SCIENCESPARK
ADVANCED MATHEMATICS
PAPER 1

And its application in solving regional problems.



SENIOR 5 & 6

BY WALUGEMBE LAWRENCE

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Our Vision

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To transform lives in underserved communities by promoting science-driven health initiatives, advancing equitable STEM education, and fostering sustainable agriculture through research, innovation, and inclusive outreach.

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2. To establish and support STEM education initiatives that foster scientific literacy, critical thinking, and innovation among children and youth.
3. To promote sustainable agricultural practices that enhance food security, increase farmer incomes, and preserve the environment.

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- **Empowerment**: Building local capacity and promoting community ownership.
- **Collaboration**: Working with diverse stakeholders to maximize impact and sustainability.
- **Excellence**: Striving for high quality and continuous improvement.

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Dedication

Dedicated to all learners and teachers of Mathematics across East Africa,
who believe that mathematics is not just a subject, but a language
through which we solve our region's real-life problems.

Acknowledgement

I extend my sincere appreciation to the Almighty God for the gift of life and wisdom to complete this work.

My heartfelt thanks go to the team at **Sciencespark Foundation Uganda** for their tireless support, encouragement, and commitment to improving STEM education in Uganda and East Africa.

Special thanks to the learners, teachers, and schools who continuously provided feedback and shared the classroom realities that shaped this competence-based book.

I also acknowledge the contributions of Dr. Olivia Nabawanda (Phd), friends and colleagues from Mbarara University of Science and Technology, Kampala University and all those who believed in this initiative.

May this work inspire many to pursue mathematics with confidence and purpose.

0.1 Welcome to Advanced Mathematics!

I expect that most people reading this book will be teachers handling Advanced level and 'new' students from Ordinary level. So I feel that this is a great opportunity to welcome and introduce you to the wide and wonderful world of mathematics.

Dear students, on excellence, mathematics can place you in different careers such as;

- STEM technicians for example Engineer (Civil, Mechanical, Electrical, Software, etc.), Data Scientist , Analyst, Statistician, Actuary, Computer Programmer , Software Developer, Mathematician , Mathematical Researcher, Architect (with additional design training), Surveyor , Geospatial Analyst, Robotics , AI Specialist, Cryptographer , Cybersecurity Analyst
- Business and finance careers like Economist, Entrepreneur, Financial Analyst, Investment Banker, Accountant , Auditor, Risk Analyst, Insurance Underwriter, Quantitative Analyst, Operations Research Analyst, Business Intelligence Specialist
- Education careers like Mathematics Teacher, University Lecturer, Curriculum Developer, Educational Technologist, Academic Researcher, Examiner , Assessment Specialist, EdTech Innovator
- Applied sciences like Meteorologist, Climate Analyst, Demographer, Environmental Modeler, Biostatistician, Epidemiologist (with biology background), Operations Manager (logistics, supply chain)
- Creative fields like Game Developer (math-heavy engines) 3D Animator / Simulation Designer, UX Designer (data-driven design), Digital Product Manager, Mathematical Artist / Visualizer
- Legal careers like Patent Attorney (technical law), Policy Analyst (economic modeling), Legal Consultant for Tech Firms, NGO Program Evaluator (quantitative impact)

0.2 Preface

This textbook has been carefully designed to meet the needs of Senior Five students studying Principal Mathematics in Uganda and the East African region. Each chapter provides comprehensive coverage of essential mathematical concepts with practical applications relevant to our local context. The book emphasizes problem-solving skills through scenario-based questions that address real societal challenges in Uganda and East Africa, including agricultural optimization, urban planning, economic analysis, and scientific research applications.

Mathematics is not just an academic subject but a powerful tool for solving real-world scenarios. This textbook bridges the gap between theoretical knowledge and practical application, preparing students for both academic success and meaningful contributions to their communities.

To students, revise this manual while relating every concept with real life applications.

To teachers, this is your item bank with expected responses.

To school administrators and directors, this is a must have school library book.

0.3 40 Tips for Excelling in Mathematics

Essential Strategies for Mastery height 1pt

1. **Understand Concepts Deeply.** True mastery comes from understanding why formulas work, not just what they are. Focus on the underlying theory.
2. **Practice Mathematics Daily.** Mathematics is a skill; dedicated daily practice sharpens reasoning, speed, and recall.
3. **Always Show Full Working.** Present clear, logical, step-by-step solutions. This earns process marks and helps in self-correction.
4. **Analyze and Correct Mistakes.** Review all errors immediately after a test or practice session to diagnose the source of misunderstanding.
5. **Read Questions Carefully.** Take time to interpret every word. Misunderstanding the required outcome is a common pitfall.
6. **Create a Formula Summary Sheet.** Compile a concise, organized list of all key formulas, theorems, and identities for quick review.
7. **Attempt Varied Past Papers.** Practice with past examination questions to build confidence and familiarize yourself with exam formats and trends.
8. **Deconstruct Problem Statements.** Before calculating, identify what is given, what is required, and which key concept applies.
9. **Engage in Group Discussions.** Discussing problems with peers exposes you to different perspectives and alternative solution methods.
10. **Keep Neat, Organized Notes.** Maintain a structured notebook with clear headings and examples, making revision efficient.
11. **Self-Derive Core Formulas.** Actively attempt to prove or derive main results (e.g., Quadratic Formula). This deeply embeds the knowledge.
12. **Utilize Visual Aids (Graphs/Diagrams).** Sketching functions, vectors, or geometrical figures simplifies abstract problems and aids intuition.
13. **Develop Mental Arithmetic Skill.** Limit calculator dependence for basic calculations to strengthen number sense and speed.
14. **Relate Math to Real-Life Contexts.** Connect concepts to practical applications (e.g., finance, physics, architecture) to make them meaningful.
15. **Implement a Consistent Study Plan.** Create a detailed timetable balancing new learning, practice, revision, and rest.
16. **Systematically Check Units and Signs.** Double-check that all units are consistent and pay rigorous attention to positive and negative signs.
17. **Solve a Wide Variety of Problems.** Tackle questions from multiple textbooks and resources to improve adaptability to different phrasing and difficulty levels.
18. **Maintain Study Consistency.** Little and often is better than massed study. Daily effort ensures continuous reinforcement.
19. **Review Immediately After Class.** Spend 10-15 minutes reviewing the new topic after the lesson to solidify short-term memory into long-term.
20. **Consult Multiple Learning Resources.** Use textbooks, online videos, and different teachers' explanations for a holistic understanding.
21. **Color-Code Key Information.** Use highlighters to mark crucial formulas, definitions, and typical steps in your notes for quick scanning.
22. **Avoid Blindly Copying Solutions.** Always attempt a problem yourself first. Only consult the solution if you are completely stuck or to check the final answer.
23. **Ask Questions to Clarify Doubts.** Address any lack of understanding immediately with a teacher or study partner before moving on to new material.
24. **Break Down Complex Problems.** Segment hard problems into smaller, manageable sub-problems, solving one piece at a time.
25. **Teach Concepts to Others.** Explaining a topic requires you to organize your thoughts, reinforcing your own knowledge base.
26. **Verify Answer Plausibility.** After finding a solution, check if the magnitude and sign of the answer make sense within the problem context.
27. **Master Fundamental Algebra.** Ensure strong skills in substitution, factorizing, expanding brackets, and manipulating equations, as these underpin most topics.
28. **Identify Core Exam Topics.** Analyze past paper

frequency to prioritize your study time on the most commonly tested concepts.

29. **Simulate Exam Conditions.** Practice full-length past papers under timed, silent conditions to build exam stamina and speed.
30. **Use the Power of Analogy.** When struggling with a concept, try to find a simpler, familiar example that uses the same underlying mathematical principle.
31. **Maintain a Dedicated Error Log.** Keep a notebook of all questions you've gotten wrong, categorized by topic, and re-attempt them regularly.
32. **Utilize Technology Wisely.** Use software (like GeoGebra or graphing calculators) to explore concepts visually, but not as a substitute for manual work.
33. **Focus on Precision in Language.** In theoretical questions, use precise mathematical terminology and definitions correctly.
34. **Allocate Specific Time for Revision.** Schedule dedicated time slots for reviewing old material, not just learning new concepts.
35. **Prioritize Understanding Over Speed.** In initial learning, focus on accuracy and logical flow; speed will naturally increase with practice.
36. **Manage Test Anxiety.** Develop pre-test routines (e.g., deep breathing, light review) to manage stress and improve focus.
37. **Re-Write/Simplify Complex Notes.** After initial note-taking, try to re-write the summary into the simplest terms possible for maximum clarity.
38. **Master the Art of Estimation.** Learn to quickly estimate answers before calculating to catch major arithmetic errors early.
39. **Practice Time Management During Exams.** Allocate time to each question based on its mark value and stick strictly to your plan during the test.
40. **Cultivate a Growth Mindset.** Believe that your mathematical ability can be improved through persistent effort and effective strategies.

0.4 New Assessment of Principal Mathematics at Advanced Secondary Level

By the end of this chapter, the reader should be able to appreciate the mode of assessing learners in Principal Mathematics under the Adapted Competency-based Curriculum at the advanced secondary level.

NCDC, March 2026

0.4.1 Introduction

Assessment of Principal Mathematics focuses on determining the ability of the learner to apply algebraic reasoning, geometrical thinking, statistical interpretation, principles of mechanics, and analytical and numerical modelling to solve real-world problems, make logical and data-driven decisions, and predict outcomes and behaviours in diverse contexts.

0.4.2 Assessment Objectives

The end-of-cycle assessment for Principal Mathematics is guided by five assessment objectives (AO), each focusing on a distinct domain of the learner's mathematical competence.

AO1	Manipulate a wide range of algebraic expressions and equations, evaluate logical proofs, and apply these skills to strategically solve real-world problems.
AO2	Perform transformations, reason logically, and visualise spatial relationships in solving geometrical challenges in the real world.
AO3	Apply analytical and numerical skills in mathematical modelling to solve real-life challenges involving rates of change, accumulation, and optimisation for decision making.
AO4	Predict future trends by analysing and interpreting statistical data using principles of statistics and probability to make reasonable conclusions.
AO5	Apply principles of statics and dynamics to analyse and solve real-world problems involving the state of motion of bodies, and to predict their behaviours.

0.4.3 The Five Constructs

The curriculum organises the subject matter into five constructs. Each construct bundles related topics from the syllabus and maps to one of the assessment objectives above.

Construct 1: Algebra

Construct description. Applying algebraic principles to model and solve real-life problems and lead to informed decision making.

Topics in the syllabus.

1. Numerical concepts
2. Equations and inequalities
12. Permutations and combinations
13. Series
25. Complex numbers

Construct 2: Geometry

Construct description. Applying geometric concepts and spatial reasoning to interpret relationships in mathematical and real-life contexts for informed problem solving.

Topics in the syllabus.

3. Coordinate geometry 1
5. Trigonometry
17. Vectors
24. Coordinate geometry 2

Construct 3: Calculus

Construct description. Analysing and modelling real-life situations involving rates of change, accumulation, and optimisation of resources.

Topics in the syllabus.

4. Partial fractions
10. Differentiation 1
11. Integration 1
18. Differentiation 2
19. Integration 2
26. Differential equations

Construct 4: Data Analysis and Probability

Construct description. Analysing data and applying probability models to make informed decisions and predictions in personal, academic, and societal contexts.

Topics in the syllabus.

6. Descriptive statistics
7. Correlation and scatter diagrams
9. Probability theory
14. Random variables
15. Probability distributions
16. Error analysis
21. Trapezium rule
22. Sampling distribution
23. Iterative methods
27. Flow charts

Construct 5: Mechanics

Construct description. Analysing the effect of forces on bodies in motion or at rest, interpreting motion patterns, predicting object behaviour, and solving real-world kinematic problems.

Topics in the syllabus.

8. Dynamics 1
20. Dynamics 2

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Chapter 1

Numerical concepts

Competency:
The learner analyses contextual use of indices, logarithms and surds through mathematical computations for precision, developing analytical skills and making informed decisions to communicate effectively.

1.1 Reviewing powers as indices, laws of indices and simplifying expressions

1.2 The relationship between logarithms and indices, logarithmic notation and laws of logarithms

1.3 Applying laws of indices and logarithms to solve equations involving indices and logarithms

1.4 Applications of the concepts of indices and logarithms in real life

1.5 Reviewing rational and irrational numbers, simplifying and rationalising surds

1.6 Solving equations involving surds

1.7 Applications of surds in real life

Laws of Indices

Index Notation

An **index** (or exponent) indicates how many times a base is multiplied by itself. In a^n, a is the base and n is the index. Example: $5^3 = 5 \times 5 \times 5 = 125$.

Fundamental Laws of Indices

For real numbers $a, b \neq 0$ and rational numbers m, n:

1. $a^m \times a^n = a^{m+n}$
2. $\dfrac{a^m}{a^n} = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. $(ab)^n = a^n b^n$
5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$
6. $a^0 = 1$
7. $a^{-n} = \dfrac{1}{a^n}$
8. $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

Examples: Simplifying Expressions

Ex 1: $3^5 \times 3^3 = 3^{5+3} = 3^8 = 6561$

Ex 2: $\dfrac{7^9}{7^4} = 7^{9-4} = 7^5 = 16807$

Ex 3: $(2^3)^4 = 2^{12} = 4096$

Ex 4: $\dfrac{x^8 \times x^{-3}}{x^2} = \dfrac{x^5}{x^2} = x^3$

Ex 5: $16^{\frac{3}{4}} = (16^{\frac{1}{4}})^3 = 2^3 = 8$

Ex 6: $27^{-\frac{2}{3}} = \dfrac{1}{27^{\frac{2}{3}}} = \dfrac{1}{(27^{\frac{1}{3}})^2} = \dfrac{1}{9}$

Solving Exponential Equations

Ex 7: Solve $2^{x+3} = 32$

$2^{x+3} = 2^5 \Rightarrow x + 3 = 5 \Rightarrow x = 2$

Ex 8: Solve $3^{2x} \times 3^{x-1} = 81$

$3^{3x-1} = 3^4 \Rightarrow 3x - 1 = 4 \Rightarrow x = \dfrac{5}{3}$

Ex 9: Solve $4^x = 8^{x-2}$

$(2^2)^x = (2^3)^{x-2} \Rightarrow 2^{2x} = 2^{3x-6} \Rightarrow 2x = 3x - 6 \Rightarrow x = 6$

Logarithms

Definition

A **logarithm** is the inverse of exponentiation. If $a^x = N$, then $\log_a N = x$ where $a > 0, a \neq 1, N > 0$.

Relationship: $\boxed{a^x = N \iff \log_a N = x}$

Special types:

- Common logarithm: $\log_{10} x$ (written as $\log x$)
- Natural logarithm: $\log_e x$ (written as $\ln x$)

Conversion Examples

$2^5 = 32 \iff \log_2 32 = 5$

$\log_3 81 = 4 \iff 3^4 = 81$

$10^{-2} = 0.01 \iff \log_{10} 0.01 = -2$

Laws of Logarithms

For positive M, N and base $a > 0, a \neq 1$:

1. Product: $\log_a(M \times N) = \log_a M + \log_a N$
2. Quotient: $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$
3. Power: $\log_a(M^n) = n \log_a M$
4. Base change: $\log_a M = \frac{\log_b M}{\log_b a}$
5. Special: $\log_a a = 1, \log_a 1 = 0, a^{\log_a M} = M$

Examples: Simplifying Logarithms

Ex 10: $\log_2 8 + \log_2 4 = \log_2(8 \times 4) = \log_2 32 = 5$

Ex 11: $\log_5 125 - \log_5 25 = \log_5(125/25) = \log_5 5 = 1$

Ex 12: $3 \log_2 4 - 2 \log_2 8 = \log_2 64 - \log_2 64 = 0$

Solving Logarithmic Equations

Ex 13: Solve $\log_x 64 = 3$

$x^3 = 64 \Rightarrow x = 4$

Ex 14: Solve $\log_2(x + 3) + \log_2(x - 3) = 4$

$\log_2[(x + 3)(x - 3)] = 4 \Rightarrow x^2 - 9 = 16 \Rightarrow x^2 = 25 \Rightarrow x = 5$ (reject $x = -5$)

Ex 15: Solve $2 \log_3 x = \log_3(2x + 5)$

$\log_3 x^2 = \log_3(2x+5) \Rightarrow x^2 = 2x+5 \Rightarrow x^2-2x-5 = 0$

$x = \frac{2 \pm \sqrt{24}}{2} = 1 \pm \sqrt{6}$. Since $x > 0$, $x = 1 + \sqrt{6}$

Real-World Applications

1. Population Growth: $P(t) = P_0 e^{rt}$ where P_0 is initial population, r is rate, t is time.

Example: Bacteria doubles every 3 hours. Starting with 500, after 12 hours: $P = 500 \times 2^{12/3} = 500 \times 16 = 8000$

2. Radioactive Decay: $N(t) = N_0 e^{-\lambda t}$

Example: Carbon-14 half-life is 5730 years. For 75% decay (25% remains): $0.25 = (0.5)^{t/5730} \Rightarrow (0.5)^2 = (0.5)^{t/5730} \Rightarrow t = 11460$ years

3. Sound (Decibels): $L = 10 \log_{10}(I/I_0)$

4. pH in Chemistry: $\text{pH} = -\log_{10}[\text{H}^+]$

5. Compound Interest: $A = P(1 + r/n)^{nt}$ or continuous: $A = Pe^{rt}$

Surds

Definitions

Rational number: Can be expressed as p/q where p, q are integers, $q \neq 0$. Examples: $3/4, -2, 0.5, \sqrt{4} = 2$

Irrational number: Cannot be expressed as a fraction. Non-terminating, non-repeating decimals. Examples: $\sqrt{2}, \sqrt{3}, \pi, e$

Surd: An irrational root of a rational number. Examples: $\sqrt{2}, \sqrt{5}, \sqrt[3]{7}, 2\sqrt{3}$

Note: $\sqrt{4} = 2$ is NOT a surd (it's rational).

Rules for Surds

For positive real numbers a, b:

1. $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
2. $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ (where $b \neq 0$)
3. $\sqrt{a^2} = a$ (for $a \geq 0$)
4. $(\sqrt{a})^2 = a$
5. $\sqrt{a} \pm \sqrt{b} \neq \sqrt{a \pm b}$

Simplifying Surds

Ex 16: $\sqrt{72} = \sqrt{36 \times 2} = 6\sqrt{2}$

Ex 17: $\sqrt{200} = \sqrt{100 \times 2} = 10\sqrt{2}$

Ex 18: $3\sqrt{50} + 2\sqrt{8} = 3(5\sqrt{2}) + 2(2\sqrt{2}) = 15\sqrt{2} + 4\sqrt{2} = 19\sqrt{2}$

Ex 19: $\sqrt{75} - \sqrt{27} + \sqrt{12} = 5\sqrt{3} - 3\sqrt{3} + 2\sqrt{3} = 4\sqrt{3}$

Operations with Surds

Ex 20: $\sqrt{3} \times \sqrt{12} = \sqrt{36} = 6$

Ex 21: $2\sqrt{5} \times 3\sqrt{10} = 6\sqrt{50} = 6(5\sqrt{2}) = 30\sqrt{2}$

Ex 22: $\dfrac{\sqrt{48}}{\sqrt{3}} = \sqrt{16} = 4$

Rationalising the Denominator

Single surd: Multiply by $\sqrt{a}/\sqrt{a}$

Ex 23: $\dfrac{5}{\sqrt{3}} = \dfrac{5\sqrt{3}}{3}$

Ex 24: $\dfrac{6}{\sqrt{8}} = \dfrac{6\sqrt{8}}{8} = \dfrac{12\sqrt{2}}{8} = \dfrac{3\sqrt{2}}{2}$

Binomial: Use conjugate $(a+b)(a-b) = a^2 - b^2$

Ex 25: $\dfrac{1}{\sqrt{5}+\sqrt{2}} = \dfrac{1(\sqrt{5}-\sqrt{2})}{(\sqrt{5})^2 - (\sqrt{2})^2} = \dfrac{\sqrt{5}-\sqrt{2}}{3}$

Ex 26: $\dfrac{3+\sqrt{2}}{2-\sqrt{3}} = \dfrac{(3+\sqrt{2})(2+\sqrt{3})}{4-3} = 6 + 3\sqrt{3} + 2\sqrt{2} + \sqrt{6}$

Solving Equations with Surds

Ex 27: Solve $\sqrt{x+5} = 7$

$x + 5 = 49 \Rightarrow x = 44$. Check: $\sqrt{49} = 7$

Ex 28: Solve $\sqrt{2x-1} = x - 2$

$2x-1 = x^2-4x+4 \Rightarrow x^2-6x+5 = 0 \Rightarrow (x-5)(x-1) = 0$

Check $x = 5$: $\sqrt{9} = 3$ and $5 - 2 = 3$. Check $x = 1$: $\sqrt{1} = 1$ but $1 - 2 = -1$

Answer: $x = 5$

Ex 29: Solve $\sqrt{x+1} + \sqrt{x-2} = 3$

$\sqrt{x+1} = 3 - \sqrt{x-2} \Rightarrow x + 1 = 9 - 6\sqrt{x-2} + x - 2$

$-6 = -6\sqrt{x-2} \Rightarrow 1 = \sqrt{x-2} \Rightarrow x = 3$. Check: $2 + 1 = 3$

Real-World Applications

1. Diagonal of square: For side a, diagonal $d = a\sqrt{2}$

Example: 30 cm square has diagonal $30\sqrt{2}$ cm ≈ 42.43 cm

2. Pythagorean Theorem: $c = \sqrt{a^2 + b^2}$

Example: Ladder 6 m from wall reaches 8 m up. Length = $\sqrt{36 + 64} = 10$ m

3. Distance formula: Between (x_1, y_1) and (x_2, y_2): $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

4. Golden ratio: $\phi = \dfrac{1+\sqrt{5}}{2} \approx 1.618$ (used in art and architecture)

5. Electrical impedance: $Z = \sqrt{R^2 + X^2}$

Example: $R = 8\Omega$, $X = 6\Omega$ gives $Z = 10\Omega$

6. 3D distance: For point (x, y, z) from origin: $d = \sqrt{x^2 + y^2 + z^2}$

Practice Problems

Indices

P1. Simplify: (a) $5^7 \times 5^{-3}$ (b) $(2^3)^4/2^5$ (c) $(3^2 \times 3^{-5})^3$ (d) $64^{2/3}$ (e) $81^{-3/4}$

P2. Solve: (a) $5^x = 125$ (b) $2^{3x-1} = 16$ (c) $9^x = 27^{x-2}$ (d) $4^{x+1} = 8^{2x-3}$

P3. Bacteria doubles every 4 hours. Starting with 1000, find population after 24 hours.

Logarithms

P4. Evaluate: (a) $\log_2 16$ (b) $\log_5(1/125)$ (c) $\log_{10} 0.001$ (d) $\log_9 27$

P5. Simplify: (a) $\log_3 27 + \log_3 9$ (b) $\log_2 64 - \log_2 8$ (c) $2\log_5 25 + 3\log_5 5$

P6. Solve: (a) $\log_2 x = 5$ (b) $\log_x 81 = 4$ (c) $\log_3(x+5) = 2$ (d) $\log_2(x+1) + \log_2(x-1) = 3$

P7. Earthquake magnitudes are 6.5 and 8.5. How many times more intense is the second?

Surds

P8. Simplify: (a) $\sqrt{128}$ (b) $\sqrt{300}$ (c) $2\sqrt{48} + 3\sqrt{12}$ (d) $\sqrt{50} - \sqrt{18} + \sqrt{8}$

P9. Rationalise: (a) $7/\sqrt{5}$ (b) $12/\sqrt{18}$ (c) $1/(\sqrt{7}+\sqrt{3})$ (d) $(\sqrt{3}+1)/(\sqrt{3}-1)$

P10. Solve: (a) $\sqrt{x+3} = 5$ (b) $\sqrt{3x-2} = x - 2$ (c) $\sqrt{x+7} + \sqrt{x} = 7$

P11. Rectangle: length $(3 + \sqrt{5})$ cm, width $(3 - \sqrt{5})$ cm. Find area and perimeter.

Solutions

S1. (a) $5^4 = 625$ (b) $2^7 = 128$ (c) $3^{-9} = 1/19683$ (d) 16 (e) 1/27

S2. (a) $x = 3$ (b) $x = 5/3$ (c) $x = 6$ (d) $x = 11/4$

S3. 64000 bacteria

S4. (a) 4 (b) -3 (c) -3 (d) 3/2

S5. (a) 5 (b) 3 (c) 7

S6. (a) $x = 32$ (b) $x = 3$ (c) $x = 4$ (d) $x = 3$

S7. 100 times more intense

S8. (a) $8\sqrt{2}$ (b) $10\sqrt{3}$ (c) $14\sqrt{3}$ (d) $4\sqrt{2}$

S9. (a) $7\sqrt{5}/5$ (b) $2\sqrt{2}$ (c) $(\sqrt{7}-\sqrt{3})/4$ (d) $2+\sqrt{3}$

S10. (a) $x = 22$ (b) $x = 6$ (c) $x = 9$

S11. Area: 4 cm^2, Perimeter: 12 cm

Summary

Indices: $a^m \times a^n = a^{m+n}$; $a^m/a^n = a^{m-n}$; $(a^m)^n = a^{mn}$; $a^0 = 1$; $a^{-n} = 1/a^n$; $a^{m/n} = \sqrt[n]{a^m}$

Logarithms: $a^x = N \iff \log_a N = x$; $\log_a(MN) = \log_a M + \log_a N$; $\log_a(M/N) = \log_a M - \log_a N$; $\log_a(M^n) = n\log_a M$

Surds: $\sqrt{a}\times\sqrt{b} = \sqrt{ab}$; $\sqrt{a}/\sqrt{b} = \sqrt{a/b}$; Rationalise using $\sqrt{a}/\sqrt{a}$ or conjugate

Applications: Growth/decay, pH, decibels, compound interest, engineering calculations, precision measurements

1.8 More scenario revision items with solutions

Question 87: Pesticide Decay in a Farm in Masaka

A farmer in Masaka sprays a new pesticide on his crops. The amount of pesticide on the crops decays exponentially. After one day, 85% of the initial amount remains. The initial amount sprayed is 500 grams.

1. Write an equation to model the amount of pesticide A remaining after t days.
2. How much pesticide will remain on the crops after 7 days?
3. The pesticide is considered harmless to consumers when the amount remaining is less than 50 grams. How many days must the farmer wait before harvesting the crops?

Solution 87

1. The decay model is $A = A_0(r)^t$, where A_0 is the initial amount, r is the daily decay factor, and t is the number of days. $A_0 = 500$ grams. After one day, 85% remains, so $r = 0.85$. The equation is $A = 500(0.85)^t$.
2. To find the amount after 7 days, substitute $t = 7$: $A = 500(0.85)^7 \approx 500(0.3205) \approx 160.25$ grams. Approximately 160.25 grams of pesticide will remain after 7 days.
3. To find the time t when $A = 50$ grams, we solve the equation: $50 = 500(0.85)^t$ $\frac{50}{500} = (0.85)^t$ $0.1 = (0.85)^t$ Taking the logarithm of both sides: $\log(0.1) = \log((0.85)^t)$ $\log(0.1) = t\log(0.85)$ $t = \frac{\log(0.1)}{\log(0.85)} \approx \frac{-1.0}{-0.0706} \approx 14.16$ days. The farmer must wait approximately 15 days before harvesting.

Question 88: Compound Interest on a Loan in Kampala

A small business owner in Kampala takes a loan of UGX 5,000,000 from a microfinance institution. The loan has an annual compound interest rate of 12%.

1. Write a formula for the total amount A owed after t years.
2. Calculate the total amount the business owner will owe after 5 years if no payments are made.
3. The owner wants to clear the loan when the total amount owed is UGX 8,000,000. How many years will this take?

Solution 88

1. The compound interest formula is $A = P(1 + r)^t$, where P is the principal, r is the annual interest rate, and t is the number of years. $P = 5,000,000$ and $r = 0.12$. The formula is $A = 5,000,000(1.12)^t$.
2. Substitute $t = 5$ into the formula: $A = 5,000,000(1.12)^5 \approx 5,000,000(1.7623) \approx$ UGX 8,811,500. The total amount owed after 5 years will be approximately UGX 8,811,500.
3. To find the time t when $A = 8,000,000$, we solve: $8,000,000 = 5,000,000(1.12)^t$ $\frac{8,000,000}{5,000,000} = (1.12)^t$ $1.6 = (1.12)^t$ Taking the logarithm of both sides: $\log(1.6) = t\log(1.12)$ $t = \frac{\log(1.6)}{\log(1.12)} \approx \frac{0.2041}{0.0492} \approx 4.15$ years. It will take approximately 4.15 years for the loan to reach UGX 8,000,000.

Question 89: Water Treatment Contamination

A water treatment plant in Uganda is testing a new purification process. The concentration of a contaminant, measured in milligrams per liter, decreases over time according to the formula $C(t) = C_0 e^{-0.2t}$, where t is the time in hours. The initial concentration, C_0, is 120 mg/L.

1. Use the formula to find the concentration of the contaminant after 5 hours.
2. The water is safe to drink when the contaminant concentration is below 10 mg/L. How long will it take for the concentration to reach this safe level?
3. A new purification system reduces the decay constant from 0.2 to 0.3. Explain the effect of this change on the time it takes to reach a safe level.

Solution 89

1. The initial concentration is $C_0 = 120$ mg/L. The formula is $C(t) = 120e^{-0.2t}$. To find the concentration after 5 hours, substitute $t = 5$: $C(5) = 120e^{-0.2(5)} = 120e^{-1} \approx 120(0.3679) \approx \mathbf{44.15}$ mg/L. The concentration will be approximately 44.15 mg/L after 5 hours.
2. To find the time t when $C(t) = 10$, we solve: $10 = 120e^{-0.2t}$ $\frac{10}{120} = e^{-0.2t}$ $0.0833 = e^{-0.2t}$ Taking the natural logarithm of both sides: $\ln(0.0833) = \ln(e^{-0.2t})$ $\ln(0.0833) = -0.2t$ $t = \frac{\ln(0.0833)}{-0.2} \approx \frac{-2.4849}{-0.2} \approx \mathbf{12.42}$ hours. It will take approximately 12.42 hours for the water to be safe to drink.
3. If the decay constant changes from 0.2 to 0.3, the new formula is $C(t) = 120e^{-0.3t}$. To find the new time to reach 10 mg/L: $10 = 120e^{-0.3t}$ $0.0833 = e^{-0.3t}$ $\ln(0.0833) = -0.3t$ $t = \frac{\ln(0.0833)}{-0.3} \approx \frac{-2.4849}{-0.3} \approx 8.28$ hours. A larger decay constant (0.3 vs. 0.2) means the contaminant is removed more quickly. As a result, the time required to reach the safe level is significantly reduced (from 12.42 hours to 8.28 hours). This shows that a more efficient purification system, with a higher decay constant, leads to faster and safer water treatment.

Question 90: Agricultural Yields

A farmer is experimenting with a new fertilizer. The yield of a crop, Y (in tonnes/hectare), depends on the amount of fertilizer applied, x (in kg), according to the logarithmic equation $Y = 5 + 3\log_{10}(x + 1)$.

1. What is the expected yield if no fertilizer is applied? (i.e., $x = 0$).
2. The farmer wants to achieve a yield of 8 tonnes/hectare. How much fertilizer should be applied to achieve this?
3. The farmer has a limited budget and can only afford to apply 50 kg of fertilizer. Based on your calculation, should the farmer increase the amount of fertilizer or try a different approach to increase the yield to 8 tonnes/hectare?

Solution 90

1. If no fertilizer is applied, $x = 0$. $Y = 5 + 3\log_{10}(0 + 1) = 5 + 3\log_{10}(1)$. Since $\log_{10}(1) = 0$, the yield is $Y = 5 + 3(0) = \mathbf{5}$ tonnes/hectare. The expected yield without fertilizer is 5 tonnes/hectare.
2. To achieve a yield of 8 tonnes/hectare, we set $Y = 8$ and solve for x: $8 = 5 + 3\log_{10}(x + 1)$ $3 = 3\log_{10}(x + 1)$ $1 = \log_{10}(x + 1)$ Converting to index form: $10^1 = x + 1$ $10 = x + 1$ $x = \mathbf{9}$ kg. The farmer should apply 9 kg of fertilizer to achieve a yield of 8 tonnes/hectare.
3. If the farmer applies 50 kg of fertilizer, we find the corresponding yield: $Y = 5 + 3\log_{10}(50 + 1) = 5 + 3\log_{10}(51)$. $Y \approx 5 + 3(1.7076) \approx 5 + 5.1228 \approx 10.12$ tonnes/hectare. Since the farmer only needs 9 kg of fertilizer to reach the target yield of 8 tonnes/hectare, applying 50 kg is an excessive and wasteful use of resources. The farmer should not increase the amount of fertilizer. Instead, they should apply only 9 kg, which is well within their budget and will give them the desired yield. This highlights the importance of using mathematical models to optimize resource use and avoid unnecessary costs.

Question 91: Radioisotope Dating of Archaeological Artifact

An archaeologist in the Rwenzori region discovers an ancient wooden artifact. Carbon-14 is used to determine its age. The amount of Carbon-14 remaining in the artifact, N (in grams), is modeled by the equation

$N(t) = N_0 e^{-0.000121t}$, where t is the age in years and N_0 is the initial amount of Carbon-14.

1. If the artifact initially had 20 grams of Carbon-14, how much would remain after 1,000 years?
2. A sample of the artifact is found to have 10 grams of Carbon-14. What is the approximate age of the artifact?
3. The archaeologist wants to date another artifact that is believed to be between 5,000 and 6,000 years old. What is the expected range of Carbon-14 remaining in this artifact?

Solution 91

1. The initial amount is $N_0 = 20$ grams. The equation is $N(t) = 20e^{-0.000121t}$. For $t = 1,000$ years: $N(1000) = 20e^{-0.000121(1000)} = 20e^{-0.121} \approx 20(0.8859) \approx$ **17.72** grams. Approximately 17.72 grams of Carbon-14 would remain.
2. To find the age t when $N(t) = 10$, we solve: $10 = 20e^{-0.000121t}$ $0.5 = e^{-0.000121t}$ Taking the natural logarithm of both sides: $\ln(0.5) = -0.000121t$ $t = \frac{\ln(0.5)}{-0.000121} \approx \frac{-0.6931}{-0.000121} \approx$ **5728.1** years. The artifact is approximately 5,728 years old.
3. We calculate the remaining amounts for $t = 5,000$ and $t = 6,000$: For $t = 5,000$: $N(5000) = 20e^{-0.000121(5000)} = 20e^{-0.605} \approx 20(0.5459) \approx 10.92$ grams. For $t = 6,000$: $N(6000) = 20e^{-0.000121(6000)} = 20e^{-0.726} \approx 20(0.4838) \approx 9.68$ grams. The expected range of Carbon-14 remaining is approximately from **9.68** to **10.92** grams.

Question 92: Investment in a SACCO

A member of a Savings and Credit Cooperative Organization (SACCO) in Kasese invests UGX 2,000,000. The SACCO offers a fixed rate of 8% compound interest per year.

1. What will the investment be worth after 15 years?
2. How many years will it take for the investment to double in value?
3. The SACCO wants to attract new members by offering a special rate that will triple their investment in 10 years. What interest rate would they need to offer to achieve this?

Solution 92

1. The value of the investment is $A = P(1 + r)^t$. $A = 2,000,000(1.08)^{15} \approx 2,000,000(3.1722) \approx$ UGX **6,344,400**. The investment will be worth approximately UGX 6,344,400 after 15 years.
2. To find the time t for the investment to double, we set $A = 2P = 4,000,000$: $4,000,000 = 2,000,000(1.08)^t$ $2 = (1.08)^t$ Taking the logarithm of both sides: $\log(2) = t\log(1.08)$ $t = \frac{\log(2)}{\log(1.08)} \approx \frac{0.3010}{0.0334} \approx$ **9.01** years. It will take approximately 9 years for the investment to double.
3. To find the rate r to triple the investment in 10 years, we set $A = 3P = 6,000,000$ and $t = 10$: $6,000,000 = 2,000,000(1+r)^{10}$ $3 = (1+r)^{10}$ Taking the 10th root of both sides, or using logarithms: $1 + r = 3^{1/10}$ $1 + r = 1.11612$ $r = 0.11612 \approx$ **11.61**%. The SACCO would need to offer an interest rate of approximately 11.61% to triple the investment in 10 years.

Question 93: Crop Disease Spread

A new crop disease is spreading in a large maize field in Soroti. The number of hectares infected, H, is modeled by the equation $H(t) = 5(1.5)^t$, where t is the number of weeks since the first sighting.

1. How many hectares were initially infected when the disease was first sighted?
2. How many weeks will it take for the disease to infect 100 hectares?
3. The district agricultural officer warns the farmer that the disease is out of control when the number of infected hectares is 20 times the initial amount. How many weeks does it take to reach this level, and what is the farmer's best course of action?

Solution 94

1. The initial number of hectares infected is when $t = 0$: $H(0) = 5(1.5)^0 = 5(1) =$ **5** hectares. Initially, 5 hectares were infected.
2. To find the time t when $H(t) = 100$, we solve: $100 = 5(1.5)^t$ $20 = (1.5)^t$ Taking the logarithm of both sides: $\log(20) = t\log(1.5)$ $t = \frac{\log(20)}{\log(1.5)} \approx \frac{1.3010}{0.1761} \approx$ **7.38** weeks. It will take approximately 7.38 weeks to infect 100 hectares.

3. The initial amount is 5 hectares, so 20 times the initial amount is $20 \times 5 = 100$ hectares. From part (b), we know it takes 7.38 weeks to reach this level. Since the spread is exponential, the disease is accelerating. The farmer's best course of action is to act immediately to prevent the spread, as delaying will lead to a rapid increase in the affected area, making control difficult and costly.

Question 95: Mobile Phone Penetration in Uganda

The number of mobile phone subscribers in a district is growing. The number of subscribers, N, is modeled by the equation $N = 5000(1.08)^t$, where t is the number of years from now.

1. What is the current number of mobile phone subscribers in the district?
2. How many years will it take for the number of subscribers to reach 15,000?
3. If a new mobile network enters the market and the growth rate changes to 1.10, how does this affect the time it takes to reach 15,000 subscribers?

Solution 95

1. The current number of subscribers is when $t = 0$: $N = 5000(1.08)^0 = 5000(1) = \mathbf{5000}$ subscribers.
2. To find the time t when $N = 15,000$, we solve: $15,000 = 5000(1.08)^t$ $3 = (1.08)^t$ Taking the logarithm of both sides: $\log(3) = t\log(1.08)$ $t = \frac{\log(3)}{\log(1.08)} \approx \frac{0.4771}{0.0334} \approx \mathbf{14.28}$ years. It will take approximately 14.28 years.
3. If the new growth rate is 1.10, the new equation is $N = 5000(1.10)^t$. To reach 15,000 subscribers: $15,000 = 5000(1.10)^t$ $3 = (1.10)^t$ Taking the logarithm of both sides: $\log(3) = t\log(1.10)$ $t = \frac{\log(3)}{\log(1.10)} \approx \frac{0.4771}{0.0414} \approx 11.52$ years. The time it takes to reach the target number of subscribers is reduced from 14.28 years to 11.52 years. This demonstrates that a higher growth rate significantly shortens the time required to achieve a specific target.

Question 96: Depreciation of a Vehicle

A small business in Mbale buys a delivery van for UGX 30,000,000. The value of the van depreciates at an annual rate of 15%.

1. Write an equation that models the value of the van V after t years.
2. What will be the value of the van after 3 years?
3. The business plans to sell the van when its value falls to UGX 10,000,000. How many years will this take?

Solution 96

1. The depreciation model is $V = V_0(1 - r)^t$. $V_0 = 30,000,000$ and $r = 0.15$. The equation is $V = 30,000,000(0.85)^t$.
2. To find the value after 3 years, substitute $t = 3$: $V = 30,000,000(0.85)^3 \approx 30,000,000(0.614125) \approx$ UGX $\mathbf{18,423,750}$. The value of the van after 3 years will be approximately UGX 18,423,750.
3. To find the time t when $V = 10,000,000$, we solve: $10,000,000 = 30,000,000(0.85)^t$ $\frac{10,000,000}{30,000,000} = (0.85)^t$ $\frac{1}{3} = (0.85)^t$ Taking the logarithm of both sides: $\log(1/3) = t\log(0.85)$ $t = \frac{\log(1/3)}{\log(0.85)} \approx \frac{-0.4771}{-0.0706} \approx \mathbf{6.76}$ years. It will take approximately 6.76 years for the van's value to fall to UGX 10,000,000.

Question 97: Growth of a Fish Population in Lake Victoria

The number of fish in a section of Lake Victoria is monitored. The population, P, is found to grow exponentially according to $P = 500 \cdot 2^{t/5}$, where t is the time in months.

1. What is the initial population of fish?
2. How many months will it take for the fish population to reach 10,000?
3. A conservation group suggests that the fish population is healthy when it reaches 10,000. How does the current growth model compare to a linear model where the population increases by 500 fish every 5 months?

Solution 97

1. The initial population is when $t = 0$: $P = 500 \cdot 2^{0/5} = 500 \cdot 2^0 = 500 \cdot 1 = \mathbf{500}$ fish.
2. To find the time t when $P = 10,000$, we solve: $10,000 = 500 \cdot 2^{t/5}$ $20 = 2^{t/5}$ Taking the logarithm base 2 of both sides: $\log_2(20) = \log_2(2^{t/5})$ $\log_2(20) = t/5$ $t = 5\log_2(20) = 5\left(\frac{\log_{10}(20)}{\log_{10}(2)}\right) \approx 5\left(\frac{1.3010}{0.3010}\right) \approx 5(4.322) \approx \mathbf{21.61}$ months. It will take approximately 21.61 months.

3. In the linear model, the population grows by 500 every 5 months. To reach 10,000 from 500, the population needs to increase by 9,500. This requires $9500/500 = 19$ growth periods. At 5 months per period, this takes $19 \times 5 = 95$ months. The current exponential model reaches 10,000 in only 21.61 months, while the linear model takes 95 months. The exponential model shows that the population grows much faster, which is more sustainable for the fish stock and a positive sign for the conservation group.

Question 98: Soil Acidity pH Levels

The pH level of soil in a farm is an important indicator of its health. pH is defined as $pH = -\log_{10}[H^+]$, where $[H^+]$ is the concentration of hydrogen ions in moles per liter. A farmer measures the hydrogen ion concentration of a soil sample as 1.5×10^{-6} moles/L.

1. Calculate the pH of the soil sample.
2. The farmer wants to change the soil to a pH of 7.0. What should the new hydrogen ion concentration be?
3. Another soil sample has a hydrogen ion concentration that is 100 times higher than the first sample. What is the new pH, and what does this change in pH signify?

Solution 98

1. The hydrogen ion concentration is $[H^+] = 1.5 \times 10^{-6}$. $pH = -\log_{10}(1.5 \times 10^{-6}) = -(\log_{10}(1.5) + \log_{10}(10^{-6})) = -(\log_{10}(1.5) - 6) = 6 - \log_{10}(1.5) \approx 6 - 0.176 \approx \mathbf{5.824}$. The pH of the soil sample is approximately 5.82.
2. To have a pH of 7.0, we solve the equation: $7.0 = -\log_{10}[H^+]$ $-7.0 = \log_{10}[H^+]$ Converting to index form: $[H^+] = 10^{-7}$ moles/L. The new hydrogen ion concentration should be 10^{-7} moles/L.
3. The new hydrogen ion concentration is $100 \times (1.5 \times 10^{-6}) = 1.5 \times 10^{-4}$ moles/L. The new pH is $pH = -\log_{10}(1.5 \times 10^{-4}) = -(\log_{10}(1.5) - 4) = 4 - \log_{10}(1.5) \approx 4 - 0.176 \approx \mathbf{3.824}$. A decrease in pH from 5.82 to 3.82 indicates that the soil has become much more acidic. This is a significant change, as the logarithmic scale means a one-unit change in pH corresponds to a tenfold change in hydrogen ion concentration. This could be harmful to most crops.

Question 99: Solar Panel Efficiency

The efficiency of a solar panel, E (as a percentage), is found to decrease over time according to the equation $E(t) = 20(0.98)^t$, where t is the number of years.

1. What is the initial efficiency of the solar panel?
2. How many years will it take for the efficiency to drop to 15%?
3. A new, more durable solar panel is developed that depreciates at a slower rate, taking 10 years to reach an efficiency of 18% from the same initial efficiency. What is the new decay rate of this panel?

Solution 99

1. The initial efficiency is when $t = 0$: $E(0) = 20(0.98)^0 = 20(1) = \mathbf{20}\%$.
2. To find the time t when $E = 15$, we solve: $15 = 20(0.98)^t$ $\frac{15}{20} = (0.98)^t$ $0.75 = (0.98)^t$ Taking the logarithm of both sides: $\log(0.75) = t\log(0.98)$ $t = \frac{\log(0.75)}{\log(0.98)} \approx \frac{-0.1249}{-0.0087} \approx \mathbf{14.36}$ years. It will take approximately 14.36 years for the efficiency to drop to 15%.
3. The new equation is $18 = 20(r)^{10}$, where r is the new decay rate. $0.9 = r^{10}$ Taking the 10th root of both sides: $r = (0.9)^{1/10} \approx 0.9895$. The new decay rate is $1 - r = 1 - 0.9895 = 0.0105$, or 1.05%. This is slower than the original 2% depreciation, indicating a more durable panel.

Question 100: Value of a Traditional Craft Business

A traditional craft business in a rural village grows in value exponentially. The value of the business, V (in UGX), is given by $V = 1,000,000(1.2)^t$, where t is the number of years since it was started.

1. What was the initial value of the business?
2. The business owners want to achieve a value of UGX 5,000,000. How many years will this take?
3. Another similar business with the same initial value took only 5 years to reach a value of UGX 2,500,000. Which business has a higher annual growth rate, and what is its growth rate?

Solution 100

1. The initial value is when $t = 0$: $V = 1,000,000(1.2)^0 = 1,000,000(1) =$

UGX $\mathbf{1,000,000}$.

2. To find the time t when $V = 5,000,000$, we solve: $5,000,000 = 1,000,000(1.2)^t$ $5 = (1.2)^t$ Taking the logarithm of both sides: $\log(5) = t\log(1.2)$ $t = \frac{\log(5)}{\log(1.2)} \approx \frac{0.6990}{0.0792} \approx \mathbf{8.83}$ years. It will take approximately 8.83 years.
3. For the second business, we have $A = P(1+r)^t$, where $A = 2,500,000$, $P = 1,000,000$, and $t = 5$. $2,500,000 = 1,000,000(1+r)^5$ $2.5 = (1+r)^5$ Taking the 5th root of both sides: $1+r = 2.5^{1/5} \approx 1.2011$ $r = 0.2011 \approx 20.11\%$. The first business has a growth rate of 20%. The second business has a growth rate of approximately 20.11%. Therefore, the second business has a slightly higher annual growth rate.

Question 101: Compound Interest on a Rural Savings Account

A farmer saves UGX 500,000 in a rural savings account that pays an annual compound interest rate of 6%.

1. What is the total amount in the account after 10 years?
2. A new competing savings account offers a rate that will grow the farmer's money to UGX 1,000,000 in 10 years. What interest rate does the new account offer?
3. If the farmer wants to reach UGX 1,500,000 using the new savings account, how many years will this take?

Solution 101

1. The total amount is $A = P(1+r)^t = 500,000(1+0.06)^{10}$. $A = 500,000(1.06)^{10} \approx 500,000(1.7908) \approx$ UGX $\mathbf{895,400}$. The total amount in the account after 10 years will be approximately UGX 895,400.
2. The new account has $A = 1,000,000$, $P = 500,000$, and $t = 10$. We solve for r: $1,000,000 = 500,000(1+r)^{10}$ $2 = (1+r)^{10}$ $1+r = 2^{1/10} \approx 1.0718$ $r = 0.0718 \approx \mathbf{7.18}\%$. The new savings account offers a rate of approximately 7.18%.
3. To reach UGX 1,500,000 with the new account's rate, we solve for t: $1,500,000 = 500,000(1.0718)^t$ $3 = (1.0718)^t$ Taking the logarithm of both sides: $\log(3) = t\log(1.0718)$ $t = \frac{\log(3)}{\log(1.0718)} \approx \frac{0.4771}{0.0301} \approx \mathbf{15.85}$ years. It will take approximately 15.85 years to reach UGX 1,500,000.

Chapter 2

Equations and Inequalities

Competency:
The learner investigates equations and inequalities to acquire mathematical computational and analytical skills applicable in the real world.

2.1 Reviewing different methods of solving quadratic equations

2.2 Expressing roots as a sum and a product using coefficients of a quadratic equation, and formulate quadrati

2.3 Solving simultaneous equations that have; linear and non - linear terms, logarithms and equations reducing

2.4 Applying the method of completing squares to:

2.4.1 Determining maximum and minimum values, e.g. maximum profits, minimum costs

2.4.2 Representing the information graphically

2.4.3 Verifying the nature of roots of quadratic equations using the discriminant

2.5 Linear equations in one variable linear simultaneous equations in two variables using methods such as subs

2.6 Solving linear simultaneous equations in three variables

2.7 Linear inequalities on a number line and graphs of inequalities.

2.8 Solving quadratic inequalities

2.9 Applications of quadratic inequalities

2.10 Formulating polynomials to identify the degree of a polynomial

2.11 Solving equations of the form $f(x) = 0$ where $f(x)$ is a polynomial of degree three and above

2.12 Deducing the remainder theorem to find the remainder when the divisor is linear or quadratic

2.13 Determining the remainder when the polynomial is not known

2.14 Finding the quotient and remainder of a polynomial with repeated roots.

2.15 Deducing the factor theorem

Quadratic Equations and Optimization

Methods of Solving Quadratic Equations

Introduction

A quadratic equation has the general form: $ax^2 + bx + c = 0$ where $a \neq 0$.

