technique of resolving a complicated fraction into partial fractions is often useful in integration.

Using partial fractions in integration

The ideal format for integrating a fraction is:

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$$

Partial fractions give us the tool to tackle fractions that are not ideal in this form.

Example 1

Find $\int \frac{1}{x^2-1}\,dx$

Solution

$$\frac{1}{x^2-1} \equiv \frac{A}{x+1} + \frac{B}{x-1} \equiv \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$$

$$\therefore 1 = A(x-1) + B(x+1)$$

Let $x = 1 \quad \Rightarrow \quad 1 = 2B \quad \therefore\ B = \frac{1}{2}$

Let $x = -1 \quad \Rightarrow \quad 1 = -2A \quad \therefore\ A = -\frac{1}{2}$

$$\int \frac{1}{x^2-1}\,dx = \int \frac{1}{2(x-1)} - \frac{1}{2(x+1)}\,dx$$

$$= \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + c$$

$$= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + c$$

Example 2

Find $\int \frac{5(x+1)}{(x-1)(x+4)}\,dx$

Solution

$$\frac{5(x+1)}{(x-1)(x+4)} \equiv \frac{A}{x-1} + \frac{B}{x+4} \equiv \frac{A(x+4)+B(x-1)}{(x-1)(x+4)}$$

$$5(x+1) = A(x+4) + B(x-1)$$

Let $x = -4 \quad \Rightarrow \quad -15 = -5B \quad \therefore B = 3$

Let $x = 1 \quad \Rightarrow \quad 10 = 5A \quad \therefore A = 2$

$$\int \frac{5(x+1)}{(x-1)(x+4)}\,dx = \int \frac{2}{x-1}\,dx + \int \frac{3}{x+4}\,dx$$

$$= 2\int \frac{1}{x-1}\,dx + 3\int \frac{1}{x+4}\,dx$$

$$= 2\ln|x-1| + 3\ln|x+4| + c$$

Example 3

Calculate the value of $\int_1^4 \frac{1}{x(x-5)}\,dx$

Solution

$$\text{Let } \frac{1}{x(x-5)} \equiv \frac{A}{x} + \frac{B}{x-5} \equiv \frac{A(x-5)+Bx}{x(x-5)}$$

$$1 = A(x-5) + Bx$$

Let $x = 5, \quad 5B = 1 \Rightarrow B = \frac{1}{5}$

Let $x = 0, \quad -5A = 1 \Rightarrow A = -\frac{1}{5}$

$$\therefore \frac{1}{x(x-5)} = -\frac{1}{5x} + \frac{1}{5(x-5)}$$

$$\int_1^4 \frac{1}{x(x-5)}\,dx = \frac{1}{5}\int_1^4 \left(-\frac{1}{x}\right) + \frac{1}{(x-5)}\,dx$$

$$= \frac{1}{5}[-\ln|x| + \ln|x-5|]_1^4$$

$$= \frac{1}{5}[(-\ln 4 + \ln|4-5|) - (-\ln 1 + \ln|1-5|)]$$

$$= \frac{1}{5}[(-\ln 4 + \ln 1) - (\ln 1 - \ln 4)]$$

$$= \frac{1}{5}(-2\ln 4)$$

$$= -\frac{2}{5}\ln 4$$

$$= \frac{1}{5}\ln\left(\frac{1}{16}\right)$$

Example 4

Find $\int \frac{1}{(x+1)(2x+3)}\,dx$

Solution

$$\frac{1}{(x+1)(2x+3)} \equiv \frac{A}{(x+1)} + \frac{B}{(2x+3)}$$

$$= \frac{A(2x+3)+B(x+1)}{(x+1)(2x+3)}$$

$$1 = A(2x+3) + B(x+1)$$

$x = -\frac{3}{2}, \quad -\frac{1}{2}B = 1 \Rightarrow \quad B = -2$

$x = -1, \quad -2A + 3 = 1 \Rightarrow A = 1$

$$\frac{1}{(x+1)(2x+3)} = \frac{1}{(x+1)} - \frac{2}{(2x+3)}$$

$$\int \frac{1}{(x+1)(2x+3)}\,dx = \int \frac{1}{(x+1)} - \frac{2}{(2x+3)}\,dx$$

$$= \ln(x+1) - \frac{2}{2}\ln(2x+3) + c$$

$$= \ln\left[\frac{x+1}{2x+3}\right] + c$$

$$= \left[3\ln(x+3) + \frac{2}{(x+3)} + \frac{3}{(x+3)^2}\right]_{-2}^{1}$$
$$= \left(3\ln 4 + \frac{2}{4} + \frac{3}{16}\right) - \left(3\ln 1 + \frac{2}{1} + \frac{3}{1}\right)$$
$$= -0.1536$$

Example 11

Find $\int \frac{3+6x+4x^2-2x^3}{x^2(x^2+3)}dx$

Solution

$$\frac{3+6x+4x^2-2x^3}{x^2(x^2+3)} \equiv \frac{2}{x} + \frac{1}{x^2} + \frac{3-4x}{(x^2+3)}$$
$$\int \frac{3+6x+4x^2-2x^3}{x^2(x^2+3)}dx = \int \left\{\frac{2}{x} + \frac{1}{x^2} + \frac{3-4x}{(x^2+3)}\right\}dx$$
$$= \int \left\{\frac{2}{x} + \frac{1}{x^2} + \frac{3}{(x^2+3)} - \frac{4x}{(x^2+3)}\right\}dx$$
$$\int \frac{3}{(x^2+3)}dx = 3\int \frac{1}{x^2+(\sqrt{3})^2}dx = \frac{3}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}$$

Hence

$$\int \left\{\frac{2}{x} + \frac{1}{x^2} + \frac{3}{(x^2+3)} - \frac{4x}{(x^2+3)}\right\}dx$$
$$= 2\ln x - \frac{1}{x} + \frac{3}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 2\ln(x^2+3) + c$$
$$= \ln\left(\frac{x}{x^2+3}\right)^2 - \frac{1}{x} + \sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}} + c$$

Example 12

Determine $\int \frac{1}{(x^2-a^2)}dx$

Solution

$$\text{Let } \frac{1}{(x^2-a^2)} \equiv \frac{A}{(x-a)} + \frac{B}{(x+a)}$$
$$\equiv \frac{A(x+a)+B(x-a)}{(x+a)(x-a)}$$

Equating the numerators gives:

$$1 = A(x+a) + B(x-a)$$

Let $x = a$, then $A = \frac{1}{2a}$ and let $x = -a$, then $B = -\frac{1}{2a}$

$$\int \frac{1}{(x^2-a^2)}dx = \int \frac{1}{2a}\left[\frac{1}{(x-a)} - \frac{1}{(x+a)}\right]dx$$
$$= \frac{1}{2a}[\ln(x-a) - \ln(x+a)] + c$$
$$= \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + c$$

Example 13

Find $\int \frac{5x+7}{x^2+4x+8}dx$

Solution

$$\frac{d}{dx}(x^2+4x+8) = 2x+4$$

Let $5x+7 \equiv A(2x+4) + B$

$$\Rightarrow A = \tfrac{5}{2}, B = -3$$
$$\frac{5x+7}{x^2+4x+8} = \frac{\frac{5}{2}(2x+4)-3}{x^2+4x+8}$$
$$\int \frac{5x+7}{x^2+4x+8}dx = \int \left\{\frac{\frac{5}{2}(2x+4)}{x^2+4x+8} - \frac{3}{x^2+4x+8}\right\}dx$$
$$= \frac{5}{2}\ln(x^2+4x+8) - 3\int \frac{1}{(x+2)^2+4}dx$$
$$= \frac{5}{2}\ln(x^2+4x+8) - \frac{3}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + c$$

Example 14

Evaluate $\int_0^1 \frac{x+1}{x^2+x+1}dx$

Solution

$$\frac{d}{dx}(x^2+x+1) = 2x+1 \text{ and } x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$$

Now

$$x+1 = \frac{1}{2}(2x+1) + \frac{1}{2}$$

Hence

$$\int_0^1 \frac{x+1}{x^2+x+1}dx = \int_0^1 \left[\frac{\frac{1}{2}(2x+1)}{x^2+x+1} + \frac{\frac{1}{2}}{x^2+x+1}\right]dx$$
$$= \left[\frac{1}{2}\ln(x^2+x+1)\right]_0^1 + \frac{1}{2}\int_0^1 \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx$$
$$= \frac{1}{2}\ln 3 + \frac{1}{2}\left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\right]_0^1$$
$$= \frac{1}{2}\ln 3 + \left(\frac{1}{\sqrt{3}}\tan^{-1}\sqrt{3} - \frac{1}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}\right)$$
$$= \frac{1}{2}\ln 3 + \frac{\pi}{6\sqrt{3}}$$

Example 15

Find $\int \frac{x-1}{x+1}dx$

Solution

$$\frac{x-1}{x+1} = 1 - \frac{2}{x+1}$$
$$\int \frac{x-1}{x+1}dx = \int 1 - \frac{2}{x+1}dx$$
$$= x - 2\ln|x+1| + c$$

Example 16

Find $\int \frac{x^3+2}{x-1}dx$

Solution

$$\frac{x^3+2}{x-1} \equiv \frac{x^3-1+3}{x-1} = \frac{x^3-1}{x-1} + \frac{3}{x-1}$$

$$= \frac{(x-1)(x^2+x+1)}{(x-1)} + \frac{3}{x-1}$$
$$= x^2+x+1+\frac{3}{x-1}$$
$$\int \frac{x^3+2}{x-1}dx = \int x^2+x+1+\frac{3}{x-1}dx$$
$$= \frac{x^3}{3}+\frac{x^2}{2}+x+3\ln(x-1)+c$$

Self-Evaluative exercise

1. Find the following integrals
   (a) $\int \frac{17-4x}{(x-2)(x+1)}dx$ [Ans: $3\ln|x-2|-7\ln|x+1|+c$]
   (b) $\int \frac{18x-1}{(2x+1)(3x-1)}dx$ [Ans: $2\ln|2x+1|+\ln|3x-1|+c$]
   (c) $\int \frac{7x-9}{x^2-2x-15}dx$ [Ans: $5\ln|x+3|+2\ln|x-5|+c$]
   (d) $\int \frac{x^2+14x+1}{(x+3)(x-5)(x+7)}dx$ [Ans: $\ln\left|\frac{x^2-2x-15}{x+7}\right|+c$]
   (e) $\int \frac{7x+4}{(x-2)(x+1)^2}dx$ [Ans: $2\ln\left|\frac{x-2}{x+1}\right|-\frac{1}{x+1}+c$]
   (f) $\int \frac{2x^2+x+8}{(x-2)(x+1)^2}dx$ [Ans: $2\ln|x-2|+\frac{3}{x+1}+c$]
   (g) $\int \frac{x+1}{9x^2-1}dx$ [Ans: $\frac{2}{9}\ln|3x-1|-\frac{1}{9}\ln|3x+1|+c$]
   (h) $\int \frac{17-5x}{(2x+3)(2-x)^2}dx$ [Ans: $\ln\left|\frac{2x+3}{2-x}\right|+\frac{1}{2-x}+c$]
   (i) $\int \frac{4x^2-6x+5}{(2-x)(2x-1)^2}dx$ [Ans: $-\frac{1}{2x-1}-\ln|2-x|+c$]
   (j) $\int \frac{10x^2-23x+11}{(2-3x)(2x-1)^2}dx$ [Ans: $-\frac{2}{2x-1}-\frac{1}{3}\ln|2-3x|-\frac{1}{2}\ln|2x-1|+c$]
   (k) $\int \frac{1}{x^2(x-1)}dx$ [Ans: $\frac{1}{x}+\ln\left|\frac{x-1}{x}\right|+c$]
   (l) $\int \frac{8(x^2+1)}{(x-3)(x+1)^2}dx$ [Ans: $5\ln|x-3|+3\ln|x+1|+\frac{4}{x+1}+c$]
   (m) $\int \frac{4x^2-x+1}{(x-1)(2x-1)}dx$ [Ans: $2x+4\ln|x-1|-\frac{3}{2}\ln|2x-1|+c$]
   (n) $\int \frac{2}{x(x^2-1)}dx$ [Ans: $\ln\left|\frac{x^2-1}{x^2}\right|+c$]
   (o) $\int \frac{2x^2+5x-1}{x^3+x^2-2x}dx$ [Ans: $2\ln|x-1|+\frac{1}{2}\ln\left|\frac{x}{x+2}\right|+c$]
   (p) $\int \frac{2}{2x-x^2}dx$ [Ans: $\ln\left|\frac{2x}{2-x}\right|+c$]
2. Evaluate the following integrals
   (a) $\int_{-1}^{1} \frac{9+4x^2}{9-4x^2}dx$ [Ans: $-2+3\ln 5$]
   (b) $\int_0^1 \frac{18-4x-x^2}{(4-3x)(1+x)^2}dx$ [Ans: $\frac{7}{3}\ln 2+\frac{3}{2}$]
   (c) $\int_2^3 \frac{x^2+x+2}{x^2+2x-3}dx$ [Ans: $1+\ln\left(\frac{25}{18}\right)$]
   (d) $\int_0^{\frac{1}{4}} \frac{4}{(2x+1)(1-2x)}dx$ [Ans: $\ln 3$]
   (e) $\int_0^1 \frac{17-5x}{(3+2x)(2-x)^2}dx$ [Ans: $\frac{1}{2}+\ln\left(\frac{10}{3}\right)$]
   (f) $\int_4^9 \frac{5x^2-8x+1}{2x(x-1)^2}dx$ [Ans: $\ln\left(\frac{32}{3}\right)-\frac{5}{24}$]
   (g) $\int_0^1 \frac{x^2}{x^2-4}dx$ [Ans: $1-\ln 3$]
   (h) $\int_0^1 \frac{10}{(x+1)(x+3)(2x+1)}dx$ [Ans: $3\ln 3-3\ln 2$]
   (i) $\int_0^4 \frac{13-2x}{(x+4)(2x+1)}dx$ [Ans: $4\ln 3-3\ln 2$]
   (j) $\int_2^6 \frac{2x^2-x+11}{(x+2)(2x-3)}dx$ [Ans: $4+4\ln 3-3\ln 2$]
   (k) $\int_0^2 \frac{25x+1}{(2x-1)(x+1)^2}dx$ [Ans: $\frac{16}{3}$]
   (l) $\int_5^8 \frac{2x^2}{x^2-16}dx$ [Ans: $6+4\ln 3$]
   (m) $\int_2^3 \frac{x^2-3x+5}{(4-x)(1-x)^2}dx$ [Ans: $\frac{1}{2}+\ln 2$]
   (n) $\int_0^2 \frac{4x^3-12x^2-22x-3}{(4-x)(2x+1)}dx$ [Ans: $\frac{1}{2}\ln\left(\frac{5}{64}\right)-6$]
   (o) $\int_0^1 \frac{4t^2+9t+8}{(t+2)(t+1)^2}dt$ [Ans: 2.546]
   (p) $\int_1^2 \frac{2+\theta+6\theta^2-2\theta^3}{\theta(\theta^2+1)}d\theta$ [Ans: 1.606]

Chapter 34 Integration by Parts

Introduction
This is the equivalent of the product rule for integration. It is usually used when the product we want to integrate is not of the form $f'(x)\big(f(x)\big)^n$ and so cannot be integrated with this standard method, or by recognition or by substitution.
Integrating by parts is particularly useful for integrating the product of two types of function, such a polynomial with a trig, exponential or log function (e.g. $x\sin x$, x^2e^x, $\ln x$)

Rearranging the product rule
The rule for integrating by Parts comes from integrating the product rule.

$$\text{Product rule: } \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating w.r.t. x to get:

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$$uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

Rearranging:

$$\int \boldsymbol{u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx}$$

Choice of u & dv/dx
Care must be taken over the choice of u and dv/dx.
The aim is to ensure that it is simpler to integrate $v\frac{du}{dx}$ than the original $u\frac{dv}{dx}$. So we choose u to be easy to differentiate and when differentiated to become simpler. Choose dv/dx to be easy to integrate.
Normally, u is assigned to any polynomial in x, and if any exponential function is involved, assign this to $\frac{dv}{dx}$. However, if $\ln x$ is involved make this u, as it is easier to differentiate $\ln[f(x)]$ function than to integrate it.

Evaluating the Definite integral by parts
Use this for substituting the limits:

$$\int_a^b u\frac{dv}{dx}\,dx = [uv]_a^b - \int_a^b v\frac{du}{dx}\,dx$$

Handling the constant of integration
The method listed above suggests adding the constant of integration at the end of the calculation. Why is this?
The best way to explain this is to show an example of adding a constant after each integration, and you can see that the first one cancels out during the calculation.

Example 1

Find: $\int x\sin x\,dx$

Solution

Let $u = x$ and $\frac{dv}{dx} = \sin x$

$\frac{du}{dx} = 1$ $\quad v = \int\frac{dv}{dx} = -\cos x + k$

where k is the constant from the first integration and c is the constant from the second integrations

$$\text{Recall: } \int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int x\sin x\,dx = x(-\cos x + k) - \int(-\cos x + k)\times 1dx$$

$$= -x\cos x + kx + \int\cos x\,dx - \int k\,dx$$

$$= -x\cos x + kx + \sin x - kx + c$$

$$= -x\cos x + \sin x + c$$

Example 2

Find: $\int x\cos x\,dx$

Solution

Let $u = x$ and $\frac{dv}{dx} = \cos x$

$\frac{du}{dx} = 1$ $\quad v = \int\frac{dv}{dx} = \int\cos x = \sin x$

$$\int x\cos x\,dx = x\sin x - \int\sin x\times 1\,dx$$

$$= x\sin x + \cos x + c$$

Now look at this situation;

Let $u = \cos x$ and $\frac{dv}{dx} = x$

$\frac{du}{dx} = -\sin x$ $\quad v = \frac{x^2}{2}$

$$\int x\cos x\,dx = \sin x\cdot\frac{x^2}{2} - \int\frac{x^2}{2}(-\sin x)dx$$

$$= \frac{x^2}{2}\sin x + \int\frac{x^2}{2}\sin x\,dx$$

As you can see, this gives a more involved solution, that has another round of integration by parts. This emphasizes the importance of choosing u wisely. In this case it would be prudent to start again with $u = x$.

Solution

Following the guidelines on choice of u & $\frac{dv}{dx}$, then we would let $u = x$ and $\frac{dv}{dx} = \ln x$. However, $\ln x$ is difficult to integrate, so choose $u = \ln x$

Let $u = \ln x$ and $\frac{dv}{dx} = x^4$

$\frac{du}{dx} = \frac{1}{x}$ $v = \frac{1}{5}x^5$

$$\int x^4 \ln x \, dx = \ln x \cdot \frac{1}{5}x^5 - \int \frac{1}{5}x^5 \cdot \frac{1}{x} dx$$
$$= \frac{1}{5}x^5 \ln x - \frac{1}{5}\int x^4 \, dx$$
$$= \frac{1}{5}x^5 \ln x - \frac{1}{5} \times \frac{1}{5}x^5 + c$$
$$= \frac{1}{5}x^5 \ln x - \frac{1}{25}x^5 + c$$
$$\int x^4 \ln x \, dx = \frac{1}{25}x^5(5\ln x - 1) + c$$

Example 10

Evaluate: $\int_2^8 x \ln x \, dx$

Solution

Let $u = \ln x$ and $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{x}$ $v = \frac{x^2}{2}$

$$\int_2^8 x \ln x \, dx = \left[\frac{x^2}{2}\ln x\right]_2^8 - \int_2^8 \frac{x^2}{2} \cdot \frac{1}{x} dx$$
$$= \left[\frac{x^2}{2}\ln x\right]_2^8 - \int_2^8 \frac{x}{2} dx$$
$$= \left[\frac{x^2}{2}\ln x\right]_2^8 - \left[\frac{x^2}{4}\right]_2^8$$
$$= \left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_2^8$$
$$= (32\ln 8 - 16)(2\ln 2 - 1)$$
$$= 32\ln 8 - 2\ln 2 - 15$$
$$= 32\ln 2^3 - 2\ln 2 - 15$$
$$= 9\ln 2 - 2\ln 2 - 15$$
$$\int_2^8 x \ln x \, dx = 94\ln 2 - 15$$

Example 11

Find: $\int \sqrt{x} \ln x \, dx$

Solution

Let $u = \ln x$ and $\frac{dv}{dx} = \sqrt{x}$

$\frac{du}{dx} = \frac{1}{x}$ $v = \int \sqrt{x} = \frac{2}{3}x^{\frac{3}{2}}$

$$\int \sqrt{x} \ln x \, dx = \ln x \cdot \frac{2}{3}x^{\frac{2}{3}} - \int \frac{2}{3}x^{\frac{3}{2}} \cdot \frac{1}{x} dx$$
$$= \frac{2}{3}x^{\frac{3}{2}} \ln x - \frac{2}{3}\int x^{\frac{1}{2}} \, dx$$
$$= \frac{2}{3}x^{\frac{3}{2}} \ln x - \frac{2}{3} \times \frac{2}{3}x^{\frac{3}{2}} + c$$
$$= \frac{2}{9}\sqrt{x^3}(3\ln x - 2) + c$$

Example 12

Evaluate $\int_1^3 x^3 \ln x \, dx$

Solution

Let $u = \ln x$ $\frac{dv}{dx} = x^3$

$\frac{du}{dx} = \frac{1}{x}$ $v = \frac{x^4}{4}$

$$\int_1^3 x^3 \ln x \, dx = \left[\left(\frac{x^4}{4}\right)(\ln x)\right]_1^3 - \int_1^3 \left(\frac{x^4}{4}\right)\left(\frac{1}{x}\right) dx$$
$$= \left[\frac{x^4}{4}\ln x - \frac{x^4}{16}\right]_1^3$$
$$= \frac{81}{4}\ln 3 - \frac{81}{16} - \frac{1}{4}\ln 1 + \frac{1}{16}$$
$$= \frac{81}{4}\ln 3 - 5 = 17.2$$

Integration by Parts: Special cases

These next examples use the integration by parts twice, which generates a term that is the same as the original integral. This term can then be moved to the L.H.S, to give the final result by division.

Generally used for integrals of the form $e^{ax}\sin bx$ or $e^{ax}\cos bx$. In this form, the choice of u and $\frac{dv}{dx}$ does not matter.

Example 13

Find: $\int \frac{\ln x}{x} dx$

Solution

Let $u = \ln x$ and $\frac{dv}{dx} = \frac{1}{x}$

$\frac{du}{dx} = \frac{1}{x}$ $v = \ln x$

$$\int \frac{\ln x}{x} dx = \ln x \cdot \ln x - \int \ln x \cdot \frac{1}{x} dx$$
$$\int \frac{\ln x}{x} dx = (\ln x)^2 - \int \frac{\ln x}{x} dx$$
$$2\int \frac{\ln x}{x} dx = (\ln x)^2$$
$$\therefore \int \frac{\ln x}{x} dx = \frac{1}{2}(\ln x)^2 + c$$

Note: $(\ln x)^2$ is not the same as $\ln x^2$

Example 14

Find: $\int x^2 \sin x \, dx$

$$u = \sin x \quad \text{and} \quad \frac{dv}{dx} = e^x$$
$$\frac{du}{dx} = \cos x \qquad v = e^x$$
$$\int e^x \cos x\, dx = e^x \cos x + \left[\sin x \,.\, e^x - \int e^x \cos x\, dx\right]$$
$$\int e^x \cos x\, dx = e^x(\cos x + \sin x) - \int e^x \cos x\, dx$$
$$2\int e^x \cos x\, dx = e^x(\cos x + \sin x) + c$$
$$\therefore \int e^x \cos x\, dx = \frac{1}{2}e^x(\cos x + \sin x) + c$$

Example 18

Find: $\int e^{2x} \sin 4x\, dx$

Solution

Let: $u = \sin 4x$ and $\frac{dv}{dx} = e^{2x}$

$$\frac{du}{dx} = 4\cos 4x \qquad v = \frac{1}{2}e^{2x}$$
$$\int e^{2x}\sin 4x\, dx = \sin 4x \cdot \frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}\cdot 4\cos 4x\, dx$$
$$\int e^{2x}\sin 4x\, dx = \frac{1}{2}e^{2x}\sin 4x - 2\int e^{2x}\cos 4x\, dx$$