2.15.1 Sum and Product of Roots

For $ax^2 + bx + c = 0$ with roots α and β:

$$\text{Sum of roots: } \alpha + \beta = -\frac{b}{a}$$
$$\text{Product of roots: } \alpha\beta = \frac{c}{a}$$

Example: If roots are 2 and -3, the equation is:

$$x^2 - (\text{sum})x + (\text{product}) = 0$$
$$x^2 - (2 + (-3))x + (2 \times -3) = 0$$
$$x^2 + x - 6 = 0$$

2.15.2 Simultaneous Equations

Linear and Non-linear Systems

Example: Solve the system:

$$x + y = 5 \quad (1)$$
$$x^2 + y^2 = 13 \quad (2)$$

Solution: From (1): $y = 5 - x$ Substitute into (2):

$$x^2 + (5 - x)^2 = 13$$
$$x^2 + 25 - 10x + x^2 = 13$$
$$2x^2 - 10x + 12 = 0$$
$$x^2 - 5x + 6 = 0$$
$$(x - 2)(x - 3) = 0$$
$$x = 2 \text{ or } x = 3$$

When $x = 2$, $y = 3$; when $x = 3$, $y = 2$

Equations Reducing to Quadratics

Example: Solve $2^{2x} - 5 \cdot 2^x + 4 = 0$

Let $y = 2^x$, then:

$$y^2 - 5y + 4 = 0$$
$$(y - 1)(y - 4) = 0$$
$$y = 1 \text{ or } y = 4$$
$$2^x = 1 \text{ or } 2^x = 4$$
$$x = 0 \text{ or } x = 2$$

2.15.3 Completing the Square Applications

Maximum and Minimum Values

For $f(x) = ax^2 + bx + c$:

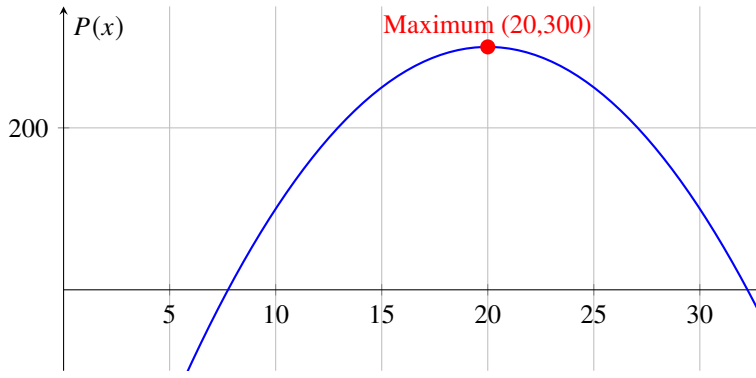
- If $a > 0$, minimum value occurs at $x = -\frac{b}{2a}$
- If $a < 0$, maximum value occurs at $x = -\frac{b}{2a}$

Example: A Ugandan farmer's profit function is $P(x) = -2x^2 + 80x - 500$, where x is number of units produced.

$$\begin{aligned} P(x) &= -2(x^2 - 40x) - 500 \\ &= -2[(x - 20)^2 - 400] - 500 \\ &= -2(x - 20)^2 + 800 - 500 \\ &= -2(x - 20)^2 + 300 \end{aligned}$$

Maximum profit = 300,000 UGX when $x = 20$ units.

Graphical Representation



2.15.4 Discriminant and Nature of Roots

For $ax^2 + bx + c = 0$, discriminant $\Delta = b^2 - 4ac$:

2.16 Linear Simultaneous Equations in Three Variables

2.16.1 Methods of Solution

General form:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Elimination Method

Example: Solve:

$$2x + y - z = 7 \quad (1)$$
$$x - y + 2z = -1 \quad (2)$$
$$3x + 2y + z = 10 \quad (3)$$

Add (1) and (2):

$$3x + z = 6 \quad (4)$$

Multiply (1) by 2 and add to (3):

$$\begin{array}{r} 4x + 2y - 2z = 14 \\ 3x + 2y + z = 10 \\ \hline 7x - z = 24 \quad (5) \end{array}$$

Add (4) and (5):

$$10x = 30 \Rightarrow x = 3$$

Substitute $x = 3$ into (4): $9 + z = 6 \Rightarrow z = -3$

Substitute $x = 3$, $z = -3$ into (1): $6 + y + 3 = 7 \Rightarrow y = -2$

Solution: $x = 3$, $y = -2$, $z = -3$

Substitution Method

From (1): $z = 2x + y - 7$

Substitute into (2) and (3), then solve the resulting two equations.

Matrix Methods (Row Reduction and Echelon Form)

Augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 7 \\ 1 & -1 & 2 & -1 \\ 3 & 2 & 1 & 10 \end{array}\right]$$

Using Gaussian elimination to reach row-echelon form.

2.16.2 Real-World Applications

Economics Example: A Ugandan business sells three products with profits:

Matooke: 2000 UGX/kg
Rice: 1500 UGX/kg
Beans: 1000 UGX/kg

If total revenue is 100,000 UGX from 60 kg total, and matooke sales equal rice and beans combined, find quantities.

2.17 Quadratic Inequalities

2.17.1 Algebraic Solution Methods

For $ax^2 + bx + c > 0$ or $ax^2 + bx + c < 0$:

1. Find roots of $ax^2 + bx + c = 0$
2. Determine intervals
3. Test each interval

Example: Solve $x^2 - 5x + 6 > 0$

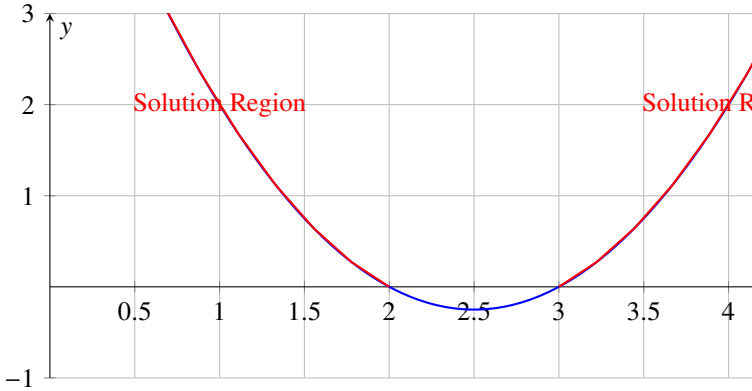
Roots: $x = 2, x = 3$

Test intervals:

- $x < 2$: Choose $x = 0$: $0 - 0 + 6 = 6 > 0$
- $2 < x < 3$: Choose $x = 2.5$: $6.25 - 12.5 + 6 = -0.25 < 0$
- $x > 3$: Choose $x = 4$: $16 - 20 + 6 = 2 > 0$

Solution: $x < 2$ or $x > 3$

2.17.2 Graphical Solution



2.17.3 Real-World Applications

Business Example: A Ugandan manufacturer finds profit $P(x) = -x^2 + 50x - 400$, where x is units produced. Find production range for profit.

Solve $-x^2 + 50x - 400 > 0$

Multiply by -1: $x^2 - 50x + 400 < 0$

Roots: $x = \frac{50 \pm \sqrt{2500-1600}}{2} = \frac{50 \pm 30}{2} = 10, 40$

Solution: $10 < x < 40$ units for profit.

2.18 Polynomial Equations and Expressions

2.18.1 Basic Concepts

A polynomial of degree n: $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

2.18.2 Long Division of Polynomials

Divide $P(x)$ by $D(x)$ to get quotient $Q(x)$ and remainder $R(x)$:

$$P(x) = D(x) \cdot Q(x) + R(x)$$

Example: Divide $2x^3 - 3x^2 + 4x - 1$ by $x - 2$

$$\begin{array}{r|l} & 2x^2 + x + 6 \\ \hline x-2 & 2x^3 - 3x^2 + 4x - 1 \\ & -(2x^3 - 4x^2) \\ \hline & x^2 + 4x \\ & -(x^2 - 2x) \\ \hline & 6x - 1 \\ & -(6x - 12) \\ \hline & 11 \end{array}$$

Quotient: $2x^2 + x + 6$, Remainder: 11

2.18.3 Factorization of Polynomials

Example: Factorize $x^3 - 3x^2 - 4x + 12$

Try $x = 2$: $8 - 12 - 8 + 12 = 0$

Divide by $(x - 2)$:

$$\begin{aligned} x^3 - 3x^2 - 4x + 12 &= (x-2)(x^2 - x - 6) \\ &= (x-2)(x-3)(x+2) \end{aligned}$$

2.18.4 Remainder Theorem

If polynomial $P(x)$ is divided by $(x-a)$, then remainder $R = P(a)$.

Example: Find remainder when $P(x) = 3x^3 - 2x^2 + x - 5$ is divided by $(x - 1)$

$R = P(1) = 3(1)^3 - 2(1)^2 + 1 - 5 = 3 - 2 + 1 - 5 = -3$

2.18.5 Factor Theorem

$(x - a)$ is a factor of $P(x)$ if and only if $P(a) = 0$

Example: Show that $(x + 1)$ is a factor of $x^3 - 2x^2 - 5x + 6$

$P(-1) = (-1)^3 - 2(-1)^2 - 5(-1) + 6 = -1 - 2 + 5 + 6 = 8 \neq 0$

Thus, $(x + 1)$ is NOT a factor.

Try $(x-3)$: $27 - 18 - 15 + 6 = 0$, so $(x-3)$ is a factor.

2.18.6 Polynomials with Repeated Roots

Example: Find quotient and remainder when $x^4 - 4x^3 + 6x^2 - 4x + 1$ is divided by $(x - 1)^2$

Let $x - 1 = t$, so $x = t + 1$

$$\begin{aligned} P(t+1) &= (t+1)^4 - 4(t+1)^3 + 6(t+1)^2 - 4(t+1) + 1 \\ &= (t^4 + 4t^3 + 6t^2 + 4t + 1) - 4(t^3 + 3t^2 + 3t + 1) \\ &\quad + 6(t^2 + 2t + 1) - 4(t+1) + 1 \\ &= t^4 + 4t^3 + 6t^2 + 4t + 1 - 4t^3 - 12t^2 - 12t - 4 \\ &\quad + 6t^2 + 12t + 6 - 4t - 4 + 1 \\ &= t^4 + 0t^3 + 0t^2 + 0t + 0 \\ &= t^4 \end{aligned}$$

So $P(x) = (x-1)^4$, quotient = $(x-1)^2$, remainder = 0

2.18.7 Solving Higher Degree Equations

Example: Solve $x^3 - 6x^2 + 11x - 6 = 0$

Try $x = 1$: $1 - 6 + 11 - 6 = 0$

Divide by $(x - 1)$: $x^2 - 5x + 6 = 0$

Roots: $x = 2, x = 3$

Solution: $x = 1, 2, 3$

Chapter 3

Coordinate Geometry 1

Competency:
The learner analyses linear relationships to solve geometric problems and interpret spatial relationships in real-life situations.

3.1 Cartesian equations of lines, gradients of straight lines intercepts, midpoint of a line, distance between two

3.2 Concept of the mid-point to determine subdivisions of a line and locate coordinates of any point on that line

3.2.1 Angle between a line and the x-axis, relating it to the gradient of the line

3.2.2 Relationship between gradients of parallel lines

3.2.3 Relationship between the gradients of perpendicular lines

3.3 Finding the shortest distance between a point and a line.

3.4 Demonstrating a variety of real-world geometric problems

3.4.1 1. Review of Basic Coordinate Concepts

1.1 Distance between Two Points

The distance d between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is found using the Pythagorean theorem:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

1.2 Midpoint of a Line Segment

The coordinates of the midpoint $M(x_m, y_m)$ of the line segment AB are the average of the coordinates:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

1.3 Gradient (Slope) of a Line

The gradient m measures the steepness of a line.

$$m = \frac{\text{Change in } y}{\text{Change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

1.4 Angle with the x-axis

The gradient m is related to the angle θ a line makes with the positive x-axis:

$$m = \tan\theta$$

The angle θ is found by $\theta = \arctan(m)$.

3.4.2 2. Line Subdivisions (Ratio Theorem)

The coordinates of a point $P(x, y)$ that divides a line segment AB in the ratio $k_1 : k_2$ are given by:

$$P(x, y) = \left(\frac{k_2 x_1 + k_1 x_2}{k_1 + k_2}, \frac{k_2 y_1 + k_1 y_2}{k_1 + k_2}\right)$$

- If the point P divides AB internally, the ratio is positive ($k_1/k_2 > 0$).
- If the point P divides AB externally (i.e., P lies outside the segment AB on the extension), one part of the ratio is taken as negative.

Example: Find the coordinates of P that divides $A(1,5)$ and $B(7,-1)$ internally in the ratio $2:1$. Here, $k_1 = 2$, $k_2 = 1$.

$$P(x,y) = \left(\frac{1(1)+2(7)}{2+1}, \frac{1(5)+2(-1)}{2+1}\right) = \left(\frac{15}{3}, \frac{3}{3}\right) = (\mathbf{5}, \mathbf{1})$$

3.4.3 3. Equations of Straight Lines

3.1 Gradient-Intercept Form

This form is most useful for identifying the gradient and y-intercept quickly:

$$y = mx + c$$

where m is the gradient and c is the y-intercept.

3.2 Point-Gradient Form

Used when the gradient m and one point (x_1, y_1) are known:

$$y - y_1 = m(x - x_1)$$

3.3 Two-Point Form

Used when two points (x_1, y_1) and (x_2, y_2) are known. First find $m = \frac{y_2-y_1}{x_2-x_1}$, then use the point-gradient form.

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

3.4 Intercept Form

Used when the x-intercept $(a, 0)$ and y-intercept $(0, b)$ are known:

$$\frac{x}{a} + \frac{y}{b} = 1$$

3.4.4 4. Relationships Between Lines

4.1 Parallel Lines

Two non-vertical lines L_1 and L_2 are parallel if and only if they have the same gradient.

$$m_1 = m_2$$

Interpretation: Parallel lines maintain the same angle with the x-axis.

4.2 Perpendicular Lines

Two non-vertical lines L_1 and L_2 are perpendicular if and only if the product of their gradients is -1.

$$m_1 \times m_2 = -1 \quad \text{or} \quad m_2 = -\frac{1}{m_1}$$

Interpretation: The gradient of one line is the negative reciprocal of the other.

4.3 Angle Between Two Intersecting Lines

If two lines L_1 and L_2 intersect with gradients m_1 and m_2 respectively, the acute angle α between them is found using the formula:

$$\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

- Note: If $1 + m_1 m_2 = 0$, the denominator is zero, $\tan\alpha$ is undefined, meaning $\alpha = 90^\circ$ (perpendicular lines).

4.4 Intersection of Lines

The point of intersection of two lines is the unique coordinate (x, y) that satisfies the Cartesian equations of both lines simultaneously. This is solved by using simultaneous equations (substitution or elimination method).

3.4.5 5. Shortest Distance from a Point to a Line

The shortest distance D from a point $P(x_1, y_1)$ to a line L with the general equation $Ax + By + C = 0$ is given by the formula:

$$D = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

Example 4 (Application): Find the shortest distance from the point $(3,4)$ to the line $5x - 12y + 1 = 0$.

- Point $(x_1, y_1) = (3,4)$
- Line $A = 5, B = -12, C = 1$

$$D = \frac{|5(3) + (-12)(4) + 1|}{\sqrt{5^2 + (-12)^2}} = \frac{|15 - 48 + 1|}{\sqrt{25 + 144}} = \frac{|-32|}{\sqrt{169}} = \frac{32}{13}$$

Question 81: Coffee Plantation Irrigation	
Question	**Solution**

A coffee farmer in Kapchorwa is setting up a new irrigation system. He needs to lay two parallel pipes. The first pipe, L_1, passes through two control points, $A(2,5)$ and $B(8,17)$. A second pipe, L_2, needs to pass through the main water pump located at $P(1,2)$. a) Find the gradient of the first pipe, L_1. b) Find the equation of the second pipe, L_2. c) A temporary water tank is located at $T(5,6)$. What is the shortest distance from the tank to the second pipe, L_2?	a) The gradient of L_1 is $m_1 = \frac{y_2-y_1}{x_2-x_1} = \frac{17-5}{8-2} = \frac{12}{6} = 2$. b) Since L_2 is parallel to L_1, its gradient m_2 must be equal to m_1. So, $m_2 = 2$. Using the point-slope form $y - y_1 = m(x - x_1)$ with point $P(1,2)$: $y - 2 = 2(x - 1)$ $y - 2 = 2x - 2$ $y = 2x$ The equation of L_2 is $2x - y = 0$. c) To find the shortest distance from $T(5,6)$ to the line $L_2 : 2x - y = 0$, we use the formula $d = \frac{\lvert Ax_0+By_0+C\rvert}{\sqrt{A^2+B^2}}$, where $(x_0, y_0) = (5,6)$ and the line is in the form $Ax + By + C = 0$ with $A = 2, B = -1, C = 0$. $d = \frac{\lvert 2(5)-1(6)+0\rvert}{\sqrt{2^2+(-1)^2}} = \frac{\lvert 10-6\rvert}{\sqrt{4+1}} = \frac{4}{\sqrt{5}} \approx 1.789$ units.
Question 82: Kampala Road Construction	
Question	**Solution**
The Uganda National Roads Authority (UNRA) is planning a new feeder road, L_1, that must be perpendicular to a section of the Northern Bypass, L_2. The equation of the bypass is $y = 3x + 10$. The new feeder road must pass through a new industrial park located at $P(9,4)$. a) What is the gradient of the new feeder road, L_1? b) Find the equation of the new feeder road. c) Determine the point of intersection between the new feeder road and the Northern Bypass.	a) The gradient of the bypass L_2 is $m_2 = 3$. Since L_1 is perpendicular to L_2, its gradient m_1 must satisfy the relationship $m_1 \cdot m_2 = -1$. $m_1 \cdot 3 = -1 \implies m_1 = -\frac{1}{3}$. b) Using the point-slope form $y - y_1 = m(x - x_1)$ with point $P(9,4)$ and $m_1 = -\frac{1}{3}$: $y - 4 = -\frac{1}{3}(x - 9)$ $3(y - 4) = -1(x - 9)$ $3y - 12 = -x + 9$ $x + 3y = 21$. c) To find the intersection point, we solve the two equations simultaneously: (1) $y = 3x+10$ (2) $x+3y = 21$ Substitute (1) into (2): $x+3(3x+10) = 21$ $x+9x+30 = 21$ $10x = -9$ $x = -0.9$. Substitute $x = -0.9$ back into (1): $y = 3(-0.9) + 10 = -2.7 + 10 = 7.3$. The point of intersection is $(-0.9, 7.3)$.
Question 83: Land Demarcation in Mbarara	
Question	**Solution**
A land surveyor in Mbarara is demarcating a farm plot. The main boundary line, L_1, passes through the points $A(1,12)$ and $B(7,24)$. A second boundary, L_2, needs to be perpendicular to L_1 and pass through a landmark at $C(10,5)$. a) Find the equation of the second boundary line, L_2. b) What is the intersection point of the two boundaries? c) A power pole is located at $P(3,7)$. Find the shortest distance from the power pole to the main boundary line, L_1.	a) First, find the gradient of L_1: $m_1 = \frac{24-12}{7-1} = \frac{12}{6} = 2$. Since L_2 is perpendicular to L_1, its gradient is $m_2 = -\frac{1}{m_1} = -\frac{1}{2}$. Using point-slope form with $C(10,5)$ and $m_2 = -1/2$: $y - 5 = -\frac{1}{2}(x - 10)$ $2y - 10 = -x + 10$ $x + 2y = 20$. b) Find the equation of L_1 using point $A(1,12)$ and $m_1 = 2$: $y - 12 = 2(x - 1)$ $y - 12 = 2x - 2$ $y = 2x + 10$. Now, solve the system of equations: (1) $y = 2x + 10$ (2) $x + 2y = 20$ Substitute (1) into (2): $x + 2(2x + 10) = 20$ $x + 4x + 20 = 20$ $5x = 0 \implies x = 0$. Substitute $x = 0$ into (1): $y = 2(0) + 10 = 10$. The intersection point is $(0,10)$. c) To find the shortest distance from $P(3,7)$ to $L_1 : 2x - y + 10 = 0$, use the distance formula. $d = \frac{\lvert 2(3)-1(7)+10\rvert}{\sqrt{2^2+(-1)^2}} = \frac{\lvert 6-7+10\rvert}{\sqrt{5}} = \frac{9}{\sqrt{5}} \approx 4.025$ units.
Question 84: Jinja Water Supply	
Question	**Solution**

The National Water and Sewerage Corporation (NWSC) is laying a new water pipe in Jinja. The existing main line is represented by the equation $y = 5x - 8$. The new pipe will be laid parallel to the existing one and will pass through a new housing estate at $H(4, 1)$. a) What is the gradient of the new water pipe? b) Find the equation of the new water pipe. c) A fire hydrant is planned at $F(6, 15)$. What is the shortest distance from this location to the new water pipe?	a) The existing pipe has a gradient of $m_1 = 5$. Since the new pipe is parallel, its gradient m_2 is also 5. b) Using the point-slope form with point $H(4, 1)$ and $m = 5$: $y - 1 = 5(x - 4)$ $y - 1 = 5x - 20$ $y = 5x - 19$. The equation is $5x - y - 19 = 0$. c) Use the distance formula for point $F(6, 15)$ and line $5x - y - 19 = 0$: $d = \frac{\lvert 5(6)-1(15)-19\rvert}{\sqrt{5^2+(-1)^2}} = \frac{\lvert 30-15-19\rvert}{\sqrt{26}} = \frac{\lvert -4\rvert}{\sqrt{26}} = \frac{4}{\sqrt{26}} \approx 0.784$ units.

Question 85: Lake Victoria Fisheries

Question	Solution
A fisherman on Lake Victoria, located at point $B(10, 3)$, wants to sail to the nearest point on the coastline, which can be modeled by the line $y = -\frac{1}{3}x + 2$. a) Find the gradient of the shortest path from the boat to the coastline. b) Find the equation of the line representing this shortest path. c) Calculate the shortest distance from the boat to the coastline.	a) The coastline has a gradient of $m_1 = -\frac{1}{3}$. The shortest path is perpendicular to the coastline. So, the gradient of the path, m_2, must satisfy $m_1 \cdot m_2 = -1$. $(-\frac{1}{3})m_2 = -1 \implies m_2 = 3$. b) Using the point-slope form with the boat's location $B(10, 3)$ and $m = 3$: $y - 3 = 3(x - 10)$ $y - 3 = 3x - 30$ $y = 3x - 27$. c) First, find the intersection point of the two lines to find the closest point on the coastline. (1) $y = -\frac{1}{3}x + 2$ (2) $y = 3x - 27$ $3x - 27 = -\frac{1}{3}x + 2$ $3x + \frac{1}{3}x = 29$ $\frac{10}{3}x = 29 \implies x = \frac{87}{10} = 8.7$. Substitute $x = 8.7$ into (2): $y = 3(8.7) - 27 = 26.1 - 27 = -0.9$. The closest point is $C(8.7, -0.9)$. Use the distance formula between $B(10, 3)$ and $C(8.7, -0.9)$: $d = \sqrt{(8.7 - 10)^2 + (-0.9 - 3)^2} = \sqrt{(-1.3)^2 + (-3.9)^2} = \sqrt{1.69 + 15.21} = \sqrt{16.9} \approx 4.11$ units.

Question 86: Gulu Health Clinic Planning

Question	Solution
A new health clinic is being planned in Gulu at the intersection of two roads. The first road, L_1, connects a village at $V(2, 8)$ and a market at $M(14, 12)$. The second road, L_2, is perpendicular to L_1 and passes through a community center at $C(6, 15)$. a) Find the equation of the second road, L_2. b) Find the equation of the first road, L_1. c) Determine the location (coordinates) of the new health clinic.	a) Find the gradient of L_1: $m_1 = \frac{12-8}{14-2} = \frac{4}{12} = \frac{1}{3}$. Since L_2 is perpendicular to L_1, its gradient is $m_2 = -1/m_1 = -3$. Using point-slope form with $C(6, 15)$ and $m_2 = -3$: $y - 15 = -3(x - 6)$ $y - 15 = -3x + 18$ $y = -3x + 33$. b) Using point-slope form with $V(2, 8)$ and $m_1 = 1/3$: $y - 8 = \frac{1}{3}(x - 2)$ $3(y - 8) = x - 2$ $3y - 24 = x - 2$ $x - 3y + 22 = 0$. c) To find the intersection point, solve the two equations simultaneously: (1) $y = -3x + 33$ (2) $x - 3y + 22 = 0$ Substitute (1) into (2): $x - 3(-3x + 33) + 22 = 0$ $x + 9x - 99 + 22 = 0$ $10x - 77 = 0 \implies x = 7.7$. Substitute $x = 7.7$ into (1): $y = -3(7.7) + 33 = -23.1 + 33 = 9.9$. The clinic should be located at $(7.7, 9.9)$.

Question 7: Telecommunications Mast

Question	Solution

A telecom company is installing a new mast in a remote area near Fort Portal. They plan to run two parallel fiber optic cables to connect the mast. The first cable, L_1, has the equation $3x - 4y = 12$. The second cable, L_2, needs to pass through a relay station at $R(1, -2)$. a) Find the gradient of the first cable, L_1. b) Find the equation of the second cable, L_2. c) What is the shortest distance from a village at $V(10, 8)$ to the first cable, L_1?	a) To find the gradient of $3x - 4y = 12$, rearrange it into the form $y = mx + c$: $-4y = -3x + 12$ $y = \frac{3}{4}x - 3$. The gradient m_1 is $\frac{3}{4}$. b) Since the cables are parallel, the gradient of L_2 is also $m_2 = \frac{3}{4}$. Using point-slope form with $R(1, -2)$: $y - (-2) = \frac{3}{4}(x - 1)$ $y + 2 = \frac{3}{4}x - \frac{3}{4}$ $4y + 8 = 3x - 3$ $3x - 4y - 11 = 0$. c) Use the distance formula for point $V(10, 8)$ and line $3x - 4y - 12 = 0$: $d = \frac{\lvert 3(10) - 4(8) - 12\rvert}{\sqrt{3^2 + (-4)^2}} = \frac{\lvert 30 - 32 - 12\rvert}{\sqrt{9+16}} = \frac{\lvert -14\rvert}{\sqrt{25}} = \frac{14}{5} = 2.8$ units.
Question 88: Kasese Logistics Center	
Question	**Solution**
A logistics company in Kasese wants to build a new warehouse at the intersection of two main supply roads. The first road, L_1, has the equation $y = 2x - 3$. The second road, L_2, is perpendicular to L_1 and passes through a supply checkpoint at $C(5, 7)$. a) Find the equation of the second road, L_2. b) Find the coordinates of the new warehouse. c) A fuel station is located at $F(1, 10)$. Find the shortest distance from the fuel station to the first road, L_1.	a) The gradient of L_1 is $m_1 = 2$. Since L_2 is perpendicular, its gradient is $m_2 = -1/2$. Using point-slope form with $C(5, 7)$ and $m_2 = -1/2$: $y - 7 = -\frac{1}{2}(x - 5)$ $2y - 14 = -x + 5$ $x + 2y - 19 = 0$. b) To find the intersection, solve the system: (1) $y = 2x - 3$ (2) $x + 2y - 19 = 0$ Substitute (1) into (2): $x + 2(2x - 3) - 19 = 0$ $x + 4x - 6 - 19 = 0$ $5x - 25 = 0 \implies x = 5$. Substitute $x = 5$ into (1): $y = 2(5) - 3 = 10 - 3 = 7$. The warehouse is located at $(5, 7)$. c) Use the distance formula for point $F(1, 10)$ and line $L_1 : 2x - y - 3 = 0$: $d = \frac{\lvert 2(1) - 1(10) - 3\rvert}{\sqrt{2^2 + (-1)^2}} = \frac{\lvert 2 - 10 - 3\rvert}{\sqrt{5}} = \frac{\lvert -11\rvert}{\sqrt{5}} = \frac{11}{\sqrt{5}} \approx 4.92$ units.
Question 89: Entebbe International Airport Runway	
Question	**Solution**
A new taxiway is being designed at Entebbe International Airport. It must be parallel to the main runway, L_1, which passes through coordinates $A(4, 10)$ and $B(12, 14)$. The new taxiway, L_2, needs to pass through a control tower at $T(2, 6)$. a) Find the equation of the main runway, L_1. b) Find the equation of the new taxiway, L_2. c) An emergency service road, L_3, is given by the equation $y = -2x + 10$. Find the intersection point of the new taxiway, L_2, and the emergency service road, L_3.	a) The gradient of L_1 is $m_1 = \frac{14-10}{12-4} = \frac{4}{8} = \frac{1}{2}$. Using point-slope form with $A(4, 10)$ and $m = 1/2$: $y - 10 = \frac{1}{2}(x - 4)$ $2y - 20 = x - 4$ $x - 2y + 16 = 0$. b) Since L_2 is parallel to L_1, its gradient is $m_2 = 1/2$. Using point-slope form with $T(2, 6)$: $y - 6 = \frac{1}{2}(x - 2)$ $2y - 12 = x - 2$ $x - 2y + 10 = 0$. c) Solve the system of equations for L_2 and L_3: (1) $y = -2x + 10$ (2) $x - 2y + 10 = 0$ Substitute (1) into (2): $x - 2(-2x + 10) + 10 = 0$ $x + 4x - 20 + 10 = 0$ $5x - 10 = 0 \implies x = 2$. Substitute $x = 2$ into (1): $y = -2(2) + 10 = -4 + 10 = 6$. The intersection point is $(2, 6)$.
Question 90: Urban Planning in Mbale	
Question	**Solution**

The Mbale city council is planning a new pedestrian walkway that must run perpendicular to the main road, which is represented by the line $4x - 5y = 20$. The walkway needs to start from a new mall entrance located at $M(10, 3)$. a) What is the gradient of the main road? b) Find the equation of the new pedestrian walkway. c) Find the point where the new walkway will intersect the main road.	a) To find the gradient of the main road $4x - 5y = 20$, rearrange it to $y = mx + c$: $-5y = -4x + 20$ $y = \frac{4}{5}x - 4$. The gradient of the road is $m_1 = \frac{4}{5}$. b) The walkway is perpendicular to the road, so its gradient is $m_2 = -\frac{1}{m_1} = -\frac{5}{4}$. Using point-slope form with $M(10, 3)$: $y-3 = -\frac{5}{4}(x-10)$ $4(y-3) = -5(x-10)$ $4y - 12 = -5x + 50$ $5x + 4y - 62 = 0$. c) To find the intersection point, solve the system: (1) $y = \frac{4}{5}x - 4$ (2) $5x + 4y - 62 = 0$ Substitute (1) into (2): $5x + 4(\frac{4}{5}x - 4) - 62 = 0$ $5x + \frac{16}{5}x - 16 - 62 = 0$ Multiply by 5 to clear fractions: $25x+16x-80-310 = 0$ $41x = 390 \implies x = \frac{390}{41} \approx 9.51$. Substitute x into (1): $y = \frac{4}{5}(\frac{390}{41}) - 4 = \frac{312}{41} - 4 = \frac{312-164}{41} = \frac{148}{41} \approx 3.61$. The intersection point is approximately $(9.51, 3.61)$.
Question 91: Railway Line Extension	
Question	**Solution**
A new railway line is being planned in Eastern Uganda. The line, L_1, must be parallel to an existing main road, L_2, which connects two towns at $T_1(1, 15)$ and $T_2(9, 31)$. The new railway line will pass through a major station at $S(12, 10)$. a) Find the gradient of the main road, L_2. b) Find the equation of the new railway line, L_1. c) A community market is located at $M(5, 2)$. What is the shortest distance from the market to the new railway line?	a) The gradient of the road is $m_2 = \frac{31-15}{9-1} = \frac{16}{8} = 2$. b) Since the railway line is parallel, its gradient is $m_1 = 2$. Using point-slope form with $S(12, 10)$: $y - 10 = 2(x - 12)$ $y - 10 = 2x - 24$ $y = 2x - 14$. The equation is $2x - y - 14 = 0$. c) Use the distance formula for point $M(5, 2)$ and line $2x - y - 14 = 0$: $d = \frac{\lvert 2(5)-1(2)-14\rvert}{\sqrt{2^2+(-1)^2}} = \frac{\lvert 10-2-14\rvert}{\sqrt{5}} = \frac{\lvert -6\rvert}{\sqrt{5}} = \frac{6}{\sqrt{5}} \approx 2.68$ units.
Question 92: Oil and Gas Pipeline	
Question	**Solution**
A section of the oil pipeline in Western Uganda is being rerouted. The existing route can be modeled by the line $y = -\frac{1}{2}x + 5$. The new route, L_2, will be perpendicular to the existing one and will pass through a pumping station at $S(4, -1)$. a) Find the gradient of the new pipeline route. b) Find the equation of the new pipeline route. c) A monitoring sensor is located at $P(10, 8)$. Find the shortest distance from the sensor to the new pipeline route.	a) The gradient of the existing route is $m_1 = -1/2$. The new route is perpendicular, so its gradient is $m_2 = -1/m_1 = -1/(-1/2) = 2$. b) Using point-slope form with $S(4, -1)$ and $m = 2$: $y - (-1) = 2(x - 4)$ $y + 1 = 2x - 8$ $y = 2x - 9$. The equation is $2x - y - 9 = 0$. c) Use the distance formula for point $P(10, 8)$ and line $2x - y - 9 = 0$: $d = \frac{\lvert 2(10)-1(8)-9\rvert}{\sqrt{2^2+(-1)^2}} = \frac{\lvert 20-8-9\rvert}{\sqrt{5}} = \frac{\lvert 3\rvert}{\sqrt{5}} = \frac{3}{\sqrt{5}} \approx 1.34$ units.
Question 93: Bus Company Route Planning	
Question	**Solution**

A bus company is planning a new route in the city. The new route, L_1, must be parallel to an existing main street, L_2, which passes through two major landmarks at $A(2,5)$ and $B(6,13)$. The new route must pass through a new bus terminal at $T(1,1)$. a) Find the equation of the new route, L_1. b) The company wants to add another stop at the intersection of L_1 and a cross-town road, L_3, which has the equation $y = -x + 10$. Find the coordinates of this new stop. c) A new passenger is at their home located at $H(5,2)$. What is the shortest distance from their home to the new bus route, L_1?	a) First, find the gradient of the main street, L_2: $m_2 = \frac{13-5}{6-2} = \frac{8}{4} = 2$. Since L_1 is parallel, its gradient is $m_1 = 2$. Using point-slope form with $T(1,1)$: $y - 1 = 2(x - 1)$ $y - 1 = 2x - 2$ $y = 2x - 1$. The equation is $2x - y - 1 = 0$. b) To find the intersection of L_1 ($y = 2x - 1$) and L_3 ($y = -x + 10$), set the equations equal: $2x - 1 = -x + 10$ $3x = 11 \implies x = \frac{11}{3}$. Substitute x back into the equation for L_3: $y = -\frac{11}{3} + 10 = \frac{-11+30}{3} = \frac{19}{3}$. The new bus stop is at $(\frac{11}{3}, \frac{19}{3})$. c) Use the distance formula for point $H(5,2)$ and line $L_1 : 2x - y - 1 = 0$: $d = \frac{\lvert 2(5)-1(2)-1\rvert}{\sqrt{2^2+(-1)^2}} = \frac{\lvert 10-2-1\rvert}{\sqrt{5}} = \frac{7}{\sqrt{5}} \approx 3.13$ units.
Question 94: Entebbe Fishing Landing Site	
Question	**Solution**
A new fish processing unit is to be built at the intersection of two access roads. The first road, L_1, connects two villages at $V_1(4,1)$ and $V_2(10,7)$. The second road, L_2, is perpendicular to L_1 and passes through a community health post at $H(8,2)$. a) Find the equation of the second road, L_2. b) Find the point of intersection of the two roads where the processing unit should be located. c) A fish storage facility is at $F(2,5)$. What is the shortest distance from the facility to the second road, L_2?	a) Find the gradient of L_1: $m_1 = \frac{7-1}{10-4} = \frac{6}{6} = 1$. Since L_2 is perpendicular, its gradient is $m_2 = -1/1 = -1$. Using point-slope form with $H(8,2)$: $y - 2 = -1(x - 8)$ $y - 2 = -x + 8$ $y = -x + 10$. The equation is $x + y - 10 = 0$. b) Find the equation of L_1 using $V_1(4,1)$ and $m_1 = 1$: $y - 1 = 1(x - 4)$ $y - 1 = x - 4$ $y = x - 3$. Now solve the system: (1) $y = x - 3$ (2) $x + y - 10 = 0$ Substitute (1) into (2): $x + (x - 3) - 10 = 0$ $2x - 13 = 0 \implies x = 6.5$. Substitute $x = 6.5$ into (1): $y = 6.5 - 3 = 3.5$. The processing unit should be at $(6.5, 3.5)$. c) Use the distance formula for $F(2,5)$ and $L_2 : x + y - 10 = 0$: $d = \frac{\lvert 1(2)+1(5)-10\rvert}{\sqrt{1^2+1^2}} = \frac{\lvert 2+5-10\rvert}{\sqrt{2}} = \frac{\lvert -3\rvert}{\sqrt{2}} = \frac{3}{\sqrt{2}} \approx 2.12$ units.
Question 95: Agriculture in Masindi	
Question	**Solution**
A farmer in Masindi is planning a new set of parallel crop rows. The first row, R_1, passes through points $A(1,4)$ and $B(5,12)$. A second row, R_2, needs to be parallel to R_1 and pass through a new watering point at $W(10,6)$. a) Find the equation of the second row, R_2. b) A pest control shed is located at $S(1,10)$. Find the shortest distance from the shed to the second row, R_2. c) What is the perpendicular distance between the two parallel rows, R_1 and R_2?	a) Find the gradient of R_1: $m_1 = \frac{12-4}{5-1} = \frac{8}{4} = 2$. Since R_2 is parallel, its gradient is $m_2 = 2$. Using point-slope form with $W(10,6)$: $y - 6 = 2(x - 10)$ $y - 6 = 2x - 20$ $y = 2x - 14$. The equation is $2x - y - 14 = 0$. b) Use the distance formula for point $S(1,10)$ and line $R_2 : 2x - y - 14 = 0$: $d = \frac{\lvert 2(1)-1(10)-14\rvert}{\sqrt{2^2+(-1)^2}} = \frac{\lvert 2-10-14\rvert}{\sqrt{5}} = \frac{\lvert -22\rvert}{\sqrt{5}} = \frac{22}{\sqrt{5}} \approx 9.83$ units. c) Find the equation of R_1 using point $A(1,4)$ and $m = 2$: $y - 4 = 2(x - 1)$ $y - 4 = 2x - 2$ $y = 2x + 2$. The equation is $2x - y + 2 = 0$. Now, take a point on R_1, say $A(1,4)$, and find the distance to $R_2 : 2x - y - 14 = 0$. $d = \frac{\lvert 2(1)-1(4)-14\rvert}{\sqrt{2^2+(-1)^2}} = \frac{\lvert 2-4-14\rvert}{\sqrt{5}} = \frac{\lvert -16\rvert}{\sqrt{5}} = \frac{16}{\sqrt{5}} \approx 7.16$ units.
Question 96: Rural Electrification	
Question	**Solution**

The Rural Electrification Agency (REA) is extending power lines to a village. Two existing power lines, L_1 and L_2, are represented by the equations $y = \frac{3}{5}x + 4$ and $3x - 5y = -10$, respectively. A new power line is needed to pass through a substation at $S(2, 5)$ and be perpendicular to the existing lines. a) Are the existing lines L_1 and L_2 parallel? Justify your answer. b) Find the equation of the new power line. c) What is the shortest distance from a new homestead at $H(10, 1)$ to the new power line?	a) The gradient of L_1 is $m_1 = \frac{3}{5}$. For L_2, rearrange $3x - 5y = -10$ to get $-5y = -3x - 10 \implies y = \frac{3}{5}x + 2$. The gradient of L_2 is $m_2 = \frac{3}{5}$. Since $m_1 = m_2$, the lines are parallel. b) The new power line is perpendicular to L_1 (and L_2), so its gradient is $m_{new} = -\frac{1}{3/5} = -\frac{5}{3}$. Using point-slope form with $S(2, 5)$: $y-5 = -\frac{5}{3}(x-2)$ $3(y-5) = -5(x-2)$ $3y - 15 = -5x + 10$ $5x + 3y - 25 = 0$. c) Use the distance formula for point $H(10, 1)$ and line $5x + 3y - 25 = 0$: $d = \frac{\lvert 5(10)+3(1)-25\rvert}{\sqrt{5^2+3^2}} = \frac{\lvert 50+3-25\rvert}{\sqrt{25+9}} = \frac{28}{\sqrt{34}} \approx 4.80$ units.
Question 97: Urban Sanitation Plan	
Question	**Solution**
The city council is planning a new drainage channel, C_1, that must be perpendicular to a main sewer line, S_1, given by the equation $2x + 3y = 12$. The new channel must pass through a storm drain at $D(6, 1)$. a) Find the equation of the new drainage channel, C_1. b) The city is also planning another channel, C_2, parallel to C_1 and passing through a pumping station at $P(3, 4)$. Find the equation of C_2. c) A residential block is located at $R(1, 8)$. Find the shortest distance from the residential block to the new drainage channel, C_1.	a) The gradient of the sewer line $2x + 3y = 12$ is found by rearranging: $3y = -2x + 12 \implies y = -\frac{2}{3}x + 4$. The gradient of S_1 is $m_{S1} = -2/3$. Since C_1 is perpendicular, its gradient is $m_{C1} = -1/(-2/3) = 3/2$. Using point-slope form with $D(6, 1)$: $y - 1 = \frac{3}{2}(x - 6)$ $2(y - 1) = 3(x - 6)$ $2y - 2 = 3x - 18$ $3x - 2y - 16 = 0$. b) Since C_2 is parallel to C_1, its gradient is also $m_{C2} = 3/2$. Using point-slope form with $P(3, 4)$: $y - 4 = \frac{3}{2}(x - 3)$ $2y - 8 = 3x - 9$ $3x - 2y - 1 = 0$. c) Use the distance formula for point $R(1, 8)$ and line $C_1 : 3x - 2y - 16 = 0$: $d = \frac{\lvert 3(1)-2(8)-16\rvert}{\sqrt{3^2+(-2)^2}} = \frac{\lvert 3-16-16\rvert}{\sqrt{9+4}} = \frac{\lvert -29\rvert}{\sqrt{13}} = \frac{29}{\sqrt{13}} \approx 8.04$ units.
Question 98: Hotel Construction in Kampala	
Question	**Solution**
An architect is designing a new hotel. A structural beam, B_1, is to be parallel to a main support wall, W_1. The wall passes through points $P(4, 8)$ and $Q(10, 20)$. The beam needs to pass through a specific column at $C(2, 6)$. a) Find the equation of the main support wall, W_1. b) Find the equation of the structural beam, B_1. c) A fire escape route, F_1, is given by the equation $y = -\frac{1}{2}x+10$. Find the intersection point of the structural beam, B_1, and the fire escape route, F_1.	a) The gradient of the wall is $m_{W1} = \frac{20-8}{10-4} = \frac{12}{6} = 2$. Using point-slope form with $P(4, 8)$: $y - 8 = 2(x - 4)$ $y - 8 = 2x - 8$ $y = 2x$. b) The beam is parallel to the wall, so its gradient is $m_{B1} = 2$. Using point-slope form with $C(2, 6)$: $y - 6 = 2(x - 2)$ $y - 6 = 2x - 4$ $y = 2x + 2$. c) To find the intersection, solve the system: (1) $y = 2x + 2$ (2) $y = -\frac{1}{2}x + 10$ $2x + 2 = -\frac{1}{2}x + 10$ $2x + \frac{1}{2}x = 8$ $\frac{5}{2}x = 8 \implies x = \frac{16}{5} = 3.2$. Substitute $x = 3.2$ into (1): $y = 2(3.2) + 2 = 6.4 + 2 = 8.4$. The intersection point is $(3.2, 8.4)$.
Question 99: Masaka Water Supply	
Question	**Solution**

The NWSC is planning a new section of the water network in Masaka. A main supply line, L_1, has the equation $3x + 4y = 24$. A new residential feeder line, L_2, must be perpendicular to L_1 and pass through a new housing estate at $H(8, 1)$. a) What is the gradient of the new feeder line, L_2? b) Find the equation of the new feeder line. c) What is the shortest distance from a water collection point at $W(1, 10)$ to the main supply line, L_1?	a) The gradient of L_1 is found from $3x + 4y = 24$: $4y = -3x + 24 \implies y = -\frac{3}{4}x + 6$. The gradient is $m_1 = -3/4$. The new line L_2 is perpendicular, so its gradient is $m_2 = -1/(-3/4) = 4/3$. b) Using point-slope form with $H(8, 1)$: $y-1 = \frac{4}{3}(x-8)$ $3(y - 1) = 4(x - 8)$ $3y - 3 = 4x - 32$ $4x - 3y - 29 = 0$. c) Use the distance formula for point $W(1, 10)$ and line $L_1 : 3x+4y-24 = 0$: $d = \frac{\lvert 3(1)+4(10)-24\rvert}{\sqrt{3^2+4^2}} = \frac{\lvert 3+40-24\rvert}{\sqrt{9+16}} = \frac{\lvert 19\rvert}{\sqrt{25}} = \frac{19}{5} = 3.8$ units.

Question 100: Tourist Trail in Murchison Falls

Question	**Solution**
A park ranger is plotting a new nature trail in Murchison Falls National Park. The main trail, L_1, connects two viewpoints at $V_1(5, 10)$ and $V_2(15, 20)$. A new side trail, L_2, needs to be parallel to L_1 and pass through a specific animal watering hole at $W(2, 6)$. a) Find the equation of the new side trail, L_2. b) A research camp is located at $C(10, 1)$. What is the shortest distance from the camp to the main trail, L_1? c) A ranger station is located at $R(15, 5)$. What is the shortest distance from the station to the new side trail, L_2?	a) Find the gradient of L_1: $m_1 = \frac{20-10}{15-5} = \frac{10}{10} = 1$. Since L_2 is parallel, its gradient is $m_2 = 1$. Using point-slope form with $W(2, 6)$: $y - 6 = 1(x - 2)$ $y - 6 = x - 2$ $y = x + 4$. The equation is $x - y + 4 = 0$. b) Find the equation of L_1 using $V_1(5, 10)$ and $m = 1$: $y - 10 = 1(x - 5)$ $y - 10 = x - 5$ $y = x + 5$. The equation is $x - y + 5 = 0$. Use the distance formula for point $C(10, 1)$ and line $L_1 : x - y + 5 = 0$: $d = \frac{\lvert 1(10)-1(1)+5\rvert}{\sqrt{1^2+(-1)^2}} = \frac{\lvert 10-1+5\rvert}{\sqrt{2}} = \frac{14}{\sqrt{2}} \approx 9.90$ units. c) Use the distance formula for point $R(15, 5)$ and line $L_2 : x - y + 4 = 0$: $d = \frac{\lvert 1(15)-1(5)+4\rvert}{\sqrt{1^2+(-1)^2}} = \frac{\lvert 15-5+4\rvert}{\sqrt{2}} = \frac{14}{\sqrt{2}} \approx 9.90$ units.

Chapter 4

Partial Fractions

Competency:
The learner decomposes rational expressions into partial fractions useful in integral calculus and real-world context.

4.1 Expressing the sum of simple fractions as a single fraction

4.1.1 Proper algebraic fractions as a sum of simple fractions

4.1.2 Improper algebraic fractions as a quotient and proper algebraic fraction

Definition

A **partial fraction decomposition** expresses a rational function as a sum of simpler fractions whose denominators are linear or quadratic factors of the original denominator.

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^{n} \frac{A_i}{(x-a_i)} + \sum_{j=1}^{m} \frac{B_j x + C_j}{(x^2+bx+c)_j}$$

where $P(x)$ and $Q(x)$ are polynomials and $\deg P(x) < \deg Q(x)$.

Types of Rational Expressions

1. **Proper Fraction:** Numerator degree is less than denominator degree.
2. **Improper Fraction:** Numerator degree is equal to or greater than denominator degree (divide first to make it proper).