Now integrate by parts again and then one final integration;

$$u = \cos 4x \qquad \frac{dv}{dx} = e^{2x}$$
$$\frac{du}{dx} = -4\sin 4x \qquad v = \frac{1}{2}e^{2x}$$
$$\int e^{2x}\sin 4x\, dx$$
$$= \frac{1}{2}e^{2x}\sin 4x - 2\left[\cos 4x \cdot \frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}\cdot(-4\sin 4x)\, dx\right]$$
$$= \frac{1}{2}e^{2x}\sin 4x - 2\left[\frac{1}{2}e^{2x}\cos 4x + 2\int e^{2x}\sin 4x\, dx\right]$$
$$= \frac{1}{2}e^{2x}\sin 4x - e^{2x}\cos 4x - 4\int e^{2x}\sin 4x\, dx$$
$$5\int e^{2x}\sin 4x\, dx = \frac{1}{2}e^{2x}\sin 4x - e^{2x}\cos 4x + c$$
$$5\int e^{2x}\sin 4x\, dx = \frac{1}{2}e^{2x}(\sin 4x - 2\cos 4x) + c$$
$$\int e^{2x}\sin 4x\, dx = \frac{1}{10}e^{2x}(\sin 4x - 2\cos 4x) + c$$

Example 19

Find: $\int x^2 e^{4x}\, dx$

Solution

Let $u = x^2$ and $\frac{dv}{dx} = e^{4x}$

$$\frac{du}{dx} = 2x \qquad v = \frac{1}{4}e^{4x}$$
$$\int x^2 e^{4x}\, dx = x^2 \cdot \frac{1}{4}e^{4x} - \int \frac{1}{4}e^{4x}\cdot 2x\, dx$$
$$= \frac{1}{4}x^2 e^{4x} - \frac{1}{2}\int xe^{4x}\, dx$$
$$= \frac{1}{4}x^2 e^{4x} - \frac{1}{2}\int u\frac{dv}{dx}dx$$

Now integrate by parts again and then one final integration;

Let $u = x$ and $\frac{dv}{dx} = e^{4x}$

$$\frac{du}{dx} = 1 \qquad v = \frac{1}{4}e^{4x}$$
$$\int xe^{4x}dx = x\cdot\frac{1}{4}e^{4x} - \int \frac{1}{4}e^{4x}\, dx$$
$$= \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x}$$

Substituting back into the original

$$\int x^2 e^{4x}\, dx = \frac{1}{4}x^2 e^{4x} - \frac{1}{2}\left(\frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x}\right) + c$$
$$= \frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{32}e^{4x} + c$$
$$= e^{4x}\left(\frac{1}{4}x^2 - \frac{1}{8}x + \frac{1}{32}\right) + c$$
$$\int x^2e^{4x}\, dx = \frac{1}{32}e^{4x}(8x^2 - 4x + 1) + c$$

Example 20

Evaluate $\int_0^1 x\tan^{-1}x\, dx$, giving the answer to 2 s.f

Solution

Let $u = \tan^{-1}x$ and $\frac{dv}{dx} = x^2$

$$\frac{du}{dx} = \frac{1}{1+x^2} \qquad v = \frac{x^3}{3}$$
$$\int_0^1 x\tan^{-1}x\, dx = \left[\frac{x^3}{3}\tan^{-1}x\right]_0^1 - \int \frac{x^3}{3}\frac{1}{1+x^2}dx$$
$$= \left(\frac{1}{3}\right)\left(\frac{\pi}{4}\right) - \frac{1}{3}\int_0^1\left(x - \frac{x}{1+x^2}\right)dx$$
$$= \frac{\pi}{12} - \frac{1}{3}\left[\frac{x^2}{2} - \frac{1}{2}\ln(1+x^2)\right]_0^1$$
$$= \frac{\pi}{12} - \frac{1}{6} + \frac{1}{6}\ln 2 = 0.2$$

Example 21

Find: $\int x^2\sin^{-1}x\, dx$

Solution

By parts;

Let $u = \sin^{-1}x$ $\qquad \frac{dv}{dx} = x^2$

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad v = \frac{x^3}{3}$$
$$\int x^2\sin^{-1}x\, dx = \frac{x^3}{3}\sin^{-1}x - \int\left(\frac{x^3}{3}\right)\frac{1}{\sqrt{1-x^2}}\ dx$$

Let $1 - x^2 = w$, then $-2x\frac{dx}{dw} = 2w \Rightarrow x\frac{dx}{dw} = -w$

$$\int \frac{x^3dx}{3\sqrt{1-x^2}} = \frac{1}{3}\int\frac{(1-w^2)}{w}(-w)dw = -\frac{1}{3}\int(1-w^2)\,dw$$
$$= -\frac{w}{9}(3-w^2) = -\frac{1}{9}(2+x^2)\sqrt{1-x^2}$$
$$\int x^2\sin^{-1}x\, dx = \frac{x^3}{3}\sin^{-1}x + \left(\frac{2+x^2}{9}\right)\sqrt{1-x^2} + c$$

Example 22

Find $\int x\sin 3x\cos 2x\,dx$

Solution

From $\sin A\cos B=\frac{1}{2}\{\sin(A+B)+\sin(A-B)\}$

$$\sin 3x\cos 2x=\tfrac{1}{2}[\sin(3x+2x)+\sin(3x-2x)]$$
$$=\frac{1}{2}[\sin 5x+\sin x]$$
$$\int x\sin 3x\cos 2x\,dx=\frac{1}{2}\int x(\sin 5x+\sin x)\,dx$$

Applying integrating by parts;

$$u=x \qquad \frac{dv}{dx}=\sin 5x+\sin x$$
$$\frac{du}{dx}=1 \qquad v=-\frac{\cos 5x}{5}-\cos x$$
$$\int x\sin 3x\cos 2x\,dx$$
$$=\tfrac{1}{2}\left[x\left(-\tfrac{\cos 5x}{5}-\cos x\right)-\int\left(-\tfrac{\cos 5x}{5}-\cos x\right)dx\right]$$
$$=\frac{1}{2}\left[x\left(-\frac{\cos 5x}{5}-\cos x\right)+\int\left(\frac{\cos 5x}{5}+\cos x\right)dx\right]$$
$$=\frac{1}{2}\left[-x\left(\frac{\cos 5x}{5}+\cos x\right)+\left(\frac{\sin 5x}{5\times 5}+\sin x\right)\right]+c$$
$$\int x\sin 3x\cos 2x\,dx$$
$$=\frac{1}{2}\left[-x\left(\frac{\cos 5x}{5}+\cos x\right)+\frac{\sin 5x}{25}+\sin x\right]+c$$

Example 23

Find $\int x\,5^x\,dx$

Solution

Let $u=x$ and $\frac{dv}{dx}=5^x$

$$\frac{du}{dx}=1 \qquad v=\frac{5^x}{\ln 5}$$
$$\int x\,5^x\,dx=\frac{x\,5^x}{\ln 5}-\int\frac{5^x}{\ln 5}dx$$
$$=\frac{x\,5^x}{\ln 5}-\frac{1}{\ln 5}\cdot\frac{5^x}{\ln 5}+c$$
$$\int x\,5^x\,dx=\frac{x\,5^x}{\ln 5}-\frac{5^x}{(\ln 5)^2}+c$$

Example 24

Find $\int\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$

Solution

Let $\sin^{-1}x=t\ \Rightarrow x=\sin t$

$$dx=\cos t\,dt$$
$$dx=\sqrt{1-\sin^2 t}\,dt$$
$$dx=\sqrt{1-x^2}dt$$

Now

$$\int\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx=\int x\frac{t}{\sqrt{1-x^2}}\left(\sqrt{1-x^2}dt\right)$$
$$=\int xt\ dt$$
$$=\int t\sin t\,dt$$

Applying integration by parts;

$$u=t \quad\text{and}\quad \frac{dv}{dt}=\sin t$$
$$\frac{du}{dt}=1 \qquad v=-\cos t$$
$$\int\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx=\int t\sin t\,dt=t(-\cos t)-\int(-\cos t)\,dt$$
$$=-t\cos t+\int\cos t\,dt$$
$$=-t\cos t+\sin t+c$$
$$=-(\sin^{-1}x)\left(\sqrt{1-x^2}\right)+x+c$$
$$\int\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx=x-\sqrt{1-x^2}\sin^{-1}x+c$$

Example 25

Find $\int\tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$

Solution

Let $x=\tan\theta\ \Rightarrow dx=\sec^2\theta\,d\theta$

$$\int\tan^{-1}\left(\frac{2x}{1-x^2}\right)dx=\int\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)\sec^2\theta\,d\theta$$
$$=\int\tan^{-1}(\tan 2\theta)\sec^2\theta\,d\theta$$
$$=\int 2\theta\sec^2\theta\,d\theta$$

Applying integration by parts;

Let $u=2\theta$ and $\frac{dv}{d\theta}=\sec^2\theta$

$$\frac{du}{d\theta}=2 \qquad v=\tan\theta$$
$$\int 2\theta\sec^2\theta\,d\theta=2\theta\tan\theta-\int 2\tan\theta\,d\theta$$
$$=2\theta\tan\theta-2\ln\sec\theta+c$$
$$\int\tan^{-1}\left(\frac{2x}{1-x^2}\right)dx=2x\tan^{-1}x-2\ln\sqrt{1+x^2}+c$$

Example 26

Find $\int\sec^3 x\,dx$

Solution

$$\int\sec^3 x\,dx=\int(\sec x)(\sec^2 x)\,dx$$

Applying integration by parts;

Let $u=\sec x$ and $\frac{dv}{dx}=\sec^2 x$

$$\frac{du}{dx}=\sec x\tan x \qquad v=\tan x$$
$$\int\sec^3 x\,dx=\sec x\tan x-\int(\sec x)(\sec x\tan x)\,dx$$
$$=\sec x\tan x-\int\tan^2 x\sec x\,dx$$
$$=\sec x\tan x-\int(\sec^2 x-1)\sec x\,dx$$

Chapter 35 Differential Equations

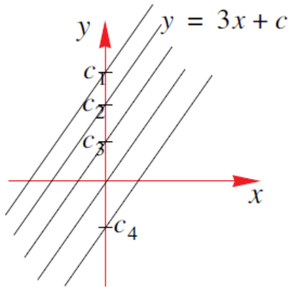
Introduction

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation (often, simply referred to as a d.e.). If this differential equation involves ordinary derivatives of one or more dependent variables with respect to a single independent variable, then we have an ordinary differential equation.

Two examples of differential equations are $\frac{dy}{dx} = 4x$, $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 1 = 0$.

A solution to such equations is an equation relating x and y and containing no differential coefficients)

For example, if $\frac{dy}{dx} = 3$, we obtain the **general solution** $y = 3x + c$. From a graphical point of view, a differential equation describes a property of a family of curves. In our case, it describes a family of curves whose gradient is always 3. In turn, we obtain a general solution $y = 3x + c$, which represents the family of straight line curves with gradient 3.



A particular solution is the equation of one particular member of that family of curves. In our case, if we also know that $x = 1, y = 5$, we would then have the particular solution $y = 3x + 2$ (i.e. solving for x we have: $5 = 3(1) + c \Rightarrow c = 2$)

In general,

1. A differential equation defines some property common to a family of curves.
2. The general solution, involving one or more arbitrary constants, is the equation of any member of the family.
3. A particular solution is the equation to only one member of the family.

Order and degree of a differential equation

The **order** of a differential equation is determined by the highest differential coefficient. The following serve as examples of differential equations of different order:

1. $\frac{dy}{dx} = 4x$ is of order 1
2. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 1 = 0$ is of order 2
3. $\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right) = 2$ is of order 1

The **degree** of a differential equation is the degree of the highest order derivative which occurs in it, after the derivative has been made free from radicals and fractions as far as the derivatives are concerned.

The degree of a differential equation does not require variables $r, s, t, \ldots$ to be free from radicals and fractions.

Example 1

Find the order and degree of the following differential equations.