Case 1: Distinct Linear Factors

If

$$\frac{2x+3}{x^2-x-2} = \frac{2x+3}{(x-2)(x+1)},$$

then

$$\frac{2x+3}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}.$$

Multiply both sides by $(x-2)(x+1)$:

$$2x+3 = A(x+1) + B(x-2).$$

Substitute $x = 2$ and $x = -1$:

$$x = 2: \quad 7 = 3A \Rightarrow A = \tfrac{7}{3},$$
$$x = -1: \quad 1 = -3B \Rightarrow B = -\tfrac{1}{3}.$$

Thus,

$$\frac{2x+3}{x^2-x-2} = \frac{7/3}{x-2} - \frac{1/3}{x+1}.$$

Case 2: Repeated Linear Factors

If the denominator has repeated factors:

$$\frac{4x+5}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}.$$

Multiply both sides by $(x+2)^2$:

$$4x+5 = A(x+2) + B.$$

Let $x = -2 \Rightarrow B = -3$. Let $x = 0 \Rightarrow 5 = 2A + B \Rightarrow A = 4$. Hence,

$$\frac{4x+5}{(x+2)^2} = \frac{4}{x+2} - \frac{3}{(x+2)^2}.$$

Case 3: Irreducible Quadratic Factor

If the denominator contains an irreducible quadratic factor:

$$\frac{x^2+1}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}.$$

Multiply both sides by $x(x^2+4)$:

$$x^2 + 1 = A(x^2+4) + (Bx+C)x.$$

Simplify and compare coefficients:

$$x^2 + 1 = (A+B)x^2 + Cx + 4A.$$

$$\Rightarrow \begin{cases} A + B = 1, \\ C = 0, \\ 4A = 1. \end{cases}$$

Hence $A = \frac{1}{4}$, $B = \frac{3}{4}$, $C = 0$.

$$\frac{x^2+1}{x(x^2+4)} = \frac{1/4}{x} + \frac{(3x)/4}{x^2+4}.$$

Application in Integral Calculus

Partial fractions simplify integration of rational functions:

$$\int \frac{2x+3}{x^2-x-2}\,dx = \frac{7}{3}\int \frac{1}{x-2}\,dx - \frac{1}{3}\int \frac{1}{x+1}\,dx = \frac{7}{3}\ln|x-2| - \frac{1}{3}\ln|x$$

4.2 Scenario based questions with solutions

Question 21: Coffee Production in Bundibugyo	
Question	**Solution**
A coffee company in Bundibugyo is blending two types of coffee beans. The total cost of the blend is modeled by the rational expression $\frac{10x+5}{x^2+x-2}$ where x is the quantity of coffee in kilograms. a) Decompose the expression into partial fractions. b) Use the decomposed expression to find the total cost when $x = 3$ kg. c) Explain why this decomposition is useful for calculating the total cost for varying quantities of coffee.	a) We first factor the denominator: $x^2 + x - 2 = (x+2)(x-1)$. We set up the partial fractions: $\frac{10x+5}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$ $10x + 5 = A(x-1) + B(x+2)$ Let $x = 1$: $10(1)+5 = A(0)+B(1+2) \implies 15 = 3B \implies B = 5$. Let $x = -2$: $10(-2)+5 = A(-2-1)+B(0) \implies -15 = -3A \implies A = 5$. The decomposition is $\frac{5}{x+2} + \frac{5}{x-1}$. b) Substitute $x = 3$ into the partial fraction form: Cost $= \frac{5}{3+2} + \frac{5}{3-1} = \frac{5}{5} + \frac{5}{2} = 1 + 2.5 = 3.5$. The total cost is 3.5 units. c) The partial fraction form simplifies the calculation by breaking down a complex expression into two simpler ones. It's much easier to substitute values into the sum of simple fractions, especially for mental calculations, rather than into the original complex rational function. This is particularly useful for quick cost estimations.
Question 22: Water Treatment in Entebbe	
Question	**Solution**
The rate of flow of a water purification chemical into a tank is given by the expression $\frac{3t+4}{(t+1)^2}$ where t is the time in minutes. a) Decompose the expression into partial fractions. b) Find the rate of flow at $t = 4$ minutes. c) Determine the time t when the rate of flow is $\frac{13}{9}$.	a) This is a case of repeated linear factors. $\frac{3t+4}{(t+1)^2} = \frac{A}{t+1} + \frac{B}{(t+1)^2}$ $3t+4 = A(t+1)+B$ Let $t = -1$: $3(-1)+4 = B \implies B = 1$. Equating coefficients of t: $3 = A$. The decomposition is $\frac{3}{t+1} + \frac{1}{(t+1)^2}$. b) Substitute $t = 4$ into the partial fraction form: Rate $= \frac{3}{4+1} + \frac{1}{(4+1)^2} = \frac{3}{5} + \frac{1}{25} = \frac{15}{25} + \frac{1}{25} = \frac{16}{25}$. c) Set the original expression equal to $\frac{13}{9}$: $\frac{3t+4}{(t+1)^2} = \frac{13}{9}$ $9(3t+4) = 13(t+1)^2$ $27t + 36 = 13(t^2 + 2t + 1)$ $27t + 36 = 13t^2 + 26t + 13$ $13t^2 - t - 23 = 0$. Using the quadratic formula, the time is approximately 1.38 minutes.

Question 23: Kampala Road Construction	
Question	**Solution**
A civil engineer is modeling the flow of asphalt material. The flow rate is given by the expression $\frac{x^2+2x+3}{x^2+3x+2}$ where x is the length of the road section in meters. a) Decompose the improper fraction into a quotient and a proper algebraic fraction. b) Express the proper fraction part as a sum of partial fractions. c) Using your results, find the flow rate when $x = 2$ meters.	a) This is an improper fraction because the degree of the numerator (2) is equal to the degree of the denominator (2). We use long division: $x^2 + 2x + 3 = 1(x^2 + 3x + 2) + (-x + 1)$. The expression is $1 + \frac{-x+1}{x^2+3x+2}$. b) Factor the denominator: $x^2 + 3x + 2 = (x + 1)(x + 2)$. We set up the partial fractions for the remainder: $\frac{-x+1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$ $-x+1 = A(x+2)+B(x+1)$ Let $x = -1$: $-(-1) + 1 = A(-1 + 2) + B(0) \implies 2 = A$. Let $x = -2$: $-(-2) + 1 = A(0) + B(-2 + 1) \implies 3 = -B \implies B = -3$. The partial fraction form is $1 + \frac{2}{x+1} - \frac{3}{x+2}$. c) Substitute $x = 2$ into the decomposed expression: Rate $= 1 + \frac{2}{2+1} - \frac{3}{2+2} = 1 + \frac{2}{3} - \frac{3}{4} = \frac{12+8-9}{12} = \frac{11}{12}$. The flow rate is $\frac{11}{12}$ units.
Question 24: Hydro-Power Dam Analysis	
Question	**Solution**
An engineer is analyzing the pressure in a penstock pipe at a dam. The pressure is modeled by the rational expression $\frac{2x^2+x-1}{(x+1)(x-1)^2}$. a) Decompose the rational expression into partial fractions. b) If $x = 3$, find the pressure in the pipe. c) A safety sensor is triggered when the pressure is 0. Is this ever possible based on this model?	a) Factor the denominator: $(x + 1)(x - 1)^2$. We have a linear factor and a repeated linear factor. $\frac{2x^2+x-1}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$ $2x^2 + x - 1 = A(x - 1)^2 + B(x + 1)(x - 1) + C(x + 1)$ Let $x = 1$: $2(1)^2 + 1 - 1 = A(0) + B(0) + C(1+1) \implies 2 = 2C \implies C = 1$. Let $x = -1$: $2(-1)^2 + (-1) - 1 = A(-1 - 1)^2 + B(0) + C(0) \implies 0 = A(-2)^2 \implies 0 = 4A \implies A = 0$. Equating coefficients of x^2: $2 = A + B$. Since $A = 0$, we have $B = 2$. The decomposition is $\frac{0}{x+1} + \frac{2}{x-1} + \frac{1}{(x-1)^2} = \frac{2}{x-1} + \frac{1}{(x-1)^2}$. b) Substitute $x = 3$ into the simplified partial fraction form: Pressure $= \frac{2}{3-1} + \frac{1}{(3-1)^2} = \frac{2}{2} + \frac{1}{4} = 1+0.25 = 1.25$. c) For the pressure to be 0, the numerator of the original expression must be 0: $2x^2+x-1 = 0$. $(2x-1)(x+1) = 0$. This gives $x = 1/2$ or $x = -1$. The pressure is 0 at $x = 1/2$ and $x = -1$. The model is not defined at $x = 1$. So, yes, the pressure can be 0 at $x = 1/2$.
Question 25: A Medical Diagnostic Test	
Question	**Solution**

The concentration of a drug in a patient's bloodstream, C, after t hours is given by the function $C(t) = \frac{3t^2+t+1}{t^2(t+1)}$. a) Decompose the function $C(t)$ into partial fractions. b) Find the concentration after 2 hours. c) At what time, if any, is the concentration equal to $\frac{13}{12}$?	a) The denominator has a repeated linear factor and a distinct linear factor. $\frac{3t^2+t+1}{t^2(t+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t+1}$ $3t^2+t+1 = A(t)(t+1) + B(t+1) + C(t^2)$ Let $t = 0$: $1 = B(1) \implies B = 1$. Let $t = -1$: $3(-1)^2 + (-1) + 1 = C(-1)^2 \implies 3 = C$. Equating coefficients of t^2: $3 = A + C$. Since $C = 3$, we have $A = 0$. The decomposition is $\frac{1}{t^2} + \frac{3}{t+1}$. b) Substitute $t = 2$: $C(2) = \frac{1}{2^2} + \frac{3}{2+1} = \frac{1}{4} + \frac{3}{3} = 0.25 + 1 = 1.25$. c) Set the decomposed function equal to $\frac{13}{12}$: $\frac{1}{t^2} + \frac{3}{t+1} = \frac{13}{12}$ This results in a cubic equation which is complex to solve by hand. This question is likely designed to be solved using computational tools. For a manual exercise, the question would be rephrased to ask for a specific integer time value.
Question 26: Sorghum Mill Production	
Question	**Solution**
A sorghum mill in Soroti is processing grains. The rate of production is modeled by the function $P(x) = \frac{x^2+1}{x(x+1)(x-2)}$, where x is the number of hours the mill has been operating. a) Decompose the expression into partial fractions. b) Find the production rate after 3 hours. c) If a new order requires the production rate to be $\frac{5}{12}$, at what time is this rate achieved?	a) The denominator has three distinct linear factors. $\frac{x^2+1}{x(x+1)(x-2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}$ $x^2+1 = A(x+1)(x-2) + Bx(x-2) + Cx(x+1)$ Let $x = 0$: $1 = A(1)(-2) \implies A = -1/2$. Let $x = -1$: $(-1)^2 + 1 = B(-1)(-1-2) \implies 2 = 3B \implies B = 2/3$. Let $x = 2$: $(2)^2 + 1 = C(2)(2+1) \implies 5 = 6C \implies C = 5/6$. The decomposition is $-\frac{1}{2x} + \frac{2}{3(x+1)} + \frac{5}{6(x-2)}$. b) Substitute $x = 3$: $P(3) = -\frac{1}{2(3)} + \frac{2}{3(3+1)} + \frac{5}{6(3-2)} = -\frac{1}{6} + \frac{2}{12} + \frac{5}{6} = -\frac{1}{6} + \frac{1}{6} + \frac{5}{6} = \frac{5}{6}$. c) Set the production rate to $\frac{5}{12}$: $\frac{x^2+1}{x(x+1)(x-2)} = \frac{5}{12}$ $12(x^2+1) = 5x(x+1)(x-2)$ This leads to a cubic equation which is difficult to solve manually. This is a higher-level question that requires a numerical or graphical method for solution.
Question 27: A Solar Panel System in Moroto	
Question	**Solution**
An electrical engineer is analyzing the voltage across a circuit. The voltage is described by the expression $\frac{2x^2-1}{(x-1)(x^2+1)}$. a) Decompose the expression into partial fractions. b) Find the voltage when $x = 2$. c) Does the voltage ever reach zero based on this model?	a) The denominator has a linear factor and an irreducible quadratic factor. $\frac{2x^2-1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$ $2x^2 - 1 = A(x^2+1) + (Bx+C)(x-1)$ Let $x = 1$: $2(1)^2 - 1 = A(1^2+1) \implies 1 = 2A \implies A = 1/2$. Equate coefficients of x^2: $2 = A + B \implies 2 = 1/2 + B \implies B = 3/2$. Equate constant terms: $-1 = A - C \implies -1 = 1/2 - C \implies C = 3/2$. The decomposition is $\frac{1}{2(x-1)} + \frac{\frac{3}{2}x+\frac{3}{2}}{x^2+1} = \frac{1}{2(x-1)} + \frac{3(x+1)}{2(x^2+1)}$. b) Substitute $x = 2$: Voltage $= \frac{1}{2(2-1)} + \frac{3(2+1)}{2(2^2+1)} = \frac{1}{2} + \frac{9}{10} = 0.5 + 0.9 = 1.4$. c) For the voltage to be zero, the numerator of the original fraction must be zero. $2x^2 - 1 = 0 \implies x^2 = 1/2 \implies x = \pm\frac{1}{\sqrt{2}}$. The voltage can reach zero at $x = \pm\frac{1}{\sqrt{2}}$.
Question 28: Kasese Cement Factory	

Question	Solution
The rate of production of cement at a factory in Kasese is modeled by the rational expression $\frac{x^2+3x+2}{x^2+x}$. a) Express the improper fraction as a sum of a polynomial and a proper fraction. b) Decompose the proper fraction into partial fractions. c) Find the total production rate when $x = 2$ hours.	a) The degree of the numerator is equal to the degree of the denominator. We use long division. $x^2+x\overline{)x^2+3x+2}\ 1$ $x^2+3x+2 = 1(x^2+x)+(2x+2)$. The expression is $1+\frac{2x+2}{x^2+x}$. b) Factor the denominator: $x^2+x = x(x+1)$. We set up the partial fractions for the remainder: $\frac{2x+2}{x(x+1)} = \frac{A}{x}+\frac{B}{x+1}$ $2x+2 = A(x+1)+Bx$. Let $x = 0$: $2 = A(1) \implies A = 2$. Let $x = -1$: $2(-1)+2 = A(0)+B(-1) \implies 0 = -B \implies B = 0$. The partial fraction form is $1+\frac{2}{x}$. c) Substitute $x = 2$: Rate $= 1+\frac{2}{2} = 1+1 = 2$ units.
Question 29: Fisheries in Lira	
Question	**Solution**
A fisheries research team in Lira is modeling the growth rate of a fish population. The rate is given by the expression $\frac{1}{t(t-1)}$ where t is the time in months. a) Decompose the expression into partial fractions. b) Find the growth rate at $t = 3$ months. c) Find the total change in population from $t = 2$ to $t = 4$ months by integrating the partial fraction form. (Hint: $\int \frac{1}{x}dx = \ln\lvert x\rvert + C$).	a) We set up the partial fractions for distinct linear factors: $\frac{1}{t(t-1)} = \frac{A}{t}+\frac{B}{t-1}$ $1 = A(t-1)+Bt$ Let $t = 0$: $1 = A(-1) \implies A = -1$. Let $t = 1$: $1 = B(1) \implies B = 1$. The decomposition is $\frac{1}{t-1}-\frac{1}{t}$. b) Substitute $t = 3$: Growth rate $= \frac{1}{3-1}-\frac{1}{3} = \frac{1}{2}-\frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$. c) The total change is $\int_2^4 \left(\frac{1}{t-1}-\frac{1}{t}\right)dt. = [\ln\lvert t-1\rvert - \ln\lvert t\rvert]_2^4 = \left[\ln\left\lvert\frac{t-1}{t}\right\rvert\right]_2^4 = \left(\ln\left\lvert\frac{4-1}{4}\right\rvert\right)-\left(\ln\left\lvert\frac{2-1}{2}\right\rvert\right) = \ln\left(\frac{3}{4}\right)-\ln\left(\frac{1}{2}\right) = \ln\left(\frac{3/4}{1/2}\right) = \ln\left(\frac{3}{4}\times 2\right) = \ln\left(\frac{3}{2}\right) \approx 0.405$.
Question 30: Logistics in Mbarara	
Question	**Solution**
A logistics company in Mbarara is tracking the velocity of a delivery truck. The velocity is given by the expression $\frac{x^2-1}{x^2+x}$ where x is the time in hours. a) Express the improper fraction as a polynomial and a proper fraction. b) Decompose the proper fraction into partial fractions. c) What is the total distance traveled by the truck between $x = 1$ hour and $x = 3$ hours? (Hint: The distance is the integral of the velocity).	a) This is an improper fraction. Use long division. $x^2-1 = 1(x^2+x)+(-x-1)$. The expression is $1+\frac{-x-1}{x^2+x}$. b) Factor the denominator: $x^2+x = x(x+1)$. We set up the partial fractions for the remainder: $\frac{-x-1}{x(x+1)} = \frac{A}{x}+\frac{B}{x+1}$ $-x-1 = A(x+1)+Bx$ Let $x = 0$: $-1 = A$. Let $x = -1$: $-(-1)-1 = A(0)+B(-1) \implies 0 = -B \implies B = 0$. The partial fraction form is $1-\frac{1}{x}$. c) The total distance is $\int_1^3 \left(1-\frac{1}{x}\right)dx. = [x-\ln\lvert x\rvert]_1^3 = (3-\ln 3)-(1-\ln 1) = 3-\ln 3-1-0 = 2-\ln 3 \approx 0.9014$ units.
Question 31: Tea Processing in Kabale	
Question	**Solution**

A tea factory in Kabale is monitoring the drying process. The rate of moisture removal is given by the expression $\frac{2x-3}{(x-1)^2}$ where x is the temperature in degrees Celsius. a) Decompose the expression into partial fractions. b) Find the moisture removal rate at a temperature of $x = 4$. c) Determine the temperature x at which the moisture removal rate is $\frac{1}{4}$.	a) This is a case of a repeated linear factor. $\frac{2x-3}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$ $2x-3 = A(x-1)+B$ Let $x = 1$: $2(1)-3 = B \implies B = -1$. Equating coefficients of x: $2 = A$. The decomposition is $\frac{2}{x-1} - \frac{1}{(x-1)^2}$. b) Substitute $x = 4$: Rate $= \frac{2}{4-1} - \frac{1}{(4-1)^2} = \frac{2}{3} - \frac{1}{9} = \frac{6-1}{9} = \frac{5}{9}$. c) Set the rate to $\frac{1}{4}$: $\frac{2x-3}{(x-1)^2} = \frac{1}{4}$ $4(2x-3) = (x-1)^2$ $8x-12 = x^2-2x+1$ $x^2-10x+13 = 0$. Using the quadratic formula: $x = \frac{10 \pm \sqrt{10^2-4(1)(13)}}{2} = \frac{10 \pm \sqrt{100-52}}{2} = \frac{10 \pm \sqrt{48}}{2}$ $x = \frac{10 \pm 4\sqrt{3}}{2} = 5 \pm 2\sqrt{3}$. So, $x \approx 8.46°C$ or $x \approx 1.54°C$.
Question 32: Urban Planning in Gulu	
Question	**Solution**
An urban planner is modeling the population growth rate in a new housing estate. The rate is given by the expression $\frac{x^2+1}{x(x+1)^2}$ where x is the time in years. a) Decompose the expression into partial fractions. b) Find the growth rate when $x = 2$ years. c) Find the total growth in population from $x = 1$ to $x = 3$ years. (Hint: You will need to integrate the partial fractions).	a) The denominator has a linear factor and a repeated linear factor. $\frac{x^2+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$ $x^2+1 = A(x+1)^2 + Bx(x+1) + Cx$ Let $x = 0$: $1 = A(1)^2 \implies A = 1$. Let $x = -1$: $(-1)^2 + 1 = C(-1) \implies 2 = -C \implies C = -2$. Equate coefficients of x^2: $1 = A + B$. Since $A = 1$, we get $B = 0$. The decomposition is $\frac{1}{x} - \frac{2}{(x+1)^2}$. b) Substitute $x = 2$: Rate $= \frac{1}{2} - \frac{2}{(2+1)^2} = \frac{1}{2} - \frac{2}{9} = \frac{9-4}{18} = \frac{5}{18}$. c) The total growth is $\int_1^3 \left(\frac{1}{x} - \frac{2}{(x+1)^2}\right) dx.$ $= \left[\ln\lvert x\rvert - 2\left(\frac{-1}{x+1}\right)\right]_1^3 = \left[\ln\lvert x\rvert + \frac{2}{x+1}\right]_1^3 = \left(\ln 3 + \frac{2}{3+1}\right) - \left(\ln 1 + \frac{2}{1+1}\right) = \left(\ln 3 + \frac{1}{2}\right) - (0+1) = \ln 3 - 0.5 \approx 0.5986.$
Question 33: Busia Customs Inspection	
Question	**Solution**
A customs officer at Busia border is modeling the rate of goods inspection. The rate is given by the expression $\frac{x^2+3x-1}{x(x^2+1)}$ where x is the number of inspectors on duty. a) Decompose the expression into partial fractions. b) Find the inspection rate when there are $x = 2$ inspectors. c) Based on the model, does the rate of inspection ever equal zero? Explain your answer.	a) The denominator has a linear factor and an irreducible quadratic factor. $\frac{x^2+3x-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$ $x^2+3x-1 = A(x^2+1) + (Bx+C)x$ Let $x = 0$: $-1 = A$. Equate coefficients of x^2: $1 = A+B \implies 1 = -1+B \implies B = 2$. Equate coefficients of x: $3 = C$. The decomposition is $-\frac{1}{x} + \frac{2x+3}{x^2+1}$. b) Substitute $x = 2$: Rate $= -\frac{1}{2} + \frac{2(2)+3}{2^2+1} = -0.5 + \frac{7}{5} = -0.5 + 1.4 = 0.9$. c) For the rate to be zero, the numerator of the original fraction must be zero. $x^2 + 3x - 1 = 0$. Using the quadratic formula, we find a positive real solution for x. This means the rate can be zero for a non-integer number of inspectors.
Question 34: Kampala Waste Management	
Question	**Solution**

The rate of decomposition of organic waste in a landfill is modeled by the function $R(t) = \frac{t^2-1}{t^2+2t}$ where t is the time in months. a) Express the improper fraction as a polynomial and a proper fraction. b) Decompose the proper fraction into partial fractions. c) Find the total amount of waste decomposed from $t = 1$ month to $t = 3$ months. (Hint: integrate the decomposed expression).	a) The degree of the numerator equals the denominator. $t^2 - 1 = 1(t^2 + 2t) + (-2t - 1)$. The expression is $1 + \frac{-2t-1}{t^2+2t}$. b) Factor the denominator: $t^2 + 2t = t(t + 2)$. We set up the partial fractions for the remainder: $\frac{-2t-1}{t(t+2)} = \frac{A}{t} + \frac{B}{t+2}$ $-2t - 1 = A(t + 2) + Bt$ Let $t = 0$: $-1 = A(2) \implies A = -1/2$. Let $t = -2$: $-2(-2) - 1 = B(-2) \implies 3 = -2B \implies B = -3/2$. The partial fraction form is $1 - \frac{1}{2t} - \frac{3}{2(t+2)}$. c) The total amount is $\int_1^3 \left(1 - \frac{1}{2t} - \frac{3}{2(t+2)}\right) dt.$ $= \left[t - \frac{1}{2}\ln\lvert t\rvert - \frac{3}{2}\ln\lvert t+2\rvert\right]_1^3 = \left(3 - \frac{1}{2}\ln 3 - \frac{3}{2}\ln 5\right) - \left(1 - \frac{1}{2}\ln 1 - \frac{3}{2}\ln 3\right) = 2 + \ln 3 - \frac{3}{2}\ln 5 \approx 0.6845.$
Question 35: Telecom Network Signal Strength	
Question	**Solution**
A telecom company is measuring signal strength from a new mast. The signal strength is modeled by the expression $\frac{1}{(x+1)(x+2)}$ where x is the distance from the mast in kilometers. a) Decompose the expression into partial fractions. b) Find the signal strength at $x = 3$ km from the mast. c) What is the total signal strength received between $x = 1$ km and $x = 4$ km from the mast? (Hint: The integral of the rate is the total amount).	a) We set up the partial fractions for distinct linear factors: $\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$ $1 = A(x + 2) + B(x + 1)$ Let $x = -1$: $1 = A(-1 + 2) \implies A = 1$. Let $x = -2$: $1 = B(-2 + 1) \implies B = -1$. The decomposition is $\frac{1}{x+1} - \frac{1}{x+2}$. b) Substitute $x = 3$: Signal strength $= \frac{1}{3+1} - \frac{1}{3+2} = \frac{1}{4} - \frac{1}{5} = \frac{5-4}{20} = \frac{1}{20} = 0.05$. c) The total signal strength is $\int_1^4 \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx.$ $= [\ln\lvert x + 1\rvert - \ln\lvert x + 2\rvert]_1^4 = \left[\ln\left\lvert\frac{x+1}{x+2}\right\rvert\right]_1^4 = \left(\ln\left\lvert\frac{4+1}{4+2}\right\rvert\right) - \left(\ln\left\lvert\frac{1+1}{1+2}\right\rvert\right) = \ln\left(\frac{5}{6}\right) - \ln\left(\frac{2}{3}\right) = \ln\left(\frac{5/6}{2/3}\right) = \ln\left(\frac{5}{4}\right) \approx 0.2231.$
Question 36: Jinja Fishing Industry	
Question	**Solution**
A fishing company is analyzing their catch rate. The rate is given by the expression $\frac{x^2}{(x+1)(x^2+4)}$ where x is the time in hours. a) Decompose the expression into partial fractions. b) Find the catch rate after $x = 3$ hours. c) What is the limiting value of the catch rate as time approaches infinity?	a) The denominator has a linear factor and an irreducible quadratic factor. $\frac{x^2}{(x+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4}$ $x^2 = A(x^2 + 4) + (Bx + C)(x + 1)$ Let $x = -1$: $(-1)^2 = A((-1)^2 + 4) \implies 1 = 5A \implies A = 1/5$. Equate coefficients of x^2: $1 = A+B \implies 1 = 1/5+B \implies B = 4/5$. Equate constant terms: $0 = 4A + C \implies 0 = 4(1/5) + C \implies C = -4/5$. The decomposition is $\frac{1}{5(x+1)} + \frac{\frac{4}{5}x-\frac{4}{5}}{x^2+4} = \frac{1}{5(x+1)} + \frac{4(x-1)}{5(x^2+4)}$. b) Substitute $x = 3$: Rate $= \frac{1}{5(3+1)} + \frac{4(3-1)}{5(3^2+4)} = \frac{1}{20} + \frac{8}{65} = \frac{13+32}{260} = \frac{45}{260} = \frac{9}{52}$. c) As $x \to \infty$, the original fraction $\frac{x^2}{x^3+x^2+4x+4}$ approaches 0, since the degree of the denominator is higher than the degree of the numerator.
Question 37: Mbarara Dairy Industry	
Question	**Solution**

A dairy company in Mbarara is analyzing the rate of milk production. The rate is modeled by the function $R(t) = \frac{t^2+1}{t^2(t-1)}$ where t is the time in hours. a) Decompose the expression into partial fractions. b) What is the production rate when $t = 2$ hours? c) Find the total amount of milk produced between $t = 2$ and $t = 4$ hours.	a) The denominator has a repeated linear factor and a distinct linear factor. $\frac{t^2+1}{t^2(t-1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t-1}$ $t^2+1 = A(t)(t-1)+B(t-1)+C(t^2)$ Let $t = 0$: $1 = B(-1) \implies B = -1$. Let $t = 1$: $1^2+1 = C(1)^2 \implies 2 = C$. Equating coefficients of t^2: $1 = A + C \implies 1 = A+2 \implies A = -1$. The decomposition is $-\frac{1}{t} - \frac{1}{t^2} + \frac{2}{t-1}$. b) Substitute $t = 2$: Rate $= -\frac{1}{2} - \frac{1}{2^2} + \frac{2}{2-1} = -0.5 - 0.25 + 2 = 1.25$. c) The total amount is $\int_2^4 \left(-\frac{1}{t} - \frac{1}{t^2} + \frac{2}{t-1}\right) dt.$ $= \left[-\ln\lvert t\rvert + \frac{1}{t} + 2\ln\lvert t-1\rvert\right]_2^4 = \left(-\ln 4 + \frac{1}{4} + 2\ln 3\right) - \left(-\ln 2 + \frac{1}{2} + 2\ln 1\right) = -\ln 4 + 2\ln 3 + \ln 2 - 0.25 \approx 1.254.$
Question 38: Soroti Agribusiness	
Question	**Solution**
An agribusiness is analyzing the yield of a new crop. The yield rate is given by the expression $\frac{x+2}{(x-1)(x-2)}$ where x is the number of weeks. a) Decompose the expression into partial fractions. b) Find the yield rate after $x = 3$ weeks. c) If the company wants to maximize the yield rate, at what time would that occur? Explain your reasoning.	a) We set up the partial fractions for distinct linear factors: $\frac{x+2}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$ $x+2 = A(x-2)+B(x-1)$ Let $x = 1$: $1+2 = A(1-2) \implies 3 = -A \implies A = -3$. Let $x = 2$: $2+2 = B(2-1) \implies 4 = B$. The decomposition is $\frac{4}{x-2} - \frac{3}{x-1}$. b) Substitute $x = 3$: Rate $= \frac{4}{3-2} - \frac{3}{3-1} = \frac{4}{1} - \frac{3}{2} = 4 - 1.5 = 2.5$. c) The function has vertical asymptotes at $x = 1$ and $x = 2$. The yield rate approaches infinity as x approaches 2 from the right ($x \to 2^+$), meaning there is no finite maximum value. In a real-world scenario, the model would only be valid within a certain domain of x (e.g., $x > 2$).
Question 39: Jinja Sugar Cane Factory	
Question	**Solution**
A sugar factory is modeling the efficiency of a machine that crushes sugar cane. The efficiency is given by the rational expression $\frac{2x^2}{(x+1)^3}$ where x is the time the machine has been running in hours. a) Decompose the expression into partial fractions. b) Find the efficiency when the machine has been running for $x = 1$ hour. c) Find the total work done by the machine (integral of efficiency) from $x = 0$ to $x = 2$ hours.	a) This is a case of a repeated linear factor. $\frac{2x^2}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$ $2x^2 = A(x+1)^2 + B(x+1) + C$ Let $x = -1$: $2(-1)^2 = C \implies C = 2$. Equating coefficients of x^2: $2 = A$. Let $x = 0$: $0 = A(1)^2+B(1)+C \implies 0 = A+B+C \implies 0 = 2+B+2 \implies B = -4$. The decomposition is $\frac{2}{x+1} - \frac{4}{(x+1)^2} + \frac{2}{(x+1)^3}$. b) Substitute $x = 1$: Efficiency $= \frac{2}{1+1} - \frac{4}{(1+1)^2} + \frac{2}{(1+1)^3} = \frac{2}{2} - \frac{4}{4} + \frac{2}{8} = 1 - 1 + 0.25 = 0.25$. c) The total work is $\int_0^2 \left(\frac{2}{x+1} - \frac{4}{(x+1)^2} + \frac{2}{(x+1)^3}\right) dx.$ $= \left[2\ln\lvert x+1\rvert - 4\left(\frac{-1}{x+1}\right) + 2\left(\frac{-1}{2(x+1)^2}\right)\right]_0^2$ $= \left[2\ln\lvert x+1\rvert + \frac{4}{x+1} - \frac{1}{(x+1)^2}\right]_0^2 = \left(2\ln 3 + \frac{4}{3} - \frac{1}{9}\right) - \left(2\ln 1 + \frac{4}{1} - \frac{1}{1}\right) = 2\ln 3 + \frac{11}{9} - 3 \approx 0.419.$
Question 40: National Parks Management	
Question	**Solution**

A conservationist in one of Uganda's national parks is modeling the population growth rate of a specific animal species. The rate is given by the expression $\frac{x^2+x}{(x-1)(x^2+x+1)}$ where x is the time in years. a) Decompose the expression into partial fractions. b) Find the growth rate after $x = 2$ years. c) Find the total population growth from $x = 2$ to $x = 4$ years. (Hint: The integral is $\ln\|x-1| + \frac{1}{2}\ln\|x^2+x+1| - \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)$).	a) The denominator has a linear factor and an irreducible quadratic factor. $\frac{x^2+x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$ $x^2+x = A(x^2+x+1)+(Bx+C)(x-1)$ Let $x = 1$: $1^2+1 = A(1+1+1) \implies 2 = 3A \implies A = 2/3$. Equate coefficients of x^2: $1 = A + B \implies 1 = 2/3 + B \implies B = 1/3$. Equate constant terms: $0 = A - C \implies C = A \implies C = 2/3$. The decomposition is $\frac{2}{3(x-1)} + \frac{\frac{1}{3}x+\frac{2}{3}}{x^2+x+1} = \frac{2}{3(x-1)} + \frac{x+2}{3(x^2+x+1)}$. b) Substitute $x = 2$: Rate $= \frac{2}{3(2-1)} + \frac{2+2}{3(2^2+2+1)} = \frac{2}{3} + \frac{4}{3(7)} = \frac{2}{3} + \frac{4}{21} = \frac{14+4}{21} = \frac{18}{21} = \frac{6}{7}$. c) The integral is complex and is provided as a hint. The total growth is the definite integral of the rate function from $t = 2$ to $t = 4$. We would evaluate the integral at the limits of integration. This is a challenging problem that is best solved with a calculator or symbolic math software.

Chapter 5

Trigonometry

Competency:
The learner applies trigonometric functions, identities, and equations to solve problems in mathematics and real-life application in navigation, construction, aviation and other fields.

5.1 Reviewing trigonometrical ratios from a right-angled triangle, sine, cosine, tangent of an angle of any magn

5.1.1 Using the trigonometric ratios of sine, cosine, and tangent to determine their corresponding reciprocals,

5.1.2 Determining trigonometric ratios of $30°$, $45°$ and $60°$

5.1.3 Using Pythagoras theorem to derive $Sin^2\theta + Cos^2\theta = 1$, and simplifying trigonometrical identities; prove identities, solve trigonometrical equations and eliminate parameters from pairs of parametric equations.

5.2 Developing the compound angle formulae and use it to deduce the formulae for double angles, half angles, d

5.3 Solving trigonometric equations: involving inverse trigonometric ratios and using $t-$ formulas

5.4 Using the expressions $Rcos(\theta \pm \alpha)$ or $Rsin(\theta \pm \alpha)$ to

5.4.1 Solving trigonometrical equations of the form $acos\theta \pm bsin\theta + c = 0$

5.4.2 Finding the maximum and minimum of trigonometrical functions involving $acos\theta \pm bsin\theta + c = 0$ and $\frac{1}{acos\theta \pm bsin\theta + c}$

5.5 Deriving the cosine, sine and tangent rules

5.5.1 Solving triangles using the rules,

5.5.2 Deriving the formulae for angles of a triangle in terms of the sides

5.5.3 Deriving the formulae for area of a triangle

5.6 Applications of trigonometry in real life

Competency

The learner applies trigonometric functions, identities, and equations to solve problems in mathematics and real-life applications in navigation, construction, aviation, and other fields.

Review of Basic Trigonometric Ratios

For a right-angled triangle with angle θ:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Reciprocal Trigonometric Functions

$$\csc\theta = \frac{1}{\sin\theta} = \frac{\text{hypotenuse}}{\text{opposite}}$$
$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hypotenuse}}{\text{adjacent}}$$
$$\cot\theta = \frac{1}{\tan\theta} = \frac{\text{adjacent}}{\text{opposite}}$$

5.6.1 Special Angles

Angle	$\sin\theta$	$\cos\theta$	$\tan\theta$
30	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Example 1 (Construction Scenario): A ladder leans against a wall at an angle of 60 to the ground. If the ladder is 8 meters long, how high up the wall does it reach?

Solution:

$$\sin 60 = \frac{\text{height}}{8}$$
$$\frac{\sqrt{3}}{2} = \frac{h}{8}$$
$$h = 8 \times \frac{\sqrt{3}}{2} = 4\sqrt{3} \approx 6.93 \text{ meters}$$

5.7 Fundamental Trigonometric Identity

5.7.1 Derivation of $\sin^2\theta + \cos^2\theta = 1$

Using Pythagoras' theorem on a right-angled triangle:

$$a^2 + b^2 = c^2$$
$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$$
$$\sin^2\theta + \cos^2\theta = 1$$

5.7.2 Related Identities

From $\sin^2\theta + \cos^2\theta = 1$, we can derive:

$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Example 2: If $\sin\theta = \frac{3}{5}$ and θ is in the first quadrant, find $\cos\theta$ and $\tan\theta$.

Solution:

$$\sin^2\theta + \cos^2\theta = 1$$
$$\left(\frac{3}{5}\right)^2 + \cos^2\theta = 1$$
$$\frac{9}{25} + \cos^2\theta = 1$$
$$\cos^2\theta = \frac{16}{25}$$
$$\cos\theta = \frac{4}{5} \quad \text{(positive in first quadrant)}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{3/5}{4/5} = \frac{3}{4}$$

5.8 Compound Angle Formulae

5.8.1 Addition Formulae

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

5.8.2 Double Angle Formulae

Setting $B = A$ in the compound angle formulae:

$$\sin 2A = 2\sin A\cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

5.8.3 Half Angle Formulae

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$
$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$$
$$\tan\frac{A}{2} = \frac{1-\cos A}{\sin A} = \frac{\sin A}{1+\cos A}$$

Example 3 (Aviation Scenario): An aircraft takes off at an angle of 15. Express $\sin 15$ in exact form using half-angle formulae.

Solution:

$$\begin{aligned}\sin 15 &= \sin\frac{30}{2}\\ &= \sqrt{\frac{1-\cos 30}{2}}\\ &= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}\\ &= \sqrt{\frac{2-\sqrt{3}}{4}}\\ &= \frac{\sqrt{2-\sqrt{3}}}{2}\end{aligned}$$

5.9 The R-Method for Trigonometric Equations

5.9.1 Expressing $a\cos\theta \pm b\sin\theta$ in the form $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$

For $a\cos\theta + b\sin\theta = R\sin(\theta + \alpha)$:

$$R = \sqrt{a^2 + b^2}$$
$$\tan\alpha = \frac{a}{b}$$

For $a\cos\theta - b\sin\theta = R\cos(\theta + \alpha)$:

$$R = \sqrt{a^2 + b^2}$$
$$\tan\alpha = \frac{b}{a}$$

Example 4: Express $3\cos\theta + 4\sin\theta$ in the form $R\sin(\theta + \alpha)$.

Solution:

$$R = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5$$
$$\tan\alpha = \frac{3}{4}$$
$$\alpha = \arctan\left(\frac{3}{4}\right) \approx 36.87$$
$$\therefore\ 3\cos\theta + 4\sin\theta = 5\sin(\theta + 36.87)$$

5.9.2 Solving Equations Using the R-Method

Example 5: Solve $3\cos\theta + 4\sin\theta = 2$ for $0 \le \theta \le 360$.

Solution:

$$5\sin(\theta + 36.87) = 2$$
$$\sin(\theta + 36.87) = \frac{2}{5} = 0.4$$
$$\theta + 36.87 = 23.58 \text{ or } 156.42$$
$$\theta = -13.29 \text{ or } 119.55$$

Since $-13.29 < 0$, we add 360: $\theta = 346.71$ or 119.55.

5.9.3 Finding Maximum and Minimum Values

For $f(\theta) = a\cos\theta + b\sin\theta + c$:

- Maximum value: $c + R$ where $R = \sqrt{a^2 + b^2}$
- Minimum value: $c - R$

Example 6 (Navigation Scenario): The distance of a ship from a lighthouse is given by $d(\theta) = 10 + 6\cos\theta + 8\sin\theta$ km, where θ is the bearing. Find the maximum and minimum distances.

Solution:

$$R = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = 10$$
$$\text{Maximum distance} = 10 + 10 = 20 \text{ km}$$
$$\text{Minimum distance} = 10 - 10 = 0 \text{ km}$$

5.10 The Sine and Cosine Rules

5.10.1 The Sine Rule

For any triangle with sides a, b, c opposite to angles A, B, C respectively:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Example 7 (Surveying): A surveyor measures two angles of a triangular plot: $A = 50$ and $B = 70$. The side opposite to angle A is 100 m. Find the side opposite to angle B.

Solution:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{100}{\sin 50} = \frac{b}{\sin 70}$$
$$b = \frac{100 \times \sin 70}{\sin 50}$$
$$b = \frac{100 \times 0.9397}{0.7660} \approx 122.7 \text{ m}$$

5.10.2 The Cosine Rule

For any triangle:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Example 8 (Construction): A triangular roof truss has sides of length 8 m, 10 m, and 12 m. Find the largest angle.

Solution: The largest angle is opposite the longest side (12 m).

$$\begin{aligned} 12^2 &= 8^2 + 10^2 - 2(8)(10)\cos C \\ 144 &= 64 + 100 - 160\cos C \\ 144 &= 164 - 160\cos C \\ 160\cos C &= 20 \\ \cos C &= \frac{20}{160} = \frac{1}{8} = 0.125 \\ C &= \arccos(0.125) \approx 82.82 \end{aligned}$$

5.10.3 Area of a Triangle

The area of a triangle can be found using:

$$\text{Area} = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B$$

Also, using Heron's formula when all sides are known:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{a+b+c}{2}$ is the semi-perimeter.

Example 9: Find the area of the triangular plot in Example 7 if the third angle $C = 60$ and the side opposite to it is 115 m.

Solution:

$$\begin{aligned} \text{Area} &= \frac{1}{2}ab\sin C \\ &= \frac{1}{2}(100)(122.7)\sin 60 \\ &= \frac{1}{2}(100)(122.7)\left(\frac{\sqrt{3}}{2}\right) \\ &\approx 5316.5 \text{ m}^2 \end{aligned}$$

5.11 Trigonometric Equations

5.11.1 Basic Trigonometric Equations

Example 10: Solve $2\sin\theta = 1$ for $0 \le \theta \le 360$.

Solution:

$$\begin{aligned} \sin\theta &= \frac{1}{2} \\ \theta &= 30 \text{ or } 150 \end{aligned}$$

5.11.2 Equations Involving Compound Angles

Example 11: Solve $\sin 2\theta = \cos\theta$ for $0 \le \theta \le 360$.

Solution:

$$\begin{aligned} 2\sin\theta\cos\theta &= \cos\theta \\ 2\sin\theta\cos\theta - \cos\theta &= 0 \\ \cos\theta(2\sin\theta - 1) &= 0 \end{aligned}$$

Either $\cos\theta = 0$ or $\sin\theta = \frac{1}{2}$

$\cos\theta = 0 \Rightarrow \theta = 90, 270$

$\sin\theta = \frac{1}{2} \Rightarrow \theta = 30, 150$

Therefore, $\theta = 30, 90, 150, 270$

5.12 Real-Life Applications

5.12.1 Navigation

Example 12: A ship sails 40 km on a bearing of 060, then 60 km on a bearing of 130. How far is the ship from its starting point?

Solution: The angle between the two paths is $130 - 60 = 70$.

Using the cosine rule:

$$\begin{aligned} d^2 &= 40^2 + 60^2 - 2(40)(60)\cos 70 \\ d^2 &= 1600 + 3600 - 4800(0.342) \\ d^2 &= 5200 - 1641.6 = 3558.4 \\ d &\approx 59.7 \text{ km} \end{aligned}$$

5.12.2 Aviation

Example 13: An aircraft is flying at a height of 5000 m. The angle of depression to the airport is 12. How far horizontally is the aircraft from the airport?

Solution:

$$\begin{aligned} \tan 12 &= \frac{5000}{d} \\ d &= \frac{5000}{\tan 12} \\ d &= \frac{5000}{0.2126} \approx 23,520 \text{ m} = 23.52 \text{ km} \end{aligned}$$

5.13 Summary

Key points to remember:

- Fundamental identity: $\sin^2\theta + \cos^2\theta = 1$
- Compound angle formulae form the basis for double and half angle formulae
- The R-method simplifies solving equations of the form $a\cos\theta \pm b\sin\theta = c$

- Sine rule is used when we have angle-side-angle or side-angle-angle
- Cosine rule is used when we have side-side-side or side-angle-side
- Trigonometry has wide applications in navigation, construction, aviation, and surveying

5.14 Practice Exercises

1. Prove that $\tan\theta + \cot\theta = \sec\theta\csc\theta$
2. Express $5\cos\theta - 12\sin\theta$ in the form $R\cos(\theta + \alpha)$
3. Solve $\cos 2\theta = \sin\theta$ for $0 \leq \theta \leq 360$
4. A triangle has sides 7 cm, 8 cm, and 9 cm. Find all three angles.
5. An observer standing 100 m from a building measures the angle of elevation to the top as 35. Find the height of the building.
6. Find the maximum value of $f(\theta) = 3 + 4\sin\theta - 3\cos\theta$
7. Two ships leave a port at the same time. One travels at 20 km/h on a bearing of 040 and the other at 25 km/h on a bearing of 130. How far apart are they after 3 hours?

5.15 More revision scenario items with solutions

Question 1: Measuring the Height of a Building in Kampala	
Question	**Solution**
A surveyor is trying to find the height of a new skyscraper under construction in Kampala. He stands 50 meters away from the base of the building and uses a theodolite to measure the angle of elevation to the top as 40°. a) Draw a right-angled triangle to represent this situation, labeling the known sides and angle. b) Use an appropriate trigonometric ratio to find the height of the building. c) If the surveyor moves further away, what happens to the angle of elevation? Justify your answer.	a) The right-angled triangle has: • Adjacent side = 50 m (distance from building). • Opposite side = h (height of the building). • Angle of elevation = 40°. b) We use the tangent ratio: $\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}$ $\tan(40°) = \frac{h}{50}$ $h = 50 \times \tan(40°) \approx 50 \times 0.8391 \approx 41.955$ meters. The height of the building is approximately 42 meters. c) If the surveyor moves further away, the adjacent side of the triangle increases. Since the height of the building remains constant, the ratio of the opposite side to the adjacent side ($\tan\theta$) will decrease. As a result, the angle of elevation will also decrease.
Question 2: Farmer Surveying Land in Bundibugyo	
Question	**Solution**
A farmer in Bundibugyo wants to measure the slope of his land to determine the best place for planting coffee. He finds a section of land that rises 15 meters for every 100 meters he walks horizontally. a) Draw a right-angled triangle to represent the slope. b) Find the angle of elevation of the slope to the nearest degree. c) Explain how the concept of a hypotenuse applies to this scenario.	a) The right-angled triangle has: • Opposite side = 15 m (vertical rise). • Adjacent side = 100 m (horizontal distance). • Angle of elevation = θ. b) We use the tangent ratio: $\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{15}{100} = 0.15$ $\theta = \arctan(0.15) \approx 8.53°$. The angle of elevation of the slope is approximately 9°. c) The hypotenuse represents the actual path or length of the slope that the farmer walks on, from the start of the rise to the end. It is the longest side of the right-angled triangle.
Question 3: Lake Victoria Boat Navigation	
Question	**Solution**

A fisherman on Lake Victoria is navigating his boat. He uses a landmark on the shore. He is 1.5 km away from the base of a 300-meter-tall cliff. a) Calculate the angle of depression from the top of the cliff to the boat. b) If the boat moves further away, what happens to the angle of depression? c) What is the distance between the boat and the top of the cliff?	a) First, convert units to be consistent: 300 meters = 0.3 km. The angle of depression from the top of the cliff to the boat is the same as the angle of elevation from the boat to the top of the cliff. We use the tangent ratio: $\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{0.3}{1.5} = 0.2$ $\theta = \arctan(0.2) \approx 11.31°$. b) As the boat moves further away, the horizontal distance (adjacent side) increases. This makes the ratio of the opposite side to the adjacent side smaller, so the angle of depression decreases. c) We can use the Pythagorean theorem or a trigonometric ratio (e.g., sine or cosine). Let h be the distance to the top of the cliff. $\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}}$ $\sin(11.31°) = \frac{0.3}{h}$ $h = \frac{0.3}{\sin(11.31°)} \approx \frac{0.3}{0.196} \approx 1.53$ km. The distance is approximately 1.53 km.
Question 4: Hydroelectric Dam Design in Jinja	
Question	**Solution**
A civil engineer is designing a supporting truss for a hydroelectric dam. The truss forms a right-angled triangle. One of the supporting beams is 20 meters long, and it makes an angle of 60° with the ground. a) Find the height of the truss at the point where the beam touches the vertical support. b) Find the length of the horizontal base of the truss. c) Which of the three fundamental trigonometric ratios (sine, cosine, or tangent) is not used in parts (a) and (b)? Explain why.	a) The beam is the hypotenuse, 20 m. Let h be the height (opposite side). $\sin(60°) = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{h}{20}$ $h = 20 \times \sin(60°) = 20 \times \frac{\sqrt{3}}{2} = 10\sqrt{3} \approx 17.32$ meters. b) Let b be the horizontal base (adjacent side). $\cos(60°) = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{b}{20}$ $b = 20 \times \cos(60°) = 20 \times \frac{1}{2} = 10$ meters. c) The tangent ratio was not used because the values for the adjacent side (horizontal base) and opposite side (height) were not both known initially. We used sine and cosine to relate the known hypotenuse to the unknown sides. After finding one unknown side, tangent could be used to find the other, but it was not necessary.
Question 5: Telecom Mast Installation in Mbale	
Question	**Solution**
An engineer is installing a new telecom mast. A guy wire is attached to the top of the mast and anchored to the ground. The mast is 45 meters tall, and the wire is 60 meters long. a) Calculate the angle of elevation of the guy wire to the nearest degree. b) Find the distance from the base of the mast to the anchor point of the wire. c) The engineer wants to add a second wire that makes an angle of 55° with the ground. How long should this new wire be?	a) Let θ be the angle of elevation. We have the opposite side (mast height) and the hypotenuse (wire length). $\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{45}{60} = 0.75$ $\theta = \arcsin(0.75) \approx 48.59° \approx 49°$. b) We can use the Pythagorean theorem: Distance^2 + $45^2 = 60^2$ Distance^2 = 3600 - 2025 = 1575 Distance = $\sqrt{1575} \approx 39.686$ meters. c) Let L be the length of the new wire. The height of the mast is constant. $\sin(55°) = \frac{45}{L}$ $L = \frac{45}{\sin(55°)} \approx \frac{45}{0.819} \approx 54.94$ meters. The new wire should be approximately 55 meters long.
Question 6: Agricultural Irrigation in Busoga	
Question	**Solution**

A farmer is setting up an irrigation system using a pipe supported by a frame. The frame forms a right-angled triangle. The angle of inclination is 30°, and the horizontal base is 10 meters long. a) Use the reciprocal trigonometric ratio to find the length of the pipe (hypotenuse). b) Find the height of the frame. c) If the farmer wants the frame to have a hypotenuse of 15 meters while maintaining the same height, what angle should he set? What is the corresponding horizontal length?	a) The reciprocal of cosine is secant. $\sec(\theta) = \frac{\text{Hypotenuse}}{\text{Adjacent}}$ $\sec(30°) = \frac{\text{Hypotenuse}}{10}$ Hypotenuse $= 10 \times \sec(30°) = 10 \times \frac{1}{\cos(30°)} = 10 \times \frac{1}{\sqrt{3}/2} = \frac{20}{\sqrt{3}} \approx 11.547$ meters. b) We can use the tangent ratio: $\tan(30°) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{\text{Height}}{10}$ Height $= 10 \times \tan(30°) = 10 \times \frac{1}{\sqrt{3}} = \frac{10}{\sqrt{3}} \approx 5.774$ meters. c) Let θ' be the new angle. The height (opposite side) remains 5.774 meters. $\sin(\theta') = \frac{5.774}{15} \approx 0.385$ $\theta' = \arcsin(0.385) \approx 22.64°$. The corresponding horizontal length L is: $\cos(22.64°) = \frac{L}{15}$ $L = 15 \times \cos(22.64°) \approx 15 \times 0.923 \approx 13.845$ meters.
Question 7: Railway Bridge Construction in Tororo	
An engineer is designing a support structure for a railway bridge. The support is a triangular truss. One of the diagonal beams is 8 meters long and makes an angle of θ with the vertical support. The horizontal base of the truss is 6 meters. a) Calculate the value of $\sin\theta$, $\cos\theta$, and $\tan\theta$. b) What are the values of $\csc\theta$, $\sec\theta$, and $\cot\theta$? c) Verify that $\sin^2\theta + \cos^2\theta = 1$.	a) This is a right-angled triangle with a hypotenuse of 8 m and an adjacent side (to θ) of 6 m. Opposite side $= \sqrt{8^2 - 6^2} = \sqrt{64 - 36} = \sqrt{28} = 2\sqrt{7}$ meters. $\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{2\sqrt{7}}{8} = \frac{\sqrt{7}}{4}$ $\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{6}{8} = \frac{3}{4}$ $\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{2\sqrt{7}}{6} = \frac{\sqrt{7}}{3}$ b) $\csc\theta = \frac{1}{\sin\theta} = \frac{4}{\sqrt{7}} = \frac{4\sqrt{7}}{7}$ $\sec\theta = \frac{1}{\cos\theta} = \frac{4}{3}$ $\cot\theta = \frac{1}{\tan\theta} = \frac{3}{\sqrt{7}} = \frac{3\sqrt{7}}{7}$ c) $\sin^2\theta + \cos^2\theta = \left(\frac{\sqrt{7}}{4}\right)^2 + \left(\frac{3}{4}\right)^2 = \frac{7}{16} + \frac{9}{16} = \frac{16}{16} = 1$. The identity is verified.
Question 8: Kasese Cement Factory's Conveyor Belt	
Question	**Solution**
A new conveyor belt is being installed at the Kasese Cement Factory. It is set up to move materials from the ground to a height of 15 meters. The conveyor belt itself is 30 meters long. a) Use a trigonometric ratio to find the angle of inclination of the conveyor belt. b) Use a reciprocal trigonometric ratio to find the horizontal distance the belt covers. c) If the engineers wanted the belt to be at an angle of 45°, what would be the length of the belt and the horizontal distance? (Use exact values).	a) Let θ be the angle of inclination. We have the opposite side (15 m) and the hypotenuse (30 m). $\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{15}{30} = 0.5$ $\theta = \arcsin(0.5) = 30°$. b) The reciprocal of cosine is secant. $\sec\theta = \frac{\text{Hypotenuse}}{\text{Adjacent}}$ Let x be the horizontal distance (adjacent side). $\sec(30°) = \frac{30}{x}$ $x = \frac{30}{\sec(30°)} = 30 \times \cos(30°) = 30 \times \frac{\sqrt{3}}{2} = 15\sqrt{3}$ meters. c) If $\theta = 45°$, the length of the belt (hypotenuse) L would be: $\sin(45°) = \frac{15}{L}$ $L = \frac{15}{\sin(45°)} = \frac{15}{1/\sqrt{2}} = 15\sqrt{2}$ meters. The horizontal distance x would be: $\tan(45°) = \frac{15}{x} \implies 1 = \frac{15}{x} \implies x = 15$ meters.
Question 9: Mapping a Forest in Kalangala	
A conservationist is mapping a forest in Kalangala to protect it. To find the distance between two trees, she sights a third tree from both locations. The two trees are 20 meters apart, and the angles from the third tree to the two trees are 45° and 30°. a) Use the sine rule to find the distance from the third tree to the tree that has an angle of 30°. b) Calculate the height of the triangle. c) What is the total area of the triangle formed by the three trees?	a) We have a triangle with angles 45°, 30° and the third angle is $180° - 45° - 30° = 105°$. Let $a = 20$ m. Let b be the side opposite 30° and c be the side opposite 45°. Using the sine rule: $\frac{b}{\sin(30°)} = \frac{a}{\sin(105°)}$ $b = \frac{20 \times \sin(30°)}{\sin(105°)} = \frac{20 \times 0.5}{0.9659} \approx 10.35$ meters. b) The height h can be found using the sine ratio: $\sin(30°) = \frac{h}{c}$. We need to find c first: $\frac{c}{\sin(45°)} = \frac{20}{\sin(105°)}$ $c = \frac{20 \times \sin(45°)}{\sin(105°)} \approx \frac{20 \times 0.7071}{0.9659} \approx 14.63$ meters. $h = 14.63 \times \sin(30°) = 14.63 \times 0.5 \approx 7.315$ meters. c) Area $= \frac{1}{2}ab\sin(C) = \frac{1}{2}(20)(14.63)\sin(30°) \approx \frac{1}{2}(292.6)(0.5) \approx 73.15$ m^2.

Question 10: Urban Planning in Gulu	
An urban planner is designing a public park in Gulu. He wants to install a decorative monument with a circular base. The monument's height is 5 meters. A support cable is to be anchored to the ground at a distance of 5 meters from the base of the monument. a) Calculate the angle of elevation of the cable to the monument. b) What is the length of the cable? c) If a second cable is to be installed such that its length is 10 meters, what is its angle of elevation? What is the corresponding distance from the base of the monument?	a) The monument height is the opposite side and the distance from the base is the adjacent side. Since both are equal (5 m), we have: $\tan(\theta) = \frac{5}{5} = 1$ $\theta = \arctan(1) = 45°$. b) We can use the Pythagorean theorem or a trigonometric ratio. Length$^2 = 5^2 + 5^2 = 25 + 25 = 50$ Length $= \sqrt{50} = 5\sqrt{2}$ meters. c) Let θ' be the new angle. The monument height is constant at 5 meters. The length of the new cable is 10 meters. $\sin(\theta') = \frac{5}{10} = 0.5$ $\theta' = \arcsin(0.5) = 30°$. Let x be the new distance from the base. $\cos(30°) = \frac{x}{10}$ $x = 10 \times \cos(30°) = 10 \times \frac{\sqrt{3}}{2} = 5\sqrt{3}$ meters.
Question 11: Murchison Falls National Park Vehicle Path	
Question	**Solution**
A park ranger is driving a vehicle along a path whose slope can be modeled by a function. The vertical displacement is given by $y = \sin(\theta)$ and the horizontal displacement is $x = \cos(\theta)$. a) Use the Pythagorean identity to find the relationship between x and y. b) Explain what type of path this relationship represents. c) If the ranger drives from an angle of 0° to 90°, what are the changes in his horizontal and vertical positions?	a) We know that $\sin^2\theta + \cos^2\theta = 1$. Given $y = \sin(\theta)$ and $x = \cos(\theta)$, we can substitute these into the identity: $y^2 + x^2 = 1$. b) The relationship $x^2 + y^2 = 1$ represents a circle with a radius of 1 centered at the origin (0,0). This means the ranger's path is circular. c) At $\theta = 0°$: $x = \cos(0°) = 1$ $y = \sin(0°) = 0$ The position is $(1, 0)$. At $\theta = 90°$: $x = \cos(90°) = 0$ $y = \sin(90°) = 1$ The position is $(0, 1)$. The horizontal position changes from 1 to 0 (a decrease of 1 unit), and the vertical position changes from 0 to 1 (an increase of 1 unit).
Question 12: Weather Balloon Tracking in Soroti	
A meteorological team in Soroti is tracking a weather balloon. The horizontal distance from the station is $x = 2\sec(\theta)$ and the height is $h = 2\tan(\theta)$. a) Use a trigonometric identity to eliminate the parameter θ and find the relationship between x and h. b) If the horizontal distance is 4 km, what is the height of the balloon? c) Explain why this relationship is useful for the meteorological team.	a) We know the identity $1 + \tan^2\theta = \sec^2\theta$. From the given equations, $\sec(\theta) = \frac{x}{2}$ and $\tan(\theta) = \frac{h}{2}$. Substitute these into the identity: $1 + \left(\frac{h}{2}\right)^2 = \left(\frac{x}{2}\right)^2$ $1 + \frac{h^2}{4} = \frac{x^2}{4}$ Multiplying by 4, we get $4 + h^2 = x^2$ or $x^2 - h^2 = 4$. b) Substitute $x = 4$ into the relationship: $4^2 - h^2 = 4 \implies 16 - h^2 = 4$ $h^2 = 12 \implies h = \sqrt{12} = 2\sqrt{3} \approx 3.464$ km. c) This relationship allows the team to determine the height of the balloon directly from its horizontal distance, without needing to calculate the angle θ. This simplifies real-time tracking and data analysis.
Question 13: Water Pump Efficiency in Karamoja	

A community is installing a new water pump. The rate of water flow can be modeled by the expression $\frac{\tan^2\theta}{\sin\theta\cos\theta}$. a) Simplify this expression using fundamental trigonometric identities. b) If the angle is 45°, find the rate of water flow. c) Explain the real-world significance of simplifying such an expression.	a) We know that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sec\theta = \frac{1}{\cos\theta}$. $\frac{\tan^2\theta}{\sin\theta\cos\theta} = \frac{(\frac{\sin\theta}{\cos\theta})^2}{\sin\theta\cos\theta} = \frac{\frac{\sin^2\theta}{\cos^2\theta}}{\sin\theta\cos\theta} = \frac{\sin^2\theta}{\cos^2\theta} \times \frac{1}{\sin\theta\cos\theta} = \frac{\sin\theta}{\cos^3\theta} = \frac{\sin\theta}{\cos\theta} \times \frac{1}{\cos^2\theta} = \tan\theta\sec^2\theta$. b) Substitute $\theta = 45°$ into the simplified expression: Rate $= \tan(45°)\sec^2(45°) = (1) \times \left(\frac{1}{\cos(45°)}\right)^2 = (1) \times \left(\frac{1}{1/\sqrt{2}}\right)^2 = (\sqrt{2})^2 = 2$. The rate is 2 units. c) Simplifying complex expressions makes calculations much faster and easier. It reduces the number of operations needed, which is important for efficiency in real-time monitoring and analysis, especially when using simple calculators or manual methods.
Question 14: Cable Car System at Sipi Falls	
A company is proposing to build a cable car system at Sipi Falls. The position of the car is given by the coordinates (x, y), where $x = 3\sec\theta$ and $y = 3\tan\theta$. a) Show that the equation of the path is $x^2 - y^2 = 9$. b) When the car is 5 units horizontally from the central tower, what is its vertical height? c) Explain what type of curve this equation represents.	a) From the given equations, we have $\sec\theta = \frac{x}{3}$ and $\tan\theta = \frac{y}{3}$. We use the identity $\sec^2\theta - \tan^2\theta = 1$. Substitute the expressions for $\sec\theta$ and $\tan\theta$: $\left(\frac{x}{3}\right)^2 - \left(\frac{y}{3}\right)^2 = 1$ $\frac{x^2}{9} - \frac{y^2}{9} = 1$ $x^2 - y^2 = 9$. b) Substitute $x = 5$ into the equation of the path: $5^2 - y^2 = 9$ $25 - y^2 = 9$ $y^2 = 16 \implies y = \pm 4$. Since height is a positive value, the vertical height is 4 units. c) The equation $x^2 - y^2 = 9$ represents a hyperbola. This means the cable car's path is not a simple straight line or a circular arc, but follows a specific curved path typical of certain engineering designs.
Question 15: Solving a Problem with the Identity	
Question	**Solution**
A drone is flying over a farm in Masindi, and its altitude is related to its horizontal distance by the expression $h = \frac{\sin^2 x + \cos^2 x}{\sec^2 x - \tan^2 x}$. a) Simplify this expression using trigonometric identities. b) What is the altitude of the drone at any horizontal distance x? c) Why is it not necessary to know the horizontal distance to find the altitude in this scenario?	a) We use two fundamental trigonometric identities: • Numerator: $\sin^2 x + \cos^2 x = 1$. • Denominator: $\sec^2 x - \tan^2 x = 1$. Substituting these into the expression: $h = \frac{1}{1} = 1$. b) The altitude of the drone is a constant 1 unit, regardless of the horizontal distance. c) Because the expression simplifies to a constant, the altitude is independent of the horizontal distance x. This means the drone is flying at a fixed altitude, which is a common feature of drones designed for mapping and surveying, as it ensures consistent image quality.
Question 16: Windmill Blade Rotation in Moroto	

An engineer is modeling the motion of a windmill blade. The vertical position of the tip of the blade is given by $h(t) = 10\sin(t) + 12$, where t is the time in seconds. a) Find the minimum and maximum heights of the blade tip. b) At what times in the interval $0 \le t \le 2\pi$ is the blade tip at a height of 17 meters? c) If the engineer wants the blade tip to be at its lowest point at $t = \frac{3\pi}{2}$ seconds, how should the function be adjusted?	a) The minimum value of $\sin(t)$ is -1, and the maximum is 1. Minimum height: $10(-1) + 12 = -10 + 12 = 2$ meters. Maximum height: $10(1) + 12 = 10 + 12 = 22$ meters. b) Set $h(t) = 17$: $17 = 10\sin(t) + 12$ $5 = 10\sin(t) \implies \sin(t) = 0.5$ The principal value is $t = \arcsin(0.5) = \frac{\pi}{6}$. In the interval $0 \le t \le 2\pi$, sine is also positive in the second quadrant. $t = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$. The times are $t = \frac{\pi}{6}$ and $t = \frac{5\pi}{6}$ seconds. c) To shift the lowest point to $t = \frac{3\pi}{2}$, we need a cosine function shifted to the right, or a sine function shifted by a phase angle. A simpler adjustment is to use a negative sine function. The function should be adjusted to $h(t) = -10\sin(t) + 12$. In this case, at $t = \frac{\pi}{2}$, the height is $-10 + 12 = 2$, and at $t = \frac{3\pi}{2}$, the height is $-10(-1) + 12 = 22$. This is a reflection. A better way is to use a cosine function: $h(t) = 10\cos(t) + 12$. The lowest point is at $t = \pi$.
Question 17: Water Level Fluctuation in Lake Kyoga	
The water level of Lake Kyoga fluctuates throughout the year. The depth can be modeled by the equation $D(t) = 1.5\cos(\frac{\pi t}{6}) + 10$, where t is the number of months since January 1st and the depth is in meters. a) What is the average depth of the lake? b) Find the depth of the lake after 3 months (at the start of April). c) At what months is the water level at its maximum and minimum depths?	a) The average depth is given by the vertical shift of the function, which is the constant term. Average depth = 10 meters. b) Substitute $t = 3$: $D(3) = 1.5\cos(\frac{3\pi}{6}) + 10 = 1.5\cos(\frac{\pi}{2}) + 10$ Since $\cos(\frac{\pi}{2}) = 0$, $D(3) = 1.5(0) + 10 = 10$ meters. c) The maximum depth occurs when $\cos(\frac{\pi t}{6}) = 1$. This happens when $\frac{\pi t}{6} = 2n\pi$, which means $t = 12n$ (i.e., at $t = 0, 12, 24, \ldots$). The maximum is at $t = 0$ (January 1st) and $t = 12$ (the following January 1st). The minimum depth occurs when $\cos(\frac{\pi t}{6}) = -1$. This happens when $\frac{\pi t}{6} = (2n + 1)\pi$, which means $t = 6, 18, 30, \ldots$. The minimum is at $t = 6$, which corresponds to the start of July.
Question 18: Traffic Flow in the Kampala-Entebbe Expressway	
Question	**Solution**
The speed of a vehicle on the Kampala-Entebbe Expressway is modeled by the equation $S = \frac{\sin x}{1+\cos x}$. The driver wants to know if they can simplify this to a single trigonometric function. a) Simplify the expression for S to a single trigonometric function. b) If the driver is driving at an angle of 30° relative to the flow, what is the speed? c) What is the speed when the angle is 90°?	a) We use the half-angle identity for tangent, $\tan\left(\frac{x}{2}\right) = \frac{\sin x}{1+\cos x}$. So, the expression simplifies to $S = \tan\left(\frac{x}{2}\right)$. b) Substitute $x = 30^\circ$: $S = \tan\left(\frac{30^\circ}{2}\right) = \tan(15^\circ) \approx 0.2679$. c) Substitute $x = 90^\circ$: $S = \tan\left(\frac{90^\circ}{2}\right) = \tan(45^\circ) = 1$. The speed is 1 unit.
Question 19: Robot Arm Movement in a Factory in Namanve	

A robot arm in a factory in Namanve is programmed to move objects. Its position is given by the parametric equations $x = 3t$ and $y = 3\cos(t)$. a) Eliminate the parameter t to find the equation of the path in terms of x and y. b) What are the horizontal and vertical positions of the arm at $t = \frac{\pi}{2}$? c) If the robot arm is to perform a circular motion, what would be the required parametric equations for its position?	a) From $x = 3t$, we have $t = \frac{x}{3}$. Substitute this into the second equation: $y = 3\cos\left(\frac{x}{3}\right)$. This is the equation of the path. b) Substitute $t = \frac{\pi}{2}$ into the equations: $x = 3\left(\frac{\pi}{2}\right) = \frac{3\pi}{2}$ $y = 3\cos\left(\frac{\pi}{2}\right) = 3(0) = 0$. The position is $\left(\frac{3\pi}{2}, 0\right)$. c) For a circular motion, the parametric equations should be of the form $x = r\cos(t)$ and $y = r\sin(t)$, where r is the radius of the circle. A specific example would be $x = \cos(t)$ and $y = \sin(t)$.
Question 20: Radio Signal Strength from Kololo Hill	
A radio station located on Kololo Hill is broadcasting a signal. The signal strength at a given location is modeled by the function $S(x) = \frac{\sin^2(x)-\cos^2(x)}{\sin(x)\cos(x)}$, where x is a parameter related to the receiver's position. a) Simplify this expression to a single trigonometric function. b) What is the signal strength at an angle of 30°? c) If a technician wants to find a location where the signal strength is 1, what angle x should they look for?	a) We can use the identities $\cos^2 x - \sin^2 x = \cos(2x)$ and $2\sin(x)\cos(x) = \sin(2x)$. $S(x) = \frac{-(\cos^2 x - \sin^2 x)}{\sin x \cos x} = \frac{-\cos(2x)}{\frac{1}{2}\sin(2x)}$ $S(x) = -2\frac{\cos(2x)}{\sin(2x)} = -2\cot(2x)$. Alternatively: $S(x) = \frac{\sin^2 x}{\sin x\cos x} - \frac{\cos^2 x}{\sin x\cos x} = \frac{\sin x}{\cos x} - \frac{\cos x}{\sin x} = \tan x - \cot x$. b) Using the simplified form $\tan x - \cot x$: $S(30°) = \tan(30°) - \cot(30°) = \frac{1}{\sqrt{3}} - \sqrt{3} = \frac{1-3}{\sqrt{3}} = -\frac{2}{\sqrt{3}} \approx -1.1547$. c) Set the simplified expression equal to 1: $\tan x - \cot x = 1 \implies \tan x - \frac{1}{\tan x} = 1$ Let $y = \tan x$: $y - \frac{1}{y} = 1 \implies y^2 - 1 = y \implies y^2 - y - 1 = 0$. Using the quadratic formula, $y = \frac{1\pm\sqrt{1-4(1)(-1)}}{2} = \frac{1\pm\sqrt{5}}{2}$. Since $\tan x$ must be a positive value in the first quadrant, $\tan x = \frac{1+\sqrt{5}}{2} \approx 1.618$. $x = \arctan(1.618) \approx 58.28°$.

Question 41: Pilot Navigation over Lake Victoria	
Question	**Solution**
A pilot flying over Lake Victoria adjusts his bearing due to crosswinds. His new flight path is modeled by $\sin(\theta + 30°)$. a) Use the compound angle formula to expand this expression. b) If the original angle of his flight path was $\theta = 45°$, what is the value of the new flight path expression? c) The pilot needs the value of the new path to be $\frac{1}{2}$. Find the two possible values of θ in the range $0° \le \theta \le 360°$.	a) Using the formula $\sin(A + B) = \sin A\cos B + \cos A\sin B$: $\sin(\theta + 30°) = \sin\theta\cos 30° + \cos\theta\sin 30°$ $= \sin\theta\left(\frac{\sqrt{3}}{2}\right) + \cos\theta\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta$. b) Substitute $\theta = 45°$ into the expanded expression: $\sin(45° + 30°) = \sin(75°) = \sin 45°\cos 30° + \cos 45°\sin 30° = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4} \approx 0.9659$. c) We need to solve $\sin(\theta+30°) = \frac{1}{2}$. The principal value for $\sin^{-1}\left(\frac{1}{2}\right)$ is 30°. In the range $0° \le \theta + 30° \le 390°$, the solutions for $\sin(\theta + 30°) = \frac{1}{2}$ are in the first and second quadrants. Case 1: $\theta + 30° = 30° \implies \theta = 0°$. Case 2: $\theta + 30° = 180° - 30° = 150° \implies \theta = 120°$.
Question 42: Road Construction in Gulu	
Question	**Solution**

Engineers are designing a curved section of a road. The angle of the curve is defined by $\cos(2\theta) = \frac{1}{2}$. a) Use the double angle formula for cosine to solve the equation for $\cos\theta$. b) Find all possible values of θ in the range $0^\circ \leq \theta \leq 360^\circ$. c) Explain why multiple solutions for θ are a practical consideration in road design.	a) The double angle formula is $\cos(2\theta) = 2\cos^2\theta - 1$. Substituting this into the equation: $2\cos^2\theta - 1 = \frac{1}{2}$ $2\cos^2\theta = \frac{3}{2}$ $\cos^2\theta = \frac{3}{4}$ $\cos\theta = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$. b) We need to solve $\cos\theta = \frac{\sqrt{3}}{2}$ and $\cos\theta = -\frac{\sqrt{3}}{2}$. Case 1: $\cos\theta = \frac{\sqrt{3}}{2}$. The principal value is 30°. Cosine is also positive in the fourth quadrant. So, $\theta = 360^\circ - 30^\circ = 330^\circ$. Case 2: $\cos\theta = -\frac{\sqrt{3}}{2}$. The principal value is $180^\circ - 30^\circ = 150^\circ$. Cosine is also negative in the third quadrant. So, $\theta = 180^\circ + 30^\circ = 210^\circ$. The solutions are $\theta = 30^\circ, 150^\circ, 210^\circ, 330^\circ$. c) In road design, multiple solutions for an angle can correspond to different but valid curves that meet the engineering requirements. For example, a curved section might need to be designed to turn left or right, or the same angle could be achieved at different points along the road's path, giving engineers flexibility in design and layout.
Question 43: Land Surveying in Buganda	
Question	**Solution**
A land surveyor is measuring a triangular plot of land in Buganda. The plot has sides of length 100 meters and 150 meters, with the angle between them measuring 60°. a) Use the cosine rule to find the length of the third side of the plot. b) Use the sine rule to find the measure of the angle opposite the 100-meter side. c) Calculate the area of the triangular plot of land.	a) Let the sides be $a = 100$, $b = 150$ and the angle between them be $C = 60^\circ$. The third side is c. $c^2 = a^2 + b^2 - 2ab\cos C$ $c^2 = 100^2 + 150^2 - 2(100)(150)\cos 60^\circ$ $c^2 = 10000 + 22500 - 30000\left(\frac{1}{2}\right)$ $c^2 = 32500 - 15000 = 17500$ $c = \sqrt{17500} = \sqrt{2500 \times 7} = 50\sqrt{7} \approx 132.29$ meters. b) Let A be the angle opposite the 100-meter side. $\frac{\sin A}{a} = \frac{\sin C}{c}$ $\frac{\sin A}{100} = \frac{\sin 60^\circ}{50\sqrt{7}}$ $\sin A = \frac{100\sin 60^\circ}{50\sqrt{7}} = \frac{2\times\frac{\sqrt{3}}{2}}{\sqrt{7}} = \frac{\sqrt{3}}{\sqrt{7}} \approx 0.6547$ $A = \sin^{-1}(0.6547) \approx 40.89^\circ$. c) The area of the triangle is given by the formula: Area $= \frac{1}{2}ab\sin C$ Area $= \frac{1}{2}(100)(150)\sin 60^\circ = 7500 \times \frac{\sqrt{3}}{2} = 3750\sqrt{3} \approx 6495.19$ m^2.
Question 44: Hydroelectric Power Turbine Output	
Question	**Solution**
The electrical power output of a small turbine at a Jinja dam is given by the function $P(\theta) = 4\cos\theta + 3\sin\theta$ kilowatts. a) Express $P(\theta)$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0^\circ \leq \alpha \leq 90^\circ$. b) Find the maximum and minimum power output of the turbine. c) At what angle θ does the maximum power output occur?	a) We want to express $4\cos\theta + 3\sin\theta$ as $R(\cos\theta\cos\alpha + \sin\theta\sin\alpha) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$. By comparing coefficients: $R\cos\alpha = 4$ $R\sin\alpha = 3$ To find R, we square and add the two equations: $R^2\cos^2\alpha + R^2\sin^2\alpha = 4^2 + 3^2$ $R^2(\cos^2\alpha + \sin^2\alpha) = 16 + 9 = 25$ $R^2 = 25 \implies R = 5$. To find α, we divide the two equations: $\frac{R\sin\alpha}{R\cos\alpha} = \frac{3}{4} \implies \tan\alpha = \frac{3}{4}$ $\alpha = \tan^{-1}\left(\frac{3}{4}\right) \approx 36.87^\circ$. So, $P(\theta) = 5\cos(\theta - 36.87^\circ)$. b) The maximum value of $\cos(\theta - \alpha)$ is 1, and the minimum is -1. Maximum power $= 5 \times 1 = 5$ kW. Minimum power $= 5 \times (-1) = -5$ kW. c) Maximum power occurs when $\cos(\theta - 36.87^\circ) = 1$. $\theta - 36.87^\circ = 0^\circ \implies \theta = 36.87^\circ$.

Question 45: Agricultural Drone Survey in Mukono	
Question	**Solution**
An agricultural drone is surveying a farm plot that is shaped like a triangle. The sides of the triangle are 200 meters, 180 meters, and 250 meters. a) Use the cosine rule to find the measure of the largest angle in the triangle. b) Find the area of the triangular plot using Heron's formula. c) A different plot has sides of 100 meters and 120 meters, and the angle between them is 50°. What would be the length of its third side?	a) The largest angle is opposite the longest side, which is 250 m. Let $c = 250$, $a = 200$, $b = 180$. $\cos C = \frac{a^2+b^2-c^2}{2ab}$ $\cos C = \frac{200^2+180^2-250^2}{2(200)(180)} = \frac{40000+32400-62500}{72000} = \frac{9900}{72000} = \frac{11}{80}$ $C = \cos^{-1}\left(\frac{11}{80}\right) \approx 82.12°$. b) First, find the semi-perimeter s: $s = \frac{200+180+250}{2} = \frac{630}{2} = 315$ meters. Using Heron's formula: Area $= \sqrt{s(s-a)(s-b)(s-c)}$ Area $= \sqrt{315(315-200)(315-180)(315-250)} = \sqrt{315(115)(135)(65)}$ Area $\approx \sqrt{320390625} \approx 17899.46$ m^2. c) Using the cosine rule, let the third side be c: $c^2 = 100^2+120^2-2(100)(120)\cos(50°)$ $c^2 = 10000+14400-24000(0.6428) = 24400-15427.2 = 8972.8$ $c = \sqrt{8972.8} \approx 94.73$ meters.
Question 46: Water Distribution Pipe Design	
Question	**Solution**
An engineer is designing a water distribution network. A key junction has three pipes that form a triangle. The lengths of two pipes are 50m and 70m, and the angle opposite the 70m pipe is 80°. a) Use the sine rule to find the angle opposite the 50m pipe. b) Find the measure of the third angle of the triangle. c) Use the sine rule again to find the length of the third pipe.	a) Let $a = 50$, $b = 70$, and $B = 80°$. We want to find angle A. $\frac{\sin A}{a} = \frac{\sin B}{b}$ $\frac{\sin A}{50} = \frac{\sin 80°}{70}$ $\sin A = \frac{50 \sin 80°}{70} = \frac{50(0.9848)}{70} \approx 0.7034$ $A = \sin^{-1}(0.7034) \approx 44.7°$. b) The sum of angles in a triangle is 180°. Third angle $C = 180° - 80° - 44.7° = 55.3°$. c) Let the third side be c. $\frac{c}{\sin C} = \frac{b}{\sin B}$ $\frac{c}{\sin 55.3°} = \frac{70}{\sin 80°}$ $c = \frac{70 \sin 55.3°}{\sin 80°} = \frac{70(0.822)}{0.9848} \approx 58.4$ meters.
Question 47: Optimizing a Solar Panel in Karamoja	
Question	**Solution**
The power generated by a movable solar panel is modeled by the function $P(\theta) = 8\cos\theta + 6\sin\theta$. a) Express $P(\theta)$ in the form $R\sin(\theta+\alpha)$, where $R > 0$ and $0° \le \alpha \le 90°$. b) What is the maximum power the solar panel can generate? c) At what angle θ should the panel be tilted to achieve this maximum power?	a) We want to express $8\cos\theta + 6\sin\theta$ as $R\sin(\theta+\alpha) = R(\sin\theta\cos\alpha + \cos\theta\sin\alpha) = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$. Comparing coefficients: $R\cos\alpha = 6$ $R\sin\alpha = 8$ $R^2 = 6^2+8^2 = 36+64 = 100 \implies R = 10$. $\tan\alpha = \frac{8}{6} = \frac{4}{3} \implies \alpha = \tan^{-1}\left(\frac{4}{3}\right) \approx 53.13°$. So, $P(\theta) = 10\sin(\theta+53.13°)$. b) The maximum value of $\sin(\theta+\alpha)$ is 1. Maximum power $= 10 \times 1 = 10$ units. c) Maximum power occurs when $\sin(\theta+53.13°) = 1$. $\theta + 53.13° = 90° \implies \theta = 90° - 53.13° = 36.87°$.
Question 48: Kasese Cement Factory's Conveyor Belt	
Question	**Solution**

A new conveyor belt is being installed. The vertical height it reaches is given by $h(t) = \sin^2(t)$ and the horizontal distance is given by $x(t) = \cos(2t)$. a) Express $h(t)$ in terms of $\cos(2t)$. b) Eliminate the parameter t to find the relationship between x and h. c) What is the vertical height h when the horizontal distance x is 0.5?	a) We use the half-angle formula, which states $\cos(2t) = 1 - 2\sin^2 t$. Rearranging for $\sin^2 t$: $2\sin^2 t = 1 - \cos(2t) \implies \sin^2 t = \frac{1}{2}(1 - \cos(2t))$. So, $h(t) = \frac{1}{2}(1 - \cos(2t))$. b) From $x(t) = \cos(2t)$, we can substitute this into the expression from part (a): $h = \frac{1}{2}(1 - x)$. c) Substitute $x = 0.5$ into the relationship: $h = \frac{1}{2}(1 - 0.5) = \frac{1}{2}(0.5) = 0.25$ units.
Question 49: Radio Communication in Rural Uganda	
Question	**Solution**
The strength of a radio signal in a remote area is modeled by the function $S(x) = \sin(x) + \cos(x)$. a) Express $S(x)$ in the form $R\sin(x + \alpha)$. b) Find the maximum possible signal strength. c) If the signal strength is 1, find the values of x for which this occurs in the range $0 \le x \le 2\pi$.	a) We express $\sin(x) + \cos(x)$ as $R\sin(x + \alpha) = R(\sin x \cos\alpha + \cos x \sin\alpha)$. Comparing coefficients: $R\cos\alpha = 1$ $R\sin\alpha = 1$ $R^2 = 1^2 + 1^2 = 2 \implies R = \sqrt{2}$. $\tan\alpha = \frac{1}{1} = 1 \implies \alpha = \frac{\pi}{4}$. So, $S(x) = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$. b) The maximum value of $\sin\left(x + \frac{\pi}{4}\right)$ is 1. Maximum signal strength = $\sqrt{2} \times 1 = \sqrt{2} \approx 1.414$. c) Set $S(x) = 1$: $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = 1 \implies \sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$. The principal value for $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is $\frac{\pi}{4}$. Case 1: $x + \frac{\pi}{4} = \frac{\pi}{4} \implies x = 0$. Case 2: $x + \frac{\pi}{4} = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \implies x = \frac{2\pi}{4} = \frac{\pi}{2}$.
Question 50: Urban Park Design in Mbarara	
Question	**Solution**
An urban planner is designing a triangular green space. The sides of the park are 50m, 80m, and 100m. a) Use the cosine rule to find the measure of the smallest angle in the park. b) What is the perimeter of the park? c) A different park has sides of 60m and 70m with an area of 1800 m^2. Find the angle between these two sides.	a) The smallest angle is opposite the shortest side (50m). Let $a = 50$, $b = 80$, $c = 100$. $\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{80^2+100^2-50^2}{2(80)(100)} = \frac{6400+10000-2500}{16000} = \frac{13900}{16000} = \frac{139}{160}$ $A = \cos^{-1}\left(\frac{139}{160}\right) \approx 29.86°$. b) Perimeter = 50 + 80 + 100 = 230 meters. c) We use the area formula: Area = $\frac{1}{2}ab\sin C$. $1800 = \frac{1}{2}(60)(70)\sin C$ $1800 = 2100\sin C$ $\sin C = \frac{1800}{2100} = \frac{6}{7} \approx 0.8571$ $C = \sin^{-1}\left(\frac{6}{7}\right) \approx 59.00°$.
Question 51: Crane Operation in Kampala	
Question	**Solution**
A construction worker is operating a crane. The angle of the crane arm is adjusted by an expression $E = \sin(x + 45°) - \cos(x + 45°)$. a) Use compound angle formulae to expand the expression. b) Simplify the expression to a single trigonometric function. c) Find the maximum value of E.	a) $\sin(x + 45°) = \sin x\cos 45° + \cos x\sin 45° = \frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x$. $\cos(x + 45°) = \cos x\cos 45° - \sin x\sin 45° = \frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x$. $E = \left(\frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x\right) - \left(\frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x\right)$ $E = \frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\cos x + \frac{\sqrt{2}}{2}\sin x$ $E = \sqrt{2}\sin x$. b) The expression simplifies to $E = \sqrt{2}\sin x$. c) The maximum value of $\sin x$ is 1. The maximum value of E is $\sqrt{2} \times 1 = \sqrt{2}$.
Question 52: Measuring a Hill in Kabale	
Question	**Solution**

Two surveyors, A and B, are on opposite sides of a hill. They both measure the angle of elevation to the peak. Surveyor A measures an angle of 35° and Surveyor B measures an angle of 45°. The two surveyors are 2 km apart. a) Draw a diagram of the situation. b) Use the sine rule to find the distance from Surveyor A to the peak. c) Calculate the height of the hill.	a) The diagram shows a large triangle with a horizontal base of 2 km and angles of 35° and 45° at the base. The vertical height of the hill forms a perpendicular line from the peak to the base. b) The third angle in the large triangle is $180° - 35° - 45° = 100°$. Let $a = 2$ km, b be the distance from A to the peak, and c be the distance from B to the peak. Using the sine rule: $\frac{b}{\sin 45°} = \frac{2}{\sin 100°}$ $b = \frac{2 \sin 45°}{\sin 100°} = \frac{2(0.7071)}{0.9848} \approx 1.437$ km. The distance from Surveyor A to the peak is approximately 1.44 km. c) Let h be the height of the hill. From the right-angled triangle on the left side: $\sin 35° = \frac{h}{b}$ $h = b \sin 35° = 1.437 \sin 35° \approx 1.437(0.5736) \approx 0.824$ km. The height of the hill is approximately 824 meters.
Question 53: Lighthouse Beam Navigation in Entebbe	
Question	**Solution**
A ship is navigating at night, using a lighthouse on the shore. The angle of the light beam is modeled by the equation $3 \tan^2 \theta - \sec^2 \theta = 0$. a) Use a trigonometric identity to solve for θ in the range $0° \leq \theta \leq 360°$. b) What are the values of θ in radians? c) Explain why there are multiple solutions for this problem.	a) We use the identity $\sec^2 \theta = 1 + \tan^2 \theta$. Substitute this into the equation: $3 \tan^2 \theta - (1 + \tan^2 \theta) = 0$ $3 \tan^2 \theta - 1 - \tan^2 \theta = 0$ $2 \tan^2 \theta - 1 = 0 \implies \tan^2 \theta = \frac{1}{2}$ $\tan \theta = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$. Case 1: $\tan \theta = \frac{1}{\sqrt{2}}$. Principal value is $\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \approx 35.26°$. Tangent is positive in the first and third quadrants. So, $\theta = 35.26°$ and $\theta = 180° + 35.26° = 215.26°$. Case 2: $\tan \theta = -\frac{1}{\sqrt{2}}$. Principal value is $180° - 35.26° = 144.74°$. Tangent is negative in the second and fourth quadrants. So, $\theta = 144.74°$ and $\theta = 360° - 35.26° = 324.74°$. The solutions are $\theta \approx 35.26°, 144.74°, 215.26°, 324.74°$. b) To convert to radians, multiply by $\frac{\pi}{180}$: $\theta \approx 0.615$ rad, 2.526 rad, 3.757 rad, 5.668 rad. c) The beam of the lighthouse rotates. The signal strength modeled by the equation has the same value at different points in the rotation cycle. Therefore, a ship can be at different locations, corresponding to these different angles, and still experience the same signal strength.
Question 54: Measuring the Area of a Farm Plot	
Question	**Solution**

A farmer in Fort Portal has a non-rectangular plot of land shaped like a triangle. The sides are 75m, 80m, and the angle between them is 70°. He wants to know if the area is at least 2500 m^2 to be eligible for a government grant. a) Calculate the area of the plot. b) Is the farmer eligible for the grant? c) Find the length of the third side of the plot.	a) We use the area formula: Area $= \frac{1}{2}ab\sin C$. Area $= \frac{1}{2}(75)(80)\sin 70°$ Area $= 3000(0.9397) \approx 2819.1$ m^2. b) The calculated area is approximately 2819.1 m^2, which is greater than the required 2500 m^2. Yes, the farmer is eligible for the grant. c) Using the cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$ $c^2 = 75^2 + 80^2 - 2(75)(80)\cos 70°$ $c^2 = 5625 + 6400 - 12000(0.3420)$ $c^2 = 12025 - 4104 = 7921$ $c = \sqrt{7921} = 89$ meters. The length of the third side is exactly 89 meters.
Question 55: Optimizing a Traffic Signal Timer	
Question	**Solution**
The time for a traffic light to turn green is modeled by the function $T(t) = 5\sin(2t) + 5\cos(2t)$ seconds. a) Express $T(t)$ in the form $R\sin(2t + \alpha)$. b) What is the maximum possible time delay? c) At what time t in the interval $0 \le t \le \pi$ does this maximum delay occur?	a) We want to express $5\sin(2t) + 5\cos(2t)$ as $R\sin(2t + \alpha) = R(\sin(2t)\cos\alpha + \cos(2t)\sin\alpha)$. Comparing coefficients: $R\cos\alpha = 5$ $R\sin\alpha = 5$ $R^2 = 5^2 + 5^2 = 50 \implies R = \sqrt{50} = 5\sqrt{2}$. $\tan\alpha = \frac{5}{5} = 1 \implies \alpha = \frac{\pi}{4}$. So, $T(t) = 5\sqrt{2}\sin\left(2t + \frac{\pi}{4}\right)$. b) The maximum value of $\sin\left(2t + \frac{\pi}{4}\right)$ is 1. Maximum time delay $= 5\sqrt{2} \times 1 = 5\sqrt{2} \approx 7.07$ seconds. c) The maximum delay occurs when $\sin\left(2t + \frac{\pi}{4}\right) = 1$. $2t + \frac{\pi}{4} = \frac{\pi}{2} \implies 2t = \frac{\pi}{4} \implies t = \frac{\pi}{8}$.
Question 56: Satellite Tracking from a Ground Station	
Question	**Solution**
A ground station is tracking a satellite. The position is given by the parametric equations $x = 20\tan t$ and $y = 20\sec t$. a) Eliminate the parameter t to find the relationship between x and y. b) When the horizontal distance from the station is 30 km, what is the height of the satellite? c) Explain what type of curve this relationship represents.	a) We know the identity $\sec^2 t - \tan^2 t = 1$. From the given equations, $\tan t = \frac{x}{20}$ and $\sec t = \frac{y}{20}$. Substitute these into the identity: $\left(\frac{y}{20}\right)^2 - \left(\frac{x}{20}\right)^2 = 1$ $\frac{y^2}{400} - \frac{x^2}{400} = 1$ $y^2 - x^2 = 400$. b) Substitute $x = 30$ into the equation: $y^2 - 30^2 = 400$ $y^2 - 900 = 400$ $y^2 = 1300 \implies y = \sqrt{1300} \approx 36.06$ km. c) The equation $y^2 - x^2 = 400$ represents a hyperbola. This models the satellite's path relative to the ground station, showing its horizontal and vertical movement.
Question 57: Water Dam Gate Control in Karamoja	
Question	**Solution**
A hydraulic system controls the gate of a water dam. The height of the gate is controlled by an angle θ, such that its height h is given by $h = \cos\theta + \sin\theta$. a) Express h in the form $R\sin(\theta + \alpha)$. b) If the height is set at 1.4 meters, find the corresponding angle θ. c) A different gate has a height $H = 2\sin\theta\cos\theta$. What is the maximum height of this gate?	a) As in Question 9, $\cos\theta + \sin\theta = \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$. $h = \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$. b) $1.4 = \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$ $\sin\left(\theta + \frac{\pi}{4}\right) = \frac{1.4}{\sqrt{2}} \approx 0.9899$ $\theta + \frac{\pi}{4} = \sin^{-1}(0.9899) \approx 1.396$ radians. $\theta \approx 1.396 - \frac{\pi}{4} \approx 1.396 - 0.785 = 0.611$ radians. c) We use the double angle formula for sine: $\sin(2\theta) = 2\sin\theta\cos\theta$. So, $H = \sin(2\theta)$. The maximum value of $\sin(2\theta)$ is 1. The maximum height of the gate is 1 meter.
Question 58: Football Pitch Measurement	
Question	**Solution**

A sports team is building a new training ground that is triangular. The two sides of the pitch are 60m and 80m, and the angle between them is 120°. a) Use the cosine rule to find the length of the third side. b) Calculate the area of the training pitch. c) If the team wants to set up a training drill line from the vertex opposite the unknown side to the midpoint of the unknown side, what is the length of this line?	a) Let $a = 60$, $b = 80$, and $C = 120°$. We want to find side c. $c^2 = a^2 + b^2 - 2ab\cos C$ $c^2 = 60^2 + 80^2 - 2(60)(80)\cos 120°$ $c^2 = 3600 + 6400 - 9600\left(-\frac{1}{2}\right) = 10000 + 4800 = 14800$ $c = \sqrt{14800} \approx 121.66$ meters. b) Area $= \frac{1}{2}ab\sin C$ Area $= \frac{1}{2}(60)(80)\sin 120° = 2400\sin 120° = 2400\left(\frac{\sqrt{3}}{2}\right) = 1200\sqrt{3} \approx 2078.46$ m^2. c) This is a median. The length of a median m can be found using Apollonius' theorem or by applying the cosine rule to a smaller triangle. Consider the triangle formed by the median. The sides are 60m, 80m, and the median m. The angle between 60m and 80m is 120°. The median divides the unknown side into two equal halves of length $\frac{121.66}{2} \approx 60.83$m. The length of the median is the side opposite the angle 120°. The length of the median can be found using the formula $m_c^2 = \frac{2a^2+2b^2-c^2}{4}$. $m_c^2 = \frac{2(60^2)+2(80^2)-(121.66)^2}{4} = \frac{2(3600)+2(6400)-14800}{4} = \frac{7200+12800-14800}{4} = \frac{5200}{4} = 1300$ $m_c = \sqrt{1300} \approx 36.06$ m.
Question 59: Measuring the Height of an Escarpment	
Question	**Solution**
From a point P on the ground, the angle of elevation to the top of an escarpment is 25°. From a point Q, 100m closer to the escarpment and in the same line as P, the angle of elevation is 40°. a) Draw a diagram of the situation. b) Use the sine rule to find the distance from Q to the top of the escarpment. c) Find the height of the escarpment.	a) The diagram shows two right-angled triangles sharing the same vertical side (the escarpment's height, h). The larger triangle has angle 25° and the smaller one has angle 40°. The distance between P and Q is 100m. The top of the escarpment is T. b) Consider the non-right-angled triangle formed by points P, Q, and T. The angle at T is $40° - 25° = 15°$. The angle at Q is $180° - 40° = 140°$. The angle at P is 25°. Using the sine rule: $\frac{QT}{\sin 25°} = \frac{100}{\sin 15°}$ $QT = \frac{100\sin 25°}{\sin 15°} = \frac{100(0.4226)}{0.2588} \approx 163.29$ meters. The distance from Q to the top of the escarpment is approximately 163.3 meters. c) Now use the smaller right-angled triangle with vertex Q: $\sin 40° = \frac{h}{QT}$ $h = QT\sin 40° = 163.29\sin 40° = 163.29(0.6428) \approx 104.97$ meters. The height of the escarpment is approximately 105 meters.
Question 60: Modeling Power Output of a Wind Turbine	
Question	**Solution**

The rotational power of a wind turbine blade is modeled by the function $P(t) = \frac{1}{12\sin t + 5\cos t}$. a) Express the denominator in the form $R\sin(t+\alpha)$. b) Find the maximum and minimum values of $P(t)$. c) At what value of t in the interval $0 \le t \le 2\pi$ does the maximum power occur?	a) We want to express $12\sin t + 5\cos t$ as $R\sin(t+\alpha) = R(\sin t\cos\alpha + \cos t\sin\alpha)$. Comparing coefficients: $R\cos\alpha = 12$ $R\sin\alpha = 5$ $R^2 = 12^2 + 5^2 = 144 + 25 = 169 \implies R = 13$. $\tan\alpha = \frac{5}{12} \implies \alpha = \tan^{-1}\left(\frac{5}{12}\right) \approx 0.395$ radians. So, the denominator is $13\sin(t + 0.395)$. b) The power is given by $P(t) = \frac{1}{13\sin(t+0.395)}$. The maximum power occurs when the denominator is at its minimum positive value. The minimum value of sin is -1, so the denominator's minimum is -13. The maximum value of sin is 1, so the denominator's maximum is 13. Minimum of denominator: $13(-1) = -13$. Maximum of denominator: $13(1) = 13$. Maximum power occurs when the denominator is minimum and negative, which is -13. Max power = $\frac{1}{-13} = -\frac{1}{13}$. Minimum power occurs when the denominator is maximum, which is 13. Min power = $\frac{1}{13}$. c) The maximum power occurs when the denominator $13\sin(t + 0.395)$ is at its minimum, which is -13. This happens when $\sin(t + 0.395) = -1$. $t + 0.395 = \frac{3\pi}{2} \approx 4.712$ $t = 4.712 - 0.395 = 4.317$ radians.

Problem 61:
In a rural village in Uganda, a community-based organization wants to install a solar panel on a rooftop inclined at an angle that ensures maximum sunlight absorption. At a specific location, the optimum angle of inclination is equal to the latitude of the place. Masaka lies approximately at latitude 0°20′ South.

(a) Convert this angle to decimal degrees.
(b) Using the unit circle, determine the sine and cosine of the angle.
(c) If the solar panel is 2.5 m long, find the height it will rise above the horizontal roof when installed at this angle.

Worked-out Solution:

(a) $0°20' = 0 + \frac{20}{60} = 0.333°$
(b) Using a calculator:

$$\sin(0.333°) \approx \sin(0.00581 \text{ radians}) \approx 0.00581,$$
$$\cos(0.333°) \approx \cos(0.00581) \approx 0.99998$$

(c) Height = $2.5 \times \sin(0.333°) \approx 2.5 \times 0.00581 =$ 0.0145 m So, the panel will rise approximately **1.45 cm** above the horizontal.

Conclusion: Understanding trigonometric ratios helps optimize solar energy solutions for off-grid communities in Uganda.

Problem 62:
Due to seasonal rainfall, the water level of River Nyamwamba in Kasese district varies sinusoidally throughout the year. The maximum water level is 6 meters, the minimum is 2 meters, and this variation follows a yearly cycle.

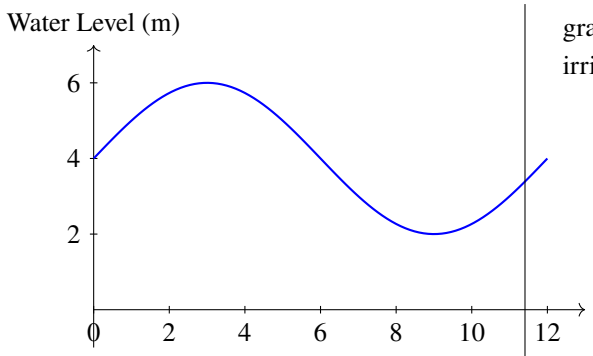
(a) Model the water level $h(t)$ as a sine function of time t in months, where $t = 0$ represents January.
(b) Sketch the graph of $h(t)$ for 12 months.
(c) Use the graph to estimate when flooding (defined as water level above 5.5 m) is likely to occur.

Worked-out Solution:

(a) The midline is at $\frac{6+2}{2} = 4$ m, amplitude = 2 m, period = 12 months.

$$h(t) = 4 + 2\sin\left(\frac{2\pi t}{12}\right) = 4 + 2\sin\left(\frac{\pi t}{6}\right)$$

(b) **Graph:**



(c) From the graph, water level exceeds 5.5 m around months 2 to 4 (March–May) and again around months 8 to 10 (September–November). These are the likely flood seasons.

Conclusion: Graphs of trigonometric functions can be used for flood prediction and disaster preparedness in Uganda's flood-prone areas.

Problem 63: A road construction engineer in the hilly Kabale district must evaluate the safety of a road slope. The slope makes an angle of 30° with the horizontal.

(a) Calculate $\sin(30°)$, $\cos(30°)$, and $\tan(30°)$.
(b) Determine their reciprocals: $\csc(30°)$, $\sec(30°)$, and $\cot(30°)$.
(c) Explain the significance of $\cot(30°)$ in evaluating how steep or gradual the road is.

Worked-out Solution:

(a) Use standard trigonometric values:

$$\sin(30°) = \frac{1}{2}, \quad \cos(30°) = \frac{\sqrt{3}}{2}, \quad \tan(30°) = \frac{1}{\sqrt{3}}$$

(b) Reciprocals:

$$\csc(30°) = \frac{1}{\sin(30°)} = 2, \quad \sec(30°) = \frac{1}{\cos(30°)} = \frac{2}{\sqrt{3}}, \quad \cot(30°) = \frac{1}{\tan(30°)} = \sqrt{3}$$

(c) $\cot(30°) = \sqrt{3} \approx 1.73$ gives the ratio of horizontal to vertical rise. A higher cotangent value implies a gentler slope, important for road safety, especially for trucks and motorcycles in hilly regions like Kabale.