(a) $\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^5 + y = 7$

(b) $y = 4\frac{dy}{dx} + 3x\frac{dx}{dy}$

(c) $\frac{d^2y}{dx^2} = \left[4 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{4}}$

(d) $(1 + y')^2 = y'^2$

Solution

(a) The order of the highest derivative in this equation is 3. The degree of the highest order is 1.

$$\therefore \text{(order, degree)} = (3, 1)$$

(b) $y = 4\frac{dy}{dx} + 3x\frac{dx}{dy}$

$$\Rightarrow y = 4\left(\frac{dy}{dx}\right) + 3x\frac{1}{\left(\frac{dy}{dx}\right)}$$

Making the above equation free from fractions involving $\frac{dy}{dx}$, we get

$$y \cdot \frac{dy}{dx} = 4\left(\frac{dy}{dx}\right)^2 + 3x$$

Highest order = 1

Degree of highest order = 2

(order, degree) = (1, 2)

(c) $\frac{d^2y}{dx^2} = \left[4 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{4}}$

To eliminate the radical in the above equation, raising to the power 4 on both sides we get

$$R\frac{dy}{dx}+RPy=RQ \quad ...(1)$$

is an exact equation, and it is apparent from the first term that the L.H.S of (1) is

$$\frac{d}{dx}(Ry)=R\frac{dy}{dx}+y\frac{dR}{dx}$$

Thus (1) may also be written as

$$R\frac{dy}{dx}+y\frac{dR}{dx}=RQ \quad ...(2)$$

Equating the second terms on the L.H.S of (1) and (2),

$$y\frac{dR}{dx}=RPy$$

$$\therefore \frac{dR}{dx}=RP$$

Separating the variables,

$$\int \frac{1}{R}dR=\int P\,dx$$

$$\ln R=\int P\,dx$$

$$R=e^{\int P\,dx}$$

Thus the required integrating factor is $e^{\int P\,dx}$. The initial assumption that an integrating factor exists is therefore justified provided that it is possible to find $\int P\,dx$

Example 8

Solve the differential equation $\frac{dy}{dx}+3y=e^{2x}$, given that $y=\frac{6}{5}$ when $x=0$.

Solution

The integrating factor is $e^{\int 3dx}=e^{3x}$. Multiplying each side of the equation by e^{3x};

$$e^{3x}\frac{dy}{dx}+3e^{3x}y=e^{5x}$$

$$\frac{d}{dx}(e^{3x}y)=e^{5x}$$

$$e^{3x}y=\int e^{5x}\,dx$$

$$e^{3x}y=\frac{1}{5}e^{5x}+c$$

Example 9

Solve $\frac{dy}{dx}+y\cot x=2\cos x$

Solution

The given equation is of the form $\frac{dy}{dx}+Py=Q$. This is linear in y

Here $P=\cot x$ and $Q=2\cos x$

The integrating factor,

$$R=e^{\int Pdx}=e^{\int \cot x dx}=e^{\ln \sin x}=\sin x$$

Multiplying both sides by $\sin x$ makes the L.H.S exact

$$\sin x\frac{dy}{dx}+y\cos x=2\sin x\cos x$$

$$\frac{d}{dx}(y\sin x)=\sin 2x$$

$$y\sin x=\int \sin 2x\,dx$$

$$y\sin x=-\frac{\cos 2x}{2}+c$$

$$2y\sin x+\cos 2x=c$$

Example 10

Solve: $(1-x^2)\frac{dy}{dx}+2xy=x\sqrt{(1-x^2)}$

Solution

$$\frac{dy}{dx}+\left(\frac{2x}{1-x^2}\right)y=\frac{x}{\sqrt{(1-x^2)}}$$

$$\int P\,dx=\int \frac{2x}{1-x^2}dx=-\ln(1-x^2)$$

Integrating factor $=e^{\int Pdx}=e^{-\ln(1-x^2)}=\frac{1}{1-x^2}$

$$y\cdot\frac{1}{1-x^2}=\int \frac{x}{\sqrt{1-x^2}}\times\frac{1}{1-x^2}dx$$

Let $1-x^2=t \Rightarrow\ -2x\,dx=dt$

$$\frac{y}{1-x^2}=-\frac{1}{2}\int t^{-\frac{3}{2}}dt$$

$$\frac{y}{1-x^2}=t^{-\frac{1}{2}}+c$$

$$\frac{y}{1-x^2}=\frac{1}{\sqrt{1-x^2}}+c$$

Example 11

Solve: $(1+y^2)dx=(\tan^{-1}y-x)dy$

Solution

The given equation can be written as

$$\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{\tan^{-1}y}{1+y^2}$$

This is linear in x. Therefore, we have

$$\int P\,dy=\int \frac{1}{1+y^2}dy=\tan^{-1}y$$

Integrating factor $=e^{\int P\,dy}=e^{\tan^{-1}y}$

The required solution s

$$xe^{\tan^{-1}y}=\int e^{\tan^{-1}y}\left(\frac{\tan^{-1}1}{1+y^2}\right)dy$$

$$\tan^{-1}y=t$$

$$\frac{dy}{1+y^2}=dt$$

$$xe^{\tan^{-1}y}=\int e^t.t\,dt$$

$$xe^{\tan^{-1}y}=te^t-e^t+c$$

$$xe^{\tan^{-1}y}=e^{\tan^{-1}y}(\tan^{-1}y-1)+c$$

Example 12

Solve: $(x+1)\frac{dy}{dx}-y=e^x(x+1)^2$

Solution

The given equation can be written as

$$\frac{dy}{dx} - \frac{y}{x+1} = e^x(x+1)$$

This is linear in y

$$\text{Here} \int P\,dx = -\int \frac{1}{x+1}dx = -\ln(x+1)$$

Integrating factor is

$$e^{\int Pdx} = e^{-\ln(x+1)} = \frac{1}{x+1}$$

The required solution is

$$y \cdot \frac{1}{x+1} = \int e^x(x+1)\frac{1}{x+1}dx$$

$$\frac{y}{x+1} = \int e^x\,dx$$

$$\frac{y}{x+1} = e^x + c$$

Example 13

Solve: $\dfrac{dy}{dx} + 2y\tan x = \sin x$

<u>Solution</u>

This differential equation is linear in y

$$\int P\,dx = \int 2\tan x\,dx = 2\ln\sec x$$

Integrating factor, $R = e^{\int Pdx} = e^{\ln\sec^2 x} = \sec^2 x$

The required solution is given by

$$y\sec^2 x = \int \sec^2 x \sin x\,dx$$

$$y\sec^2 x = \int \tan x \sec x\,dx$$

$$y\sec^2 x = \sec x + c$$

$$y = \cos x + c\cos^2 x$$

Example 14

Solve the differential equation

$$\frac{dy}{dx}\sin x + 2y\cos x = 4\sin^2 x\cos x\,,\quad y\left(\frac{1}{6}\pi\right) = \frac{17}{4}$$

<u>Solution</u>

$$\frac{dy}{dx}\sin x + 2y\cos x = 4\sin^2 x\cos x$$

$$\frac{dy}{dx} + 2y\cot x = 4\sin x\cos x$$

Integrating factor $= e^{\int 2\cot x dx} = e^{2\ln\sin x} = \sin^2 x$

$$\frac{d}{dx}[y\sin^2 x] = (4\sin x\cos x)\sin^2 x$$

$$y\sin^2 x = \int 4\sin^3 x\cos x\,dx$$

$$y\sin^2 x = \sin^4 x + c$$

Now $y\left(\frac{\pi}{16}\right) = \frac{17}{4}$

$$\frac{17}{4} \times \frac{1}{4} = \frac{1}{16} + c$$

$$c = 4$$

$$y\sin^2 x = \sin^4 x + 4$$

$$y = \sin^2 x + 4\,\text{cosec}^2 x$$

First order homogeneous equations

A differential equation of first order and first degree is said to be homogeneous if it can be put in the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ or $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$

Working rule for solving homogeneous equations:

By definition the given equation can be put in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad \ldots(1)$$

To solve (1) put $\quad y = vx \quad \ldots(2)$

Differentiating (2) with respect to x gives

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \ldots(3)$$

Using (2) and (3) in (1), we have

$$v + x\frac{dv}{dx} = f(v)$$

$$\text{or } x\frac{dv}{dx} = f(v) - v$$

Separating the variables x and v we have

$$\frac{dx}{x} = \frac{dv}{f(v) - v}$$

$$\ln x + c = \int \frac{dv}{f(v) - v}$$

where c is an arbitrary constant. After integration, replace v by $\frac{y}{x}$

Example 15

Solve $\quad xy\dfrac{dy}{dx} = x^2 + y^2$

<u>Solution</u>

Dividing each side by x^2,

$$\frac{y}{x}\frac{dy}{dx} = 1 + \left(\frac{y}{x}\right)^2$$

Let $y = ux$, then

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$\therefore u\left(u + x\frac{du}{dx}\right) = 1 + u^2$$

$$ux\frac{du}{dx} = 1$$

Separating the variables,

$$\int u\,du = \int \frac{1}{x}dx$$

$$\frac{1}{2}u^2 = \ln(Bx)$$

$$\left(\frac{y}{x}\right)^2 = 2\ln(Bx)$$

$$\left(\frac{y}{x}\right)^2 = \ln(Ax^2) \text{ where } A = B^2$$

Therefore the general solution is

$$y^2 = x^2\ln(Ax^2)$$

Applications of differential equations

In this section, we solve problems on differential equations which have direct impact on real life situation. Solving these types of problems involve

(i) Construction the mathematical model describing the given situation
(ii) Seeking solution for the model formulated in (i) using the methods discussed earlier.

Illustration:

Let A be any population at time t. The rate of change of population is directly proportional to initial population i.e.

$$\frac{dA}{dt} \propto A \text{ i.e} \frac{dA}{dt} = kA$$

where k is called the constant of proportionality

(1) If $k > 0$, we say that A grows exponentially with growth constant k (growth problem).
(2) If $k < 0$ we say that A decreases exponentially with decreasing k (decay problem).

This linear equation can be solved in three ways i.e. (i) variable separation (ii) linear (using integrating factor) (iii) by using characteristic equation with single root k. In all the ways we get the solution as $A = ce^{kt}$ where c s the arbitrary constant and k is the constant of proportionality.

Examples of applications leading to differential equations

1. Population Growth and Decay

Although the number of members of a population (people in a given country, bacteria in a laboratory culture, wildflowers in a forest, etc.) at any given time t is necessarily an integer, models that use differential equations to describe the growth and decay of populations usually rest on the simplifying assumption that the rate of change of the number of members of the population is proportional to the number already present.

2. Newton's law of cooling

According to Newton's law of cooling, the temperature of a body changes at a rate proportional to the difference between the temperature of the body and the temperature of the surrounding medium. Thus, if T_m is the temperature of the medium and $T = T(t)$ is the temperature of the body at time t, then

$$T' = -k(T - T_m)$$

where k is a positive constant and the minus sign indicates that; the temperature of the body increases with time if it is less than the temperature of the medium, or decreases if it is greater. If T_m is constant, then

$$T = T_m + (T_0 - T_m)e^{-kt}$$

where T_0 is the temperature of the body when $t = 0$.

3. Glucose absorption by the body

Glucose is absorbed by the body at a rate proportional to the amount of glucose present in the blood stream. Let λ denote the (positive) constant of proportionality. Suppose there are G_0 units of glucose in the bloodstream when $t = 0$, and let $G = G(t)$ be the number of units in the bloodstream a time $t > 0$. Then since the glucose being absorbed by the body is leaving the bloodstream, G satisfies the equation

$$G' = -\lambda G$$

If c is an arbitrary constant, then

$$G = ce^{-\lambda t}$$

Setting $t = 0$ and requiring that $G(0) = G_0$ yields $c = G_0$

$$G(t) = G_0 e^{-\lambda t}$$

Now lets complicate matters by injecting glucose intravenously at a constant rate of r units of glucose per unit of time. Then the rate of change of the amount of glucose in the bloodstream per unit time is

$$G' = -\lambda G + r$$

where the first term on the right is due to the absorption of the glucose by the body and the second term is due to the injection.