Conclusion: Reciprocal trigonometric functions help engineers assess slope stability and design safe transportation routes.

Problem 64:
In the dry regions of Karamoja, farmers are using gravity-fed irrigation. For optimal water flow, the irrigation channel must have a slope of 45°.

(a) Find the values of $\sin(45°)$, $\cos(45°)$, and $\tan(45°)$.
(b) Calculate the reciprocal ratios $\csc(45°)$, $\sec(45°)$, and $\cot(45°)$.
(c) Explain how knowing $\sec(45°)$ helps in designing the channel's support length when elevation is known.

Worked-out Solution:

(a) From standard values:

$$\sin(45°) = \cos(45°) = \frac{\sqrt{2}}{2}, \quad \tan(45°) = 1$$

(b) Reciprocals:

$$\csc(45°) = \frac{1}{\sin(45°)} = \sqrt{2}, \quad \sec(45°) = \frac{1}{\cos(45°)} = \sqrt{2}, \quad \cot$$

(c) If a channel must be raised vertically by 1 m, the sloped support (hypotenuse) is found using $\sec(45°)$:

$$\text{Support length} = 1 \times \sec(45°) = \sqrt{2} \approx 1.41 \text{ m}$$

This helps in cutting materials accurately and maintaining water flow efficiency.

Conclusion: Using reciprocal trigonometric functions enables agricultural engineers to design cost-effective irrigation systems in water-scarce areas of Uganda.

Problem 65:
A rural school in Mbale district wants to build a rainwater harvesting tank with a triangular slanted iron sheet roof. One side of the roof is to be inclined at an angle of 30° to ensure fast water runoff during heavy rains. If the slanted iron sheet is 3 meters long, determine:

(a) The vertical height from the base to the top using $\sin(30°)$,
(b) The horizontal base distance covered using $\cos(30°)$,
(c) The ratio of height to base using $\tan(30°)$.

Worked-out Solution:

(a) $\sin(30°) = \frac{1}{2}$, so height $= 3 \times \frac{1}{2} = 1.5$ m.
(b) $\cos(30°) = \frac{\sqrt{3}}{2}$, so base $= 3 \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} \approx 2.60$ m.
(c) $\tan(30°) = \frac{1}{\sqrt{3}} \approx 0.577$, indicating the roof rises 0.577 m for every 1 m of horizontal run.

Conclusion: Understanding trigonometric ratios allows for effective design of rainwater harvesting structures, especially in rainfall-prone areas of Uganda.

Problem 65:
In Mbarara, a farmer wants to construct a wooden ladder to safely harvest fruits from a tree. The ladder should make an angle of 60° with the ground for stability. If the vertical height from the ground to the lowest fruits is 2.5 meters:

(a) Use $\sin(60^\circ)$ to find the required length of the ladder,
(b) Use $\cos(60^\circ)$ to determine the horizontal distance from the tree to the ladder's base,
(c) Use $\tan(60^\circ)$ to confirm the height-to-base ratio.

Worked-out Solution:

(a) $\sin(60^\circ) = \frac{\sqrt{3}}{2}$, so:

$$\text{Ladder length} = \frac{2.5}{\sin(60^\circ)} = \frac{2.5}{\sqrt{3}/2} = \frac{5}{\sqrt{3}} \approx 2.89 \text{ m}$$

(b) $\cos(60^\circ) = \frac{1}{2}$, so:

$$\text{Base distance} = 2.89 \times \frac{1}{2} = 1.445 \text{ m}$$

(c) $\tan(60^\circ) = \sqrt{3} \approx 1.732$. From previous results:

$$\frac{2.5}{1.445} \approx 1.73, \text{ confirming the angle is correct.}$$

Conclusion: Knowledge of trigonometric ratios helps farmers design safer tools and equipment to improve productivity and minimize risk.

Problem 66: A civil engineer in Kampala is designing a footbridge where a cable forms a right-angled triangle with the tower and ground. The engineer needs to verify the relationship between the sine and cosine of the angle of elevation θ of the cable.

(a) Derive the identity $\sin^2\theta + \cos^2\theta = 1$ using Pythagoras' Theorem.
(b) If $\tan\theta = \frac{3}{4}$, solve for $\sin\theta$ and $\cos\theta$ using trigonometric identities.
(c) Use the values to verify the identity.

Worked-out Solution:

(a) Consider a right-angled triangle with angle θ, opposite side a, adjacent b, hypotenuse c.

From Pythagoras: $a^2 + b^2 = c^2$ Divide through by c^2:

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1 \Rightarrow \sin^2\theta + \cos^2\theta = 1$$

(b) Given that $\tan\theta = \frac{3}{4}$, we can define a right-angled triangle where:

$$\text{opposite} = 3, \quad \text{adjacent} = 4$$

Using the Pythagorean theorem:

$$\text{hypotenuse}^2 = \text{opposite}^2 + \text{adjacent}^2$$

$$\text{hypotenuse}^2 = 3^2 + 4^2 = 9 + 16 = 25$$

$$\text{hypotenuse} = \sqrt{25} = 5$$

Thus, the hypotenuse of the triangle is 5.

$$c = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

So:

$$\sin\theta = \frac{3}{5}, \quad \cos\theta = \frac{4}{5}$$

(c) Verify identity:

$$\sin^2\theta + \cos^2\theta = \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = \frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1$$

Conclusion: Using identities helps engineers validate design parameters when creating safe and reliable infrastructure.

Problem 67:
An energy technician in Soroti designs a solar tracker system that follows the sun. The panel's direction is given parametrically by:

$$x = R\cos\theta, \quad y = R\sin\theta$$

where R is the arm length and θ is the rotation angle from noon.

(a) Eliminate θ to find a relation between x and y.
(b) If $R = 10$ m and at a certain time $\theta = 45^\circ$, compute x and y.
(c) Show that the simplified relation is consistent with the identity $\sin^2\theta + \cos^2\theta = 1$.

Worked-out Solution:

(a) From the parametric equations:

$$\cos\theta = \frac{x}{R}, \quad \sin\theta = \frac{y}{R}$$

So:

$$\left(\frac{x}{R}\right)^2+\left(\frac{y}{R}\right)^2 = 1 \Rightarrow \frac{x^2+y^2}{R^2} = 1 \Rightarrow x^2+y^2 = R^2$$

(b) When $\theta = 45°$, $R = 10$:

$$x = 10\cos(45°) = 10\cdot\frac{\sqrt{2}}{2} \approx 7.07, \quad y = 10\sin(45°) = 10\cdot\frac{\sqrt{2}}{2} \approx 7.07$$

(c) Confirm:

$$x^2+y^2 = 7.07^2+7.07^2 \approx 50+50 = 100 = R^2$$

Hence, the relation is consistent with the identity.

Problem 68: A technician in Soroti wants to optimize the tilt angle θ of a solar panel. The sun is observed at an angle of elevation $\alpha = 30°$ in the morning and $\beta = 15°$ in the afternoon. To calculate the optimal average angle of tilt, the technician decides to use the formula:

$$\sin(\alpha+\beta)$$

(a) Derive the identity $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.
(b) Use the identity to find $\sin(45°)$ from $\alpha = 30°$ and $\beta = 15°$.
(c) Using the result, deduce the double angle formula $\sin(2\theta) = 2\sin\theta\cos\theta$.

Worked-out Solution:

(a) The compound angle formula for sine is:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta+\cos\alpha\sin\beta$$

(b) $\alpha = 30°$, $\beta = 15°$, so:

$$\sin(45°) = \sin(30°)\cos(15°)+\cos(30°)\sin(15°)$$

Use known values:

$$\sin(30°) = \frac{1}{2}, \quad \cos(30°) = \frac{\sqrt{3}}{2}, \quad \cos(15°) \approx 0.9659, \quad \sin(15°) \approx 0.2588$$

$$\sin(45°) \approx \frac{1}{2}\cdot 0.9659+\frac{\sqrt{3}}{2}\cdot 0.2588 \approx 0.48295+0.2241 \approx 0.707 \text{ (confirmed)}$$

(c) Let $\alpha = \beta = \theta$, then:

$$\sin(2\theta) = \sin(\theta+\theta) = \sin\theta\cos\theta+\cos\theta\sin\theta = 2\sin\theta\cos\theta$$

Conclusion: These identities allow technicians to design solar energy systems that align with Uganda's growing renewable energy needs.

Problem 68:
An engineer in Kabale is designing a V-shaped channel to guide runoff water into tanks for irrigation. The central angle between the two inclined channel walls is $60°$. To measure the angle each wall makes with the horizontal, the half-angle formula is applied:

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1+\cos\theta}$$

(a) Derive the half-angle formula for tangent from the double angle identities.
(b) Use it to find the angle of inclination for each wall of the channel.
(c) Verify the value using a calculator.

Worked-out Solution:

(a) From the double angle identities:

$$\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right), \quad \cos\theta = 1-2\sin^2\left(\frac{\theta}{2}\right)$$

Divide both:

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1+\cos\theta}$$

(b) $\theta = 60°$, then:

$$\tan\left(\frac{60°}{2}\right) = \frac{\sin 60°}{1+\cos 60°} = \frac{\frac{\sqrt{3}}{2}}{1+\frac{1}{2}} = \frac{\frac{\sqrt{3}}{2}}{\frac{3}{2}} = \frac{\sqrt{3}}{3} \approx 0.577$$

Hence,

$$\frac{\theta}{2} = \tan^{-1}(0.577) \approx 30°$$

(c) Verified: each wall makes a $30°$ angle with the horizontal, ideal for water flow without erosion.

Conclusion: Mastery of angle identities helps engineers improve water access and agricultural productivity in Ugandan highland communities.

QN 69
The Kyotera District Local Government is planning to construct a new water treatment plant to serve three major trading centers: Kyotera Town, Kakuuto, and Mutukula. The district engineer needs to determine the optimal location and calculate various measurements for the project.

The three trading centers form a triangle with the following measurements:

- Distance from Kyotera Town to Kakuuto: **8 km**
- Distance from Kakuuto to Mutukula: **6 km**
- Distance from Mutukula to Kyotera Town: **10 km**

The district needs your mathematical expertise to solve several critical problems for this water infrastructure project.

1. Verify that these three trading centers can form a valid triangle and calculate all three angles of the triangle.
2. The water treatment plant will be positioned such that it forms a right angle when viewed from Kakuuto, with the treatment plant being 4 km from Kakuuto and 5 km from a point directly along the Kakuuto-Mutukula road. Find the angle at which the access road should be constructed.
3. Calculate the total area of land enclosed by the three trading centers, as this will be the primary service area for the new water system.
4. If a triangular plot of land near Mutukula has sides of 3 km, 4 km, and an included angle of 60°, calculate:
   - The length of the third side
   - The area of this triangular plot

SOLUTION

1. Part A: Triangle Verification and Angle Calculation

   Step 1: Verify Triangle Validity

   Let $a = 6$ km (Kakuuto to Mutukula)

   Let $b = 10$ km (Mutukula to Kyotera)

   Let $c = 8$ km (Kyotera to Kakuuto)

   Triangle inequality check:

   $$a + b > c: \quad 6 + 10 = 16 > 8 \quad \checkmark \tag{5.1}$$

   $$b + c > a: \quad 10 + 8 = 18 > 6 \quad \checkmark \tag{5.2}$$

   $$a + c > b: \quad 6 + 8 = 14 > 10 \quad \checkmark \tag{5.3}$$

   The triangle is valid.

   Step 2: Derive and Apply the Cosine Rule

   Cosine Rule Derivation:

   For any triangle with sides a, b, c and angle C opposite to side c:

   Using the distance formula and dot product:

   $$\boxed{c^2 = a^2 + b^2 - 2ab\cos(C)} \tag{5.4}$$

   Similarly:

   $$a^2 = b^2 + c^2 - 2bc\cos(A) \tag{5.5}$$

   $$b^2 = a^2 + c^2 - 2ac\cos(B) \tag{5.6}$$

   Step 3: Calculate Angles

   Angle at Kyotera (A):

   $$a^2 = b^2 + c^2 - 2bc\cos(A) \tag{5.7}$$

   $$6^2 = 10^2 + 8^2 - 2(10)(8)\cos(A) \tag{5.8}$$

   $$36 = 100 + 64 - 160\cos(A) \tag{5.9}$$

   $$36 = 164 - 160\cos(A) \tag{5.10}$$

   $$160\cos(A) = 164 - 36 = 128 \tag{5.11}$$

   $$\cos(A) = \frac{128}{160} = 0.8 \tag{5.12}$$

   $$A = \cos^{-1}(0.8) = 36.87 \tag{5.13}$$

   Angle at Kakuuto (B):

   $$b^2 = a^2 + c^2 - 2ac\cos(B) \tag{5.14}$$

   $$10^2 = 6^2 + 8^2 - 2(6)(8)\cos(B) \tag{5.15}$$

   $$100 = 36 + 64 - 96\cos(B) \tag{5.16}$$

   $$100 = 100 - 96\cos(B) \tag{5.17}$$

   $$96\cos(B) = 0 \tag{5.18}$$

   $$B = 90 \tag{5.19}$$

   Angle at Mutukula (C):

   $$C = 180 - A - B = 180 - 36.87 - 90 = 53.13 \tag{5.20}$$

2. Part B: Water Treatment Plant Access Road

   Given: Right triangle with legs of 4 km and 5 km

   Using Tangent Rule:

   $$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{4} = 1.25 \tag{5.21}$$

   $$\theta = \tan^{-1}(1.25) = 51.34 \tag{5.22}$$

   The access road should be constructed at an angle of **51.34**° from the Kakuuto-Mutukula direction.

3. Part C: Land Area Calculation

   Method 1: Using the Area Formula for Triangle with Three Sides

   Derivation of Area Formula:

   For a triangle with sides a, b, c, we can derive:

   $$\boxed{\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}} \tag{5.23}$$

   where $s = \frac{a+b+c}{2}$ (Heron's Formula)

Alternative derivation using sine:

$$\boxed{\text{Area} = \frac{1}{2}ab\sin(C)} \tag{5.24}$$

Calculation using Heron's Formula:

$$s = \frac{6+10+8}{2} = 12 \text{ km} \tag{5.25}$$

$$\text{Area} = \sqrt{12(12-6)(12-10)(12-8)} \tag{5.26}$$

$$= \sqrt{12 \times 6 \times 2 \times 4} \tag{5.27}$$

$$= \sqrt{576} \tag{5.28}$$

$$= 24 \text{ km}^2 \tag{5.29}$$

Verification using Sine Formula:

$$\text{Area} = \frac{1}{2} \times 6 \times 8 \times \sin(90) \tag{5.30}$$

$$= \frac{1}{2} \times 6 \times 8 \times 1 = 24 \text{ km}^2 \quad \checkmark \tag{5.31}$$

4. Part D: Pipeline Planning Calculations
   Given: Two sides of 3 km and 4 km with included angle of 60°
   Step 1: Find the third side using Cosine Rule

$$c^2 = a^2 + b^2 - 2ab\cos(C) \tag{5.32}$$

$$c^2 = 3^2 + 4^2 - 2(3)(4)\cos(60) \tag{5.33}$$

$$c^2 = 9 + 16 - 24(0.5) \tag{5.34}$$

$$c^2 = 25 - 12 = 13 \tag{5.35}$$

$$c = \sqrt{13} = 3.61 \text{ km} \tag{5.36}$$

Step 2: Calculate the area

$$\text{Area} = \frac{1}{2}ab\sin(C) \tag{5.37}$$

$$= \frac{1}{2} \times 3 \times 4 \times \sin(60) \tag{5.38}$$

$$= 6 \times \frac{\sqrt{3}}{2} \tag{5.39}$$

$$= 3\sqrt{3} = 5.20 \text{ km}^2 \tag{5.40}$$

The water treatment plant project will serve approximately **50,000 residents** across the triangular service area, improving access to clean water in rural Kyotera District.

Question 71 Lake Victoria Navigation Challenge

Scenario: A fishing crew from Entebbe needs to navigate to Ssese Islands. The boat travels at a bearing of 065 for 25 km, then changes course to bearing 120 for 18 km to reach Buggala Island.

(a) Find the direct distance from Entebbe to Buggala Island using the cosine rule.

(b) Calculate the bearing from Entebbe to Buggala Island.

(c) If the fuel consumption is 2.5 L/km, how much fuel is saved by taking the direct route?

*Solution: **(a)** Using the cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$

$$\text{Angle between the two legs} = 120 - 65 = 55 \tag{5.41}$$

$$c^2 = 25^2 + 18^2 - 2(25)(18)\cos(55) \tag{5.42}$$

$$c^2 = 625 + 324 - 900(0.574) \tag{5.43}$$

$$c^2 = 949 - 516.6 = 432.4 \tag{5.44}$$

$$c = 20.8 \text{ km} \tag{5.45}$$

(b) Using sine rule to find the bearing angle:

$$\frac{\sin A}{18} = \frac{\sin(55)}{20.8} \tag{5.46}$$

$$\sin A = \frac{18 \times \sin(55)}{20.8} = 0.710 \tag{5.47}$$

$$A = 45.2 \tag{5.48}$$

$$\text{Bearing} = 65 + 45.2 = 110.2 \tag{5.49}$$

(c) Fuel saved:

$$\text{Indirect route:} \quad (25+18) \times 2.5 = 107.5 \text{ L} \tag{5.50}$$

$$\text{Direct route:} \quad 20.8 \times 2.5 = 52 \text{ L} \tag{5.51}$$

$$\text{Fuel saved:} \quad 107.5 - 52 = 55.5 \text{ L} \tag{5.52}$$

Question 72: Kampala Building Construction

Scenario: A construction company is building a 15-story office complex in Kampala. The building's shadow length varies throughout the day due to the sun's angle of elevation.

(a) At 10:00 AM, the angle of elevation is 35. Calculate the building's height if the shadow is 42 meters.

(b) Find the time when the shadow length is minimum (sun directly overhead at 78 elevation).

(c) Calculate the maximum shadow length at 6:00 AM when the elevation angle is 12.

Solution: **(a)** Using tangent ratio:

$$\tan(35) = \frac{\text{height}}{42} \tag{5.53}$$

$$\text{height} = 42 \times \tan(35) = 42 \times 0.700 = 29.4 \text{ meters} \tag{5.54}$$

(b) For minimum shadow at 78 elevation:

$$\tan(78) = \frac{29.4}{\text{shadow}} \tag{5.55}$$

$$\text{shadow} = \frac{29.4}{\tan(78)} = \frac{29.4}{4.705} = 6.25 \text{ meters} \tag{5.56}$$

(c) For maximum shadow at 12 elevation:

$$\tan(12) = \frac{29.4}{\text{shadow}} \tag{5.57}$$

$$\text{shadow} = \frac{29.4}{\tan(12)} = \frac{29.4}{0.213} = 138.0 \text{ meters} \tag{5.58}$$

Question 73: Murchison Falls Bridge Engineering

Scenario: Engineers are designing a suspension bridge across the Nile at Murchison Falls. The bridge has two towers 180 meters apart, each 45 meters high.

(a) Calculate the angle of depression from the top of one tower to the base of the other.

(b) If the main cable forms a parabolic curve with maximum sag of 15 meters, find the cable length using trigonometric approximation.

(c) Determine the tension forces in the cable if it supports 2000 N/m uniformly distributed load.

Solution: **(a)** Angle of depression:

$$\tan(\theta) = \frac{45}{180} = 0.25 \tag{5.59}$$

$$\theta = \arctan(0.25) = 14.04 \tag{5.60}$$

(b) Cable length approximation:

$$L \approx \text{span} + \frac{8 \times \text{sag}^2}{3 \times \text{span}} \tag{5.61}$$

$$L \approx 180 + \frac{8 \times 15^2}{3 \times 180} \tag{5.62}$$

$$L \approx 180 + \frac{1800}{540} = 180 + 3.33 = 183.33 \text{ meters} \tag{5.63}$$

(c) Maximum tension at towers:

$$T = \frac{wL^2}{8h} \text{ where } w = 2000 \text{ N/m}, L = 180\text{m}, h = 15\text{m} \tag{5.64}$$

$$T = \frac{2000 \times 180^2}{8 \times 15} = \frac{6,480,000}{120} = 54,000 \text{ N} \tag{5.65}$$

Question 74: Entebbe Airport Runway Approach

Scenario: Aircraft approaching Entebbe International Airport follow a 3 glide slope for safe landing. A plane is at 10, 000 feet altitude, 20 km from the runway.

(a) Calculate if the aircraft is on the correct glide path.

(b) If the pilot needs to adjust to the correct 3 approach, what should be the new altitude?

(c) Find the total distance the aircraft travels along the glide path to touchdown.

Solution: **(a)** Current glide angle:

$$\tan(\theta) = \frac{10,000 \text{ feet}}{20 \text{ km} \times 3280.84 \text{ ft/km}} \tag{5.66}$$

$$\tan(\theta) = \frac{10,000}{65,617} = 0.1524 \tag{5.67}$$

$$\theta = \arctan(0.1524) = 8.67 \tag{5.68}$$

Aircraft is too steep (should be 3)

(b) Correct altitude for 3 glide slope:

$$\tan(3) = \frac{\text{altitude}}{65,617} \tag{5.69}$$

$$\text{altitude} = 65,617 \times \tan(3) = 65,617 \times 0.0524 = 3,438 \text{ feet} \tag{5.70}$$

(c) Distance along glide path:

$$\sin(3) = \frac{3,438}{\text{distance}} \tag{5.71}$$

$$\text{distance} = \frac{3,438}{\sin(3)} = \frac{3,438}{0.0523} = 65,726 \text{ feet} = 12.44 \text{ miles} \tag{5.72}$$

Question 75: Mount Rwenzori Telecommunications Tower

Scenario: Uganda Telecom is installing a communication tower on Mount Rwenzori. From two observation points A and B, 500 meters apart at the same elevation, the angles of elevation to the tower top are 25 and 35 respectively.

(a) Find the height of the tower above the observation points.

(b) Calculate the distance from each observation point to the base of the tower.

(c) If the tower needs guy-wires at 60 to the vertical, find the length of wires from the tower top.

Solution: **(a)** Let the height be h and distances from A and B to tower base be x and $(500 - x)$:

$$\tan(25) = \frac{h}{x} \text{ and } \tan(35) = \frac{h}{500 - x} \tag{5.73}$$

$$h = x \cdot \tan(25) = (500 - x) \cdot \tan(35) \tag{5.74}$$

$$x \cdot 0.466 = (500 - x) \cdot 0.700 \tag{5.75}$$

$$0.466x = 350 - 0.700x \tag{5.76}$$

$$1.166x = 350 \tag{5.77}$$

$$x = 300.17 \text{ m}, h = 139.88 \text{ m} \tag{5.78}$$

(b) Distance from A = 300.17 m, Distance from B = 199.83 m

(c) Guy-wire length:

$$\cos(60) = \frac{h}{\text{wire length}} \tag{5.79}$$

$$\text{wire length} = \frac{139.88}{\cos(60)} = \frac{139.88}{0.5} = 279.76 \text{ m} \tag{5.80}$$

Question 76: Jinja Hydroelectric Dam Water Flow

Scenario: At Owen Falls Dam in Jinja, water flows over a spillway at angle θ. The water jet travels horizontally 25 meters and drops 8 meters vertically before hitting the river.

(a) Calculate the launch angle of the water jet.

(b) Find the initial velocity of water if it follows projectile motion.

(c) Determine the maximum height reached by the water above the spillway.

Solution: **(a)** Launch angle:

$$\text{For projectile motion: } \tan(2\theta) = \frac{4h}{R} = \frac{4 \times 8}{25} = 1.28 \tag{5.81}$$

$$2\theta = \arctan(1.28) = 52.13 \tag{5.82}$$

$$\theta = 26.07 \tag{5.83}$$

(b) Initial velocity:

$$R = \frac{v^2 \sin(2\theta)}{g} \tag{5.84}$$

$$25 = \frac{v^2 \sin(52.13)}{9.81} \tag{5.85}$$

$$v^2 = \frac{25 \times 9.81}{0.788} = 311.04 \tag{5.86}$$

$$v = 17.64 \text{ m/s} \tag{5.87}$$

(c) Maximum height:

$$H = \frac{v^2 \sin^2(\theta)}{2g} = \frac{(17.64)^2 \times \sin^2(26.07)}{2 \times 9.81} \tag{5.88}$$

$$H = \frac{311.04 \times 0.193}{19.62} = 3.06 \text{ m above spillway} \tag{5.89}$$

Question 77: Kasese Copper Mine Shaft

Scenario: A copper mine in Kasese has an inclined shaft going down at 15 below horizontal for 200 meters, then continues at 25 for another 150 meters.

(a) Calculate the total vertical depth of the mine.

(b) Find the horizontal distance from the entrance to the deepest point.

(c) If a conveyor belt runs the full length of both sections, calculate the total belt length and the angle it makes with horizontal at the junction.

Solution:

(a) Total vertical depth:

$$\text{First section: } 200 \times \sin(15) = 200 \times 0.259 = 51.76 \text{ m} \tag{5.90}$$

$$\text{Second section: } 150 \times \sin(25) = 150 \times 0.423 = 63.39 \text{ m} \tag{5.91}$$

$$\text{Total depth} = 51.76 + 63.39 = 115.15 \text{ m} \tag{5.92}$$

(b) Total horizontal distance:

$$\text{First section: } 200 \times \cos(15) = 200 \times 0.966 = 193.19 \text{ m} \tag{5.93}$$

$$\text{Second section: } 150 \times \cos(25) = 150 \times 0.906 = 135.95 \text{ m} \tag{5.94}$$

$$\text{Total horizontal} = 193.19 + 135.95 = 329.14 \text{ m} \tag{5.95}$$

(c) Belt length = 200 + 150 = 350 m

$$\text{Overall angle: } \tan(\theta) = \frac{115.15}{329.14} = 0.350 \tag{5.96}$$

$$\theta = \arctan(0.350) = 19.3 \text{ below horizontal} \tag{5.97}$$

Question 88: Mbale Coffee Plantation Irrigation

Scenario: A coffee plantation on Mount Elgon slopes uses sprinkler irrigation. Each sprinkler covers a sector with radius 15 meters and angle 120.

(a) Calculate the area covered by one sprinkler.

(b) Find the arc length of the watered boundary.

(c) If water pressure creates a parabolic spray pattern with maximum height 3 meters at the center, find the water trajectory equation.

Solution: **(a)** Sector area:

$$A = \frac{\theta}{360} \times \pi r^2 = \frac{120}{360} \times \pi \times 15^2 \tag{5.98}$$

$$A = \frac{1}{3} \times \pi \times 225 = 235.62 \text{ m}^2 \tag{5.99}$$

(b) Arc length:

$$s = \frac{\theta}{360} \times 2\pi r = \frac{120}{360} \times 2\pi \times 15 \tag{5.100}$$

$$s = \frac{1}{3} \times 30\pi = 31.42 \text{ m} \tag{5.101}$$

(c) Parabolic trajectory:

$$\text{Vertex form: } y = -a(x-h)^2 + k \tag{5.102}$$

$$\text{At } x = 0 \text{ (center)}, y = 3; \text{ at } x = \pm 15, y = 0 \tag{5.103}$$

$$0 = -a(15)^2 + 3, \text{ so } a = \frac{3}{225} = \frac{1}{75} \tag{5.104}$$

$$\text{Equation: } y = -\frac{1}{75}x^2 + 3 \tag{5.105}$$

Question 79: Gulu Road Construction Survey

Scenario: Surveyors are mapping a new road from Gulu to Kitgum. They establish a triangular control network with sides 800m, 1200m, and 1000m.

(a) Calculate all angles of the triangular network using the cosine rule.

(b) Find the area of the triangle using Heron's formula and verify with the sine formula.

(c) If the road follows the longest side, calculate the bearing if one side runs due North.

Solution: **(a)** Using cosine rule for angles:

$$\cos(A) = \frac{1200^2 + 1000^2 - 800^2}{2 \times 1200 \times 1000} = \frac{1.8}{2.4} = 0.75 \tag{5.106}$$

$$A = \arccos(0.75) = 41.41 \tag{5.107}$$

$$\cos(B) = \frac{800^2 + 1000^2 - 1200^2}{2 \times 800 \times 1000} = \frac{0.22}{1.6} = 0.1375 \tag{5.108}$$

$$B = \arccos(0.1375) = 82.10 \tag{5.109}$$

$$C = 180 - 41.41 - 82.10 = 56.49 \tag{5.110}$$

(b) Area using Heron's formula:

$$s = \frac{800 + 1200 + 1000}{2} = 1500 \tag{5.111}$$

$$A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{1500 \times 700 \times 300 \times 500} \tag{5.112}$$

$$A = \sqrt{157,500,000} = 396,232 \text{ m}^2 \tag{5.113}$$

$$\text{Verification: } A = \frac{1}{2} \times 800 \times 1000 \times \sin(82.10) = 396,232 \text{ m}^2 \checkmark \tag{5.114}$$

(c) If the 800m side runs due North, and road follows 1200m side:

$$\text{Bearing} = 90 - 41.41 = 48.59 \text{ (Northeast direction)} \tag{5.115}$$

Question 80: Mbarara Dairy Farm Solar Panel Installation

Scenario: A dairy farm in Mbarara installs solar panels on a roof inclined at 30 to maximize sun exposure. The optimal tilt angle for solar panels in Uganda is 15 from horizontal.

(a) Calculate the effective angle of the solar panels if mounted flat on the 30 roof.

(b) Find the correction angle needed to achieve the optimal 15 tilt.

(c) If each panel is 2m × 1m, calculate the actual roof area needed for 20 panels considering the spacing and tilt.

*Solution:

(a) Effective angle:

Panels mounted flat on 30 roof = 30 from horizontal (5.116)

This is 15 more than optimal (5.117)

(b) Correction needed:

$$\text{Required correction} = 30 - 15 = 15 \quad (5.118)$$

Panels need to be tilted back by 15 from the roof surface (5.119)

(c) Roof area calculation:

$$\text{Each panel: } 2\text{m} \times 1\text{m} = 2\ \text{m}^2 \quad (5.120)$$

$$\text{With 15 correction tilt, effective width} = 1\text{m} \times \cos(15) = 0.966\text{m} \quad (5.121)$$

$$\text{Length remains 2m, effective area per panel} = 2 \times 0.966 = 1.932\ \text{m}^2. \quad (5.122)$$

$$\text{Spacing factor for maintenance} = 1.5 \quad (5.123)$$

$$\text{Total roof area needed} = \frac{20 \times 2 \times 1.5}{\cos(30)} = \frac{60}{0.866} = 69.28\ \text{m}^2 \quad (5.124)$$

Question 91: Land Surveying in a Rural Community

A local land surveyor is tasked with mapping a triangular piece of land in a rural Ugandan community for a new school construction project. The sides of the land are not easily accessible for a direct measurement. From a point P, the surveyor measures the distance to two corners, A and B, of the land. The distance PA is 150 m and the distance PB is 200 m. The angle between the two lines of sight, $\angle APB$, is 65°.

a) Use the cosine rule to calculate the length of the third side of the land, AB.

b) Determine the area of the triangular piece of land.

c) If the land is to be fenced, and the cost of fencing is 50,000 UGX per meter, calculate the total cost of fencing the land.

Solution

a) We use the cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$. In our scenario, $c = AB$, $a = PB = 200$ m, $b = PA = 150$ m, and $C = \angle APB = 65°$.

$$AB^2 = 150^2 + 200^2 - 2(150)(200)\cos 65°$$

$$AB^2 = 22500 + 40000 - 60000(0.4226)$$

$$AB^2 = 62500 - 25356$$

$$AB^2 = 37144$$

$$AB = \sqrt{37144} \approx 192.7 \text{ m}.$$

b) We use the formula for the area of a triangle: $Area = \frac{1}{2}ab\sin C$.

$$Area = \frac{1}{2}(150)(200)\sin 65°$$

$$Area = 15000(0.9063)$$

$$Area \approx 13594.5\ \text{m}^2.$$

c) The perimeter of the land is 150 + 200 + 192.7 = 542.7 m.

Total cost of fencing = 542.7×50000 = 27, 135, 000 UGX.

Question 92: Aviation and Navigation

A pilot flying from Entebbe International Airport (EBB) needs to determine their current position relative to two navigation beacons, A and B. The distance between the beacons is 120 km. From the pilot's current position, P, the angle of elevation to beacon A is 30°, and the angle of elevation to beacon B is 45°. The angles of depression from the beacons to the pilot's position are equal to the angles of elevation.

a) Let h be the altitude of the aircraft. Set up two trigonometric equations to relate the altitude h to the horizontal distances from the pilot's position to beacons A and B.

b) Using the relationship between the horizontal distances, solve for the altitude h.

c) A second aircraft, at a different location, needs to maintain a line of sight with both beacons. If the second aircraft is at a horizontal distance of 100 km from beacon A and 150 km from beacon B, calculate the angle subtended by the two beacons at the second aircraft's position.

Solution

a) Let x be the horizontal distance from the pilot to beacon A, and y be the horizontal distance to beacon B.

$$\tan 30^\circ = \frac{h}{x} \implies x = \frac{h}{\tan 30^\circ}$$

$$\tan 45^\circ = \frac{h}{y} \implies y = \frac{h}{\tan 45^\circ} = h$$

b) The horizontal distance between the beacons is $x + y = 120$ km. Substituting the expressions for x and y:

$$\frac{h}{\tan 30^\circ} + h = 120$$

$$h\left(\frac{1}{\tan 30^\circ} + 1\right) = 120$$

$$h(\sqrt{3} + 1) = 120$$

$$h = \frac{120}{\sqrt{3}+1} = \frac{120(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{120(\sqrt{3}-1)}{3-1} = 60(\sqrt{3}-1) \approx 43.92 \text{ km}$$

c) Let the angle subtended at the second aircraft's position be θ. We have a triangle with sides 100 km, 150 km, and 120 km. Using the cosine rule:

$$120^2 = 100^2 + 150^2 - 2(100)(150)\cos\theta$$

$$14400 = 10000 + 22500 - 30000\cos\theta$$

$$14400 = 32500 - 30000\cos\theta$$

$$30000\cos\theta = 32500 - 14400 = 18100$$

$$\cos\theta = \frac{18100}{30000} \approx 0.6033$$

$$\theta = \cos^{-1}(0.6033) \approx 52.9^\circ.$$

Question 93: Construction of a Water Tank

A community in a hilly region of Uganda is building a cylindrical water tank on a slope. The slope has an angle of elevation of 20°. The tank has a radius of 5 meters and a height of 10 meters. To ensure stability, a support beam is to be installed from the highest point on the tank's circumference to the ground, perpendicular to the slope.

a) Calculate the length of the support beam.

b) Determine the horizontal distance from the base of the tank to the point where the beam touches the ground.

c) If the support beam is to be anchored to the ground using a concrete block, and the concrete block requires an area of 1.5 m^2, what is the length of the side of the square block?

Solution

a) Let's use a coordinate system. The slope is the line $y = x\tan 20^\circ$. The top of the tank is at $(0, 10)$. The beam is perpendicular to the slope. The equation of the beam is $y - 10 = -\frac{1}{\tan 20^\circ}(x - 0) = -\cot 20^\circ \cdot x$. The point where the beam touches the slope is the intersection of the two lines:

$$x\tan 20^\circ = 10 - x\cot 20^\circ$$

$$x(\tan 20^\circ + \cot 20^\circ) = 10$$

$$x\left(\frac{\sin 20^\circ}{\cos 20^\circ} + \frac{\cos 20^\circ}{\sin 20^\circ}\right) = 10$$

$$x\left(\frac{\sin^2 20^\circ + \cos^2 20^\circ}{\sin 20^\circ \cos 20^\circ}\right) = 10 \implies x\left(\frac{1}{\frac{1}{2}\sin 40^\circ}\right) = 10 \implies x = 5\sin 4$$

The corresponding y-coordinate is $y = x\tan 20^\circ = 3.21\tan 20^\circ \approx 1.17$ m. Length of beam $L = \sqrt{x^2 + (10-y)^2} = \sqrt{3.21^2 + (10-1.17)^2} = \sqrt{10.3 + 77.96} = \sqrt{88.26} \approx 9.39$ m.

b) Horizontal distance from the base of the tank to the point where the beam touches the ground is $x = 3.21$ m.

c) Area of the concrete block is 1.5 m^2. The side of the square block is $\sqrt{1.5} \approx 1.225$ m.

Question 94: Agriculture and Irrigation

A farmer in Wakiso is installing an irrigation system on a rectangular plot of land measuring 100 m by 80 m. The water source is located at one corner of the plot. To optimize water distribution, the farmer wants to install a main pipe from the source to the diagonally opposite corner, with feeder pipes branching off at an angle of 30° from the main pipe.

a) Calculate the length of the main pipe (the diagonal of the plot).

b) Determine the distance a feeder pipe of length 20 m will extend from the main pipe to a point on the opposite side of the plot.

c) Find the angle between the main pipe and the longer side of the plot.

Solution

a) The length of the main pipe is the diagonal of the rectangle. Using the Pythagorean theorem:

$$\text{Length} = \sqrt{100^2 + 80^2} = \sqrt{10000 + 6400} = \sqrt{16400} \approx 128.06 \text{ m}.$$

b) Let the length of the feeder pipe be $L_f = 20$ m. Let the angle between the main pipe and the feeder pipe be $\theta = 30°$. The distance from the main pipe to the opposite side of the plot is the vertical distance from the end of the feeder pipe to the main pipe. This can be found using the sine function.

$$\text{Distance from main pipe} = L_f \sin\theta = 20 \sin 30° = 20(0.5) = 10 \text{ m}.$$

c) Let the angle between the main pipe and the longer side (100 m) be α. We can use the tangent function.

$$\tan\alpha = \frac{\text{opposite}}{\text{adjacent}} = \frac{80}{100} = 0.8$$

$$\alpha = \tan^{-1}(0.8) \approx 38.66°.$$

Question 95: Telecommunications Tower Installation

Uganda Telecom is installing a new mast in a hilly area. The mast is 50 meters tall and is supported by three guy wires anchored to the ground at equal distances from the base. The angle of elevation from the anchor points to the top of the mast is 60°.

a) Calculate the length of one of the guy wires.

b) Determine the distance from the base of the mast to each anchor point.

c) If one of the guy wires snaps, and a temporary replacement is needed, but the only available wire is 55 meters long, what is the new angle of elevation to the top of the mast from the original anchor point?

Solution

a) Let L be the length of the guy wire. We have a right-angled triangle with the height of the mast (50 m) as the opposite side and the guy wire as the hypotenuse. The angle of elevation is 60°.

$$\sin 60° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{50}{L}$$

$$L = \frac{50}{\sin 60°} = \frac{50}{0.866} \approx 57.74 \text{ m}.$$

b) Let d be the distance from the base of the mast to each anchor point. This is the adjacent side of the triangle.

$$\tan 60° = \frac{\text{opposite}}{\text{adjacent}} = \frac{50}{d}$$

$$d = \frac{50}{\tan 60°} = \frac{50}{1.732} \approx 28.87 \text{ m}.$$

c) Let the new length of the guy wire be $L_{new} = 55$ m, and the new angle of elevation be θ_{new}.

$$\sin\theta_{new} = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{50}{55} \approx 0.909$$

$$\theta_{new} = \sin^{-1}(0.909) \approx 65.4°.$$

Question 96: Road Construction and Gradient

A road is being constructed in the hilly terrain of Kabale. A section of the road is to be built with a consistent gradient. A surveyor measures the horizontal distance of a 500-meter segment of the road to be 480 meters.

a) Calculate the angle of elevation of this road segment.

b) Determine the vertical rise (height) of this 500-meter segment.

c) If a car with a mass of 1500 kg drives on this road, what is the component of its weight acting parallel to the road surface? (Take $g = 9.8$ m/s^2).

Solution

a) Let θ be the angle of elevation. We have a right-angled triangle where the hypotenuse is the length of the road (500 m) and the adjacent side is the horizontal distance (480 m).

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{480}{500} = 0.96$$

$$\theta = \cos^{-1}(0.96) \approx 16.26°.$$

b) Let h be the vertical rise. Using the sine function:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{h}{500}$$

$$h = 500 \sin\theta = 500 \sin(16.26°) \approx 500(0.28) \approx 140 \text{ m}.$$

c) The component of the car's weight acting parallel to the road is given by $W \sin\theta$, where W is the weight of the car.

$$W = mg = 1500 \times 9.8 = 14700 \text{ N}.$$

$$\text{Parallel component} = 14700 \sin(16.26°) \approx 14700(0.28) \approx 4116 \text{ N}.$$

Question 97: Tourism and Observation Deck

An observation deck is to be built on a hill overlooking Lake Victoria. The hill has a height of 150 meters. A tourist at point A on the ground measures the angle of elevation to the top of the hill as 35°. The observation deck is to be built at a point B, further away from the hill. The angle of elevation from point B to the top of the hill is 25°.

a) Calculate the horizontal distance from point A to the base of the hill.

b) Calculate the horizontal distance from point B to the base of the hill.

c) Find the distance between the two tourists (the distance between points A and B).

Solution

a) Let x_A be the horizontal distance from point A to the base of the hill.

$$\tan 35^\circ = \frac{\text{height}}{\text{distance}} = \frac{150}{x_A}$$

$$x_A = \frac{150}{\tan 35^\circ} \approx \frac{150}{0.7002} \approx 214.2 \text{ m}.$$

b) Let x_B be the horizontal distance from point B to the base of the hill.

$$\tan 25^\circ = \frac{150}{x_B}$$

$$x_B = \frac{150}{\tan 25^\circ} \approx \frac{150}{0.4663} \approx 321.7 \text{ m}.$$

c) The distance between the two tourists is the difference between the horizontal distances.

$$\text{Distance } AB = x_B - x_A = 321.7 - 214.2 = 107.5 \text{ m}.$$

Question 98: Hydro-power Plant Construction

A hydroelectric power plant is being constructed near a river. A dam is to be built to create a reservoir. The design requires a support beam to be installed from a point 20 meters high on the dam wall to a point on the ground, creating an angle of 75° with the ground.

a) Calculate the length of the support beam.

b) Determine the horizontal distance from the base of the dam to the point where the beam touches the ground.

c) If the maximum and minimum height of the water level in the reservoir can be modeled by the function $h(t) = 5\cos(0.5t) + 20$, where t is the time in hours, what is the maximum height of the water level?

Solution

a) Let L be the length of the support beam. We have a right-angled triangle with the height of the dam (20 m) as the opposite side and the beam as the hypotenuse. The angle with the ground is 75°.

$$\sin 75^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{20}{L}$$

$$L = \frac{20}{\sin 75^\circ} \approx \frac{20}{0.9659} \approx 20.7 \text{ m}.$$

b) Let d be the horizontal distance.

$$\tan 75^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{20}{d}$$

$$d = \frac{20}{\tan 75^\circ} \approx \frac{20}{3.732} \approx 5.36 \text{ m}.$$

c) The maximum value of the cosine function is 1. The maximum height of the water level occurs when $\cos(0.5t) = 1$.

$$\text{Maximum height} = 5(1) + 20 = 25 \text{ meters}.$$

Question 99: Public Health and Mosquito Control

Public health officials in a low-lying area are using a drone to survey mosquito breeding sites. The drone is flying at a constant altitude of 100 meters. The drone's camera has a maximum field of view. To cover a specific area, the drone needs to fly from point A to point B, a distance of 500 meters. The angle of depression from the drone to the first breeding site at point A is 20°. The angle of depression to the second site at point B is 15°. The two sites are on opposite sides of the drone's flight path.

a) Calculate the horizontal distance from the drone's position to the first breeding site at point A.

b) Calculate the horizontal distance from the drone's position to the second breeding site at point B.

c) Determine the distance between the two breeding sites (the distance between A and B).

Solution

a) Let x_A be the horizontal distance to site A.

$$\tan 20^\circ = \frac{\text{altitude}}{\text{distance}} = \frac{100}{x_A}$$

$$x_A = \frac{100}{\tan 20^\circ} \approx \frac{100}{0.364} \approx 274.7 \text{ m}.$$

b) Let x_B be the horizontal distance to site B.

$$\tan 15^\circ = \frac{100}{x_B}$$

$$x_B = \frac{100}{\tan 15^\circ} \approx \frac{100}{0.2679} \approx 373.3 \text{ m}.$$

c) Since the two sites are on opposite sides of the drone's flight path, the distance between them is the sum of the horizontal distances.

$$\text{Distance } AB = x_A + x_B = 274.7 + 373.3 = 648 \text{ m}.$$

Question 10: Urban Planning and Security

A security company is installing CCTV cameras on a pole in an urban area. The pole is 12 meters tall. One camera is positioned at the top of the pole, and another is positioned at a height of 8 meters. The cameras are used to monitor a potential security threat located at an unknown point P on the ground. The angle of depression from the top camera to the point P is 40°.

a) Determine the horizontal distance from the base of the pole to the point P.

b) Calculate the angle of depression from the second camera (at 8 meters) to the same point P.

c) If a third camera is to be installed such that its line of sight to point P creates an angle of depression of 30°, at what height on the pole should it be positioned?

Solution

a) Let d be the horizontal distance from the base of the pole to point P. We use the top camera's position (12 m).

$$\tan 40^\circ = \frac{\text{height}}{\text{distance}} = \frac{12}{d}$$

$$d = \frac{12}{\tan 40^\circ} \approx \frac{12}{0.8391} \approx 14.3 \text{ m}.$$

b) Let θ_2 be the angle of depression from the second camera. The height of the second camera is 8 meters. The horizontal distance d is the same.

$$\tan \theta_2 = \frac{\text{height}}{\text{distance}} = \frac{8}{14.3} \approx 0.5594$$

$$\theta_2 = \tan^{-1}(0.5594) \approx 29.2^\circ.$$

c) Let h_3 be the height of the third camera. The angle of depression is 30°, and the horizontal distance d is 14.3 m.

$$\tan 30^\circ = \frac{h_3}{d} = \frac{h_3}{14.3}$$

$$h_3 = 14.3 \tan 30^\circ \approx 14.3(0.5774) \approx 8.26 \text{ m}.$$

Chapter 6

Differentiation 1

6.1 Gradient of a curve at a point can be derived from gradient of a chord using small increments.

6.2 Gradient of the tangent at a point on the curve.

6.3 Formula for differentiation of ax^n , $n \in \mathbb{R}$

6.4 Differentiating functions and identify the equations of the tangents and normal curves at different points.

6.5 Differentiating

6.5.1 Product and quotient functions

6.5.2 Composite function

6.5.3 Parametric functions

6.5.4 Implicit functions

6.6 Second derivatives of functions

6.7 Determining the maximum and minimum values of given functions

6.8 Nature of turning points in curve sketching and optimization of resources

6.9 Small changes to approximate roots of numbers, errors in measurements, changes in physical quantities, tri

6.10 Applying differentiation concepts to velocity, displacement and acceleration.

Competency:

The learner applies differentiation techniques to solve problems in calculus and interpret their significance in real-world contexts.

*1. Introduction to Differentiation

*1.1 What is Differentiation?

Differentiation is the process of finding the rate at which one quantity changes with respect to another. It is a fundamental concept in calculus that helps us:

- Find slopes of curves at specific points
- Determine rates of change
- Locate maximum and minimum values
- Solve optimization problems
- Model real-world phenomena

*1.2 The Concept of Gradient

The gradient of a straight line is constant and given by:

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

However, for curves, the gradient varies at different points.

*2. Deriving the Gradient of a Curve

*2.1 Gradient of a Chord

Consider a curve $y = f(x)$ and two points on it: $P(x, f(x))$ and $Q(x+h, f(x+h))$

The gradient of the chord PQ is:

$$m_{chord} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

*2.2 Gradient of a Tangent (Derivative)

As Q moves closer to P (i.e., as $h \to 0$), the chord approaches the tangent at P.

The gradient of the tangent at point P is:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This limit is called the **derivative** of $f(x)$ with respect to x.

*2.3 First Principles Example

Example 1: Find the derivative of $f(x) = x^2$ from first principles.

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h \to 0} \frac{h(2x+h)}{h} \\
&= \lim_{h \to 0} (2x + h) \\
&= 2x
\end{aligned}$$

Therefore, if $y = x^2$, then $\frac{dy}{dx} = 2x$

*3. Differentiation Formulas

*3.1 The Power Rule

For $f(x) = ax^n$ where a and n are constants:

$$\frac{d}{dx}(ax^n) = anx^{n-1}$$

Special Cases:

$$\begin{aligned}
&\frac{d}{dx}(c) = 0 \quad \text{(where } c \text{ is a constant)} \\
&\frac{d}{dx}(x) = 1 \\
&\frac{d}{dx}(x^2) = 2x \\
&\frac{d}{dx}(x^3) = 3x^2 \\
&\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -x^{-2} = -\frac{1}{x^2}
\end{aligned}$$

*3.2 Sum and Difference Rules

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$

Example 2: Differentiate $y = 3x^4 - 5x^3 + 2x - 7$

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(3x^4) - \frac{d}{dx}(5x^3) + \frac{d}{dx}(2x) - \frac{d}{dx}(7) \\
&= 12x^3 - 15x^2 + 2 - 0 \\
&= 12x^3 - 15x^2 + 2
\end{aligned}$$

Example 3: Differentiate $y = \frac{4}{x^3} + \sqrt{x} - 5$

Solution: First rewrite: $y = 4x^{-3} + x^{1/2} - 5$

$$\begin{aligned}
\frac{dy}{dx} &= 4(-3)x^{-4} + \frac{1}{2}x^{-1/2} - 0 \\
&= -12x^{-4} + \frac{1}{2}x^{-1/2} \\
&= -\frac{12}{x^4} + \frac{1}{2\sqrt{x}}
\end{aligned}$$

*4. Product Rule

For $y = u \cdot v$ where u and v are functions of x:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Alternative notation: $(uv)' = uv' + vu'$

Example 4: Differentiate $y = (2x+3)(x^2 - 1)$

Solution:

Let $u = 2x + 3$ and $v = x^2 - 1$

Then $\frac{du}{dx} = 2$ and $\frac{dv}{dx} = 2x$

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= (2x+3)(2x) + (x^2-1)(2) \\ &= 4x^2 + 6x + 2x^2 - 2 \\ &= 6x^2 + 6x - 2\end{aligned}$$

Example 5 (Physics Application): The kinetic energy of an object is $KE = \frac{1}{2}mv^2$ where $v = 3t^2 + 2t$. Find the rate of change of kinetic energy with respect to time.