The equation that satisfies $G(0) = G_0$ will then be given by

$$G = \frac{r}{\lambda} + \left(G_0 - \frac{r}{\lambda}\right) e^{-\lambda t}$$

4. Spread of epidemics

One model for the spread of epidemics assumes that the number of people infected changes a rate proportional to the number of people already infected and the number of people who are susceptible, but not yet infected. Therefore, if S denotes the total population of susceptible people and $I = I(t)$ denotes the number of infected people at time t, then $S - I$ is the number of people who are susceptible but not yet infected. Thus

$$I' = rI(S - I)$$

where r is a positive constant. Assuming that $I(0) = I_0$, the solution of this equation is

$$I = \frac{SI_0}{I_0 + (S - I_0)e^{-rSt}}$$

5. Exponential reduction or Radioactive decay

In exponential decay, a quantity slowly decreases in the beginning and then decreases rapidly. We use the exponential decay formula to find population decay (depreciation) and we can also use the exponential decay formula to find half-life (the amount of time for the population to become half of its size).

Assuming there are N number of atoms, decreasing over time. Using a time constant λ, the behavior of the decay is governed by the differential equation

$$\frac{dN}{dt} = -\lambda N$$

The solution of the equation above is given by

$$\frac{dP}{P^2} = k\,dt$$
$$\int P^{-2}\,dP = \int k\,dt$$
$$-P^{-1} = kt + c$$
$$\frac{1}{P} = -(kt + c)$$
$$P = -\frac{1}{(kt + c)}$$

Find c:

At time $t = 1$, $P = 1000$

$$1000 = -\frac{1}{k + c}$$
$$1000(k + c) = -1$$
$$1000k + 1000c = -1 \text{ ... (i)}$$

At time $t = 2$, $P = 2000$

$$2000 = -\frac{1}{2k + c}$$
$$2000(2k + c) = -1$$
$$4000k + 2000c = -1 \text{ ... (ii)}$$

(ii) – 2(i);

$$2000k = 1$$
$$k = \frac{1}{2000}$$

Substituting k into (i);

$$\frac{1}{2} + 1000c = -1$$
$$c = -\frac{3}{2000}$$

When population $P = 10000$,

$$10000 = -\frac{1}{\left(\frac{t}{2000} - \frac{3}{2000}\right)}$$
$$10000 = -\frac{2000}{t - 3}$$
$$t - 3 = -\frac{2}{10}$$
$$t = 2.8 \text{ hrs}$$

Example 5

The population of a small village is 1097 in the year 1566. Assuming the population, P, grows according to the differential equation below, and where t is the number of years after 1566:

$$\frac{dP}{dt} = 0.3Pe^{-0.3t}$$

(a) Find the population of the village in 1576, correct to 3 significant figures

(b) Find the maximum population the village will grow to, in the long term.

<u>Solution</u>

$$\frac{dP}{dt} = 0.03Pe^{-0.03t}$$
$$\frac{dP}{P} = 0.03e^{-0.03t}dt$$
$$\int \frac{dP}{P} = \int 0.03e^{-0.03t}dt$$
$$\ln P = -e^{-0.03t} + c$$

To find c: $P = 1097$ & $t = 0$

$$\ln 1097 = -e^{0} + c$$
$$7 = -1 + c$$
$$c = 8$$
$$\ln P = -e^{-0.03t} + 8$$
$$\ln P = 8 - e^{-0.03t}$$

To find the population in 10 years' time:

$$\ln P = 8 - e^{-0.03\times 10} = 8 - e^{-0.3} = 7.2592$$
$$P = 1420$$

To find the limiting population in the long term:

$$\ln P = 8 - e^{-0.03t}$$
$$\ln P = 8 - \frac{1}{e^{0.03t}}$$

As t increases, $\frac{1}{e^{0.03t}} \to 0$. Thus, in the long term

$$\ln P = 8 - 0$$
$$P = 2980$$

Example 6

It is thought that the rate at which a rumour spreads is jointly proportional to the number, x, of people who have heard the rumour and the number $N - x$, those who are yet to hear it. This relationship is approximated by the differential equation $\frac{dx}{dt} = kx(N - x)$.

Assuming that when $t = 0$, $x = 1$, find an equation for x as a function of time t.

<u>Solution</u>

$$\frac{dx}{dt} = kx(N - x)$$
$$\frac{1}{kx(N - x)}dx = kdt$$
$$\int \frac{1}{kx(N - x)}dx = \int k\,dt$$

Expressing $\frac{1}{x(N-x)}$ as $\frac{1}{N}\left(\frac{1}{x} + \frac{1}{N-x}\right)$

$$\frac{1}{N}\int \left(\frac{1}{x} + \frac{1}{N - x}\right)dx = \int k\,dt$$
$$\frac{1}{N}(\ln x - \ln(N - x)) = kt + c$$

When $t = 0$, $x = 1$, which gives

$$\frac{1}{N}(\ln 1 - \ln(N - 1)) = 0 + c$$
$$c = -\frac{1}{N}\ln(N - 1)$$

Thus;

$$\frac{1}{N}(\ln x - \ln(N - x)) = kt - \frac{1}{N}\ln(N - 1)$$
$$\frac{1}{N}(\ln x - \ln(N - x)) + \frac{1}{N}\ln(N - 1) = kt$$
$$\frac{1}{N}\ln\left(\frac{x(N - 1)}{N - x}\right) = kt$$

$$dA = -3t^{\frac{1}{2}}dt$$
$$\int dA = \int -3t^{\frac{1}{2}}\,dt$$
$$A = -2t^{\frac{3}{2}} + c$$

When $t = 0, A = 10$
$$\Rightarrow c = 10$$

At any time t,
$$A = 10 - 2t^{\frac{3}{2}}$$

(b) For drug free, $A = 0$
$$\Rightarrow 0 = 10 - 2t^{\frac{3}{2}}$$
$$5 = t^{\frac{3}{2}}$$
$$t^3 = 25$$
$$t = 2.9 \text{ hours}$$

Hence, the patient will be drug free in 2.9 hours or 2 hours 54 minutes.

Example 10

The number of bacteria in a yeast culture grows at a rate which is proportional to the number present. If the population of a colony of yeast bacteria triples in 1 hour. Show that the number of bacteria at the end of five hours will be 3^5 times of the population at initial time.

<u>Solution</u>

Let A be the number of bacteria at any time t
$$\frac{dA}{dt} \propto A$$
$$\frac{dA}{dt} = kA$$
$$A = ce^{kt}$$

Initially, i.e. when $t = 0$, assume that $A = A_0$
$$A_0 = ce^0 = c$$
$$\therefore A = A_0e^{kt}$$

when $t = 1, A = 3A_0$
$$\Rightarrow 3A_0 = A_0e^k$$
$$e^k = 3$$

when $t = 5, A = A_0e^{5k} = A_0(e^k)^5 = 3^5A_0$

$\therefore$ The number of bacteria at the end of 5 hours will be 3^5 times of the number of bacteria at initial time.

Example 11

A beaker containing water at 100℃ is placed in a room which has a constant temperature of 20℃. The rate of cooling at any moment s proportional to the difference between the temperature of the room and the liquid. If after 5 minutes the temperature of the water is 60℃, what will it be after 10 minutes?

<u>Solution</u>

Let the temperature of the water at any time t min be θ ℃. Then the rate of change of temperature is $d\theta/dt$, thus
$$\frac{d\theta}{dt} \propto (\theta - 20)$$
$$\frac{d\theta}{dt} = -k(\theta - 20)$$
$$\frac{d\theta}{\theta - 20} = -kdt$$
$$\int \frac{d\theta}{\theta - 20} = \int -k\,dt$$
$$\ln(\theta - 20) = -kt + c$$
$$\theta - 20 = e^{-kt+c}$$
$$\theta - 20 = e^{-kt} \times e^c$$
$$\theta - 20 = Ae^{-kt} \quad \text{where } A = e^c$$
$$\theta = 20 + Ae^{-kt}$$

Now initially, $t = 0$ and $\theta = 100$,
$$100 = 20 + A$$
$$A = 80$$
$$\theta = 20 + 80e^{-kt}$$

Also when $t = 5, \theta = 60$
$$60 = 20 + 80e^{-5k}$$
$$e^{-5k} = 0.5$$

It is possible to find k exactly from this equation but it is not necessary to find θ when $t = 10$.

When $t = 10$,
$$\theta = 20 + 80e^{-10k} = 20 + 80(e^{-5k})^2$$
$$\theta = 20 + 80(0.5)^2$$
$$\theta = 40 \text{ ℃}$$

Example 12

Water is leaking out of a tank from a tap which is located 5 cm from the bottom of the tank. The height of the water, h cm, is decreasing at a rate proportional to the square root of the difference of the height of the water and the height of the tap.

(a) Model this problem with a differential equation involving h, the time t in minutes and a suitable proportionality constant.

The initial height of the water in the tank is 230 cm and 5 minutes later it has dropped to 105 cm.

(b) Find a solution of the differential equation of part (a)

(c) Calculate the time taken for the height of the water to fall to 30 cm.

(d) State how many minutes it takes for the tank to stop leaking.

<u>Solution</u>

(a) h = height of water (cm)

t = time in minutes
$$\frac{dh}{dt} = -k\sqrt{h - 5}$$

(b) $\frac{dh}{dt} = -k(h - 5)^{\frac{1}{2}}$
$$\frac{1}{(h - 5)^{\frac{1}{2}}}dh = -kdt$$

$$t = 100\ln\left[\frac{4x}{20-x}\right]$$

(c) Rearrange the answer in part (b) to show further that

$$x = \frac{20}{1+4e^{-0.01t}}$$

(d) If a vet cannot attend the farm for 24 hours, since he infection was first discovered, find how many extra chickens will be infected by the time the vet arrives.

Solution

(a) $\frac{dx}{dt} = kx(20-x)$

(b) when $t = 0$, $x = 4$, $\frac{dx}{dt} = 0.032$

$$0.032 = k \times 4 \times 16$$
$$64k = 0.032$$
$$k = 0.0005$$
$$\Rightarrow \frac{dx}{dt} = kx(20-x)$$
$$\int \frac{1}{x(20-x)}dx = \int k\,dt$$

Partial fractions;

$$\frac{1}{x(20-x)} \equiv \frac{A}{x} + \frac{B}{20-x}$$
$$1 \equiv A(20-x) + Bx$$

If $x = 0$, $1 = 20A \Rightarrow A = \frac{1}{20}$

If $x = 20$, $1 = 20B \Rightarrow B = \frac{1}{20}$

$$\int \frac{\frac{1}{20}}{x} + \frac{\frac{1}{20}}{20-x}dx = \int k\,dt$$
$$\int \frac{1}{x} + \frac{1}{20-x}dx = \int 20k\,dt$$
$$\ln x - \ln(20-x) = 20kt + c$$
$$\ln\left(\frac{x}{20-x}\right) = \frac{1}{100}t + c$$
$$100\ln\left(\frac{x}{20-x}\right) = t + c$$

When $t = 0$, $x = 4$

$$100\ln\frac{1}{4} = c$$
$$c = -100\ln 4$$
$$100\ln\left(\frac{x}{20-x}\right) = t - 100\ln 4$$
$$t = 100\ln\left(\frac{x}{20-x}\right) + 100\ln 4$$
$$t = 100\left[\ln\left(\frac{x}{20-x}\right) + \ln 4\right]$$
$$t = 100\ln\left(\frac{4x}{20-x}\right)$$

(c)

$$\frac{t}{100} = \ln\left(\frac{4x}{20-x}\right)$$
$$e^{\frac{1}{100}t} = \frac{4x}{20-x}$$
$$20e^{\frac{1}{100}t} - xe^{\frac{1}{100}t} = 4x$$
$$20e^{\frac{1}{100}t} = 4x + xe^{\frac{1}{100}t}$$
$$20e^{\frac{1}{100}t} = x\left(4 + e^{\frac{1}{100}t}\right)$$
$$x = \frac{20e^{\frac{1}{100}t}}{4 + e^{\frac{1}{100}t}}$$

Multiply the top and bottom by $e^{-\frac{1}{100}t}$

$$x = \frac{20}{4e^{-0.01t}+1}$$

(d) When $t = 24$

$$x = \frac{20}{1+e^{-\frac{24}{100}}} = 4.8233$$

Thus 4823 chickens infected at that time

$$\therefore 4823 - 4000 = 823 \text{ extra}$$

Self-Evaluation exercise

1. Find the general solutions of the following differential equations

(a) $3y^2\frac{dy}{dx} + 2x = 1$

[Ans: $y^3 = x - x^2 + c$]

(b) $\frac{dy}{dx} = 2e^{x-y}$

[Ans: $y = \ln(2e^x + C)$

(c) $\frac{dy}{dx} = \frac{xe^x}{\sin y\cos y}$

[Ans: $\cos^2 y + e^x(x-1) = c$]