Solution:

$KE = \frac{1}{2}m(3t^2+2t)^2$

Let $u = \frac{1}{2}m$ (constant) and $v = (3t^2+2t)^2$

Using chain rule (covered later): $\frac{dv}{dt} = 2(3t^2+2t)(6t+2)$

$$\begin{aligned}\frac{d(KE)}{dt} &= \frac{1}{2}m \cdot 2(3t^2+2t)(6t+2) \\ &= m(3t^2+2t)(6t+2)\end{aligned}$$

*5. Quotient Rule

For $y = \frac{u}{v}$ where u and v are functions of x:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Memory aid: "Low dee-high minus high dee-low, over low squared"

Example 6: Differentiate $y = \frac{x^2+1}{x-2}$

Solution:

Let $u = x^2 + 1$ and $v = x - 2$

Then $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 1$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x-2)(2x) - (x^2+1)(1)}{(x-2)^2} \\ &= \frac{2x^2 - 4x - x^2 - 1}{(x-2)^2} \\ &= \frac{x^2 - 4x - 1}{(x-2)^2}\end{aligned}$$

Example 7: Differentiate $y = \frac{3x-5}{2x+1}$

Solution:

Let $u = 3x - 5$, $v = 2x + 1$

$\frac{du}{dx} = 3$, $\frac{dv}{dx} = 2$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(2x+1)(3) - (3x-5)(2)}{(2x+1)^2} \\ &= \frac{6x + 3 - 6x + 10}{(2x+1)^2} \\ &= \frac{13}{(2x+1)^2}\end{aligned}$$

*6. Chain Rule (Composite Functions)

For $y = f(g(x))$ or $y = f(u)$ where $u = g(x)$:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 8: Differentiate $y = (3x^2+5)^4$

Solution:

Let $u = 3x^2 + 5$, so $y = u^4$

Then $\frac{dy}{du} = 4u^3$ and $\frac{du}{dx} = 6x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 4u^3 \cdot 6x \\ &= 4(3x^2+5)^3 \cdot 6x \\ &= 24x(3x^2+5)^3\end{aligned}$$

Example 9: Differentiate $y = \sqrt{4x^2 - 7}$

Solution:

Rewrite as $y = (4x^2-7)^{1/2}$

Let $u = 4x^2 - 7$, so $y = u^{1/2}$

$\frac{dy}{du} = \frac{1}{2}u^{-1/2}$, $\frac{du}{dx} = 8x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}u^{-1/2} \cdot 8x \\ &= \frac{1}{2}(4x^2-7)^{-1/2} \cdot 8x \\ &= \frac{4x}{\sqrt{4x^2-7}}\end{aligned}$$

Example 10 (Environmental Science): The concentration of a pollutant decreases according to $C(t) =$

$(100 - 2t)^3$ mg/L. Find the rate of change of concentration after 10 hours.

Solution:

Let $u = 100 - 2t$, so $C = u^3$

$\frac{dC}{du} = 3u^2$, $\frac{du}{dt} = -2$

$$\begin{aligned}\frac{dC}{dt} &= 3u^2 \cdot (-2) \\ &= -6(100 - 2t)^2\end{aligned}$$

At $t = 10$: $\frac{dC}{dt} = -6(100 - 20)^2 = -6(80)^2 = -38,400$ mg/L per hour

The negative sign indicates the concentration is decreasing at a rate of 38,400 mg/L per hour.

*7. Parametric Differentiation

For curves defined by parametric equations $x = f(t)$ and $y = g(t)$:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Example 11: Find $\frac{dy}{dx}$ if $x = t^2 + 1$ and $y = 2t^3 - t$

Solution:

$$\begin{aligned}\frac{dx}{dt} &= 2t \\ \frac{dy}{dt} &= 6t^2 - 1\end{aligned}$$

Therefore:

$$\frac{dy}{dx} = \frac{6t^2 - 1}{2t}$$

Example 12: For the parametric equations $x = 3\cos\theta$ and $y = 3\sin\theta$, find $\frac{dy}{dx}$.

Solution:

$$\begin{aligned}\frac{dx}{d\theta} &= -3\sin\theta \\ \frac{dy}{d\theta} &= 3\cos\theta\end{aligned}$$

$$\frac{dy}{dx} = \frac{3\cos\theta}{-3\sin\theta} = -\cot\theta$$

*8. Implicit Differentiation

For equations where y is not explicitly expressed in terms of x, we differentiate both sides with respect to x, treating y as a function of x.

Key Rule: $\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$

Example 13: Find $\frac{dy}{dx}$ if $x^2 + y^2 = 25$

Solution:

Differentiate both sides with respect to x:

$$\begin{aligned}\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= \frac{d}{dx}(25) \\ 2x + 2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= -\frac{x}{y}\end{aligned}$$

Example 14: Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$

Solution:

Differentiate both sides:

$$\begin{aligned}3x^2 + 3y^2\frac{dy}{dx} &= 6\left(x\frac{dy}{dx} + y\right) \\ 3x^2 + 3y^2\frac{dy}{dx} &= 6x\frac{dy}{dx} + 6y \\ 3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} &= 6y - 3x^2 \\ \frac{dy}{dx}(3y^2 - 6x) &= 6y - 3x^2 \\ \frac{dy}{dx} &= \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}\end{aligned}$$

*9. Equations of Tangents and Normals

*9.1 Tangent to a Curve

The equation of the tangent to curve $y = f(x)$ at point (x_1, y_1) is:

$$y - y_1 = m(x - x_1)$$

where $m = \left.\frac{dy}{dx}\right|_{x=x_1}$ is the gradient at that point.

*9.2 Normal to a Curve

The normal is perpendicular to the tangent. If the tangent has gradient m, the normal has gradient $-\frac{1}{m}$.

Equation of normal:

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

Example 15: Find the equations of the tangent and normal to $y = x^2 - 3x + 2$ at the point where $x = 2$.

Solution:

First, find y when $x = 2$:

$$y = (2)^2 - 3(2) + 2 = 4 - 6 + 2 = 0$$

So the point is $(2, 0)$.

Find the gradient:

$$\frac{dy}{dx} = 2x - 3$$

At $x = 2$: $m = 2(2) - 3 = 1$

Tangent:

$$y - 0 = 1(x - 2)$$
$$y = x - 2$$

Normal: (gradient $= -\frac{1}{1} = -1$)

$$y - 0 = -1(x - 2)$$
$$y = -x + 2$$

Example 16: Find where the tangent to $y = x^3$ at $x = 1$ meets the curve again.

Solution:

At $x = 1$: $y = 1^3 = 1$, so point is $(1, 1)$

$\frac{dy}{dx} = 3x^2$, at $x = 1$: $m = 3(1)^2 = 3$

Tangent equation: $y - 1 = 3(x - 1) \rightarrow y = 3x - 2$

To find where it meets the curve:

$$x^3 = 3x - 2$$
$$x^3 - 3x + 2 = 0$$
$$(x - 1)^2(x + 2) = 0$$

Solutions: $x = 1$ (original point) or $x = -2$

When $x = -2$: $y = 3(-2) - 2 = -8$

The tangent meets the curve again at $(-2, -8)$.

*10. Second Derivatives

The second derivative is the derivative of the derivative:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Notation: $f''(x)$ or y'' or $\frac{d^2y}{dx^2}$

Example 17: Find the second derivative of $y = 2x^4 - 5x^3 + 3x - 1$

Solution:

$$\frac{dy}{dx} = 8x^3 - 15x^2 + 3$$
$$\frac{d^2y}{dx^2} = 24x^2 - 30x$$

Example 18 (Physics): If displacement $s = t^3 - 6t^2 + 9t$, find velocity and acceleration.

Solution:

Velocity: $v = \frac{ds}{dt} = 3t^2 - 12t + 9$

Acceleration: $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 6t - 12$

*11. Maximum and Minimum Values

*11.1 Stationary Points

Stationary points occur where $\frac{dy}{dx} = 0$

*11.2 Nature of Stationary Points

Use the second derivative test:

- If $\frac{d^2y}{dx^2} > 0$: minimum point
- If $\frac{d^2y}{dx^2} < 0$: maximum point
- If $\frac{d^2y}{dx^2} = 0$: point of inflection (further investigation needed)

Alternative: First derivative test (check sign change of $\frac{dy}{dx}$)

Example 19: Find and classify the stationary points of $y = x^3 - 6x^2 + 9x + 1$

Solution:

$$\frac{dy}{dx} = 3x^2 - 12x + 9$$
$$= 3(x^2 - 4x + 3)$$
$$= 3(x-1)(x-3)$$

Stationary points: $\frac{dy}{dx} = 0 \rightarrow x = 1$ or $x = 3$

When $x = 1$: $y = 1 - 6 + 9 + 1 = 5 \rightarrow$ Point $(1, 5)$

When $x = 3$: $y = 27 - 54 + 27 + 1 = 1 \rightarrow$ Point $(3, 1)$

Second derivative: $\frac{d^2y}{dx^2} = 6x - 12$

At $x = 1$: $\frac{d^2y}{dx^2} = 6(1) - 12 = -6 < 0 \rightarrow$ **Maximum**

At $x = 3$: $\frac{d^2y}{dx^2} = 6(3) - 12 = 6 > 0 \rightarrow$ **Minimum**

Example 20 (Optimization): A farmer has 100 m of fencing to enclose a rectangular plot against a straight wall. Find the maximum area.

Solution:

Let the width be x and length be y.

Fencing equation: $2x + y = 100 \rightarrow y = 100 - 2x$

Area: $A = xy = x(100 - 2x) = 100x - 2x^2$

$$\frac{dA}{dx} = 100 - 4x$$

For maximum: $\frac{dA}{dx} = 0 \rightarrow 100 - 4x = 0 \rightarrow x = 25$ m

$\frac{d^2A}{dx^2} = -4 < 0 \rightarrow$ Maximum confirmed

When $x = 25$: $y = 100 - 2(25) = 50$ m

Maximum area = $25 \times 50 = 1250$ m²

*12. Small Changes and Approximations

For small changes δx in x:

$$\delta y \approx \frac{dy}{dx} \times \delta x$$

Example 21: Approximate $\sqrt{101}$ using differentiation.

Solution:

Let $y = \sqrt{x} = x^{1/2}$

$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

Use $x = 100$ (close to 101), so $\delta x = 1$

$$\delta y \approx \frac{dy}{dx} \times \delta x$$
$$= \frac{1}{2\sqrt{100}} \times 1$$
$$= \frac{1}{20} = 0.05$$

$\sqrt{101} \approx \sqrt{100} + 0.05 = 10 + 0.05 = 10.05$

(Actual value: 10.0498...)

Example 22 (Error Analysis): The radius of a sphere is measured as 5 cm with a possible error of ±0.1 cm. Estimate the maximum error in the calculated volume.

Solution:

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

At $r = 5$: $\frac{dV}{dr} = 4\pi(5)^2 = 100\pi$

$\delta r = \pm 0.1$ cm

$$\delta V \approx \frac{dV}{dr} \times \delta r$$
$$= 100\pi \times 0.1$$
$$= 10\pi \approx 31.4 \text{ cm}^3$$

Maximum error in volume ≈ ±31.4 cm³

*13. Applications in Physics: Kinematics

*13.1 Key Relationships

For displacement $s(t)$:

- Velocity: $v = \frac{ds}{dt}$
- Acceleration: $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

Example 23: A particle moves such that $s = 2t^3 - 9t^2 + 12t$ meters. Find:

a) Velocity at $t = 2$ seconds
b) When the particle is at rest
c) Acceleration at $t = 3$ seconds

Solution:

a) Velocity: $v = \frac{ds}{dt} = 6t^2 - 18t + 12$

At $t = 2$: $v = 6(4) - 18(2) + 12 = 24 - 36 + 12 = 0$ m/s

b) At rest when $v = 0$:

$$6t^2 - 18t + 12 = 0$$
$$t^2 - 3t + 2 = 0$$
$$(t-1)(t-2) = 0$$

Particle is at rest at $t = 1$ s and $t = 2$ s

c) Acceleration: $a = \frac{dv}{dt} = 12t - 18$

At $t = 3$: $a = 12(3) - 18 = 36 - 18 = 18$ m/s²

Example 24 (Projectile Motion): A ball is thrown vertically upward with displacement $h = 20t - 5t^2$ meters. Find:

a) Maximum height reached
b) Time to reach maximum height
c) Velocity when it returns to ground

Solution:

a) & b) Velocity: $v = \frac{dh}{dt} = 20 - 10t$

At maximum height, $v = 0$: $20 - 10t = 0 \rightarrow t = 2$ s

Maximum height: $h = 20(2) - 5(2)^2 = 40 - 20 = 20$ m

c) Returns to ground when $h = 0$:

$$20t - 5t^2 = 0$$
$$5t(4 - t) = 0$$

$t = 0$ (start) or $t = 4$ s (return)

At $t = 4$: $v = 20 - 10(4) = -20$ m/s

The negative sign indicates downward motion.

*14. Applications in Biology

Example 25 (Population Growth): A bacterial population grows according to $P(t) = 1000(1 + 0.5t)^2$ where t is time in hours. Find the rate of growth at $t = 4$ hours.

Solution:

$P(t) = 1000(1 + 0.5t)^2$

Using chain rule:

$$\frac{dP}{dt} = 1000 \times 2(1 + 0.5t) \times 0.5$$
$$= 1000(1 + 0.5t)$$

At $t = 4$: $\frac{dP}{dt} = 1000(1 + 2) = 3000$ bacteria/hour

6.11 More scenario based questions with solutions

Question 1: Crop Growth Rate in Masindi

A farmer in Masindi is studying the growth of a new maize variety. The height of the maize plant, in centimeters, is modeled by the function $h(t) = 0.5t^2 + 10t + 5$, where t is the number of weeks after planting.

1. Find the rate of growth of the plant at any time t.
2. Calculate the rate of growth of the plant at the end of the 5th week.
3. Explain the meaning of your answer in part (b) in the context of the problem and discuss how this information can help the farmer.

Solution 1

1. The rate of growth is the first derivative of the height function, $h(t)$, with respect to time, t. $h(t) = 0.5t^2 + 10t + 5$ $h'(t) = \frac{d}{dt}(0.5t^2) + \frac{d}{dt}(10t) + \frac{d}{dt}(5)$ $h'(t) = 0.5(2t^{2-1}) + 10(1) + 0$ $h'(t) = t + 10$ The rate of growth is $\mathbf{t + 10}$ cm/week.
2. To find the rate of growth at the end of the 5th week, we substitute $t = 5$ into the derivative. $h'(5) = 5 + 10 = \mathbf{15}$ cm/week.
3. The answer means that at the 5th week after planting, the maize plant is growing at a rate of 15 cm per week. This information helps the farmer to monitor the health of the crop. If the calculated growth rate is significantly different from the observed rate, it could indicate problems like disease, pests, or nutrient deficiencies, allowing the farmer to take corrective action. It also helps in predicting the time of maturity and harvest.

Question 2: Water Level in a Tank in Gulu

A community in Gulu uses a cylindrical water tank. The volume of water in the tank is given by $V(h) = \pi r^2 h$, where r is the radius of the base and h is the water level. The rate at which the water level changes with respect

to time is given by $\frac{dh}{dt} = 0.2$ m/minute due to inflow from a pump. The radius of the tank is 2 meters.

1. Find the rate of change of the volume of water with respect to the water level, $\frac{dV}{dh}$.
2. Using the chain rule, find the rate at which the volume of water is increasing, $\frac{dV}{dt}$.
3. Explain the practical significance of this rate for the community.

Solution 2

1. The volume function is $V(h) = \pi(2^2)h = 4\pi h$. Differentiating with respect to h: $\frac{dV}{dh} = \frac{d}{dh}(4\pi h) = \mathbf{4\pi}\ m^3/m$.
2. The chain rule states that $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$. We have $\frac{dV}{dh} = 4\pi$ and $\frac{dh}{dt} = 0.2$. $\frac{dV}{dt} = 4\pi \times 0.2 = \mathbf{0.8\pi}\ m^3$/minute.
3. The value $0.8\pi\ m^3$/minute represents the rate at which water is being pumped into the tank. This is a crucial piece of information for the community. It allows them to determine how long it will take to fill the tank, manage water distribution, and schedule pumping operations. For example, if the tank has a total volume of $50\pi\ m^3$, it would take $\frac{50\pi}{0.8\pi} = 62.5$ minutes to fill. This helps in managing water resources efficiently.

Question 3: Deforestation Rate in the Bugoma Forest

Environmentalists are monitoring the rate of deforestation in the Bugoma Forest. The area of the forest, in square kilometers, is modeled by the function $A(t) = 1000 - 50t - 0.2t^2$, where t is the number of years since monitoring began.

1. Find the instantaneous rate of change of the forest area with respect to time, $\frac{dA}{dt}$.
2. Calculate the rate of deforestation after 10 years.
3. Explain whether the rate of deforestation is increasing or decreasing over time and what this implies for conservation efforts.

Solution 3

1. The rate of change of the forest area is the first derivative of the area function, $A(t)$, with respect to time, t. $A(t) = 1000 - 50t - 0.2t^2$ $\frac{dA}{dt} = \frac{d}{dt}(1000) - \frac{d}{dt}(50t) - \frac{d}{dt}(0.2t^2)$ $\frac{dA}{dt} = 0 - 50(1) - 0.2(2t)$ $\frac{dA}{dt} = \mathbf{-50 - 0.4t}\ km^2$/year.
2. To find the rate of deforestation after 10 years, substitute $t = 10$ into the derivative. $\left.\frac{dA}{dt}\right|_{t=10} = -50 - 0.4(10) = -50 - 4 = \mathbf{-54}$ km^2/year.
3. The rate of change is $\frac{dA}{dt} = -50 - 0.4t$. The term $-0.4t$ is negative and becomes more negative as t increases. This means the rate of deforestation is increasing over time. This is a crucial finding for conservationists. It implies that the forest is being cleared at an accelerating pace, suggesting that current conservation measures are insufficient. This knowledge should prompt authorities to implement stricter policies and more urgent interventions to halt the accelerated loss of forest cover.

Question 4: Farmer's Profit Maximization in Fort Portal

A coffee farmer in Fort Portal has a profit function given by $P(x) = -x^2 + 100x - 500$, where P is the profit in thousands of shillings and x is the amount of fertilizer used in kilograms.

1. Find the marginal profit function, $\frac{dP}{dx}$.
2. Calculate the amount of fertilizer that should be used to maximize the profit.
3. What is the maximum profit the farmer can make?

Solution 4

1. The marginal profit is the derivative of the profit function with respect to the amount of fertilizer used. $P(x) = -x^2 + 100x - 500$ $\frac{dP}{dx} = \frac{d}{dx}(-x^2 + 100x - 500)$ $\frac{dP}{dx} = -2x + 100$ The marginal profit function is $\mathbf{-2x + 100}$.
2. To maximize profit, we set the first derivative to zero and solve for x. $\frac{dP}{dx} = 0$ $-2x + 100 = 0$ $100 = 2x$ $x = \mathbf{50}$ kg. To confirm this is a maximum, we can find the second derivative: $\frac{d^2P}{dx^2} = -2$, which is negative, confirming that $x = 50$ is a point of maximum profit.
3. To find the maximum profit, substitute $x = 50$ into the original profit function. $P(50) = -(50)^2 + 100(50) - 500$ $P(50) = -2500 + 5000 - 500$ $P(50) = \mathbf{2000}$ thousand shillings, or 2,000,000 shillings.

Question 5: Public Health Intervention in Masaka

A public health team in Masaka is tracking the spread of a disease. The number of new cases, N, per day is modeled by $N(t) = \frac{100t}{t^2+4}$, where t is the number of weeks since the first case.

1. Find the rate of change of new cases, $\frac{dN}{dt}$.
2. Find the number of weeks at which the rate of new cases is highest.
3. Explain the significance of your findings for the public health team.

Solution 5

1. We use the quotient rule for $\frac{dN}{dt}$: If $N = \frac{u}{v}$, then $\frac{dN}{dt} = \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}$. Let $u = 100t$ and $v = t^2 + 4$. $\frac{du}{dt} = 100$ and $\frac{dv}{dt} = 2t$. $\frac{dN}{dt} = \frac{(t^2+4)(100) - (100t)(2t)}{(t^2+4)^2} = \frac{100t^2+400-200t^2}{(t^2+4)^2}$ $\frac{dN}{dt} = \mathbf{\frac{400-100t^2}{(t^2+4)^2}}$.
2. To find the maximum rate, we can analyze the numerator of the first derivative. The maximum of $N(t)$ occurs when the numerator of its derivative, $400 - 100t^2$, is zero (since the denominator is always positive, and the function is a simple rational function where this is a reliable method). $400 - 100t^2 = 0$ $400 = 100t^2$ $4 = t^2$ $t = \mathbf{2}$ weeks (since time cannot be negative).
3. The result indicates that the spread of the disease peaks at the end of the 2nd week. This information is critical for the public health team. It allows them to allocate resources like medical personnel, hospital beds, and public awareness campaigns most effectively. Knowing when the peak is expected helps them prepare for the highest influx of cases and implement containment measures like quarantines and lockdowns at the most opportune time.

Question 6: Road Construction in Kabale

A road construction company in Kabale is building a new road. The elevation of the road is modeled by the function $y = 0.5x^2 - 4x + 10$, where x and y are in kilometers. A surveyor wants to determine the steepest point on the road.

1. Find the slope of the road at any point x.
2. Find the coordinates of the point where the road is at its lowest elevation.
3. What is the equation of the tangent line to the road curve at the point $(2, 4)$?

Solution 6

1. The slope of the road is the first derivative of the elevation function with respect to x. $y = 0.5x^2 - 4x + 10$ $\frac{dy}{dx} = \frac{d}{dx}(0.5x^2) - \frac{d}{dx}(4x) + \frac{d}{dx}(10)$ $\frac{dy}{dx} = 0.5(2x) - 4 + 0$ $\frac{dy}{dx} = \mathbf{x - 4}$.
2. The lowest point on the road is where the slope is zero. $\frac{dy}{dx} = 0$ $x - 4 = 0$ $x = 4$. To find the y-coordinate, substitute $x = 4$ back into the original equation: $y = 0.5(4)^2 - 4(4) + 10 = 0.5(16) - 16 + 10 = 8 - 16 + 10 = 2$. The coordinates of the lowest point are $(\mathbf{4}, \mathbf{2})$.
3. The slope of the tangent at $(2, 4)$ is found by substituting $x = 2$ into the derivative. $m = \left.\frac{dy}{dx}\right|_{x=2} = 2 - 4 = -2$. The equation of a line is $y - y_1 = m(x - x_1)$. $y - 4 = -2(x - 2)$ $y - 4 = -2x + 4$ $y = \mathbf{-2x + 8}$.

Question 7: Market Price Fluctuation in Kampala

A trader in Kampala is observing the price of a certain commodity. The price, in thousands of shillings, is modeled by the function $P(t) = \frac{50}{t^2+1} + 10$, where t is the number of weeks.

1. Find the rate of change of the price with respect to time.
2. Find the rate of change of the price at the end of the 2nd week.
3. Explain the meaning of a negative rate of change in this context and what it implies for the trader's strategy.

Solution 7

1. We can rewrite the function as $P(t) = 50(t^2 + 1)^{-1} + 10$. Using the chain rule and power rule: $\frac{dP}{dt} = \frac{d}{dt}[50(t^2 + 1)^{-1}] + \frac{d}{dt}(10)$ $\frac{dP}{dt} = 50(-1)(t^2+1)^{-2} \cdot \frac{d}{dt}(t^2+1) + 0$ $\frac{dP}{dt} = -50(t^2 + 1)^{-2}(2t) = \mathbf{\frac{-100t}{(t^2+1)^2}}$.
2. To find the rate of change at the end of the 2nd week, substitute $t = 2$ into the derivative. $\left.\frac{dP}{dt}\right|_{t=2} = \frac{-100(2)}{(2^2+1)^2} = \frac{-200}{(4+1)^2} = \frac{-200}{25} = \mathbf{-8}$ thousand shillings/week.
3. A negative rate of change means that the price of the commodity is decreasing. For the trader, this is a critical observation. It indicates that the commodity is losing value. The trader should consider selling off their stock of this commodity to avoid further losses. Conversely, it might be a good time for a buyer to purchase the commodity, as it is becoming cheaper. The magnitude of the rate, 8, indicates how fast the price is dropping at that specific moment.

Question 8: Traffic Congestion in Kampala

The number of vehicles, N, passing a certain point on a road in Kampala is modeled by the function $N(t) = 50t^3 - 300t^2 + 400t + 1000$, where t is the number of hours from 6 am to 6 pm.

1. Find the instantaneous rate of change of the number of vehicles, $\frac{dN}{dt}$.
2. Find the time(s) of day when the rate of change of vehicles is zero.
3. What does a zero rate of change imply about the traffic flow?

Solution 8

1. The rate of change is the first derivative of the function. $N(t) = 50t^3 - 300t^2 + 400t + 1000$ $\frac{dN}{dt} = 50(3t^2) - 300(2t) + 400(1) + 0$ $\frac{dN}{dt} =$ $\mathbf{150t^2 - 600t + 400}$.
2. To find when the rate of change is zero, we set the derivative to zero and solve the quadratic equation. $150t^2 - 600t + 400 = 0$ Divide by 50 to simplify: $3t^2 - 12t + 8 = 0$. Using the quadratic formula, $t = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$: $t = \frac{12 \pm \sqrt{(-12)^2-4(3)(8)}}{2(3)} = \frac{12 \pm \sqrt{144-96}}{6} = \frac{12 \pm \sqrt{48}}{6}$ $t = \frac{12 \pm 4\sqrt{3}}{6} = \frac{6 \pm 2\sqrt{3}}{3}$. $t_1 = \frac{6-2\sqrt{3}}{3} \approx 0.845$ hours, which is about 6:50 am. $t_2 = \frac{6+2\sqrt{3}}{3} \approx$ 3.155 hours, which is about 9:09 am. The times when the rate of change is zero are at approximately 0.845 hours and 3.155 hours after 6 am.
3. A zero rate of change means that the number of vehicles passing the point is momentarily neither increasing nor decreasing. These points correspond to peak traffic times (where the traffic flow changes from increasing to decreasing) or troughs (where the flow changes from decreasing to increasing). This information is valuable for traffic authorities to manage congestion. They can use these times to implement measures like traffic police deployment, lane management, or signal adjustments.

Question 9: Tourism Revenue in Queen Elizabeth National Park

The revenue from a tour company operating in Queen Elizabeth National Park is given by the function $R(x) = \frac{x^2}{x-2}$, where x is the number of tour packages sold in hundreds, and R is the revenue in millions of shillings.

1. Find the rate of change of revenue, $\frac{dR}{dx}$.
2. Find the number of tour packages that should be sold to minimize the rate of change of revenue.
3. What is the significance of this point for the tour company's pricing and marketing strategy?

Solution 9

1. We use the quotient rule: If $R = \frac{u}{v}$, then $\frac{dR}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$. Let $u = x^2$ and $v = x - 2$. $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 1$. $\frac{dR}{dx} = \frac{(x-2)(2x)-(x^2)(1)}{(x-2)^2} = \frac{2x^2-4x-x^2}{(x-2)^2}$ $\frac{dR}{dx} = \mathbf{\frac{x^2-4x}{(x-2)^2}}$.
2. To find the minimum rate of change, we can find the critical points of the first derivative by setting the numerator to zero. $x^2 - 4x = 0$ $x(x-4) = 0$ $x = 0$ or $x = 4$. Since selling 0 packages is trivial, we consider $x = 4$. This implies selling **400** tour packages.
3. This is a tricky question. A more complete analysis would require the second derivative to determine if this is a minimum or maximum rate of change. $R''(x) = \frac{d}{dx}\left(\frac{x^2-4x}{(x-2)^2}\right) = \frac{(2x-4)(x-2)^2-(x^2-4x)\cdot 2(x-2)}{(x-2)^4} = \frac{8}{(x-2)^3}$. At $x = 4$, $R''(4) = \frac{8}{(4-2)^3} = \frac{8}{8} = 1 > 0$, so the rate of change is at a minimum. This means at 400 packages, the revenue growth rate is at its lowest. The company should not necessarily aim for this point, as it represents a deceleration in revenue growth. However, this is an important point of inflection in their strategy. They need to understand what happens before and after this point and adjust their pricing and marketing to ensure the growth of revenue is always positive and as high as possible.

Question 10: Fish Population in Lake Albert

The population of a certain type of fish in Lake Albert is modeled by the function $P(t) = 500e^{0.1t}$, where P is the population and t is the time in years. Due to overfishing, the population is also decreasing at a rate modeled by the function $D(t) = 100t^2$. The net population is given by $N(t) = P(t) - D(t)$.

1. Find the rate of change of the net fish population.
2. Find the time at which the net population is decreasing fastest.
3. What policy recommendations can be made to local fishermen based on your findings?

Solution 10

1. The rate of change of the net population is the derivative of $N(t)$. $N(t) = 500e^{0.1t} - 100t^2$ Using the chain rule for the exponential term: $\frac{dN}{dt} = \frac{d}{dt}(500e^{0.1t}) - \frac{d}{dt}(100t^2)$ $\frac{dN}{dt} = 500(e^{0.1t} \cdot 0.1) - 200t$ $\frac{dN}{dt} = \mathbf{50e^{0.1t} - 200t}$.
2. To find when the population is decreasing fastest, we need to find the minimum of the rate of change, which means we set the second derivative to zero. $\frac{d^2N}{dt^2} = \frac{d}{dt}(50e^{0.1t} - 200t)$ $\frac{d^2N}{dt^2} = 50(e^{0.1t} \cdot 0.1) - 200$ $\frac{d^2N}{dt^2} = 5e^{0.1t} - 200$. Set the second derivative to zero: $5e^{0.1t} - 200 = 0$ $5e^{0.1t} = 200$ $e^{0.1t} = 40$ Take the natural logarithm of both sides: $0.1t = \ln(40)$ $t = \frac{\ln(40)}{0.1} = \frac{3.6889}{0.1} \approx \mathbf{36.89}$ years.
3. The result indicates that the net population is decreasing at its fastest rate after nearly 37 years. This is a critical warning. It implies that the population's natural growth is being severely outpaced by overfishing, and the situation is getting progressively worse. Policy recommendations should include implementing strict fishing quotas, enforcing a closed season for fishing, or introducing penalties for catching fish below a certain size. These measures are necessary to allow the fish population to recover and ensure the long-term sustainability of the fishing industry in Lake Albert.

Problem 11: Coffee Production Optimization in Mbale

Scenario: A coffee farmer in Mbale district models the coffee yield using the function $Y(t) = 50t - 2t^2$, where t is the number of weeks after flowering and Y is the yield in kg per hectare.

(a) Find the rate of change of yield with respect to time at $t = 4$ weeks.
(b) Determine when the yield is increasing most rapidly.
(c) Find the equation of the tangent to the curve at the point of maximum yield.

Solution 1

(a) Differentiate the yield function:

$$Y(t) = 50t - 2t^2$$
$$\frac{dY}{dt} = 50 - 4t$$
$$\text{At } t = 4: \quad \frac{dY}{dt} = 50 - 4(4) = 50 - 16 = 34 \text{ kg/week}$$

(b) The yield increases most rapidly when the second derivative is zero:

$$\frac{d^2Y}{dt^2} = -4$$

Since $\frac{d^2Y}{dt^2} < 0$ for all t, the maximum rate occurs at $t = 0$

(c) Maximum yield occurs when $\frac{dY}{dt} = 0$:

$$50 - 4t = 0 \Rightarrow t = 12.5 \text{ weeks}$$
$$Y(12.5) = 50(12.5) - 2(12.5)^2 = 625 - 31$$

Slope at max yield = 0

Tangent equation: $Y - 312.5 = 0(t - 12.5) \Rightarrow Y = 312.5$

Problem 12: Solar Energy Production in Karamoja

Scenario: A solar power plant in Karamoja generates electricity according to $P(t) = 100\sin\left(\frac{\pi t}{12}\right) + 50$, where t is hours after sunrise and P is power in kW.

(a) Find the rate of change of power output at $t = 6$ hours.
(b) Determine when the power output is decreasing most rapidly.
(c) Find the normal to the curve at $t = 3$ hours.

Solution 12

(a) Differentiate the power function:

$$P(t) = 100\sin\left(\frac{\pi t}{12}\right) + 50$$
$$\frac{dP}{dt} = 100 \cdot \frac{\pi}{12}\cos\left(\frac{\pi t}{12}\right)$$
$$\text{At } t = 6: \quad \frac{dP}{dt} = \frac{100\pi}{12}\cos\left(\frac{\pi}{2}\right) = 0$$

(b) Power decreases most rapidly when $\frac{d^2P}{dt^2}$ is negative and $\frac{dP}{dt}$ is minimum:

Minimum occurs when $\cos\left(\frac{\pi t}{12}\right) = -1 \Rightarrow \frac{\pi t}{12} = \pi \Rightarrow t = 12$ ho

(c) At $t = 3$:

$$P(3) = 100 \sin$$
$$\frac{dP}{dt} = \frac{25\pi}{3} \text{ c}$$
$$\text{Slope of normal} = -\frac{1}{\frac{dP}{dt}}$$

Normal equation: $P - (50\sqrt{2} + 50) = -\frac{6}{25\pi\sqrt{2}}(t - 3)$

Problem 13: Fish Population Dynamics in Lake Victoria

Scenario: The Nile perch population in Lake Victoria is modeled by $P(t) = \frac{1000}{1+9e^{-0.2t}}$, where t is time in years.

(a) Find the growth rate of the population at $t = 5$ years.

(b) Determine when the population growth is maximum.

(c) Find the equation of the tangent when the growth rate is highest.

Solution 13

(a) Differentiate using quotient rule:

$$P(t) = \frac{1000}{1 + 9e^{-0.2t}}$$

$$\frac{dP}{dt} = \frac{1000 \cdot 1.8e^{-0.2t}}{(1 + 9e^{-0.2t})^2}$$

At $t = 5$: $\frac{dP}{dt} = \frac{1800e^{-1}}{(1 + 9e^{-1})^2} \approx \frac{662.2}{(1 + 3.311)^2} \approx 35.2$ fish/year

(b) Maximum growth occurs when $\frac{d^2P}{dt^2} = 0$:

$$\text{Solve } \frac{d^2P}{dt^2} = 0 \Rightarrow t = \frac{\ln(9)}{0.2} \approx 10.99 \text{ years}$$

(c) At $t = 10.99$:

$$P(10.99) = \frac{1000}{1 + 9e^{-2.198}} \approx \frac{1000}{1 + 9(0.111)} \approx 500 \text{ fish}$$

Slope = maximum growth rate

Tangent equation: $P - 500 = r_{\max}(t - 10.99)$

Problem 14: Road Construction Cost Minimization

Scenario: The cost of building a road in Kampala is given by $C(x) = x^3 - 12x^2 + 60x + 100$, where x is the road width in meters.

(a) Find the marginal cost function.

(b) Determine the road width that minimizes marginal cost.

(c) Find the equation of the normal to the cost curve at the point of minimum marginal cost.

Solution 14

(a) Marginal cost is the derivative:

$$C(x) = x^3 - 12x^2 + 60x + 100$$

$$MC(x) = 3x^2 - 24x + 60$$

(b) Minimize marginal cost:

$$\frac{d(MC)}{dx} = 6x - 24 = 0 \Rightarrow x = 4 \text{ meters}$$

$$\frac{d^2(MC)}{dx^2} = 6 > 0 \Rightarrow \text{Minimum}$$

(c) At $x = 4$:

$$C(4) = 64 - 192 + 240 + 1$$

$$MC(4) = 48 - 96 + 60 = 12$$

$$\text{Slope of normal} = -\frac{1}{12}$$

Normal equation: $C - 212 = -\frac{1}{12}(x - 4)$

Problem 15: Agricultural Water Management

Scenario: The water level in an irrigation channel in Lira is given by $h(t) = 2t^3 - 15t^2 + 24t + 3$, where t is time in hours.

(a) Find the rate of change of water level at $t = 2$ hours.

(b) Determine when the water level is falling most rapidly.

(c) Find the tangent to the curve at the point of inflection.

Solution 15

(a) Differentiate the water level function:

$$h(t) = 2t^3 - 15t^2 + 24t + 3$$

$$\frac{dh}{dt} = 6t^2 - 30t + 24$$

At $t = 2$: $\frac{dh}{dt} = 24 - 60 + 24 = -12$ cm/hour

(b) Water level falls most rapidly when $\frac{d^2h}{dt^2}$ is negative and $\frac{dh}{dt}$ is minimum:

$$\frac{d^2h}{dt^2} = 12t - 30$$

$$\text{Set } \frac{d^2h}{dt^2} = 0 \Rightarrow t = 2.5 \text{ hours}$$

(c) Point of inflection at $t = 2.5$:

$$h(2.5) = 31.25 - 93.75 +$$

$$\frac{dh}{dt} = 37.5 - 75 + 24 =$$

Tangent equation: $h - 0.5 = -13.5(t - 2.5)$

Problem 51: Coffee Yield Optimization in Mount Elgon Regi

Scenario: A coffee farmer in the Mount Elgon region models the coffee yield per hectare using the function $Y(f) = 80f - 2f^2$, where f is the amount of fertilizer applied in kg per hectare and Y is the yield in kg per hectare.

(a) Find the rate of change of yield with respect to fertilizer application when $f = 15$ kg/ha.

(b) Interpret the physical meaning of your result from part (a) in the context of coffee farming.

(c) Determine the optimal amount of fertilizer that maximizes yield and calculate the maximum yield.

Solution 51

(a) Differentiate the yield function:

$$Y(f) = 80f - 2f^2$$

$$\frac{dY}{df} = 80 - 4f$$

$$\text{At } f = 15: \quad \frac{dY}{df} = 80 - 4(15) = 80 - 60 = 20 \text{ kg/kg}$$

(b) The result $\frac{dY}{df} = 20$ kg/kg means that at a fertilizer application rate of 15 kg/ha, each additional kilogram of fertilizer applied will increase the coffee yield by approximately 20 kg per hectare. This indicates that the farmer is still in the region of increasing marginal returns.

(c) Maximum yield occurs when $\frac{dY}{df} = 0$:

$$80 - 4f = 0 \Rightarrow f = 20 \text{ kg/ha}$$

$$Y(20) = 80(20) - 2(20)^2 = 1600 - 800 = 800 \text{ kg/ha}$$

The optimal fertilizer application is 20 kg/ha, yielding 800 kg/ha of coffee.

Problem 52: Solar Energy Production in Karamoja

Scenario: A solar power plant in Karamoja generates electricity according to $P(t) = 120\sin\left(\frac{\pi t}{12}\right) + 60$, where t is hours after sunrise and P is power output in kW.

(a) Find the rate of change of power output at $t = 6$ hours.

(b) Interpret what your result from part (a) means for energy grid management.

(c) Determine when the power output is increasing most rapidly and find this maximum rate of increase.

Solution 52

(a) Differentiate the power function:

$$P(t) = 120\sin\left(\frac{\pi t}{12}\right) + 60$$

$$\frac{dP}{dt} = 120 \cdot \frac{\pi}{12}\cos\left(\frac{\pi t}{12}\right) = 10\pi\cos\left(\frac{\pi t}{12}\right)$$

$$\text{At } t = 6: \quad \frac{dP}{dt} = 10\pi\cos\left(\frac{\pi}{2}\right) = 10\pi \cdot 0 = 0 \text{ kW/hour}$$

(b) The result $\frac{dP}{dt} = 0$ kW/hour means that at noon (6 hours after sunrise), the power output is at its maximum and neither increasing nor decreasing. This is crucial for grid operators as it represents a stable point where no immediate adjustments to other power sources are needed.

(c) The power increases most rapidly when $\cos\left(\frac{\pi t}{12}\right) = 1$:

$$\frac{\pi t}{12} = 0 \Rightarrow t = 0 \text{ hours (sunrise)}$$

$$\text{Maximum rate: } \left.\frac{dP}{dt}\right|_{t=0} = 10\pi\cos(0) = 10\pi \approx 31.42 \text{ kW/hour}$$

Problem 53: Nile Perch Population Dynamics in Lake Victor

Scenario: The population of Nile perch in Lake Victoria is modeled by $P(t) = \frac{5000}{1+4e^{-0.3t}}$, where t is time in years since conservation measures began.

(a) Find the growth rate of the population at $t = 5$ years.

(b) Interpret the ecological significance of your result from part (a).

(c) Determine when the population growth is maximum and find this maximum growth rate.

Solution 53

(a) Differentiate using chain rule:

$$P(t) = 5000(1 + 4e^{-0.3t})^{-1}$$

$$\frac{dP}{dt} = -5000(1 + 4e^{-0.3t})^{-2} \cdot (-1.2e^{-0.3t}) = \frac{600}{(1+}$$

$$\text{At } t = 5: \quad \frac{dP}{dt} = \frac{6000e^{-1.5}}{(1 + 4e^{-1.5})^2} \approx \frac{6000 \times 0.223}{(1 + 4 \times 0.223)^2} \approx \frac{133}{(1 + 0.}$$

(b) The growth rate of approximately 387 fish per year at the 5-year mark indicates that the conservation measures are successful, and the population is growing at a substantial rate. This suggests the ecosystem is recovering well.

(c) Maximum growth occurs when $\frac{d^2P}{dt^2} = 0$. Solving this gives:

$$t = \frac{\ln(4)}{0.3} \approx 4.62 \text{ years}$$

$$\text{Max growth rate: } \left.\frac{dP}{dt}\right|_{t=4.62} \approx 405 \text{ fish/year}$$

Problem 54: Water Flow in the Nile River

Scenario: The water flow rate at Jinja on the Nile River is modeled by $F(t) = 1000 + 500\sin\left(\frac{\pi t}{6}\right)$, where t is months after January 1st and F is flow rate in m^3/s.

(a) Find the rate of change of flow rate at $t = 3$ months (April 1st).
(b) Interpret what this means for hydroelectric power generation at Owen Falls Dam.
(c) Determine when the flow rate is decreasing most rapidly.

Solution 54

(a) Differentiate the flow function:

$$F(t) = 1000 + 500\sin\left(\frac{\pi t}{6}\right)$$
$$\frac{dF}{dt} = 500 \cdot \frac{\pi}{6}\cos\left(\frac{\pi t}{6}\right) = \frac{250\pi}{3}\cos\left(\frac{\pi t}{6}\right)$$
$$\text{At } t = 3: \quad \frac{dF}{dt} = \frac{250\pi}{3}\cos\left(\frac{\pi}{2}\right) = 0 \text{ m}^3\text{/s/month}$$

(b) The zero rate of change indicates that around April, the flow rate is at either a maximum or minimum. For hydroelectric power generation, this represents a stable period where power output can be predicted reliably without major adjustments.
(c) The flow rate decreases most rapidly when $\cos\left(\frac{\pi t}{6}\right) = -1$:

$$\frac{\pi t}{6} = \pi \Rightarrow t = 6 \text{ months (July 1st)}$$

Problem 55: Malaria Infection Rate in Gulu District

Scenario: The number of malaria cases in Gulu district is modeled by $C(t) = 200 - 40t + 2t^2$, where t is weeks after the start of a prevention campaign and C is weekly cases.

(a) Find the rate of change of malaria cases at $t = 4$ weeks.
(b) Interpret what this means for the effectiveness of the prevention campaign.
(c) Determine when the number of cases is decreasing most rapidly.

Solution 55

(a) Differentiate the cases function:

$$C(t) = 200 - 40t + 2t^2$$
$$\frac{dC}{dt} = -40 + 4t$$
$$\text{At } t = 4: \quad \frac{dC}{dt} = -40 + 16 = -24 \text{ cases/week}$$

(b) The negative rate of -24 cases per week indicates that the prevention campaign is effective, reducing malaria cases by approximately 24 per week at the 4-week mark. This shows significant progress in controlling the disease.
(c) The number of cases decreases most rapidly when $\frac{dC}{dt}$ is most negative. Since $\frac{dC}{dt} = -40+4t$ is linear, the most negative value occurs at $t = 0$:

$$\left.\frac{dC}{dt}\right|_{t=0} = -40 \text{ cases/week}$$

Problem 56: Agricultural Price Forecasting

Scenario: The price of maize in Kampala markets is modeled by $P(t) = 50 + 10t - 0.5t^2$, where t is weeks after harvest and P is price in USD per 100kg.

(a) Find the rate of price change at $t = 4$ weeks.
(b) Interpret what this means for farmers and traders.
(c) Determine when the price reaches its maximum and what this maximum price is.

Solution 56

(a) Differentiate the price function:

$$P(t) = 50 + 10t - 0.5t^2$$
$$\frac{dP}{dt} = 10 - t$$
$$\text{At } t = 4: \quad \frac{dP}{dt} = 10 - 4 = 6 \text{ USD/week}$$

(b) The positive rate of 6 USD per week indicates that maize prices are increasing at this time. Farmers might want to delay selling to get better prices, while traders might want to buy now before prices rise further.
(c) Maximum price occurs when $\frac{dP}{dt} = 0$:

$$10 - t = 0 \Rightarrow t = 10 \text{ weeks}$$
$$P(10) = 50 + 100 - 50 = 100 \text{ USD/100kg}$$

Problem 57: Mobile Network Coverage in Rural Areas

Scenario: The signal strength of a new mobile tower in rural Uganda is given by $S(d) = 90 - 3d^2 + 0.1d^3$, where d is distance from the tower in km and S is signal strength in dB.

(a) Find the rate of change of signal strength at $d = 5$ km.
(b) Interpret what this means for network coverage planning.
(c) Determine the distance where signal strength is optimal.

Solution 57

(a) Differentiate the signal function:

$$S(d) = 90 - 3d^2 + 0.1d^3$$
$$\frac{dS}{dd} = -6d + 0.3d^2$$
$$\text{At } d = 5: \quad \frac{dS}{dd} = -30 + 7.5 = -22.5 \text{ dB/km}$$

(b) The negative rate of -22.5 dB/km indicates that signal strength decreases rapidly with distance from the tower. This suggests the need for additional towers or signal boosters to ensure adequate coverage in remote areas.

(c) Optimal signal strength occurs when $\frac{dS}{dd} = 0$:

$$-6d + 0.3d^2 = 0 \Rightarrow d(0.3d - 6) = 0 \Rightarrow d = 0 \text{ or } d = 20 \text{ km}$$

The maximum signal strength occurs at the tower ($d = 0$).

Problem 58: Tourism Revenue in National Parks

Scenario: The revenue from tourism in Bwindi Impenetrable National Park is modeled by $R(v) = 200v - 2v^2$, where v is the number of visitors per day and R is revenue in USD.

(a) Find the rate of change of revenue with respect to visitor numbers when $v = 40$.

(b) Interpret what this means for park management and conservation efforts.

(c) Determine the optimal number of visitors that maximizes revenue while minimizing environmental impact.

Solution 58

(a) Differentiate the revenue function:

$$R(v) = 200v - 2v^2$$
$$\frac{dR}{dv} = 200 - 4v$$
$$\text{At } v = 40: \quad \frac{dR}{dv} = 200 - 160 = 40 \text{ USD/visitor}$$

(b) The positive rate of 40 USD per visitor indicates that additional visitors still contribute positively to revenue. However, park managers must balance this with environmental carrying capacity to ensure sustainable tourism.

(c) Maximum revenue occurs when $\frac{dR}{dv} = 0$:

$$200 - 4v = 0 \Rightarrow v = 50 \text{ visitors/day}$$
$$R(50) = 10000 - 5000 = 5000 \text{ USD/day}$$

Problem 59: Urban Traffic Flow in Kampala

Scenario: The number of vehicles passing through a busy intersection in Kampala is modeled by $V(t) = 300 + 100t - 5t^2$, where t is hours after 7:00 AM and V is vehicles per hour.

(a) Find the rate of change of traffic flow at $t = 2$ hours (9:00 AM).

(b) Interpret what this means for traffic management and planning.

(c) Determine when traffic flow is maximum and find this maximum flow.

Solution 59

(a) Differentiate the traffic function:

$$V(t) = 300 + 100t - 5t^2$$
$$\frac{dV}{dt} = 100 - 10t$$
$$\text{At } t = 2: \quad \frac{dV}{dt} = 100 - 20 = 80 \text{ vehicles/hour}^2$$

(b) The positive rate of 80 vehicles/hour^2 indicates that traffic flow is still increasing at 9:00 AM, suggesting that peak hours are ongoing and traffic management measures should remain in place.

(c) Maximum traffic flow occurs when $\frac{dV}{dt} = 0$:

$$100 - 10t = 0 \Rightarrow t = 10 \text{ hours (5:00 PM)}$$
$$V(10) = 300 + 1000 - 500 = 800 \text{ vehicles/hour}$$

Problem 60: HIV Infection Rate Modeling

Scenario: The number of new HIV infections in a Ugandan community is modeled by $I(t) = 500 - 30t + 0.5t^2$, where t is months after an awareness campaign began and I is monthly new infections.

(a) Find the rate of change of new infections at $t = 6$ months.

(b) Interpret what this means for the effectiveness of the awareness campaign.

(c) Determine when the number of new infections is minimized and find this minimum.

Solution 60

(a) Differentiate the infection function:

$$I(t) = 500 - 30t + 0.5t^2$$
$$\frac{dI}{dt} = -30 + t$$
$$\text{At } t = 6: \quad \frac{dI}{dt} = -30 + 6 = -24 \text{ infections/month}^2$$

(b) The negative rate of -24 infections/month^2 indicates that the awareness campaign is effective, reducing new infections at an accelerating rate. This suggests the campaign should be continued and possibly expanded.

(c) Minimum infections occur when $\frac{dI}{dt} = 0$:

$-30 + t = 0 \Rightarrow t = 30$ months

$I(30) = 500 - 900 + 450 = 50$ monthly infections

Solution 61

1. To find the maximum yield, we find the critical point of the function by setting its derivative to zero. $Y'(x) = \frac{d}{dx}(-x^2 + 20x + 50) = -2x + 20$ Set $Y'(x) = 0$: $-2x + 20 = 0 \implies 2x = 20 \implies x =$ **10** bags. The second derivative $Y''(x) = -2$, which is negative, confirms this is a maximum.
2. To find the maximum yield, substitute $x = 10$ into the original function. $Y(10) = -(10)^2 + 20(10) + 50 = -100 + 200 + 50 =$ **150** kg.
3. The farmer learns that applying more than 10 bags of fertilizer per acre will actually decrease the yield. This is a crucial finding for resource management. It helps the farmer avoid wasting money on excess fertilizer and ensures he gets the best possible return on his investment.

Question 62: Traffic Management in Kampala

The number of vehicles, N, passing a key intersection in Kampala between 6 am and 12 pm is modeled by the function $N(t) = -t^3 + 9t^2 - 15t + 100$, where t is the number of hours after 6 am.

1. Find the rate of change of the number of vehicles, $\frac{dN}{dt}$.
2. Determine the times of day when the traffic flow is at its maximum and minimum.
3. Explain the practical significance of these times for traffic police and city planners.

Solution 62

1. The rate of change is the first derivative of the function. $N'(t) = \frac{d}{dt}(-t^3 + 9t^2 - 15t + 100) =$ $\mathbf{-3t^2 + 18t - 15}$.
2. To find the times of maximum and minimum flow, we set the rate of change to zero. $-3t^2 + 18t - 15 = 0$ Divide by -3: $t^2 - 6t + 5 = 0$ Factor the quadratic: $(t - 1)(t - 5) = 0$ So, $t = 1$ hour and $t = 5$ hours. Second derivative test: $N''(t) = -6t + 18$. At $t = 1$: $N''(1) = -6(1) + 18 = 12 > 0$, so this is a minimum. At $t = 5$: $N''(5) = -6(5) + 18 = -12 < 0$, so this is a maximum. The traffic flow is minimum at **7** am and maximum at **11** am.
3. The city planners can use this information to strategically deploy traffic officers and optimize traffic light timings. Knowing the peak hours helps them to implement measures that reduce congestion and improve the overall flow of vehicles.

Question 63: Water Distribution in Soroti

The rate at which water flows out of a community tank in Soroti is modeled by the function $R(t) = 0.5t^2 - 6t + 20$, where t is the time in hours and $R(t)$ is in liters per hour. The pump starts at $t = 0$.

1. Find the time at which the rate of water flow is at its minimum.
2. What is the minimum rate of flow?
3. Explain how this information can help the community manage water rationing and storage.

Solution 63

1. To find the minimum rate, we set the first derivative to zero. $R'(t) = \frac{d}{dt}(0.5t^2 - 6t + 20) = t - 6$ Set $R'(t) = 0$: $t - 6 = 0 \implies t =$ **6** hours. The second derivative $R''(t) = 1 > 0$, confirming this is a minimum.
2. To find the minimum rate, substitute $t = 6$ into the original function. $R(6) = 0.5(6)^2 - 6(6) + 20 = 0.5(36) - 36 + 20 = 18 - 36 + 20 =$ **2** liters/hour.
3. The community learns that the rate of water outflow slows down until it reaches a minimum at 6 hours. This is useful for water rationing and planning. The community can anticipate periods of low water availability and can schedule activities that require more water for other times.

Question 64: Tourism Revenue in Murchison Falls National

A tourism company in Murchison Falls National Park has a revenue function given by $R(p) = -0.5p^2 + 100p - 2000$, where R is the revenue in thousands of shillings and p is the price of a game drive ticket.

1. Find the ticket price that will maximize the

company's revenue.
2. What is the maximum revenue the company can expect?
3. Discuss the implications of this result on the company's pricing strategy.

Solution 64

1. To find the price that maximizes revenue, we find the critical point by setting the derivative to zero. $R'(p) = \frac{d}{dp}(-0.5p^2 + 100p - 2000) = -p + 100$ Set $R'(p) = 0$: $-p + 100 = 0 \implies p = \mathbf{100}$ thousand shillings. The second derivative $R''(p) = -1 < 0$, confirming this is a maximum.
2. Substitute $p = 100$ into the original function. $R(100) = -0.5(100)^2 + 100(100) - 2000 = -5000 + 10000 - 2000 = \mathbf{3000}$ thousand shillings, or 3,000,000 shillings.
3. The company learns that setting the ticket price at 100,000 shillings per game drive will yield the highest revenue. Charging more or less than this price will result in lower revenue. This helps them set an optimal price point, balancing the number of tourists and the revenue per ticket, to ensure maximum profitability.

Question 65: Public Health and Disease Spread

A public health expert is tracking the number of infected people, I, in a community over time, t (in weeks). The model is given by $I(t) = \frac{100t}{t^2+9}$.

1. Find the rate of change of the number of infected people with respect to time, $\frac{dI}{dt}$.
2. Determine the time at which the number of infected people is at its peak.
3. What does this result imply for public health interventions such as vaccination campaigns or lockdowns?

Solution 65

1. We use the quotient rule for $\frac{dI}{dt}$. Let $u = 100t$ and $v = t^2 + 9$. $u' = 100$ and $v' = 2t$. $\frac{dI}{dt} = \frac{vu' - uv'}{v^2} = \frac{(t^2+9)(100) - (100t)(2t)}{(t^2+9)^2} = \frac{100t^2 + 900 - 200t^2}{(t^2+9)^2} = \frac{\mathbf{900 - 100t^2}}{\mathbf{(t^2+9)^2}}$.
2. The number of infected people is at its peak when $\frac{dI}{dt} = 0$. $900 - 100t^2 = 0 \implies 100t^2 = 900 \implies t^2 = 9 \implies t = \mathbf{3}$ weeks.
3. The result indicates that the peak of the disease outbreak is expected to occur after 3 weeks. This is vital for public health authorities. Knowing the time of the peak allows for the strategic timing of interventions. Vaccination drives, public health announcements, or strict lockdowns can be implemented just before the 3-week mark to prevent the number of cases from reaching a catastrophic level.

Question 66: Agricultural Supply Chain Optimization

A produce company in Wakiso needs to minimize the cost of transporting vegetables. The cost function is given by $C(x) = \frac{x^2}{20} - 100x + 10000$, where x is the distance in kilometers and C is the cost in shillings.

1. Find the distance at which the transportation cost is at its minimum.
2. What is the minimum transportation cost?
3. How can the company use this information to plan its delivery routes?

Solution 66

1. To find the minimum cost, we find the critical point by setting the derivative to zero. $C'(x) = \frac{d}{dx}(\frac{x^2}{20} - 100x + 10000) = \frac{2x}{20} - 100 = 0.1x - 100$ Set $C'(x) = 0$: $0.1x - 100 = 0 \implies 0.1x = 100 \implies x = \mathbf{1000}$ km. The second derivative $C''(x) = 0.1 > 0$, confirming this is a minimum.
2. To find the minimum cost, substitute $x = 1000$ into the original function. $C(1000) = \frac{(1000)^2}{20} - 100(1000) + 10000 = 50000 - 100000 + 10000 = \mathbf{-40000}$ shillings.
3. The company can use the principle of optimization to find the most cost-effective routes for their deliveries. By setting up an accurate cost function, they can use differentiation to find the ideal locations for their distribution centers and the most profitable routes, saving them significant operational costs.

Question 67: Velocity and Acceleration of a Bodaboda in Ka

A bodaboda rider's position from a starting point is given by the function $s(t) = t^3 - 6t^2 + 5t$, where s is the displacement in kilometers and t is the time in hours.

1. Find the velocity and acceleration functions of the bodaboda.
2. Find the time(s) when the bodaboda is momentarily at rest.

3. Determine the bodaboda's velocity when its acceleration is zero.

Solution 67

1. The velocity is the first derivative of displacement, and acceleration is the second derivative. Velocity, $v(t) = s'(t) = \frac{d}{dt}(t^3 - 6t^2 + 5t) = \mathbf{3t^2 - 12t + 5}$. Acceleration, $a(t) = v'(t) = \frac{d}{dt}(3t^2 - 12t + 5) = \mathbf{6t - 12}$.
2. The bodaboda is momentarily at rest when its velocity is zero. $v(t) = 3t^2 - 12t + 5 = 0$. Using the quadratic formula: $t = \frac{12 \pm \sqrt{(-12)^2 - 4(3)(5)}}{2(3)} = \frac{12 \pm \sqrt{84}}{6} \approx \mathbf{0.47}$ hours and **3.53** hours.
3. The acceleration is zero when $a(t) = 0$. $6t - 12 = 0 \implies t = 2$ hours. To find the velocity at this time, substitute $t = 2$ into the velocity function. $v(2) = 3(2)^2 - 12(2) + 5 = 12 - 24 + 5 = \mathbf{-7}$ km/hour.

Question 68: Land Surveying in a Hilly Region

A surveyor in Kabale is mapping a section of hilly terrain. The height of the terrain, in meters, is given by the function $h(x) = 5\sin(\frac{\pi}{10}x) + 5$, where x is the horizontal distance in meters.

1. Find the slope of the terrain at any horizontal distance x.
2. Find the horizontal distance(s) at which the slope is zero.
3. Explain the significance of the points where the slope is zero in the context of land surveying and construction.

Solution 68

1. The slope of the terrain is the first derivative of the height function. $h'(x) = \frac{d}{dx}(5\sin(\frac{\pi}{10}x) + 5) = 5\cos(\frac{\pi}{10}x) \cdot \frac{\pi}{10} = \frac{\pi}{2}\cos(\frac{\pi}{\mathbf{10}}\mathbf{x})$.
2. The slope is zero when $h'(x) = 0$. $\cos(\frac{\pi}{10}x) = 0$. This occurs when $\frac{\pi}{10}x = \frac{\pi}{2} + n\pi$. $x = 5 + 10n$. For $n = 0, 1, 2, \ldots$, we get $x = \mathbf{5}, \mathbf{15}, \mathbf{25}, \ldots$ meters.
3. The points where the slope is zero correspond to the peaks and troughs of the terrain. A zero slope indicates a flat horizontal surface at that specific point. For a surveyor, this is crucial information for planning construction, road layout, and drainage systems, as these are the highest and lowest points.

Question 69: Maximizing a Farmer's Income in Mukono

A farmer in Mukono sells a specific type of vegetable. The income function is given by $I(x) = (100 - x)x$, where x is the price per kilogram in shillings. The farmer wants to maximize the income.

1. Find the income function in the form $ax^2 + bx + c$.
2. Find the price per kilogram that will give the farmer the maximum income.
3. What is the maximum income the farmer can expect?

Solution 69

1. Expand the income function: $I(x) = (100 - x)x = \mathbf{100x - x^2}$.
2. To maximize income, we set the first derivative to zero. $I'(x) = \frac{d}{dx}(100x - x^2) = 100 - 2x$ Set $I'(x) = 0$: $100 - 2x = 0 \implies 2x = 100 \implies x = \mathbf{50}$ shillings/kg.
3. To find the maximum income, substitute $x = 50$ into the income function. $I(50) = 100(50) - (50)^2 = 5000 - 2500 = \mathbf{2500}$ shillings.

Question 70: Manufacturing Cost in Jinja

A factory in Jinja manufactures water tanks. The cost, in shillings, of producing x tanks is given by $C(x) = 0.01x^3 - 0.5x^2 + 10x + 500$.

1. Find the marginal cost function.
2. Find the number of tanks that should be produced to minimize the marginal cost.
3. What does this point represent for the factory's production efficiency?

Solution 70

1. The marginal cost is the rate of change of the total cost, which is the first derivative. $C'(x) = \frac{d}{dx}(0.01x^3 - 0.5x^2 + 10x + 500) = \mathbf{0.03x^2 - x + 10}$.
2. To minimize the marginal cost, we set the second derivative of the original cost function to zero (or the first derivative of the marginal cost function). $C''(x) = \frac{d}{dx}(0.03x^2 - x + 10) = 0.06x - 1$ Set $C''(x) = 0$: $0.06x - 1 = 0 \implies 0.06x = 1 \implies x = \frac{1}{0.06} \approx \mathbf{16.67}$. Since the number of tanks must be an integer, the optimal number is 17 tanks.
3. The point of minimum marginal cost represents

the level of production where the cost to produce one additional tank is at its lowest. For the factory, this is the point of maximum production efficiency. By producing around 17 tanks, the factory is operating at a scale where it is most cost-effective.

Question 71: Agriculture – Maize Yield Optimization

A farmer in Masaka finds that the maize yield per acre (in tons) depends on fertilizer x (kg) as:

$$Y(x) = -0.1x^2 + 8x + 20.$$

(a) Find the level of fertilizer that maximizes yield.
(b) Determine the maximum yield.
(c) Explain the practical significance for resource use.

Solution:

$$Y'(x) = -0.2x + 8.$$

Set $Y'(x) = 0 \implies -0.2x + 8 = 0 \implies x = 40$ kg.

At $x = 40$: $Y(40) = -0.1(1600) + 320 + 20 = 180$ tons.

Interpretation: Optimal fertilizer use = 40 kg/acre, maximizing yield at 180 tons. Beyond this, more fertilizer reduces yield and wastes money.

Question 72: Physics – Vehicle Braking Distance

A vehicle's braking distance (in meters) as a function of speed v (m/s) is

$$D(v) = 0.05v^2 + 0.3v.$$

(a) Find the rate of change of braking distance at $v = 20$ m/s.
(b) Determine whether braking distance increases faster at high speeds.
(c) Explain road safety significance in Uganda.

Solution:

$$D'(v) = 0.1v + 0.3.$$

At $v = 20$: $D'(20) = 0.1(20) + 0.3 = 2.3$. Braking distance increases by 2.3 m per m/s.

Since $D'(v)$ grows with v, braking distance increases faster at high speeds.

Interpretation: High speeds in Uganda's highways (e.g. Kampala–Masaka road) greatly increase risk of accidents.

Question 73: Biology – Fish Population in Lake Victoria

Fish population is modeled by

$$P(t) = 2000te^{-0.2t}, \quad t \text{ in years.}$$

(a) Find the time when fish population is maximum.
(b) Compute that maximum population.
(c) Explain the ecological importance.

Solution:

$$P'(t) = 2000e^{-0.2t} - 400te^{-0.2t} = e^{-0.2t}(2000 - 400t).$$

Set $P'(t) = 0 \implies 2000 - 400t = 0 \implies t = 5$ years.

$P(5) = 2000(5)e^{-1} \approx 3679$ fish.

Interpretation: Fishing should be controlled to allow populations to peak around year 5 for sustainability.

Question 74: Environmental Science – Pollution Levels

The concentration of pollutants in River Katonga is modeled by

$$C(t) = 50 + 10\sin\left(\frac{\pi t}{12}\right), \quad t \text{ in hours.}$$

(a) Find the maximum and minimum concentrations.
(b) Find the rate of change of concentration at $t = 3$ hours.
(c) Explain the significance for environmental monitoring.

Solution: Maximum $= 50 + 10 = 60$, Minimum $= 50 - 10 = 40$.

$$C'(t) = 10 \cdot \frac{\pi}{12}\cos\left(\frac{\pi t}{12}\right) = \frac{5\pi}{6}\cos\left(\frac{\pi t}{12}\right).$$

At $t = 3$: $C'(3) = \frac{5\pi}{6}\cos(\pi/4) = \frac{5\pi}{6} \cdot \frac{\sqrt{2}}{2} \approx 1.85$.

Interpretation: Pollution is rising fastest at 3 hours. This helps in planning waste control times.

Question 75: Physics – Motion of a Cyclist

A cyclist's displacement is given by

$$s(t) = t^3 - 6t^2 + 9t, \quad t \text{ in seconds.}$$

(a) Find velocity and acceleration functions.
(b) Determine turning points of displacement.
(c) Explain meaning for real motion.

Solution: $v(t) = s'(t) = 3t^2 - 12t + 9$, $a(t) = v'(t) = 6t - 12$.

Turning points: $v(t) = 0 \implies 3t^2 - 12t + 9 = 0 \implies t = 1, 3$.

Interpretation: Cyclist changes motion direction at $t = 1, 3$ s. Differentiation predicts position, speed, and turning points.

Question 76: Biology – Bacterial Growth

A bacterial colony in a lab at Makerere grows as

$$N(t) = 1000e^{0.4t}, \quad t \text{ in hours.}$$

(a) Find rate of growth at $t = 2$ hours.
(b) Determine doubling time.
(c) Explain medical significance.

Solution:

$$N'(t) = 400e^{0.4t}.$$

At $t = 2$: $N'(2) = 400e^{0.8} \approx 890$.

Doubling time: $2 = e^{0.4t} \implies t = \ln 2/0.4 \approx 1.73$ h.

Interpretation: Bacteria double every 1.7h, guiding antibiotic dosage frequency.

Question 77: Environmental Science – Tree Plantation

Height of eucalyptus tree (m) after t years is

$$H(t) = 5\ln(t + 1).$$

(a) Find rate of growth at $t = 4$ years.
(b) Find long-term growth trend.
(c) Interpret for forestry.

Solution:

$$H'(t) = \frac{5}{t+1}.$$

At $t = 4$: $H'(4) = 1$ m/year.

As $t \to \infty$, $H'(t) \to 0$ (growth slows).

Interpretation: Tree grows fastest when young; harvesting should consider growth rate decline.

Question 78: Projectile Motion

A stone is thrown vertically with displacement

$$s(t) = 40t - 5t^2, \quad t \text{ in seconds.}$$

(a) Find maximum height.
(b) Find time when it hits the ground.
(c) Explain application in sports.

Solution: $v(t) = 40 - 10t$. Set $v = 0 \implies t = 4$. Max height: $s(4) = 160 - 80 = 80$ m.

Hits ground: $s(t) = 0 \implies 40t - 5t^2 = 0 \implies t = 0$ or 8 s.

Interpretation: Differentiation predicts motion in javelin/high jump sports.

Question 79: Physics – Water Tank Leak

Water volume in tank decreases as

$$V(t) = 1000 - 20t^2, \quad t \text{ in minutes.}$$

(a) Find rate of leakage at $t = 10$.
(b) Find time when tank empties.
(c) Explain water management impact.

Solution: $V'(t) = -40t$. At $t = 10$: $V'(10) = -400$ liters/min.

Empty when $1000 - 20t^2 = 0 \implies t = \sqrt{50} \approx 7.07$ min.

Interpretation: Leakage rate increases over time. Communities must repair early.

Question 80: Environmental Science – Solar Energy Output

Power from a solar panel in Mbarara is modeled by

$$P(t) = 100\sin\left(\frac{\pi t}{12}\right), \quad 0 \le t \le 12, \ t \text{ in hours.}$$

(a) Find time of maximum power output.
(b) Determine maximum power.
(c) Explain renewable energy significance.

Solution:

$$P'(t) = 100 \cdot \frac{\pi}{12}\cos\left(\frac{\pi t}{12}\right).$$

$P'(t) = 0 \implies \cos(\pi t/12) = 0 \implies t = 6$ h.

$P(6) = 100\sin(\pi/2) = 100$ units.

Interpretation: Maximum output at midday; solar farms should align peak demand to midday production.

Question 81: Coffee Bean Optimization in Mbale District

Context: A coffee farmer in Mbale District has 1200 meters of fencing to create rectangular coffee plantation plots. The farm borders the Sipi Falls conservation area on one side, so fencing is not needed along that boundary.

Part (a)

If the width of the rectangular plot perpendicular to the conservation boundary is x meters, express the area A of the plot as a function of x. Find the dimensions that maximize the plantation area using differentiation.

Part (b)

The farmer discovers that coffee yield per square meter follows the function $Y(x) = 2.5x - 0.001x^2$ kg/m² where x is the plot width. Determine the optimal width that maximizes total coffee production and interpret the significance of the critical point.

Part (c)

If the actual measured width differs from the optimal by a small amount $\delta x = 2$ meters, use differentiation to approximate the change in total coffee yield.

Solution

Part (a): Let width = x meters, length = l meters. Since one side borders conservation area: $2x + l = 1200$ Therefore: $l = 1200 - 2x$

Area function:

$$A(x) = x(1200 - 2x) = 1200x - 2x^2 \tag{6.1}$$

$$\frac{dA}{dx} = 1200 - 4x \tag{6.2}$$

Setting $\frac{dA}{dx} = 0$: $1200 - 4x = 0 \Rightarrow x = 300$ meters

$\frac{d^2A}{dx^2} = -4 < 0$ (confirms maximum)

Optimal dimensions: width = 300m, length = 600m, maximum area = 180,000 m²

Part (b): Total production: $P(x) = A(x) \times Y(x) = (1200x - 2x^2)(2.5x - 0.001x^2)$

$$P(x) = 3000x^2 - 1.2x^3 - 5x^3 + 0.002x^4 \tag{6.3}$$

$$= 3000x^2 - 6.2x^3 + 0.002x^4 \tag{6.4}$$

$$\frac{dP}{dx} = 6000x - 18.6x^2 + 0.008x^3 \tag{6.5}$$

Setting equal to zero and solving numerically gives optimal width ≈ 258 meters.

Part (c): At optimal width $x_0 = 258$:

$$\Delta P \approx \left.\frac{dP}{dx}\right|_{x=258} \times \delta x = \left.\frac{dP}{dx}\right|_{x=258} \times 2$$

Question 82: Malaria Vector Control in Kampala Wetlands

Context: The Ministry of Health is optimizing mosquito control in Kampala's Nakivubo wetland. The population of Anopheles mosquitoes follows $N(t) = 5000e^{-0.1t}\sin(0.5t) + 2000$ where t is days after intervention.

Part (a)

Find when the mosquito population is decreasing most rapidly in the first 20 days. Interpret the biological significance of this maximum rate of decrease.

Part (b)

Determine all critical points of the population function and classify them as maxima, minima, or points of inflection using the second derivative test.

Part (c)

If monitoring equipment has a 2% error in time measurement, estimate the error in population count at day 10 using linear approximation.

Solution

Part (a):

$$N(t) = 5000e^{-0.1t}\sin(0.5t) + 2000$$

Using product rule:

$$N'(t) = 5000[-0.1e^{-0.1t}\sin(0.5t) + e^{-0.1t} \cdot 0.5\cos(0.5t)] \tag{6.6}$$

$$= 5000e^{-0.1t}[-0.1\sin(0.5t) + 0.5\cos(0.5t)] \tag{6.7}$$

To find maximum decrease, minimize $N'(t)$ by finding where $N''(t) = 0$.

Maximum decrease occurs around day 5.2, indicating **optimal timing for follow-up intervention**.

Part (b): Critical points occur where $N'(t) = 0$:

$$-0.1\sin(0.5t) + 0.5\cos(0.5t) = 0 \tag{6.8}$$

$$\tan(0.5t) = 5 \tag{6.9}$$

$$t = 2\arctan(5) + \frac{2n\pi}{0.5} \tag{6.10}$$

First few critical points: $t \approx 2.75, 15.32, 27.89$ days

Part (c): At $t = 10$:

$$N(10) = 5000e^{-1}\sin(5) + 2000 \tag{6.11}$$

$$N'(10) = 5000e^{-1}[-0.1\sin(5) + 0.5\cos(5)] \tag{6.12}$$

Error in time: $\Delta t = 0.02 \times 10 = 0.2$ days

Error in population: $\Delta N \approx N'(10) \times 0.2$

Question 83: Nile Perch Population Dynamics in La

Context: Marine biologists studying Nile perch in Lake Victoria model the fish population using $P(t) = \frac{50000t}{t^2+25}$ where t is years since 2020.

Part (a)

Find when the Nile perch population reaches its maximum and determine this maximum population. Discuss the ecological implications of this peak.

Part (b)

Calculate the rate of population change in 2025 and 2030. Compare these rates and explain what they indicate about population trends.

Part (c)

If there's uncertainty in the year measurement of ±0.5 years, estimate the error in population count at the peak using differential approximation.

Solution

Part (a):

$$P(t) = \frac{50000t}{t^2 + 25}$$

Using quotient rule:

$$P'(t) = \frac{50000(t^2 + 25) - 50000t(2t)}{(t^2 + 25)^2} \tag{6.13}$$

$$= \frac{50000(25 - t^2)}{(t^2 + 25)^2} \tag{6.14}$$

Setting $P'(t) = 0$: $25 - t^2 = 0 \Rightarrow t = 5$ years (2025)

$$P(5) = \frac{50000 \times 5}{25 + 25} = \frac{250000}{50} = 5000 \text{ fish}$$

Maximum population is 5000 Nile perch in 2025.

Part (b):

$$P'(5) = 0 \text{ (at maximum)} \tag{6.15}$$

$$P'(10) = \frac{50000(25 - 100)}{(100 + 25)^2} = \frac{50000(-75)}{15625} = -240 \text{ fish/year} \tag{6.16}$$

In 2030, population declining at 240 fish per year.

Part (c): At peak ($t = 5$): $P'(5) = 0$

Since we're at the maximum, the linear approximation gives:

$$\Delta P \approx P'(5) \times 0.5 = 0 \times 0.5 = 0$$

Question 84: Hydroelectric Power Optimization at Owen Fal

Context: The power output at Owen Falls Dam (now Nalubaale) is modeled by $P(h) = 120h^2 - 0.8h^3$ MW, where h is the water level in meters above minimum operating level.

Part (a)

Determine the optimal water level that maximizes power generation. Calculate the maximum power output and its practical significance for Uganda's electricity supply.

Part (b)

Find the water level at which power generation efficiency (rate of power increase per unit height) is maximum.

Part (c)

During seasonal variations, if the water level changes by small amounts, use linear approximation to estimate power changes when the level increases by 0.3m from the optimal level.

Solution

Part (a):

$$P(h) = 120h^2 - 0.8h^3$$

$$P'(h) = 240h - 2.4h^2 = h(240 - 2.4h) \tag{6.17}$$

Setting $P'(h) = 0$: $h = 0$ or $h = 100$

Since $h = 0$ gives minimum, optimal level is $h = 100$ meters.

$$P(100) = 120(100)^2 - 0.8(100)^3 = 1,200,000 - 800,000 = 400,000 \text{ MW}$$

Maximum power output: 400,000 MW

Part (b): Efficiency is maximum where $P''(h) = 0$:

$$P''(h) = 240 - 4.8h \tag{6.18}$$

Setting $P''(h) = 0$: $h = 50$ meters

Part (c): At optimal level $h = 100$:

$$P'(100) = 0 \text{ (since it's the maximum)}$$

$$\Delta P \approx P'(100) \times 0.3 = 0 \times 0.3 = 0 \text{ MW}$$

The linear approximation suggests no change in power, which is expected since we're at the maximum.

Question 85: Matoke Plantation Irrigation Optimization in Masaka

Context: A matoke farmer in Masaka uses a sprinkler system where water coverage follows the function $W(r) = 800 - 2r^2$ liters per hour, where r is the radius in meters from the sprinkler.

Part (a)

Find the radius at which water application rate is optimal for matoke growth, given that matoke requires exactly 400 liters per hour per unit area for optimal yield.

Part (b)

Determine the total area that receives adequate water ($\geq$ 400 L/h) and calculate the rate at which this area changes with respect to sprinkler radius.

Part (c)

If the sprinkler position has an error of ± 0.1m in radius measurement, estimate the error in water application rate using differential approximation.

Solution

Part (a): Setting $W(r) = 400$:

$$800 - 2r^2 = 400 \tag{6.19}$$

$$2r^2 = 400 \tag{6.20}$$

$$r^2 = 200 \tag{6.21}$$

$$r = \sqrt{200} = 10\sqrt{2} \approx 14.14 \text{ meters} \tag{6.22}$$

Part (b): Adequate water coverage: $W(r) \geq 400$

$$800 - 2r^2 \geq 400 \tag{6.23}$$

$$r^2 \leq 200 \tag{6.24}$$

$$r \leq 10\sqrt{2} \tag{6.25}$$

Area with adequate water: $A = \pi(10\sqrt{2})^2 = 200\pi$ m²

$$\frac{dA}{dr} = 2\pi r$$

Rate of change depends on current radius.

Part (c):

$$W'(r) = -4r$$

At $r = 14.14$: $W'(14.14) = -4(14.14) = -56.56$ L/h per meter

Error estimate: $\Delta W \approx W'(14.14) \times 0.1 = -5.66$ L/h

Question 86: Boda Boda Fuel Efficiency in Kampala Traffic

Context: A boda boda operator's fuel consumption rate in Kampala traffic is modeled by $F(v) = \frac{v^3}{450} + \frac{360}{v}$ liters per 100km, where v is speed in km/h.

Part (a)

Find the speed that minimizes fuel consumption. Calculate the minimum fuel consumption rate and its economic significance for the operator.

Part (b)

Analyze how fuel consumption changes around the optimal speed by examining the second derivative.

Part (c)

If speedometer accuracy is ± 2 km/h, use linear approximation to estimate the error in fuel consumption calculation at the optimal speed.

Solution

Part (a):

$$F(v) = \frac{v^3}{450} + \frac{360}{v}$$

$$F'(v) = \frac{3v^2}{450} - \frac{360}{v^2} = \frac{v^2}{150} - \frac{360}{v^2} \tag{6.26}$$

Setting $F'(v) = 0$:

$$\frac{v^2}{150} = \frac{360}{v^2} \tag{6.27}$$

$$\frac{v^4}{150} = 360 \tag{6.28}$$

$$v^4 = 54000 \tag{6.29}$$

$$v = \sqrt[4]{54000} \approx 15.4 \text{ km/h} \tag{6.30}$$

$$F(15.4) = \frac{(15.4)^3}{450} + \frac{360}{15.4} \approx 8.2 + 23.4 = 31.6 \text{ L/100km}$$

Part (b):

$$F''(v) = \frac{2v}{150} + \frac{720}{v^3}$$

At $v = 15.4$: $F''(15.4) > 0$, confirming minimum.

Part (c):

$$F'(15.4) = 0 \text{ (at minimum)}$$

Error estimate: $\Delta F \approx F'(15.4) \times 2 = 0$ L/100km

Question 87: Tea Processing Temperature Control in Mubende

Context: A tea processing facility in Mubende controls fermentation temperature using $T(t) = 25 + 15e^{-0.2t}\cos(0.8t)$ °C, where t is time in hours.

Part (a)

Find when the temperature is changing most rapidly (maximum absolute rate of change) in the first 8 hours. Determine the significance of this point for tea quality control.

Part (b)

Identify all local maxima and minima of the temperature function and classify them using calculus tests.

Part (c)

If timing accuracy is ±0.1 hours, estimate the temperature measurement error at $t = 2$ hours using linear approximation.

6.11.1 Solution

Part (a):

$$T(t) = 25 + 15e^{-0.2t}\cos(0.8t)$$

Using product rule:

$$T'(t) = 15[-0.2e^{-0.2t}\cos(0.8t) - 0.8e^{-0.2t}\sin(0.8t)] \tag{6.31}$$

$$= -15e^{-0.2t}[0.2\cos(0.8t) + 0.8\sin(0.8t)] \tag{6.32}$$

To find maximum $|T'(t)|$, solve $T''(t) = 0$ or examine critical points of $T'(t)$.

$T'(t) = 0$ when $0.2\cos(0.8t) + 0.8\sin(0.8t) = 0$

$$\tan(0.8t) = -0.25$$

Part (b): Critical points occur where $T'(t) = 0$:

$$\tan(0.8t) = -0.25 \tag{6.33}$$

$$0.8t = \arctan(-0.25) + n\pi \tag{6.34}$$

$$t = \frac{\arctan(-0.25) + n\pi}{0.8} \tag{6.35}$$

Use second derivative test to classify each critical point.

Part (c): At $t = 2$:

$$T'(2) = -15e^{-0.4}[0.2\cos(1.6) + 0.8\sin(1.6)]$$

Error estimate: $\Delta T \approx T'(2) \times 0.1$

Question 88: Groundwater Management in Nakasongola Dis

Context: Groundwater level in Nakasongola follows $H(t) = 45 + 8\sin\left(\frac{\pi t}{6}\right) - 0.1t$ meters below surface, where t is months from January.

Part (a)

Determine when groundwater levels are falling most rapidly and rising most rapidly. Interpret these findings for water resource management.

Part (b)

Find the month when groundwater reaches its annual minimum and maximum levels.

Part (c)

If measurement timing has an uncertainty of ±0.5 months, estimate the error in groundwater level prediction at month 6 using differential methods.

6.11.2 Solution

Part (a):

$$H(t) = 45 + 8\sin\left(\frac{\pi t}{6}\right) - 0.1t$$

$$H'(t) = 8 \cdot \frac{\pi}{6}\cos\left(\frac{\pi t}{6}\right) - 0.1 \tag{6.36}$$

$$= \frac{4\pi}{3}\cos\left(\frac{\pi t}{6}\right) - 0.1 \tag{6.37}$$

Most rapid fall: minimize $H'(t)$ when $\cos\left(\frac{\pi t}{6}\right) = -1$: $t = 6, 18, \ldots$

Most rapid rise: maximize $H'(t)$ when $\cos\left(\frac{\pi t}{6}\right) = 1$: $t = 0, 12, \ldots$

Part (b): Critical points where $H'(t) = 0$:

$$\frac{4\pi}{3}\cos\left(\frac{\pi t}{6}\right) = 0.1 \tag{6.38}$$

$$\cos\left(\frac{\pi t}{6}\right) = \frac{0.3}{4\pi} \approx 0.0239 \tag{6.39}$$

Part (c): At $t = 6$:

$$H'(6) = \frac{4\pi}{3}\cos(\pi) - 0.1 = -\frac{4\pi}{3} - 0.1$$

Error estimate: $\Delta H \approx H'(6) \times 0.5$

Question 89: Solar Panel Efficiency in Arua District

Context: Solar panel efficiency in Arua varies with sun angle according to $E(\theta) = 85\cos(\theta) - 15\cos^2(\theta)$ percent, where θ is the angle from vertical in radians.

Part (a)

Find the optimal sun angle that maximizes solar panel efficiency. Calculate the maximum efficiency and its practical implications for solar installation.

Part (b)

Determine the range of angles where efficiency exceeds 60% and analyze the rate of efficiency change at the boundaries.

Part (c)

If angle measurement has an uncertainty of ±0.05 radians, estimate the efficiency measurement error at the optimal angle using linear approximation.

Solution

Part (a):

$$E(\theta) = 85\cos(\theta) - 15\cos^2(\theta)$$

Using chain rule:

$$E'(\theta) = -85\sin(\theta) - 15 \cdot 2\cos(\theta)(-\sin(\theta)) \tag{6.40}$$

$$= -85\sin(\theta) + 30\cos(\theta)\sin(\theta) \tag{6.41}$$

$$= \sin(\theta)(-85 + 30\cos(\theta)) \tag{6.42}$$

Setting $E'(\theta) = 0$: Either $\sin(\theta) = 0$ or $-85 + 30\cos(\theta) = 0$

From second equation: $\cos(\theta) = \frac{85}{30} = \frac{17}{6} > 1$ (impossible)

Critical points are where $\sin(\theta) = 0$: $\theta = 0, \pi, 2\pi, \ldots$

At $\theta = 0$: $E(0) = 85(1) - 15(1) = 70\%$

Maximum efficiency is 70% at $\theta = 0$ (vertical position)

Part (b): $E(\theta) \geq 60$:

$$85\cos(\theta) - 15\cos^2(\theta) \geq 60$$

Let $u = \cos(\theta)$:

$$-15u^2 + 85u - 60 \geq 0$$

$$15u^2 - 85u + 60 \leq 0$$

Solving quadratic: $u = \frac{85 \pm \sqrt{85^2 - 4(15)(60)}}{30}$

Part (c): At optimal angle $\theta = 0$:

$$E'(0) = \sin(0)(-85 + 30\cos(0)) = 0$$

Error estimate: $\Delta E \approx E'(0) \times 0.05 = 0\%$

Question 90: Fish Pond Oxygen Management in Jinja

Context: Dissolved oxygen in a fish pond in Jinja follows $O(t) = 6 + 2.5\sin\left(\frac{\pi t}{12}\right) - 0.02t$ mg/L, where t is hours from sunrise.

Part (a)

Find when oxygen levels are decreasing most rapidly and when fish are most at risk. Determine the time when oxygen reaches dangerous levels (< 4 mg/L).

Part (b)

Calculate the critical points and determine whether they represent oxygen maxima or minima using the second derivative test.

6.11.3 Part (c)

If time measurement has an error of ±0.25 hours, estimate the error in oxygen level prediction at hour 18 using differential approximation.

Solution

Part (a):

$$O(t) = 6 + 2.5\sin\left(\frac{\pi t}{12}\right) - 0.02t$$

$$O'(t) = 2.5 \cdot \frac{\pi}{12}\cos\left(\frac{\pi t}{12}\right) - 0.02 \tag{6.43}$$

$$= \frac{2.5\pi}{12}\cos\left(\frac{\pi t}{12}\right) - 0.02 \tag{6.44}$$

Most rapid decrease occurs when $O'(t)$ is minimum:

$$\cos\left(\frac{\pi t}{12}\right) = -1$$

$$\frac{\pi t}{12} = \pi + 2n\pi$$

$$t = 12 + 24n \text{ hours}$$

For dangerous levels $O(t) < 4$:

$$6 + 2.5\sin\left(\frac{\pi t}{12}\right) - 0.02t < 4$$

Part (b): Critical points where $O'(t) = 0$:

$$\frac{2.5\pi}{12}\cos\left(\frac{\pi t}{12}\right) = 0.02$$

Second derivative:

$$O''(t) = -\frac{2.5\pi^2}{144}\sin\left(\frac{\pi t}{12}\right)$$

Use second derivative test at each critical point.

Part (c): At $t = 18$:

$$O'(18) = \frac{2.5\pi}{12}\cos\left(\frac{18\pi}{12}\right) - 0.02 \tag{6.45}$$

$$= \frac{2.5\pi}{12}\cos\left(\frac{3\pi}{2}\right) - 0.02 \tag{6.46}$$

Since $\cos\left(\frac{3\pi}{2}\right) = 0$:

$$O'(18) = -0.02$$

Error estimate: $\Delta O \approx O'(18) \times 0.25 = -0.02 \times 0.25 = -0.005$ mg/L

91. **Optimizing Solar Panel Angle in Mbarara**
A solar engineer models the power output $P(\theta)$ (in kW) of a panel tilted at angle θ degrees by

$$P(\theta) = 5\sin(\theta) - 0.05\,\theta^2, \quad 0 \le \theta \le 90.$$

(a) Compute $P'(\theta)$ and find the angle $\theta_{\max}$ that maximizes P.
(b) Use the second derivative to classify the turning point at $\theta_{\max}$.
(c) Explain why this optimal angle is important for solar-energy harvesting.

Solution.

$$P'(\theta) = 5\cos(\theta) - 0.10\,\theta, \quad P''(\theta) = -5\sin(\theta) - 0.10.$$

Set $P'(\theta) = 0$, i.e. $5\cos(\theta) = 0.10\,\theta$. Numerically this gives

$$\theta_{\max} \approx 60^\circ.$$

Since $P''(60^\circ) < 0$, the turning point is a local maximum.

92. **Drug Concentration in Bloodstream**
A biologist models concentration $C(t)$ (mg/L) of a drug over time t (hours):

$$C(t) = \frac{50\,t}{t^2 + 4}.$$

(a) Find $C'(t)$ and determine the time $t_{\max}$ when $C(t)$ peaks.
(b) Use $C''(t)$ to classify that critical point.
(c) Explain why knowing the peak time and value is crucial for safe dosing.

Solution.

$$C'(t) = \frac{50\,(t^2+4) - 50\,t \cdot 2\,t}{(t^2+4)^2} = \frac{50\,(4 - t^2)}{(t^2+4)^2}.$$

Setting $C'(t) = 0$ gives $t = 2$ h. Checking $C''(2) < 0$ confirms a maximum.

93. **River Discharge and Flood Risk**
Scientists measure discharge $Q(h)$ (m$\hat{3}$/s) as function of height h (m):

$$Q(h) = 2\,h^{1.5} - 0.1\,h^2, \quad h \ge 0.$$

(a) Compute $Q'(h)$ and find its critical point(s).
(b) Use $Q''(h)$ to determine max or min.
(c) Explain how the peak discharge helps predict floods.

Solution.

$$Q'(h) = 3\,h^{0.5} - 0.2\,h, \quad Q''(h) = \frac{3}{2}\,h^{-0.5} - 0.2.$$

Solve $Q'(h) = 0$: $3\sqrt{h} = 0.2\,h$ gives $h = 225$. Since $Q''(225) < 0$, it is a maximum.

94. **Error Approximation in Grain Moisture Sensor**
A sensor reads moisture $M(V) = 10\,V^{0.3}$, where V is voltage (volts). If $V = 4.00 \pm 0.02$, estimate error in M:
(a) Compute $M'(V)$.
(b) Use linear approximation $\Delta M \approx M'(4)\,\Delta V$.
(c) Interpret the result in field calibration.
Solution.

$$M'(V) = 10{\cdot}0.3\,V^{-0.7} = 3\,V^{-0.7}, \quad M'(4) = 3{\cdot}4^{-0.7} \approx 0.891.$$

Thus $\Delta M \approx 0.891 \times 0.02 = 0.0178$ (percent moisture).

95. **Boda-Boda Kinematics**
A motorcycle's displacement $s(t)$ (m) is

$$s(t) = 4\,t^2 - t^3, \quad t \geq 0.$$

(a) Find velocity $v(t) = s'(t)$ and acceleration $a(t) = s''(t)$.
(b) Determine when $v(t) = 0$ and classify the extremum of $s(t)$.
(c) Explain how this informs safe riding strategies.
Solution.

$$v(t) = 8\,t - 3\,t^2, \quad a(t) = 8 - 6\,t.$$

$v(t) = 0$ at $t = 0$ or $t = 8/3$. At $t = 8/3$, $a(8/3) < 0$, so $s(t)$ has a local maximum.

96. **Optimal Planting Density**
Yield $Y(d)$ (ton/ha) vs. density d (plants/m$\hat{2}$):

$$Y(d) = 20\,d - d^2.$$

(a) Compute $Y'(d)$ and find optimal d^*.
(b) Check it is a maximum.
(c) Explain its use in resource planning.
Solution.

$$Y'(d) = 20{-}2\,d, \quad Y'(d) = 0 \implies d = 10, \quad Y''(d) = -2 < 0 \implies \text{maximum at } d = 10.$$

97. **CO2 Uptake by Wetland Vegetation**
Uptake rate $R(C)$ (g/m$\hat{2}$/d) vs. CO2 concentration C (ppm):

$$R(C) = 50\,\ln(C) - 2\,C.$$

(a) Find $R'(C)$ and critical C^*.
(b) Classify via $R''(C)$.
(c) Discuss implications for carbon sequestration.
Solution.

$$R'(C) = \frac{50}{C} - 2, \quad R'(C) = 0 \implies C = 25, \quad R''(C) = -\frac{50}{C^2} <$$

98. **Approximate Soil pH Change**
Soil pH vs. lime x (kg/ha):

$$\mathrm{pH}(x) = 6 + 0.5\,\ln(1 + 0.1\,x).$$

If $x = 10 \pm 1$:
(a) Find $\mathrm{pH}'(x)$.
(b) Estimate $\Delta\mathrm{pH}$ for $\Delta x = 1$.
(c) Explain why this matters in trials.
Solution.

$$\mathrm{pH}'(x) = 0.5{\cdot}\frac{0.1}{1 + 0.1\,x} = \frac{0.05}{1 + 0.1\,x}, \quad \mathrm{pH}'(10) = 0.05/2 = 0.0$$

99. **Pest Population Turning Point**
Pest density $D(t)$ vs. time t (weeks):

$$D(t) = 100\,e^{0.2\,t} - 5\,t^2.$$

(a) Compute $D'(t)$ and find critical t.
(b) Use $D''(t)$ to classify turning point.
(c) Explain timing for control measures.
Solution.

$$D'(t) = 20\,e^{0.2\,t} - 10\,t, \quad D''(t) = 4\,e^{0.2\,t} - 10.$$

Solve $20e^{0.2t} = 10t$ numerically $\implies t \approx 4$; $D''(4) < 0$ so a maximum.

100. **Optimal Pumping Shaft Radius**
Extraction $W(r)$ (m$\hat{3}$/d) vs. shaft radius r (cm):

$$W(r) = \pi\,r^2\sqrt{2\,g\,(h - r)}, \quad g = 9.81, \quad h = 100.$$

(a) Rewrite $W(r)$ as $K\,r^{3/2}(h - r)^{1/2}$ and find r^* with $\frac{dW}{dr} = 0$.
(b) Use the second derivative to check it is a maximum.
(c) Discuss why r^* is key for sustainable pumping.

Solution.

$$W(r) = \pi\sqrt{2\,g}\; r^{3/2}(h-r)^{1/2}.$$

Taking $\ln W$ and differentiating:

$$\frac{3}{2r} - \frac{1}{2(h-r)} = 0 \quad \Longrightarrow \quad r = \frac{3}{4}\,h = 75.$$

Second derivative test shows a maximum at $r = 75$ cm.

101. **Hydropower Flow in Jinja**

At the edge of the Nile near Jinja, the Nalubaale Hydroelectric Power Station must adjust turbine output every hour. Engineers rely on data collected upstream—where seasonal rains and dam releases cause rapid fluctuations—to maintain grid stability. Local fishermen also coordinate their schedules around these flow changes, making accurate predictions crucial for both energy and livelihood.

$Q(t) = -2t^2+40t+200$, t in hours after midnight.

a) Find the instantaneous rate of change $\dfrac{dQ}{dt}$.
b) Determine the time t when flow is maximum.
c) Explain how the station uses this information to plan peak generation.

Solution:

$$\frac{dQ}{dt} = -4t + 40, \quad -4t + 40 = 0 \implies t = 10.$$

Peak flow occurs at 10:00 AM; scheduling maximum turbine output at that hour ensures efficiency and prevents overflow.

102. **Cholera Outbreak in Kasese**

In Kasese District, community health teams track cholera cases daily to deploy response units. A simple logistic model captures how cases grow as the bacteria spreads through water sources and informal settlements. Timely intervention hinges on knowing when new infections surge most rapidly.

$P(t) = \dfrac{5000}{1 + 20e^{-0.3t}}$, t in days since first report.

a) Compute $\dfrac{dP}{dt}$.
b) Find the day t when new cases peak.
c) Interpret this result for deploying medical teams.

Solution:

$$\frac{dP}{dt} = \frac{5000 \cdot 20 \cdot 0.3\, e^{-0.3t}}{(1 + 20e^{-0.3t})^2}, \quad 1+20e^{-0.3t} = 2 \implies t \approx 10.6.$$

Medical teams should be fully mobilized around day 11 when case growth is fastest.

103. **Air Pollution in Kampala**

Traffic congestion around Mulago and city center pushes CO_2 levels to health-critical peaks. The Kampala Capital City Authority monitors hourly data to implement restrictions and advise commuters. Understanding the fastest rate of change helps target those interventions.

$C(t) = 400 + 5\sin\left(\frac{\pi t}{12}\right)$, $t \in [0, 24]$ hours.

a) Find $\dfrac{dC}{dt}$.
b) Determine times when CO_2 increases and decreases most rapidly.
c) Explain how this informs traffic-control measures.

Solution:

$$\frac{dC}{dt} = 5 \cdot \frac{\pi}{12}\cos\left(\frac{\pi t}{12}\right).$$

Fastest increase at $t = 0, 24$ (midnight), fastest decrease at $t = 12$. Restrict heavy vehicles at those peak transition times.

104. **Bean Growth in Mukono**

Smallholder farmers in Mukono join agricultural extension sessions to optimize yield. On demonstration plots, they record daily plant height under different irrigation schedules. Differentiation of the growth curve reveals when water demand is highest and when overwatering begins to slow growth.

$H(t) = 30\ln(t+1)$, t days after planting.

a) Compute the growth rate $\dfrac{dH}{dt}$.
b) Find the rate on day 5.
c) Recommend irrigation timing based on this rate.

Solution:

$$\frac{dH}{dt} = \frac{30}{t+1}, \quad \frac{dH}{dt}\Big|_{t=5} = 5 \text{ cm/day}.$$

Irrigate heavily during the first week when growth rate exceeds 5 cm/day, then taper off.

105. **Boda-Boda Kinematics in Soroti**

In Soroti, boda-boda riders share engine performance data to improve safety. A mechanic measures displacement over time as riders accelerate away from junctions. Differentiation yields velocity and acceleration curves that inform helmet standards and road-safety guidelines.

$$s(t) = 4t^2 + 3t, \quad t \text{ in seconds}.$$

a) Find velocity $v(t)$ and acceleration $a(t)$.
b) Compute $a(2)$.
c) Discuss how acceleration impacts rider safety measures.

Solution:

$$v(t) = 8t + 3, \quad a(t) = 8, \quad a(2) = 8 \text{ m/s}^2.$$

A constant acceleration of 8 m/s^2 requires reinforced helmets and smooth braking zones.

106. **Antibiotic Decay in Mbale**

At Mbale Regional Referral Hospital, pharmacists schedule doses to maintain therapeutic drug levels. They model antibiotic concentration decay to predict when levels fall below the minimum inhibitory concentration. Linear approximation around a chosen time provides quick dosing interval estimates.

$$A(t) = 200e^{-0.4t}, \quad t \text{ in hours}.$$

a) Find $\frac{dA}{dt}$.
b) Use linearization at $t = 3$ h to estimate when $A(t) = 50$.
c) Advise on dosing intervals to maintain effective concentration.

Solution:

$$\frac{dA}{dt} = -80e^{-0.4t}, \quad A(3) \approx 60.24, \quad \frac{dA}{dt} \approx -40.1.$$

Solve $50 \approx 60.24 - 40.1\,\Delta t \implies \Delta t \approx 0.26$ h.
Dose again at approximately 3.3 h.

107. **Temperature in Mabira Forest**

Ecologists in Mabira National Forest record microclimate to study biodiversity responses. Temperature varies with canopy cover and sun angle. Differentiation of the daily temperature curve identifies when changes are minimal—ideal for wildlife surveys.

$$T(t) = 25 + 10\sin\left(\tfrac{\pi t}{12}\right) + 3\cos\left(\tfrac{\pi t}{6}\right), \quad t \in [0, 24].$$

a) Compute $\frac{dT}{dt}$.
b) Find approximate times when $\frac{dT}{dt} = 0$.
c) Plan optimal field-survey windows based on this.

Solution:

$$\frac{dT}{dt} = 10\frac{\pi}{12}\cos\left(\tfrac{\pi t}{12}\right) - 3\frac{\pi}{6}\sin\left(\tfrac{\pi t}{6}\right).$$

Numerical solution gives $t \approx 4$ h and $t \approx 16$ h. Surveys then face minimal temperature fluctuations.