(d) $\frac{dy}{dx}\cos^2 x = y^2\sin^2 x$

[Ans: $y = \frac{1}{c+x-\tan x}$]

(e) $e^{x+2y}\frac{dy}{dx} + (1-x)^2 = 0$

[Ans: $y = \frac{1}{2}\ln[2e^{-x}(x^2+1) + K]$

2. Find the exact solutions of the following differential equations.

(a) $\frac{dy}{dx} + \frac{4x}{y} = 0$ when $y = 2$ at $x = 0$

[Ans: $4x^2 + y^2 = 4$]

(b) $\frac{dy}{dx} = 3x^2\sqrt{y}$ when $y = 0$ at $x = 1$

[Ans: $y = \frac{1}{4}(x^3-1)^2$]

(c) $\frac{dy}{dx} + e^{x-y} = 0$ when $y = 0$ at $x = 0$

[Ans: $e^x + e^y = 2$]

(d) $\frac{dy}{dx} = \frac{2x\ln x}{y}$ when $y = 2e$ at $x = e$

[Ans: $y^2 = x^2(2\ln x - 1) + 3e^2$]

(e) $e^y\frac{dy}{dx} + xe^x = 0$ when $y = 0$ at $x = 0$

[Ans: $y = x + \ln(1-x)$]

(f) $\frac{dy}{dx} = \frac{y}{x(x+1)^2}$; $y(1) = \frac{1}{2}$

[Ans: $\ln y = \ln\left(\frac{x}{x+1}\right) + \frac{1}{x+1} - \frac{1}{2}$]

(g) $(1+x)\frac{dy}{dx} = y(1-x)$ when $y = 1$ at $x = 0$

[Ans: $y = (x+1)^2e^{-x}$

(h) $\frac{dy}{dx} = 24\cos^2 y\cos^3 x$, when $y = \frac{\pi}{4}$ at $x = \frac{\pi}{6}$

[Ans: $\tan y = 24\sin x - 8\sin^3 x - 10$]

(i) $xy + (1+x)\frac{dy}{dx} = y$; $y(0) = 3$

[Ans: $y = 3(1+x)^2 e^{-x}$]

(j) $\frac{dy}{dx}\cot x = 1 - y^2$; $y\left(\frac{\pi}{4}\right) = 0$

[Ans: $y = (x+2)e^{-x}$]

3. Given that $\frac{dy}{dx}\sin x = \sin x \sin 2x + y\cos x$. If $y = \frac{3}{2}$ at $x = \frac{\pi}{6}$, find the exact value of y at $x = \frac{\pi}{4}$

[Ans: $1 + \sqrt{2}$]

4. Given $\frac{dy}{dx} = x + 2y$, with $y = -\frac{1}{4}$ at $x = 0$. By using a suitable substitution or otherwise, show that the solution of the differential equation is
$$y = -\frac{1}{4}(2x+1)$$

5. Solve the differential equation
$$\frac{dy}{dx} = \frac{y}{x} - \left(\frac{y}{x}\right)^2$$
subject to the condition $y = 1$ at $x = 1$

[Ans: $y = \frac{x}{1+\ln x}$]

6. By using a suitable substitution, solve the differential equation
$$xy\frac{dy}{dx} = x^2 + y^2$$
subject to the boundary condition $y = 1$ at $x = 1$

[Ans: $y = x^2(1 + 2\ln x)$]

7. By using a suitable substitution, or otherwise, solve the differential equation
$$\frac{dy}{dx} = x^2 + 2xy + y^2$$
subject to the condition $y(0) = 0$

[Ans: $y = -x + \tan x$]

8. By using a suitable substitution, solve the differential equation
$$\frac{dy}{dx} = \frac{xy + y^2}{x^2}$$
subject to the condition $y = -1$ at $x = 1$

[Ans: $y = -\frac{1}{1+\ln x}$]

9. If $\frac{dy}{dx} = \frac{x^2+3y^2}{xy}$; $y(1) = \frac{1}{\sqrt{2}}$, show that $y^2 = x^6 - \frac{1}{2}x^2$

10. By using a suitable substitution, solve the differential equation
$$2x^2\frac{dy}{dx} = x^2 + y^2$$
subject to the condition $y(1) = 0$

[Ans: $y = x - \frac{2x}{2+\ln x}$]

11. Solve the differential equation
$$x^2\frac{dy}{dx} + xy = y^2, \qquad y\left(\frac{1}{2}\right) = 2$$

[Ans: $y = \frac{2x}{1-2x^2}$]

12. Given the differential equation
$$xy\frac{dy}{dx} = (x-y)^2 + xy, \ \ y(1) = 0$$
Show that the solution of the above differential equation is
$$(x-y)e^{\frac{y}{x}} = 1$$

13. Solve the differential equation
$$x\frac{dy}{dx} + y = 4x^2y^2, \qquad y\left(\frac{1}{2}\right) = 2$$

[Ans: $y = \frac{1}{3x-4x^2}$]

14. Given the differential equation
$$x\frac{dy}{dx} + 3y = xe^{-x^2}$$
Show clearly that the general solution of the above differential equation can be written in the form
$$2yx^3 + (x^2+1)e^{-x^2} = \text{constant}$$

15. Use the substitution $y = xv$ to solve the following differential equation
$$2\frac{dy}{dx} = 1 + \frac{y^2}{x^2}, \ \ y(e) = -e$$

[Ans: $y = x - \frac{2x}{\ln x}$]

16. It is given that a curve with a certain equation passes through the point $(0, 1)$ and satisfies the differential equation
$$\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$$
By solving the differential equation, show that an equation for the curve is
$$y = e^{\frac{x^2}{2y^2}}$$

17. The number of bacterial cells N on a laboratory dish is increasing, so that the hourly rate of increase is 5 times the number of the bacteria present at that time. Initially 100 bacteria were laced on the dish.
   (a) Form a suitable differential equation to model this problem.
   (b) Find the solution of this differential equation
   (c) Find to the nearest minute, the time taken for the bacteria to reach 10000.

[Ans: $\frac{dN}{dt} = 5N$; $N = 100e^{5t}$; 55 minutes]

18. A certain brand of car is valued at £V at a time t years from new. A model for the value of the car assumes that the rate of decrease of its value is proportional to its value at that time.
   (a) By forming and solving a suitable differential equation, show that
$$V = Ae^{-kt}$$
   where A and k are positive constants.

The value of one such car when new is £30000 and this value halves after 3 years.
   (b) Find, to the nearest £100, the value of one such car after 10 years.

One such car is to be scrapped when its value drops below £500.

(c) Find after how many years this car is to be scrapped.

[Ans: £3000, $t = 17.7 \approx 18$]

19. The number x of bacterial cells in time t hours, after they were placed on a laboratory dish, is increasing at the rate proportional to the number of the bacterial cells present at that time.
(a) If x_0 is the initial number of the bacterial cells and k is a positive constant, show that
$$x = x_0 e^{kt}$$
(b) If the number of bacteria triples in 2 hours, show that $k = \ln\sqrt{3}$

20. During a car service, the motor oil is drained out of the engine. The rate in cm^3s^{-1}, at which the oil is drained out, is proportional to the volume $V\ cm^3$, of the oil still left inside the engine.
(a) Form a differential equation involving V, the time t in seconds and a proportionality constant k
Initially there were 4000 cm^3 of oil in the engine.
(b) Find a solution of the differential equation, giving the answer in terms of k
It takes T seconds to drain half the oil out of the engine
(c) Show clearly that $kT = \ln 2$

[Ans: $\frac{dV}{dt} = -kV$; $V = 4000e^{-kt}$]

21. The number, x thousands, of reported cases of an infectious disease, t months after it was reported, is now dropping. The rate at which it is dropping is proportional to the square of the reported cases.
It is assumed that x can be treated as a continuous variable.
(a) Form a differential equation in terms of x, t and a proportionality constant k
Initially there were 2500 reported cases and one month later they had dropped to 1600 cases.
(b) Solve the differential equation to show that
$$x = \frac{40}{9t + 16}$$
(c) Find after how many months there will be 250 reported cases

[Ans: $\frac{dx}{dt} = -kx^2$, $t = 16$]

22. Water is leaking out of a hole at the bottom of a tank. Let the height of the water in the tank be y cm at time t minutes.
At any given time after the leaking started, the height of the water in the tank is decreasing at a rate proportional to the cube root of the height of the water in the tank. When $t = 0$, $y = 125$ and when $t = 3$, $y = 64$.
By forming and solving a differential equation, find the value of y when $t = 7\frac{7}{12}$

[Ans: $y = 3.375$]

23. A body is moving and its distance, x metres, is measured from a fixed point O at different times, t seconds.
The body is moving in such a way, so that the rate of change of its distance x is inversely proportional to its distance x at that time.
When $t = 0$, $x = 50$ and when $t = 4$, $x = 30$
Determine the time it takes for the body to reach O.

[Ans: $t = 6.25$]

24. The temperature in a bathroom is maintained at the constant value of 20°C and the water in a hot bath is left to cool down. The rate, in °C per second, at which the temperature of the water in the bath, T°C, is cooling down, is proportional to the difference in the temperature between the bathwater and the room.
Initially the bathwater had a temperature of 40°C, and at that instant was cooling down at a rate of 0.005°C per second. Let t be the time in seconds, since he bath water was left to cool down.
(a) Show that
$$\frac{dT}{dt} = -\frac{1}{4000}(T - 20)$$
(b) Solve the differential equation of part (a), to find, correct to the nearest minute, after how long the temperature of the bathwater will drop to 36°C.

[Ans: 15 minutes]

25. Hot tea in a cup has a temperature T °C at time t minutes and it is left to cool n a room of constant temperature T_0. Newton's law of cooling asserts that the rate at which a body cools is directly proportional to the excess temperature of the body and the temperature of its immediate surroundings.
(a) Assuming the tea cooling in the cup obeys this law, form a differential equation in terms of T, T_0, t and a proportionality constant k.
(b) Show clearly that
$$T = T_0 + Ae^{-kt}$$
where A is a constant
Initially the temperature of the tea is 80°C and 10 minutes later is 60°C. The room temperature remains constant at 20°C.
(c) Find the value of t when the tea reaches a temperature of 40°C

[Ans: $\frac{dT}{dt} = -k(T - T_0)$, $t = 27.1$]

26. At time t hours, the rate of decay of the mass, x kg, of a radioactive substance is directly proportional to the mass present at that time. Initially the mass is x_0.
(a) By forming and solving a suitable differential equation, show that
$$x = x_0 e^{-kt}$$
where k is a positive constant

Chapter 36 Examination Questions

SECTION A

1. A cylindrical can of capacity $1000\ cm^3$ is made from a thin sheet of metal. The can is open at the top and closed at the bottom. The radius of the bottom is x cm. Find the value of x that will minimize the area of the sheet to be used. (Leave π in your answer).
[2024, No. 2]
2. Show that
$$\int_0^1 \left(\frac{1}{9-x^2}\right) dx = \frac{1}{6}\ln 2$$
[2024, No. 4]
3. Find the area enclosed by the curve $y = x^2$ and the line $y = x$ from $x = 0$ to $x = 1$.
[2024, No. 8]
4. Evaluate $\int_0^{\pi/3} (1 + \cos 3y)^2\, dy$
[2023, No. 5]
5. Given that $y = e^x \cos 3x$, show that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$.
[2023, No. 3]
6. Use Maclaurin's theorem to expand $\ln(2 + x)$, in ascending powers of x as far as the term in x^2
[2023, No. 8]
7. Find the equation of the tangent to the curve $x^3 + 2y^3 + 3xy = 0$ at the point $(2, -1)$.
[2022, No. 3]
8. An inverted conical container has a hole at bottom. A liquid is dripping through the hole at a rate of 2 cm^3s^{-1}. When the depth of the liquid in the container is x cm, its volume is $\frac{1}{3}\pi x^3$ cm3. Find the rate at which the level of the liquid is decreasing when x is 5 cm.
[2022, No. 6]
9. Using the substitution $y = Vx$ or otherwise, solve the differential equation
$$x\frac{dy}{dx} = 2y + x$$
[2022, No. 8]
10. Given that $y = \ln\left\{x\sqrt{(x+1)^3}\right\}$, find $\frac{dy}{dx}$
[2020, No. 3]
11. Evaluate $\int_0^{\frac{\pi}{3}} \tan^{-1}\frac{1}{2}x\, dx$
[2020, No. 5]
12. Find the gradient of the curve $x^2 \tan x - xy - 2y^2 = -2$ at the point $(0, 1)$
[2020, No. 8]
13. Using the substitution $u = \tan^{-1} x$, show that
$$\int_0^1 \frac{\tan^{-1} x}{1 + x^2} dx = \frac{\pi^2}{32}$$
[2019, No. 3]
14. Find the equation of the tangent to the curve $y = \frac{a^3}{x^2}$ at the point $P\left(\frac{a}{t}, at^2\right)$.
[2019, No. 5]
15. Find the area enclosed between the curve $y = 2x^2 - 4x$ and the x-axis.
[2019, No. 7]
16. Given that $Q = \sqrt{80 - 0.1P}$ and $E = \frac{-dQ}{dP}\cdot\frac{P}{Q}$, find E when $P = 600$
[2019, No. 8]
17. Find $\int x^2 e^x\, dx$
[2018, No. 3: Ans: $x^2e^x - 2xe^x + 2e^x + c$]
18. Determine the equation of the tangent to the curve $y^3 + y^2 - x^4 = 1$ at the point $(1, 1)$.
[2018, No. 6: Ans: $5y = 4x + 1$]
19. The region bounded by the curve $y^2 = x^2 - 2x$ and the x-axis from $x = 0$ and $x = 2$, is rotated about the x-axis. Calculate the volume of the solid formed.
[2018, No. 8: Ans: $\frac{16\pi}{15}$]
20. Differentiate $\left(\frac{1+2x}{1+x}\right)^2$ with respect to x
[2017, No. 3: Ans: $\frac{2(1+2x)}{(1+x)^4}$]
21. Show that $\int_2^4 x \ln x\, dx = 14\ln 2 - 3$
[2017, No. 6]
22. A container is in the form of an inverted right circular cone. Its height is 100 cm and base radius is 40 cm. The container is full of water and has a small hole at its vertex. Water is flowing through the hole at a rate of $10\ cm^3s^{-1}$. Find the rate at which the water level in the container is falling when the height of water in the container is halved. [2017, No. 8: Ans: 0.00796]
23. Evaluate $\int_{\frac{1}{2}}^{1} 10x\sqrt{(1-x^2)}dx$
[2016, No. 3: Ans: 2.165]
24. Solve the equation $\frac{dy}{dx} = 1 + y^2$ given that $y = 1$ when $x = 0$.
[2016, No. 4: Ans: $y = \tan\left(x + \frac{\pi}{4}\right)$]
25. Using small changes, show that $(244)^{\frac{1}{5}} = 3\frac{1}{405}$.
[2016, No. 7]
26. Differentiate $e^{-x^2}x^3 \sin x$ with respect to x

79. The distance S m of a particle from a fixed point is given by $S = t^2(t^2+6) - 4t(t-1)(t+1)$, where t is the time. Find the velocity and acceleration of the particle when $t = 1$ s.

[Mar 1998, No. 4: Ans: 8 ms^{-1}, 0 ms^{-2}]

80. Using the substitution $2x + 1 = p$, find $\int_0^1 \frac{xdx}{(2x+1)^3}$

[Mar 1998, No. 5: Ans: $\frac{1}{18}$]

81. Use Maclaurin's expansion to express $\ln(1+x)^2$ in ascending powers of x up to the term in x^4.

[Mar 1998, No. 8: Ans: $2x - x^2 + \frac{2x^3}{3} - \frac{x^4}{2}$]

82. Given that $x = \theta - \sin\theta$ and $y = 1 - \cos\theta$. Show that $\frac{dy}{dx} = \cot\theta$

[1997, No. 4]

83. Find (i) $\int \sin^2 x\, dx$

(ii) $\int \tan^3 x\, dx$

[1997, No. 6: Ans: (i) $\frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + C$ (ii) $\frac{1}{2}\tan^2 x - \ln\cos x + C$]

84. Determine the volume of the solid generated when the region bounded by the curve $y = \cos 2x$ and the x-axis for values of x between 0 and ¾ is rotated about the x-axis.

[1997, No. 8: Ans: 1.232 cubic units]

85. Differentiate with respect to x, expressing your results as simply as possible.

$$\sin^{-1}\left(\frac{3 + 5\cos x}{5 + 3\cos x}\right)$$

[1996, No. 7: Ans: $\frac{-4}{5+3\cos x}$]

86. Evaluate $\int_0^{\frac{1}{2}\pi} \sin 2x \cos x\, dx$

[1996, No. 8: Ans: $\frac{2}{3}$]

SECTION B

1. (a) Use the substitution $x = \sin\theta$ to evaluate

$$\int_0^1 \frac{1+x}{\sqrt{1-x^2}}dx$$

(b) Given that $y = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$, find $\frac{dy}{dx}$ in terms of x.

[2024, No. 11]

2. (a) Use Maclaurin's theorem to expand $\ln(1-2x)$ in ascending powers of x as far as the term in x^3.

(b) Using small changes, find the approximate value of $\tan 46°$ correct to three decimal places.

[2024, No. 13]

3. The rate at which the quantity M of a commodity is sold is proportional to the difference between the amount initially present and the quantity sold at any time t. Initially 10 tonnes of the commodity were present. After one day, 2 tonnes were sold.

(a) Form a differential equation for the quantity of the commodity sold.

(b) (i) Determine the expression for M in terms of t

(ii) Calculate the quantity sold at the end of 5 days.

[2024, No. 16]

4. Express $f(x) = \frac{3x^3+2x^2-3x+1}{x(1-x)}$ in partial fractions.

Hence find $\int f(x)\, dx$

[2023, No. 10]

5. Given the curve $y = \frac{1}{4x^2-1}$, determine the;

(a) coordinates of the turning points of the curve

(b) equation of the asymptotes

Hence sketch the curve.

[2023, No. 14]

6. The rate at which a body cools is proportional to the amount by which its temperature exceeds that of its surroundings. The body is placed in a room of temperature 25°C. After 6 minutes the temperature of the body dropped from 90°C to 60°C.

(a) Form a differential equation for the rate of cooling of the body.

(b) Find the time it takes for the body to cool from 40°C to 30°C.

[2023, No. 16]

7. Express $\frac{11x-1}{(1-x)^2(2+3x)}$ in partial fractions.

Hence evaluate $\int_0^{\frac{1}{2}} \frac{11x-1}{(1-x)^2(2+3x)}dx$ giving your answer in the form $k + \ln b$ where k is an integer and b is a fraction.

[2022, No. 10]

8. (a) Differentiate $\frac{(x^2+1)}{(x+1)^3}$ with respect to x

(b) Given that $x = \frac{3t}{t+3}$ and $y = \frac{4t+1}{t-2}$, find $\frac{d^2y}{dx^2}$ in terms of t in the simplest form.

[2008, No. 13: Ans: (a) 0.1905 (b) $\frac{\pi^2}{16}$]

48. On the same axes sketch the curves $f(x) = x^2(x+2)$ and $g(x) = \frac{1}{f(x)}$. Show the asymptotes and turning points.

[2008, No. 14]

49. (a) Solve the differential equation:
$$x\frac{dy}{dx} - y = x^3e^{x^2}$$
(b) The number of car accidents x in a year on a highway was found to approximate the differential equation $\frac{dx}{dt} = Kx$, where t is the time in years and K a constant. At the beginning of 2000 the number of recorded accidents was 50.
If the number of accidents increased to 60 at the beginning of 2002, estimate the number that was expected at the beginning of 2005.

[2008, No. 16: Ans: (a) $y = \frac{x}{2}\left(e^{x^2} + 2C\right)$ (b) 79]

50. Sketch the curve $y = \frac{4(x-3)}{x(x+2)}$

[2007, No. 10]

51. (a) $\int \frac{x^3}{(1+x^2)^{\frac{1}{2}}}dx$
(b) Use the substitution $t = \tan\frac{x}{2}$ to evaluate
$$\int_0^{\pi/2} \frac{dx}{1+\sin x+\cos x}$$

[2007, No. 13: Ans: (a) $\frac{1}{3}(1+x^2)^{\frac{1}{2}}(x^2-2)+c$ (b) 0.693]

52. (a) Given $x = r\cos\theta$ and $y = r\sin\theta$, show that $\frac{d^2y}{dx^2} = -\frac{1}{r}cosec^3\theta$
(b) Solve: $\frac{dy}{dx} + 2y\tan x = \cos^2 x$ given that $y = 2$, when $x = 0$

[2007, No. 16: Ans: (b) $y = \cos^2 x\,(2+x)$]

53. (a) Differentiate from first principles $y = \frac{x}{x^2+1}$ with respect to x
(b) (i) Determine the turning points of the curve $y = x^2(x-4)$
(ii) Sketch the curve in (i) above for $-2 \le x \le 5$
(iii) Find the area enclosed by the curve above and the x-axis

[2006, No. 10: Ans: (a) $\frac{1-x^2}{(x^2+1)^2}$ (b) (i) max $(0,0)$, min $\left(\frac{8}{3}, -\frac{256}{27}\right)$ (iii) $\frac{64}{3}$ sq. units]

54. (a) Find the first three terms of the expansion of $\frac{1}{1+x}$, using Maclaurin's theorem
(b) Use Maclaurin's theorem to expand $\tan x$ in ascending powers of x up to the term in x^3

[2006, No. 14: Ans: (a) $1 - x + x^2$ (b) $x + \frac{x^3}{3}$]

55. Find:
(a) $\int \ln x^2\,dx$
(b) $\int \frac{dx}{e^x-1}$

[2006, No. 16: Ans: (a) $2x\ln(x-1)+c$ (b) $\ln(1-e^{-x})+c$]

56. Given the curve $y = \sin 3x$, find the
(a) (i) value of $\frac{dy}{dx}$ at the point $\left(\frac{\pi}{3}, 0\right)$
(ii) equation of the tangent to the curve at this point
(b) (i) Sketch the curve $y = \sin 3x$
(ii) Calculate the area bounded by the tangent in (a) (i) above, the curve and the y-axis

[2005, No. 13: Ans: (a) (i) -3 (ii) $y = -3x + \pi$ (b) (ii) 0.9783 sq.units]

57. Express $\frac{3x^2+x+1}{(x-2)(x+1)^3}$ into partial fractions
Hence evaluate $\int_3^4 \frac{3x^2+x+1}{(x-2)(x+1)^3}dx$
Give your answer correct to 3 decimal places.

[2005, No. 15: Ans: 0.317]

58. (a) Solve the differential equation
$$\frac{1}{x}\frac{dy}{dx} = \sin x\sec^2 3y$$
(b) A hot body at a temperature of 100°C is placed in a room of temperature of 20°C. Ten minutes later, its temperature is 60°C.
(i) Write down a differential equation to represent the rate of change of temperature, θ of the body with time, t.
(ii) Determine the temperature of the body after a further ten minutes

[2005, No. 16: Ans: (a) $\frac{y}{2} + \frac{1}{12}\sin 6y = \sin x - x\cos x + c$ (b) (ii) 40°C]

59. Express $f(x) = \frac{x^2-4}{(x+1)^2(x-5)}$ in partial fractions
Hence evaluate $\int_6^7 f(x)\,dx$ correct to 4 decimal places.

[2004, No. 12: Ans: $\frac{5}{12(x+1)} + \frac{1}{2(x+1)^2} + \frac{7}{12(x-5)}$; 0.4689]

60. (a) Differentiate the following with respect to x:
(i) $(\sin x)^x$
(ii) $\frac{(x+1)^2}{(x+4)^3}$
giving your answers to their simplest forms
(b) The distance of a particle moving in a straight line from a fixed point after time t is given by
$$x = e^{-t}\sin t$$
Show that the particle is instantaneously at rest at time $t = \frac{\pi}{4}$ seconds.
Find its acceleration at $t = \frac{\pi}{4}$ seconds.

[2004, No. 14: Ans: (a) (i) $(\sin x)^x[x\cot x + \ln\sin x]$ (ii) $\frac{(5-x)(x+1)}{(x+4)^4}$ (b) -0.6447]

61. (a) Solve the differential equation

$$\tan x\frac{dy}{dx} - y = \sin^2 x$$

(b) An athlete runs at a speed proportional to the square root of the distance he still has to cover. If the athlete starts running at $10\ \text{ms}^{-1}$ and has a distance of 1600 m to cover, find how long he will take to cover this distance.

[2004, No. 16: Ans: (a) $y = \sin^2 x + c\sin x$ (b) 320 s]

62. Determine the nature of the turning points of the curve

$$y = \frac{x^2 - 6x + 5}{(2x-1)}$$

Sketch the graph of the curve for $x = -2$ to $x = 7$. State any asymptotes.

[2003, No. 13: Ans: $x = 0, y = -2$]

63. (a) Find $\int x^3 e^{x^4}\,dx$

(b) Use the substitution $t = \tan x$ to find $\int \frac{1}{1+\sin^2 x}dx$

[2003, No. 15: Ans: (a) $\frac{1}{4}e^{x^4} + c$ (b) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\tan x\right) + C$]

64. (a) Solve the differential equation $\frac{dR}{dt} = e^{2t} + t$, given that $R(0) = 3$

(b) The acceleration of a particle after time t seconds is given by $a = 5 + \cos\frac{1}{2}t$. If initially the particle is moving at $1\ \text{ms}^{-1}$, find its velocity after 2π seconds and the distance it would have covered by then.

[2003, No. 16: Ans: (a) $R = \frac{1}{2}e^{2t} + \frac{1}{2}t^2 + \frac{5}{2}$ (b) $10\pi + 1\ \text{ms}^{-1}$, $10\pi^2 + 2\pi + 4$]

65. (a) Use $t = \tan\frac{\theta}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{d\theta}{3-\cos\theta}$

(b) Integrate the following with respect to x:

(i) $\ln x$ (ii) $x^2\sin 2x$

[2002, No. 14: Ans: (a) 0.6755 (b) (i) $x(\ln x - 1) + c$ (ii) $\frac{-x^2}{2}\cos 2x + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + c$]

66. Given the curve $y = \frac{x(x-3)}{(x-1)(x-4)}$,

(a) show that the curve does not have turning points

(b) find the equations of the asymptotes. Hence sketch the curve.

[2002, No. 15: Ans: (b) $x = 1, x = 4, y = 1$]

67. (i) The volume of a water reservoir is generated by rotating the curve $y = kx^2$ about the y-axis. Show that when the central depth of the water in the reservoir is h metres, the surface area A is proportional to h and the volume v is proportional to h^2.

(ii) If the rate of loss of water from the reservoir due to evaporation is $\lambda A\ \text{m}^2$ per day, obtain a differential equation for h after t days. Hence deduce that the depth of water decreases at a constant rate.

(iii) Given that $\lambda = \frac{1}{2}$, determine how long it will take for the depth of water to decrease from 20 m to 2 m

[2002, No. 16: Ans: (i) $v = \frac{\pi}{2k}h^2$ (ii) $\frac{dh}{dt} = -\lambda$ (iii) 36 days]

68. (a) Using calculus of small increments, or otherwise, find $\sqrt{98}$ correct to one decimal place.

(b) Use Maclaurins' theorem to expand $\ln(1 + ax)$, where a is a constant.

Hence or otherwise expand $\ln\left(\frac{(1+x)}{\sqrt{1-2x}}\right)$ up to the term in x^3

For what values of x is the expansion valid?

[2001, No. 11: Ans: (a) 9.9 (b) $ax - \frac{a^2x^2}{2} + \frac{a^3x^3}{3} + \cdots$, $2x + \frac{x^2}{2} + \frac{5x^3}{3} + \cdots$, $|x| < \frac{1}{2}$]

69. (i) Find the Cartesian equation of the curve given parametrically by:

$$x = \frac{1+t}{1-t}, \qquad y = \frac{2t^2}{1-t}$$

(ii) Sketch the curve

(iii) Find the area enclosed between the curve and the line $y = 1$

[2001, No. 15: Ans: (iii) 1.9548 sq. units]

70. (a) Integrate $\frac{2x}{\sqrt{(x^2+4)}}$ with respect to x

(b) Evaluate $\int_0^{\pi/6}\sin x\sin 3x\,dx$

(c) Using the substitution $x = 3\sin\theta$, evaluate

$$\int_0^3 \sqrt{\frac{3+x}{3-x}}\,dx$$

[2001, No. 16: Ans: (a) $2\left(\sqrt{x^2+4}\right) + c$ (b) 0.1083 (c) 7.7124]

71. Express $f(x) = \frac{6x}{(x-2)(x+4)^2}$ into partial fractions

Hence evaluate $\int f(x)\,dx$

[2000, No. 14: Ans: $\frac{1}{3(x-2)} - \frac{1}{3(x+4)} + \frac{4}{(x+4)^2}$; $\ln\left(\frac{x-2}{x+4}\right)^{\frac{1}{3}} - \frac{4}{x+4} + C$]

72. Show that the tangent to the curve $4 - 2x - 2x^2$ at points $(-1, 4)$ and $\left(\frac{1}{2}, 2\frac{1}{2}\right)$ respectively, pass through the point the point $\left(-\frac{1}{4}, 5\frac{1}{2}\right)$. Calculate the area of the curve enclosed between the curve and the x-axis.

[2000, No. 15: Ans: 9sq. units]

73. (a) An inverted cone with a vertical angle of $60°$ is collecting leaking from a tap at a rate of $0.2\ \text{cm}^3\text{s}^{-1}$. If the height of water collected in the cone is 10 cm, find the rate at which the surface area of water is increasing.

(b) Given that $y = e^{\tan x}$ show that $\frac{d^2y}{dx^2} - (2\tan x + \sec^2 x)\frac{dy}{dx} = 0$

[2000, No. 16: Ans: (a) $0.12\ \text{cm}^2\text{s}^{-1}$]

[1996, No. 14: Ans: (i) max $(-3,-4)$ min $\left(3,-\frac{1}{4}\right)$ (ii) $x=-9, x=-1, y=1$]

87. (a) Find the general solution of the equation

$$x\frac{dy}{dx}-2y=(x-2)e^x$$

(b) The rate of cooling of a body is given by the equation $\frac{dT}{dt}=-k(T-10)$ where T is the temperature in degrees Centigrate, k is a constant, and t is the time in minutes.
When $t=0, T=90$ and when $t=5, T=60$. Find T when $t=10$.

[1996, No. 15: Ans: (a) $y=e^x+cx^2$ (b) 41.25℃]

88. Show that $f(x)=\frac{x(x-5)}{(x-3)(x+2)}$ has no turning points

Sketch the curve $y=f(x)$. If $g(x)=\frac{1}{f(x)}$, sketch the curve $y=g(x)$ on the same axes.
Show the asymptotes and where $f(x)$ and $g(x)$ intersect

[1995, No. 6]

89. (a) (i) Show that $\frac{d}{dx}(a^x)=a^x\ln a$

(ii) Find $\int 3^{\sqrt{2x-1}}\,dx$

(b) A shell is formed by rotating the portion of the parabola $y^2=4x$ for which $0\le x\le 1$ through two right angles about its axis.
Find
(i) the volume of the solid formed
(ii) the area of the base of the solid formed

[1995, No. 9: Ans: (a) (ii) $\frac{3^{\sqrt{2x-1}}}{\ln 3}\left(\sqrt{2x-1}-\frac{1}{\ln 3}\right)+c$ (b) (i) 6.2832 cubic units (ii) 12.5664 sq. units]

90. Express $\frac{x^3-3}{(x-2)(x^2+1)}$ as partial fractions

Hence or otherwise find $\int\frac{x^3-3}{(x-2)(x^2+1)}dx$

[1995, No. 10: Ans: $x+\ln(x-2)+\frac{1}{2}\ln(x^2+1)+\tan^{-1}x+c$]

91. (a) (i) If $x^2\sec x-xy+2y^2=15$, find $\frac{dy}{dx}$

(ii) Given that $y=\theta-\cos\theta$; $x=\sin\theta$, show that $\frac{d^2y}{dx^2}=\frac{1+\sin\theta}{\cos^3\theta}$

(b) Determine the maximum and minimum values of x^2e^{-x}

[1995, No. 11: Ans: (a) (i) $\frac{y-x^2\sec x\tan x-2x\sec x}{4y-x}$ (b) 0, 0.5413]

92. (a) Differentiate:
(i) $e^{ax}\sin bx$
(ii) $\frac{(x+1)^2(x+2)}{(x+3)^3}$
giving your answers to the simplest form

(b) Given that $y=e^{\tan^{-1}x}$, show that

$$(1+x^2)\frac{d^2y}{dx^2}+(2x-1)\frac{dy}{dx}=0$$

Hence or otherwise, determine the first four non-zero terms of the Maclaurin's expansion of y.

[1994, No. 9: Ans: (a) (i) $e^{ax}\sin bx\,(a+b\cot bx)$ (ii) $\frac{(x+1)(5x+9)}{(x+3)^4}$ (b) $1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots$]

93. (a) Determine the equation of the normal to the curve $y=\frac{1}{x}$ at the point $x=2$. Find the coordinates of the other point where the normal meets the curve again

(b) Find the area of the region bounded by the curve $y=\frac{1}{x(2x+1)}$, the x-axis and the lines $x=1, x=2$

[1994, No. 11: Ans: (a) $\left(-\frac{1}{8},-8\right)$ (b) 0.1823 sq. units]

94. Show that the curve $y=\frac{x+1}{x^2+2x}$ has no turning points. Sketch the curve. Give the equations of the asymptotes.

[1994, No. 12]

95. Use the Maclaurin's theorem to show that the expansion $e^{-x}\sin x$ up to the term in x^3 is $\frac{x}{3}(x^2-3x+3)$.

Hence evaluate $e^{-\frac{\pi}{3}}\sin\frac{\pi}{3}$ to 4 decimal places

[1993, No. 5: Ans: 0.334]

96. (a) Differentiate with respect to x:
(i) $\tan^{-1}\left(\frac{6x}{1-2x^2}\right)$
(ii) $(\cos x)^{2x}$

(b) Write down the expression for the volume v, and surface area s of a cylinder of radius r and height h.
If the surface area is kept constant, show that the volume of the cylinder will be maximum when $h=2r$

[1993, No. 6: Ans: (i) $\frac{6+12x^2}{1+32x^2+4x^4}$ (ii) $2(\cos x)^{2x}(\ln\cos x-x\tan x)$]

97. Find: (i) $\int\ln(x^2-4)\,dx$
(i) $\int\frac{dx}{3-2\cos x}$
(ii) Use the substitution $x=\frac{1}{u}$ to evaluate $\int_1^2\frac{dx}{x\sqrt{x^2-1}}$

[1993, No. 7: Ans: (i) $x\ln(x^2-4)-2x+2\left[\ln\left(\frac{x+2}{x-2}\right)\right]+c$ (ii) $\frac{2}{\sqrt{5}}\tan^{-1}\left[\sqrt{5}\tan\frac{x}{2}\right]+c$ (iii) $\frac{\pi}{3}$]

98. A curve is given by the parametric equations $x=4\cos 2t$, $y=2\sin t$
(i) Find the equation of the normal to the curve at $t=\frac{5}{6}\pi$
(ii) Sketch the curve for $-\frac{\pi}{2}<t<\frac{\pi}{2}$
(iii) Find the area enclosed by the curve and the y-axis.

[1993, No. 8: Ans: (i) $y=4x-7$ (iii) 7.6425 sq. units]

About the book

Principles of Pure Mathematics is a master piece that has been written to match the requirements of the new A-Level Curriculum as stipulated by the NCDC.

It covers in a single volume all the pure mathematics required by students taking mathematics at advanced level and further still providing a basis for mathematics encountered at higher levels of learning.

The book has been organised into five sections: Algebra, Coordinate geometry, Vectors, Trigonometry and Calculus. It has been written with a lot of experience and based on extensive research. It therefore offers most of (if not all) the needs of the learners and teachers of pure mathematics and is inevitably a must-have.

Presentation of mathematics concepts is comprehensive, clear, accurate and easy to understand. Every chapter has a relatively large number of worked examples and self-evaluative exercises. Relevant examination past paper questions have as well been included at the end of each section.

ISBN 978-9970-9920-1-0

Published by
Scofield International

+256 782 421 905