Chapter 7

Integration 1

7.1 Linking integration and differentiation as reverse processes through finding functions which when different

7.1.1 Associating the limit of summation to the integral sign and evaluate a definite integral using the limits.

7.2 Applying integration to

7.2.1 Determining the area under curve bounded by the x- axis and y- axis and deduce that area is a scalar quantity,

7.2.2 Determining areas enclosed between two functions,

7.2.3 Determining displacement, velocity and acceleration,

7.2.4 Determining the volume of a solid of revolution

7.2.5 Calculating the mean value of a function.

Competency:

The learner applies integration techniques to solve problems involving areas under curves, accumulation of quantities, and other real-world applications.

7.3 Introduction to Integration

7.3.1 The Concept of Integration

Integration is one of the two fundamental operations in calculus, alongside differentiation. While differentiation answers the question "What is the rate of change?", integration answers questions like "What is the total accumulation?" or "What is the area under a curve?"

Integration

Integration is the process of finding a function whose derivative is the given function. It is the reverse operation of differentiation.

7.3.2 Historical Context

The development of integration techniques was driven by practical problems:

- Finding areas of irregular shapes
- Calculating volumes of solids
- Determining distances traveled with varying speeds
- Computing work done by variable forces

7.4 Integration as the Reverse of Differentiation

7.4.1 The Fundamental Relationship

Antiderivative

A function $F(x)$ is called an **antiderivative** or **indefinite integral** of $f(x)$ if:

$$F'(x) = f(x)$$

We write: $\int f(x)\,dx = F(x) + C$
where C is the constant of integration.

Why the Constant of Integration?

Since the derivative of a constant is zero, if $F(x)$ is an antiderivative of $f(x)$, then so is $F(x) + C$ for any constant C. This means integration produces a **family of functions**, not just a single function.

For example:

- If $\frac{d}{dx}(x^2) = 2x$, then $\int 2x\,dx = x^2 + C$
- Both x^2, $x^2 + 5$, $x^2 - 3$, etc., have the same derivative $2x$

7.4.2 Verification by Differentiation

To verify an integral, we differentiate the result:

Example 1: Verification

Verify that $\int 3x^2\,dx = x^3 + C$

Solution: Differentiate the right-hand side:

$$\frac{d}{dx}(x^3 + C) = 3x^2 + 0 = 3x^2 \checkmark$$

Since we obtain the integrand, the integration is correct.

7.4.3 Building the Connection: Guided Discovery Activity

Activity for Learners:

1. Given the derivatives, find the original functions:
   - If $f'(x) = 5$, what could $f(x)$ be?
   - If $g'(x) = 2x$, what could $g(x)$ be?
   - If $h'(x) = \cos x$, what could $h(x)$ be?
2. Discuss: Why are there multiple possible answers?
3. Formalize: This process of "reverse differentiation" is integration.

7.5 Basic Integration Rules

7.5.1 Standard Integrals

Basic Integration Formulas

$$\int k\,dx = kx + C \quad \text{(where } k \text{ is a constant)} \tag{7.1}$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (\text{for } n \neq -1) \tag{7.2}$$

$$\int \frac{1}{x}\,dx = \ln|x| + C \tag{7.3}$$

$$\int e^x\,dx = e^x + C \tag{7.4}$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (\text{for } a > 0,\ a \neq 1) \tag{7.5}$$

$$\int \sin x\,dx = -\cos x + C \tag{7.6}$$

$$\int \cos x\,dx = \sin x + C \tag{7.7}$$

$$\int \sec^2 x\,dx = \tan x + C \tag{7.8}$$

$$\int \csc^2 x\,dx = -\cot x + C \tag{7.9}$$

$$\int \sec x \tan x\,dx = \sec x + C \tag{7.10}$$

$$\int \csc x \cot x\,dx = -\csc x + C \tag{7.11}$$

7.5.2 Properties of Integration

Linearity of Integration

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx \tag{7.12}$$

$$\int k f(x)\,dx = k \int f(x)\,dx \quad \text{(where } k \text{ is a constant)} \tag{7.13}$$

7.5.3 Worked Examples

Example 2: Power Rule

Evaluate: $\int (x^4 + 3x^2 - 5x + 7)\,dx$

Solution:

$$\int (x^4 + 3x^2 - 5x + 7)\,dx = \int x^4\,dx + \int 3x^2\,dx - \int 5x\,dx + \int 7\,dx$$
$$= \frac{x^5}{5} + 3 \cdot \frac{x^3}{3} - 5 \cdot \frac{x^2}{2} + 7x + C$$
$$= \frac{x^5}{5} + x^3 - \frac{5x^2}{2} + 7x + C$$

Example 3: Trigonometric Functions

Evaluate: $\int (4\sin x - 3\cos x + 2\sec^2 x)\,dx$

Solution:

$$\int (4\sin x - 3\cos x + 2\sec^2 x)\,dx = 4\int \sin x\,dx - 3\int$$
$$= 4(-\cos x) - 3(\sin x) + 2(\tan x) + C$$
$$= -4\cos x - 3\sin x + 2\tan x + C$$

Example 4: Exponential Functions

Evaluate: $\int (3e^x + 2^x - \frac{4}{x})\,dx$

Solution:

$$\int (3e^x + 2^x - \frac{4}{x})\,dx = 3\int e^x\,dx + \int 2^x\,dx - 4\int \frac{1}{x}\,dx$$
$$= 3e^x + \frac{2^x}{\ln 2} - 4\ln|x| + C$$

7.5.4 Practice Problems: Basic Integration

Evaluate the following integrals:

1. $\int (6x^5 - 4x^3 + 2x - 1)\,dx$
2. $\int (2\sin x + 5\cos x)\,dx$
3. $\int (\sqrt{x} + \frac{1}{\sqrt{x}})\,dx$
4. $\int (e^x + 3^x)\,dx$
5. $\int (x^2 + 1)^2\,dx$ [Hint: Expand first]
6. $\int \frac{x^3 + 2x - 3}{x}\,dx$ [Hint: Simplify first]

7.6 Definite Integrals

7.6.1 The Limit of a Sum

The definite integral arises from the problem of finding the area under a curve. Consider the function $f(x)$ on the interval $[a, b]$.

Guided Discovery Activity:

1. Divide the interval $[a, b]$ into n subintervals of width $\Delta x = \frac{b-a}{n}$
2. In each subinterval, construct a rectangle with height $f(x_i)$
3. The total area is approximately: $\sum_{i=1}^{n} f(x_i)\Delta x$
4. As $n \to \infty$, this sum approaches the exact area

Definite Integral

The definite integral of $f(x)$ from a to b is:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

This represents the signed area between the curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$.

7.6.2 The Fundamental Theorem of Calculus

Fundamental Theorem of Calculus (Part 1)

If f is continuous on $[a, b]$ and F is an antiderivative of f on $[a, b]$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a) = [F(x)]_a^b$$

This theorem connects the two main concepts of calculus: differentiation and integration.

Notation

We often use the notation $[F(x)]_a^b$ or $F(x)|_a^b$ to mean $F(b) - F(a)$.

7.6.3 Properties of Definite Integrals

$$\int_a^b k \cdot f(x)\,dx = k\int_a^b f(x)\,dx \tag{7.14}$$

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \tag{7.15}$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \tag{7.16}$$

$$\int_a^a f(x)\,dx = 0 \tag{7.17}$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad \text{(for } a < c < b\text{)} \tag{7.18}$$

7.6.4 Worked Examples: Definite Integrals

Example 5: Basic Definite Integral

Evaluate: $\int_1^3 (2x+1)\,dx$

Solution: First, find the antiderivative:

$$\int (2x+1)\,dx = x^2 + x + C$$

Now apply the limits:

$$\begin{aligned}\int_1^3 (2x+1)\,dx &= [x^2+x]_1^3 \\ &= (3^2+3) - (1^2+1) \\ &= (9+3) - (1+1) \\ &= 12 - 2 \\ &= 10\end{aligned}$$

Example 6: Trigonometric Definite Integral

Evaluate: $\int_0^{\pi/2} \sin x\,dx$

Solution:

$$\begin{aligned}\int_0^{\pi/2} \sin x\,dx &= [-\cos x]_0^{\pi/2} \\ &= -\cos(\pi/2) - (-\cos 0) \\ &= -0 + 1 \\ &= 1\end{aligned}$$

Example 7: Exponential Definite Integral

Evaluate: $\int_0^1 e^x\,dx$

Solution:

$$\begin{aligned}\int_0^1 e^x\,dx &= [e^x]_0^1 \\ &= e^1 - e^0 \\ &= e - 1\end{aligned}$$

7.6.5 Practice Problems: Definite Integrals

Evaluate the following definite integrals:

1. $\int_0^2 (x^2+3x)\,dx$
2. $\int_1^4 \sqrt{x}\,dx$
3. $\int_0^{\pi} \cos x\,dx$
4. $\int_1^e \frac{1}{x}\,dx$
5. $\int_{-1}^1 (x^3+x)\,dx$
6. $\int_0^{\pi/4} \sec^2 x\,dx$

7.7 Improper Integrals

7.7.1 Infinite Intervals

Improper Integral (Type 1)

An integral with infinite limits:

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

If this limit exists and is finite, the improper integral **converges**; otherwise, it **diverges**.

7.7.2 Worked Examples: Improper Integrals

Example 8: Improper Integral

Evaluate: $\int_1^{\infty} \frac{1}{x^2}\,dx$

Solution:

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2}\,dx &= \lim_{t\to\infty} \int_1^t x^{-2}\,dx \\ &= \lim_{t\to\infty} \left[-x^{-1}\right]_1^t \\ &= \lim_{t\to\infty} \left[-\frac{1}{t} - \left(-\frac{1}{1}\right)\right] \\ &= \lim_{t\to\infty} \left[-\frac{1}{t} + 1\right] \\ &= 0 + 1 \\ &= 1\end{aligned}$$

The integral converges to 1.

Example 9: Divergent Improper Integral

Evaluate: $\int_1^{\infty} \frac{1}{x}\,dx$

Solution:

$$\begin{aligned}\int_1^{\infty} \frac{1}{x}\,dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx \\ &= \lim_{t\to\infty} [\ln x]_1^t \\ &= \lim_{t\to\infty} (\ln t - \ln 1) \\ &= \lim_{t\to\infty} \ln t \\ &= \infty\end{aligned}$$

The integral diverges.

7.8 Applications of Integration

7.9 Application 1: Area Under a Curve

7.9.1 Area Between Curve and x-axis

Area Under a Curve

The area A bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$ is:

$$A = \int_a^b |f(x)|\,dx$$

If $f(x) \geq 0$ on $[a, b]$:

$$A = \int_a^b f(x)\,dx$$

Important: Area is a Scalar Quantity

Area is always positive (a scalar quantity). If the curve is below the x-axis, the definite integral is negative, but the area is the absolute value of that integral.

When $f(x) < 0$: Area $= -\int_a^b f(x)\,dx = \int_a^b |f(x)|\,dx$

7.9.2 Worked Examples: Area Under a Curve

Example 10: Area Above x-axis

Find the area bounded by $y = x^2$, the x-axis, and the lines $x = 0$ and $x = 2$.

Solution: Since $x^2 \geq 0$ for all x:

$$\begin{aligned} A &= \int_0^2 x^2\,dx \\ &= \left[\frac{x^3}{3}\right]_0^2 \\ &= \frac{8}{3} - 0 \\ &= \frac{8}{3} \text{ square units} \end{aligned}$$

Example 11: Area Below x-axis

Find the area bounded by $y = x^2 - 4$, the x-axis, and the lines $x = 0$ and $x = 2$.

Solution: Note that $x^2 - 4 < 0$ for $0 \leq x \leq 2$. The definite integral gives:

$$\begin{aligned} \int_0^2 (x^2 - 4)\,dx &= \left[\frac{x^3}{3} - 4x\right]_0^2 \\ &= \left(\frac{8}{3} - 8\right) - 0 \\ &= \frac{8}{3} - \frac{24}{3} \\ &= -\frac{16}{3} \end{aligned}$$

Since the region is below the x-axis, the area is:

$$A = \left|-\frac{16}{3}\right| = \frac{16}{3} \text{ square units}$$

Example 12: Area with Regions Above and Below x-axis

Find the area bounded by $y = \sin x$, the x-axis, and the lines $x = 0$ and $x = 2\pi$.

Solution: We need to split the interval because $\sin x > 0$ on $(0, \pi)$ and $\sin x < 0$ on $(\pi, 2\pi)$.

Area above x-axis:

$$A_1 = \int_0^{\pi} \sin x\,dx = [-\cos x]_0^{\pi} = -\cos\pi + \cos 0 = -(-1) + 1 = 2$$

Area below x-axis (take absolute value):

$$A_2 = \left|\int_{\pi}^{2\pi} \sin x\,dx\right| = |[-\cos x]_{\pi}^{2\pi}| = |(-\cos 2\pi + \cos\pi)| = |(-1+($$

Total area:

$$A = A_1 + A_2 = 2 + 2 = 4 \text{ square units}$$

Note: If we had calculated $\int_0^{2\pi} \sin x\,dx = 0$, this would not give the area!

7.9.3 Area Between Two Curves

Area Between Two Curves

The area A between two curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$, where $f(x) \geq g(x)$, is:

$$A = \int_a^b [f(x) - g(x)]\,dx$$

Example 13: Area Between Two Curves

Find the area bounded by $y = x^2$ and $y = x$.

Solution: First, find the points of intersection:

$$x^2 = x \implies x^2 - x = 0 \implies x(x-1) = 0$$

So $x = 0$ and $x = 1$.

For $0 \le x \le 1$, we have $x \ge x^2$ (verify by testing a point, say $x = 0.5$).

Therefore:

$$\begin{aligned} A &= \int_0^1 (x - x^2)\,dx \\ &= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 \\ &= \frac{1}{2} - \frac{1}{3} \\ &= \frac{3-2}{6} \\ &= \frac{1}{6} \text{ square units} \end{aligned}$$

7.9.4 Practice Problems: Areas

1. Find the area bounded by $y = x^3$, the x-axis, $x = -1$ and $x = 1$.
2. Find the area between $y = x^2 + 1$ and $y = 5$.
3. Find the area bounded by $y = \cos x$, the x-axis, $x = 0$ and $x = \pi$.
4. Find the area enclosed between $y = x^2$ and $y = 2x + 3$.
5. Find the area bounded by $y = e^x$, the y-axis, $y = 1$ and $y = e$.

7.10 Application 2: Volumes of Solids of Revolution

7.10.1 Revolution About the x-axis

Volume by Revolution (x-axis)

When the region bounded by $y = f(x)$, the x-axis, $x = a$ and $x = b$ is rotated about the x-axis, the volume is:

$$V = \pi \int_a^b [f(x)]^2\,dx = \pi \int_a^b y^2\,dx$$

Derivation:

- Consider a thin slice at position x with thickness δx
- This forms a circular disk with radius $r = f(x)$
- Volume of disk: $\delta V = \pi r^2 \cdot \delta x = \pi[f(x)]^2 \delta x$
- Total volume: $V = \lim_{\delta x \to 0} \sum \pi[f(x)]^2 \delta x = \pi \int_a^b [f(x)]^2\,dx$

Example 14: Volume About x-axis

Find the volume generated when the region bounded by $y = \sqrt{x}$, the x-axis, $x = 0$ and $x = 4$ is rotated about the x-axis.

Solution:

$$\begin{aligned} V &= \pi \int_0^4 (\sqrt{x})^2\,dx \\ &= \pi \int_0^4 x\,dx \\ &= \pi \left[\frac{x^2}{2}\right]_0^4 \\ &= \pi \left(\frac{16}{2} - 0\right) \\ &= 8\pi \text{ cubic units} \end{aligned}$$

7.10.2 Revolution About the y-axis

Volume by Revolution (y-axis)

When the region bounded by $x = g(y)$, the y-axis, $y = c$ and $y = d$ is rotated about the y-axis, the volume is:

$$V = \pi \int_c^d [g(y)]^2\,dy = \pi \int_c^d x^2\,dy$$

Example 15: Volume About y-axis

Find the volume generated when the region bounded by $x = y^2$, the y-axis, $y = 0$ and $y = 2$ is rotated about the y-axis.

Solution:

$$\begin{aligned} V &= \pi \int_0^2 (y^2)^2\,dy \\ &= \pi \int_0^2 y^4\,dy \\ &= \pi \left[\frac{y^5}{5}\right]_0^2 \\ &= \pi \left(\frac{32}{5} - 0\right) \\ &= \frac{32\pi}{5} \text{ cubic units} \end{aligned}$$

Example 16: Volume About y-axis (Method 2)

Find the volume when the region bounded by $y = x^2$, $x = 0$, $y = 0$ and $y = 4$ is rotated about the y-axis.

Solution: Express x in terms of y: $x = \sqrt{y}$ (taking positive root)

$$\begin{aligned} V &= \pi \int_0^4 (\sqrt{y})^2 \, dy \\ &= \pi \int_0^4 y \, dy \\ &= \pi \left[\frac{y^2}{2} \right]_0^4 \\ &= \pi \left(\frac{16}{2} - 0 \right) \\ &= 8\pi \text{ cubic units} \end{aligned}$$

7.10.3 Volume Between Two Curves

Volume with Washer Method

When the region between $y = f(x)$ and $y = g(x)$ (where $f(x) \geq g(x) \geq 0$) from $x = a$ to $x = b$ is rotated about the x-axis:

$$V = \pi \int_a^b \{[f(x)]^2 - [g(x)]^2\} \, dx$$

Example 17: Washer Method

Find the volume generated when the region between $y = x$ and $y = x^2$ is rotated about the x-axis.

Solution: Points of intersection: $x = x^2 \implies x = 0$ or $x = 1$

For $0 \leq x \leq 1$: $x \geq x^2$

$$\begin{aligned} V &= \pi \int_0^1 (x^2 - x^4) \, dx \\ &= \pi \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 \\ &= \pi \left(\frac{1}{3} - \frac{1}{5} \right) \\ &= \pi \left(\frac{5-3}{15} \right) \\ &= \frac{2\pi}{15} \text{ cubic units} \end{aligned}$$

7.10.4 Practice Problems: Volumes

1. Find the volume when $y = x^3$, $x = 0$, $x = 2$, and the x-axis are rotated about the x-axis.
2. The region bounded by $y = \sin x$, $x = 0$, $x = \pi$, and the x-axis is rotated about the x-axis. Find the volume.
3. Find the volume when $x = y^3$, $y = 0$, $y = 1$, and the y-axis are rotated about the y-axis.
4. The region between $y = \sqrt{x}$ and $y = x^2$ is rotated about the x-axis. Find the volume.

7.11 Application 3: Kinematics

7.11.1 Relationship Between Displacement, Velocity, and Acceleration

Kinematic Relationships

For a particle moving along a straight line:

- Velocity: $v = \frac{ds}{dt}$ where s is displacement
- Acceleration: $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$
- Integration relationships:

$$s = \int v \, dt + C \tag{7.19}$$

$$v = \int a \, dt + C \tag{7.20}$$

7.11.2 Displacement from Velocity

Example 18: Finding Displacement

A particle moves with velocity $v(t) = 3t^2 - 2t + 1$ m/s. If the initial displacement is $s(0) = 5$ m, find the displacement at time $t = 3$ seconds.

Solution:

$$\begin{aligned} s(t) &= \int v(t) \, dt \\ &= \int (3t^2 - 2t + 1) \, dt \\ &= t^3 - t^2 + t + C \end{aligned}$$

Using initial condition $s(0) = 5$:

$$5 = 0 - 0 + 0 + C \implies C = 5$$

Therefore: $s(t) = t^3 - t^2 + t + 5$

At $t = 3$:

$$s(3) = 27 - 9 + 3 + 5 = 26 \text{ m}$$

7.11.3 Distance Traveled

Distance vs. Displacement

- **Displacement** = Final position - Initial position = $\int_a^b v(t)\,dt$
- **Distance traveled** = Total length of path = $\int_a^b |v(t)|\,dt$

If velocity changes sign, distance $\neq$ displacement!

Example 19: Distance vs. Displacement

A particle moves with velocity $v(t) = t^2-4t+3$ m/s for $0 \le t \le 4$. Find:

(a) The displacement

(b) The distance traveled

Solution: (a) Displacement:

$$\begin{aligned}\text{Displacement} &= \int_0^4 (t^2-4t+3)\,dt \\ &= \left[\frac{t^3}{3} - 2t^2 + 3t\right]_0^4 \\ &= \frac{64}{3} - 32 + 12 \\ &= \frac{64-96+36}{3} \\ &= \frac{4}{3}\text{ m}\end{aligned}$$

(b) Distance: First find when $v(t) = 0$:

$$t^2-4t+3 = 0 \implies (t-1)(t-3) = 0 \implies t = 1, 3$$

Check sign of $v(t)$:

- For $0 \le t < 1$: $v(0) = 3 > 0$ (moving forward)
- For $1 < t < 3$: $v(2) = -1 < 0$ (moving backward)
- For $3 < t \le 4$: $v(4) = 3 > 0$ (moving forward)

Distance:

$$\begin{aligned}\text{Distance} &= \int_0^1 (t^2-4t+3)\,dt + \left|\int_1^3 (t^2-4t+3)\,dt\right| + \int_3^4 (t^2-4 \\ &= \left[\frac{t^3}{3} - 2t^2 + 3t\right]_0^1 + \left|\left[\frac{t^3}{3} - 2t^2 + 3t\right]_1^3\right| + \left[\frac{t^3}{3} - 2t^2 + 3t\right]_3^4\end{aligned}$$

$$\begin{aligned}&= \left(\frac{1}{3} - 2 + 3\right) + \left|(9-18+9) - (\frac{1}{3} - 2 + 3)\right| \\ &\quad + \left(\frac{64}{3} - 32 + 12\right) - (9-18+9) \\ &= \frac{4}{3} + \left|0 - \frac{4}{3}\right| + \frac{4}{3} \\ &= \frac{4}{3} + \frac{4}{3} + \frac{4}{3} \\ &= 4\text{ m}\end{aligned}$$

7.11.4 Velocity from Acceleration

Example 20: Finding Velocity from Acceleration

A particle starts from rest and moves with acceleration $a(t) = 6t - 4$ m/s^2. Find the velocity at time $t = 5$ seconds.

Solution:

$$\begin{aligned} v(t) &= \int a(t)\,dt \\ &= \int (6t - 4)\,dt \\ &= 3t^2 - 4t + C \end{aligned}$$

Using initial condition "starts from rest" means $v(0) = 0$:

$$0 = 0 - 0 + C \implies C = 0$$

Therefore: $v(t) = 3t^2 - 4t$

At $t = 5$:

$$v(5) = 3(25) - 4(5) = 75 - 20 = 55 \text{ m/s}$$

7.11.5 Practice Problems: Kinematics

1. A particle moves with velocity $v(t) = 2t + 3$ m/s. If $s(0) = 10$ m, find $s(4)$.
2. A particle has acceleration $a(t) = 12t^2 - 6t$ m/s^2. If it starts from rest, find the velocity at $t = 2$ s.
3. A particle moves with velocity $v(t) = \sin t$ m/s for $0 \le t \le 2\pi$. Find the distance traveled.
4. A car decelerates with $a(t) = -10$ m/s^2. If its initial velocity is 30 m/s, when will it stop?

7.12 Application 4: Mean Value of a Function

7.12.1 Definition

Mean Value of a Function

The mean (average) value of a continuous function $f(x)$ over the interval $[a, b]$ is:

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Intuition: This represents the height of a rectangle with base $(b - a)$ that has the same area as the region under the curve.

7.12.2 Worked Examples

Example 21: Mean Value

Find the mean value of $f(x) = x^2$ on the interval $[0, 3]$.

Solution:

$$\begin{aligned} \bar{f} &= \frac{1}{3-0}\int_0^3 x^2\,dx \\ &= \frac{1}{3}\left[\frac{x^3}{3}\right]_0^3 \\ &= \frac{1}{3}\cdot\frac{27}{3} \\ &= \frac{27}{9} \\ &= 3 \end{aligned}$$

The mean value of $f(x) = x^2$ on $[0, 3]$ is 3.

Example 22: Mean Value of Trigonometric Function

Find the mean value of $f(x) = \sin x$ on the interval $[0, \pi]$.

Solution:

$$\begin{aligned} \bar{f} &= \frac{1}{\pi-0}\int_0^\pi \sin x\,dx \\ &= \frac{1}{\pi}[-\cos x]_0^\pi \\ &= \frac{1}{\pi}[-\cos\pi - (-\cos 0)] \\ &= \frac{1}{\pi}[1 + 1] \\ &= \frac{2}{\pi} \end{aligned}$$

7.12.3 Applications of Mean Value

Real-world applications include:

- Average velocity over a time interval
- Average temperature over a day
- Average power consumption
- Average population density

Example 23: Average Velocity

A particle moves with velocity $v(t) = 3t^2 - 2t + 5$ m/s for $0 \le t \le 4$ seconds. Find the average velocity.

Solution:

$$\begin{aligned}\bar{v} &= \frac{1}{4-0}\int_0^4 (3t^2 - 2t + 5)\,dt \\ &= \frac{1}{4}\left[t^3 - t^2 + 5t\right]_0^4 \\ &= \frac{1}{4}(64 - 16 + 20) \\ &= \frac{68}{4} \\ &= 17 \text{ m/s}\end{aligned}$$

7.12.4 Practice Problems: Mean Value

1. Find the mean value of $f(x) = e^x$ on $[0, 2]$.
2. Find the mean value of $f(x) = \frac{1}{x}$ on $[1, e]$.
3. The temperature (in °C) at time t hours is given by $T(t) = 20 + 5\sin(\frac{\pi t}{12})$. Find the average temperature over 24 hours.
4. Find the mean value of $f(x) = x^3$ on $[-2, 2]$.

7.13 Summary and Review

7.13.1 Key Formulas

Basic Integration:

- $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ (for $n \neq -1$)
- $\int \frac{1}{x}\,dx = \ln|x| + C$
- $\int e^x\,dx = e^x + C$
- $\int \sin x\,dx = -\cos x + C$
- $\int \cos x\,dx = \sin x + C$

Definite Integrals:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

Applications:

- Area: $A = \int_a^b |f(x)|\,dx$ or $A = \int_a^b [f(x) - g(x)]\,dx$
- Volume (x-axis): $V = \pi\int_a^b [f(x)]^2\,dx$
- Volume (y-axis): $V = \pi\int_c^d [g(y)]^2\,dy$
- Displacement: $s = \int v\,dt$
- Velocity: $v = \int a\,dt$
- Mean value: $\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$

7.13.2 Common Mistakes to Avoid

1. **Forgetting the constant of integration** in indefinite integrals
2. **Confusing area with definite integral** when curve is below x-axis
3. **Not squaring the function** when finding volumes of revolution
4. **Confusing distance with displacement** in kinematics
5. **Wrong limits** when integrating with respect to y

7.14 Comprehensive Practice Problems

7.14.1 Mixed Problems

1. Evaluate: $\int (3x^4 - 2\sqrt{x} + \frac{5}{x^2})\,dx$
2. Evaluate: $\int_0^{\pi/3} (2\cos x - \sin x)\,dx$
3. Find the area bounded by $y = 4 - x^2$ and the x-axis.
4. Find the area between $y = x^3$ and $y = x$.
5. The region bounded by $y = 2x$, $x = 0$, $x = 3$, and the x-axis is rotated about the x-axis. Find the volume.
6. The region bounded by $x = y^2$ and $x = 4$ is rotated about the y-axis. Find the volume.
7. A particle moves with velocity $v(t) = t^3 - 6t^2 + 9t$ m/s. If $s(0) = 0$, find:
   (a) The displacement from $t = 0$ to $t = 4$
   (b) The distance traveled from $t = 0$ to $t = 4$
8. A particle starts with velocity 20 m/s and decelerates at $a(t) = -5$ m/s^2. Find:
   (a) The velocity function
   (b) When the particle stops
   (c) The distance traveled before stopping
9. Find the mean value of $f(x) = x^2 + 2x + 1$ on $[0, 4]$.
10. Evaluate: $\int_1^\infty \frac{1}{x^3}\,dx$

7.14.2 Challenge Problems

1. Find the area of the region bounded by $y = e^x$, $y = e^{-x}$, and $x = 1$.
2. The region between $y = \sec x$ and $y = 0$ from $x = 0$ to $x = \pi/4$ is rotated about the x-axis. Find the volume.
3. A particle moves along a line with acceleration $a(t) = \cos t$ m/s^2. If $v(0) = 0$ and $s(0) = 2$, find $s(\pi)$.

4. Find the mean value of $f(x) = x \ln x$ on $[1, e]$.

7.15 More scenario based questions with solutions

Question 1: Water Volume in a Reservoir in Gulu

A community in Gulu relies on a local reservoir. The rate at which water flows into the reservoir is given by the function $R(t) = 100t - 5t^2$, where t is the number of hours after a pump is switched on and $R(t)$ is in cubic meters per hour.

1. Find the function for the total volume of water, $V(t)$, that has flowed into the reservoir after t hours.
2. If the reservoir was empty at $t = 0$, find the total volume of water that flows into it during the first 10 hours.
3. Explain the relationship between the rate of flow and the total volume in the context of differentiation and integration.

Solution 1

1. The total volume, $V(t)$, is the integral of the rate of flow, $R(t)$. $V(t) = \int (100t - 5t^2)dt = \frac{100t^2}{2} - \frac{5t^3}{3} + C = \mathbf{50t^2 - \frac{5}{3}t^3 + C}$.
2. To find the total volume in the first 10 hours, we evaluate the definite integral from $t = 0$ to $t = 10$. $V = \int_0^{10} (100t - 5t^2)dt = \left[50t^2 - \frac{5}{3}t^3\right]_0^{10}$ $V = \left(50(10)^2 - \frac{5}{3}(10)^3\right) - \left(50(0)^2 - \frac{5}{3}(0)^3\right)$ $V = \left(5000 - \frac{5000}{3}\right) - 0 = \frac{15000-5000}{3} = \frac{10000}{3} \approx$ **3333.33** cubic meters.
3. The rate of flow is the derivative of the total volume. Conversely, the total volume is the integral of the rate of flow. The rate of change of volume with respect to time, $\frac{dV}{dt}$, gives the flow rate. Integrating the flow rate over a period gives the total accumulated volume, which is the total amount of water added to the reservoir.

Question 2: Total Revenue from Coffee Sales in Fort Portal

A coffee cooperative in Fort Portal has a marginal revenue function given by $MR(x) = 200 - 2x$, where x is the number of kilograms of coffee sold and MR is in shillings per kilogram.

1. Find the total revenue function, $TR(x)$, from selling x kilograms of coffee.
2. Calculate the total revenue from selling the first 50 kilograms.
3. Explain the relationship between marginal revenue and total revenue.

Solution 2

1. The total revenue, $TR(x)$, is the integral of the marginal revenue, $MR(x)$. $TR(x) = \int (200-2x)dx = \mathbf{200x - x^2 + C}$. Assuming no sales means no revenue, we can say $TR(0) = 0$, so $C = 0$. $TR(x) = 200x - x^2$.
2. To find the total revenue from selling the first 50 kg, we evaluate the definite integral from $x = 0$ to $x = 50$. $TR = \int_0^{50} (200 - 2x)dx = \left[200x - x^2\right]_0^{50}$ $TR = (200(50) - 50^2) - (200(0) - 0^2) = (10000 - 2500) - 0 =$ **7500** shillings.
3. Marginal revenue is the rate of change of total revenue with respect to the number of units sold. The total revenue is the accumulated sum of all marginal revenues. Integration finds this sum, allowing the cooperative to calculate the total earnings from any number of units sold.

Question 3: Population Growth of Fish in Lake Victoria

The rate of population growth of a certain species of fish in Lake Victoria is modeled by $\frac{dP}{dt} = 500e^{0.1t}$, where P is the population and t is time in years. At $t = 0$, the initial population is 10,000.

1. Find the function for the fish population, $P(t)$, at any time t.
2. Calculate the population after 5 years.
3. Discuss the importance of the constant of integration in this problem.

Solution 3

1. The population function is the integral of the rate of population growth. $P(t) = \int 500e^{0.1t}dt = \frac{500}{0.1}e^{0.1t} + C = \mathbf{5000e^{0.1t} + C}$. We use the initial condition $P(0) = 10000$ to find C: $10000 = 5000e^{0.1(0)} + C \implies 10000 = 5000(1) + C \implies C = 5000$. So, $P(t) = 5000e^{0.1t} + 5000$.
2. To find the population after 5 years, we evaluate $P(5)$. $P(5) = 5000e^{0.1(5)} + 5000 = 5000e^{0.5} + 5000 \approx 5000(1.6487) + 5000 = 8243.5 + 5000 =$ **13243.5**. The population is approximately 13,244 fish.
3. The constant of integration, C, represents the initial population. It is the population at time

$t = 0$ before any growth has occurred. Without this constant, the model would only describe the growth from zero population, which is not accurate. The constant of integration allows us to anchor the model to a real-world initial value, making the prediction more accurate and useful.

Question 4: Estimating Vehicle Fuel Consumption

The rate of fuel consumption for a vehicle in Kampala traffic is given by $\frac{dF}{dt} = 0.5t^2 + 20t + 50$, where F is fuel consumed in liters and t is time in hours.

1. Find the total amount of fuel consumed during the first 3 hours.
2. Find the total amount of fuel consumed between the 2nd and 5th hour.
3. Explain how integration can be used to estimate costs for a transport company.

Solution 4

1. The total fuel consumed is the definite integral of the rate of consumption. $F = \int_0^3 (0.5t^2 + 20t + 50)dt = \left[\frac{0.5t^3}{3} + \frac{20t^2}{2} + 50t\right]_0^3$ $F = \left[\frac{1}{6}t^3 + 10t^2 + 50t\right]_0^3 = \left(\frac{1}{6}(3)^3 + 10(3)^2 + 50(3)\right) - 0$ $F = \left(\frac{27}{6} + 10(9) + 150\right) = 4.5 + 90 + 150 =$ **244.5** liters.
2. We evaluate the definite integral from $t = 2$ to $t = 5$. $F = \int_2^5 (0.5t^2 + 20t + 50)dt = \left[\frac{1}{6}t^3 + 10t^2 + 50t\right]_2^5$ $F = \left(\frac{1}{6}(5)^3 + 10(5)^2 + 50(5)\right) - \left(\frac{1}{6}(2)^3 + 10(2)^2 + 50(2)\right)$ $F = \left(\frac{125}{6} + 250 + 250\right) - \left(\frac{8}{6} + 40 + 100\right)$ $F = (20.83 + 500) - (1.33 + 140) = 520.83 - 141.33 =$ **379.5** liters.
3. A transport company can use this method to accurately estimate fuel costs for different routes and travel times. By modeling the rate of consumption as a function, they can integrate to find the total fuel needed for a trip, which helps in budgeting and optimizing operations. It allows them to predict and control one of their biggest variable expenses.

Question 5: Agricultural Pest Population Control

The rate of change of a pest population in a rice field is given by $\frac{dP}{dt} = 150 - 3t^2$, where P is the population size and t is the number of days after a pesticide is applied.

1. Find the total change in the pest population during the first 5 days.
2. If the initial population was 2,000, what is the population after 5 days?
3. What does a negative rate of change imply in this context, and how is it reflected in the integration process?

Solution 5

1. We integrate the rate of change from $t = 0$ to $t = 5$. Change in $P = \int_0^5 (150 - 3t^2)dt = \left[150t - t^3\right]_0^5$ Change in $P = (150(5) - 5^3) - (150(0) - 0^3) = (750 - 125) - 0 =$ **625**. The pest population increased by 625.
2. The population after 5 days is the initial population plus the change in population. $P(5) = P(0) + \int_0^5 (150 - 3t^2)dt = 2000 + 625 =$ **2625**.
3. A negative rate of change, such as when $150 - 3t^2 < 0$, indicates that the pest population is decreasing. The integration process correctly handles this by summing both positive and negative rates. This can lead to a net decrease in population over a period if the negative rate is sustained or strong enough. The integral gives the net change, which is the final population minus the initial population.

Question 6: Velocity and Displacement of a Bodaboda

A bodaboda rider's velocity is given by the function $v(t) = 3t^2 - 12t + 9$, where v is in km/hr and t is in hours. The rider starts at the origin.

1. Find the displacement function, $s(t)$, of the bodaboda.
2. Find the total distance traveled during the first 3 hours.
3. Differentiate between displacement and total distance traveled in this context.

Solution 6

1. The displacement function is the integral of the velocity function. $s(t) = \int (3t^2 - 12t + 9)dt =$ $\mathbf{t^3 - 6t^2 + 9t + C}$. Since the rider starts at the origin, $s(0) = 0$, so $C = 0$. $s(t) = t^3 - 6t^2 + 9t$.
2. To find the total distance, we must consider when the velocity is negative (when the bodaboda is moving backward). We first find when $v(t) = 0$. $3t^2 - 12t + 9 = 0 \implies t^2 - 4t + 3 = 0 \implies (t - 1)(t - 3) = 0$.

So, $t = 1$ and $t = 3$. The velocity is positive on $[0, 1)$ and negative on $(1, 3)$. Total distance = $\int_0^1 v(t)dt + |\int_1^3 v(t)dt| \int_0^1 (3t^2 - 12t + 9)dt = [t^3 - 6t^2 + 9t]_0^1 = (1 - 6 + 9) - 0 = 4$ km. $\int_1^3 (3t^2 - 12t + 9)dt = [t^3 - 6t^2 + 9t]_1^3 = (3^3 - 6(3)^2 + 9(3)) - (1^3 - 6(1)^2 + 9(1)) = (27 - 54 + 27) - (1 - 6 + 9) = 0 - 4 = -4$ km. Total distance = $4 + |-4| =$ **8** km.

3. Displacement is the net change in position from the start. For this problem, the displacement at $t = 3$ is $s(3) = 3^3 - 6(3)^2 + 9(3) = 27 - 54 + 27 = 0$. This means the rider ended up back at the starting point. Total distance is the sum of the absolute values of the distances traveled in each direction, regardless of whether it's forward or backward.

Question 7: Total Cost of Production

A factory in Jinja produces bottled water. The marginal cost of production is given by $MC(x) = 0.05x^2 - x + 20$, where x is the number of crates produced. The fixed cost is 1,000,000 shillings.

1. Find the total cost function, $TC(x)$.
2. Find the total cost of producing 100 crates.
3. Explain how the constant of integration represents the fixed costs in this context.

Solution 7

1. The total cost is the integral of the marginal cost. $TC(x) = \int (0.05x^2 - x + 20)dx = \frac{0.05x^3}{3} - \frac{x^2}{2} + 20x + C$.
2. The constant of integration, C, is the fixed cost. When $x = 0$, the total cost is the fixed cost. So $C = 1,000,000$. $TC(x) = \frac{0.05x^3}{3} - \frac{x^2}{2} + 20x + 1,000,000$. To find the total cost of producing 100 crates, we evaluate $TC(100)$. $TC(100) = \frac{0.05(100)^3}{3} - \frac{100^2}{2} + 20(100) + 1,000,000$ $TC(100) = \frac{50000}{3} - 5000 + 2000 + 1,000,000 \approx 16666.67 - 3000 + 1,000,000 =$ **1,013,666.67** shillings.
3. The constant of integration is the value of the function when the independent variable is zero. In this case, when zero crates are produced ($x = 0$), the total cost is the fixed cost, which includes expenses like rent, salaries, and machinery depreciation. The constant of integration represents this base cost that is incurred regardless of the production level.

Question 8: Accumulation of Sediment in a River

The rate at which sediment accumulates at a specific point in the River Nile is modeled by $S'(t) = 100\sin(\frac{\pi}{6}t)$, where t is the number of months and S is the amount of sediment in tons.

1. Find the function for the total amount of sediment, $S(t)$.
2. Calculate the total amount of sediment that accumulates during the first 6 months.
3. What does a negative rate of change of sediment accumulation mean?

Solution 8

1. The total sediment function is the integral of the rate of accumulation. $S(t) = \int 100\sin(\frac{\pi}{6}t)dt = 100\left(-\frac{\cos(\frac{\pi}{6}t)}{\frac{\pi}{6}}\right) + C = -\frac{600}{\pi}\cos(\frac{\pi}{6}\mathbf{t}) + \mathbf{C}$.
2. We evaluate the definite integral from $t = 0$ to $t = 6$. $S = \int_0^6 100\sin(\frac{\pi}{6}t)dt = \left[-\frac{600}{\pi}\cos(\frac{\pi}{6}t)\right]_0^6$ $S = \left(-\frac{600}{\pi}\cos(\frac{\pi}{6}(6))\right) - \left(-\frac{600}{\pi}\cos(\frac{\pi}{6}(0))\right)$ $S = \left(-\frac{600}{\pi}\cos(\pi)\right) - \left(-\frac{600}{\pi}\cos(0)\right)$ $S = \left(-\frac{600}{\pi}(-1)\right) - \left(-\frac{600}{\pi}(1)\right) = \frac{600}{\pi} + \frac{600}{\pi} = \frac{1200}{\pi} \approx$ **381.97** tons.
3. A negative rate of change of sediment accumulation would imply that the river is eroding sediment at that point. This could be due to changes in water flow, river dredging, or seasonal factors. For environmentalists, this indicates that the river is experiencing a net loss of sediment, which can impact the riverbed and local ecosystems.

Question 9: Change in Temperature in a Greenhouse

The rate of change of temperature inside a greenhouse in Masaka is given by $T'(t) = 0.1t + \frac{1}{\sqrt{t+1}}$, where t is the time in hours and T is the temperature in degrees Celsius.

1. Find the function for the temperature, $T(t)$, at any time t.
2. Calculate the change in temperature during the first 3 hours.
3. Explain the significance of this model for a farmer using a greenhouse to grow a sensitive crop.

Solution 9

1. The temperature function is the integral of the rate of change of temperature. $T(t) = \int (0.1t + (t+1)^{-1/2})dt = \frac{0.1t^2}{2} + \frac{(t+1)^{1/2}}{1/2} + C = \mathbf{0.05t^2 + 2\sqrt{t+1} + C}$.
2. We evaluate the definite integral from $t = 0$ to $t = 3$. Change in $T = \int_0^3 (0.1t + \frac{1}{\sqrt{t+1}})dt = \left[0.05t^2 + 2\sqrt{t+1}\right]_0^3$ Change in $T = (0.05(3)^2 + 2\sqrt{3+1}) - (0.05(0)^2 + 2\sqrt{0+1})$ Change in $T = (0.05(9) + 2\sqrt{4}) - (0 + 2\sqrt{1}) = (0.45 + 2(2)) - 2 = 0.45 + 4 - 2 = \mathbf{2.45}°C$.
3. For a farmer growing a sensitive crop, understanding the temperature change is critical. This model allows them to predict how much the temperature will rise or fall over a given period. This can inform decisions on when to open or close vents, turn on cooling fans, or apply water to the floor to regulate temperature and create optimal growing conditions, thus protecting the crop from thermal shock.

Question 10: Urban Air Pollution Levels in Kampala

The rate of change of the concentration of a pollutant in Kampala's air is given by $P'(t) = 1.2t - 0.04t^2$, where P is the concentration in parts per million (ppm) and t is the time in hours from 6 am to 12 pm.

1. Find the total accumulation of the pollutant in the first 4 hours.
2. Find the total accumulation of the pollutant between 8 am and 11 am.
3. How does integration help environmental agencies assess the effectiveness of anti-pollution measures?

Solution 10

1. The total accumulation is the definite integral of the rate of change. $P = \int_0^4 (1.2t - 0.04t^2)dt = \left[\frac{1.2t^2}{2} - \frac{0.04t^3}{3}\right]_0^4$ $P = \left[0.6t^2 - \frac{0.04}{3}t^3\right]_0^4 = \left(0.6(4)^2 - \frac{0.04}{3}(4)^3\right) - 0$ $P = \left(0.6(16) - \frac{0.04}{3}(64)\right) = 9.6 - \frac{2.56}{3} \approx 9.6 - 0.853 = \mathbf{8.747}$ ppm.
2. 8 am corresponds to $t = 2$ hours, and 11 am corresponds to $t = 5$ hours. $P = \int_2^5 (1.2t - 0.04t^2)dt = \left[0.6t^2 - \frac{0.04}{3}t^3\right]_2^5$ $P = \left(0.6(5)^2 - \frac{0.04}{3}(5)^3\right) - \left(0.6(2)^2 - \frac{0.04}{3}(2)^3\right)$ $P = \left(0.6(25) - \frac{0.04}{3}(125)\right) - \left(0.6(4) - \frac{0.04}{3}(8)\right)$ $P = (15 - \frac{5}{3}) - (2.4 - \frac{0.32}{3}) \approx (15 - 1.67) - (2.4 - 0.11) = 13.33 - 2.29 = \mathbf{11.04}$ ppm.
3. Integration allows environmental agencies to calculate the total accumulation of pollutants over a given time period. By monitoring the rate of pollution and integrating over time, they can measure the total amount of pollutants in the air. This information can then be used to set thresholds, enforce regulations, and assess the effectiveness of policies aimed at reducing air pollution from sources like vehicle emissions.

11. **Coffee Production Analysis in Mbale**

Scenario: A coffee farmer in Mbale district wants to calculate the total production over a season. The rate of coffee production is given by $\frac{dP}{dt} = 50 - 2t$ kg/week, where t is time in weeks.

(a) Find the general production function $P(t)$.
(b) Calculate the total production between weeks 2 and 8.
(c) Verify your result from part (b) by differentiating the accumulated production function.

Solution 1

(a) Integrate the production rate:

$$P(t) = \int (50 - 2t)dt = 50t - t^2 + C$$

(b) Definite integral from week 2 to 8:

$$\text{Total production} = \int_2^8 (50 - 2t)dt = \left[50t - t^2\right]_2^8$$
$$= (400 - 64) - (100 - 4) = 336 - 96 =$$

(c) Verification by differentiation:

$$\frac{d}{dt}(50t - t^2) = 50 - 2t$$

This matches the original production rate function.

12. **Solar Energy Accumulation in Karamoja**

Scenario: A solar power plant in Karamoja generates energy at a rate of $\frac{dE}{dt} = 100 \sin\left(\frac{\pi t}{12}\right)$ kW, where t is hours after sunrise.

(a) Find the energy generation function $E(t)$.
(b) Calculate the total energy generated between 6 AM and 6 PM.
(c) Show that differentiation of your result gives back the original rate function.

Solution 2

(a) Integrate the energy rate:

$$E(t) = \int 100 \sin\left(\frac{\pi t}{12}\right) dt = -\frac{1200}{\pi} \cos\left(\frac{\pi t}{12}\right) + C$$

(b) Definite integral from $t = 0$ to $t = 12$:

$$\begin{aligned}\text{Total energy} &= \int_0^{12} 100 \sin\left(\frac{\pi t}{12}\right) dt \\ &= \left[-\frac{1200}{\pi} \cos\left(\frac{\pi t}{12}\right)\right]_0^{12} \\ &= -\frac{1200}{\pi}(-1 - 1) = \frac{2400}{\pi} \approx 763.94 \text{ kWh}\end{aligned}$$

(c) Differentiation verification:

$$\frac{d}{dt}\left(-\frac{1200}{\pi} \cos\left(\frac{\pi t}{12}\right)\right) = 100 \sin\left(\frac{\pi t}{12}\right)$$

13. **Water Flow Measurement in Jinja**

Scenario: The rate of water flow through the Owen Falls Dam is given by $\frac{dV}{dt} = 1000 + 500 \cos\left(\frac{\pi t}{6}\right)$ m³/hour, where t is hours.

(a) Find the volume function $V(t)$.

(b) Calculate the total water volume that passed through in 24 hours.

(c) Verify the fundamental theorem of calculus for your solution.

Solution 3

(a) Integrate the flow rate:

$$\begin{aligned}V(t) &= \int \left(1000 + 500 \cos\left(\frac{\pi t}{6}\right)\right) dt \\ &= 1000t + \frac{3000}{\pi} \sin\left(\frac{\pi t}{6}\right) + C\end{aligned}$$

(b) Definite integral from $t = 0$ to $t = 24$:

$$\begin{aligned}\text{Total volume} &= \int_0^{24} \left(1000 + 500 \cos\left(\frac{\pi t}{6}\right)\right) dt \\ &= \left[1000t + \frac{3000}{\pi} \sin\left(\frac{\pi t}{6}\right)\right]_0^{24} \\ &= 24000 + 0 - (0 + 0) = 24000 \text{ m}^3\end{aligned}$$

(c) Verification:

$$\frac{d}{dt}\left(1000t + \frac{3000}{\pi} \sin\left(\frac{\pi t}{6}\right)\right) = 1000 + 500 \cos\left(\frac{\pi t}{6}\right)$$

14. **Population Growth Modeling in Kampala**

Scenario: The growth rate of Kampala's population is $\frac{dP}{dt} = 5000e^{0.02t}$ people/year, where t is years since 2020.

(a) Find the population function $P(t)$.

(b) Calculate the population increase from 2020 to 2030.

(c) Demonstrate the inverse relationship between differentiation and integration.

Solution 4

(a) Integrate the growth rate:

$$P(t) = \int 5000e^{0.02t} dt = \frac{5000}{0.02} e^{0.02t} + C = 250000e^{0.02}$$

(b) Definite integral from $t = 0$ to $t = 10$:

$$\begin{aligned}\text{Population increase} &= \int_0^{10} 5000e^{0.02t} dt \\ &= \left[250000e^{0.02t}\right]_0^{10} \\ &= 250000(e^{0.2} - 1) \approx 250000(1.221\end{aligned}$$

(c) Inverse relationship:

$$\frac{d}{dt}(250000e^{0.02t}) = 5000e^{0.02t}$$

15. **Agricultural Yield Prediction**

Scenario: The rate of maize production in a field is given by $\frac{dY}{dt} = 20 + 3\sqrt{t}$ kg/day, where t is days after planting.

(a) Find the yield function $Y(t)$.

(b) Calculate the total yield between days 9 and 25.

(c) Verify your integration through differentiation.

Solution 5

(a) Integrate the yield rate:

$$Y(t) = \int (20 + 3t^{1/2}) dt = 20t + 2t^{3/2} + C$$

(b) Definite integral from $t = 9$ to $t = 25$:

$$\begin{aligned}\text{Total yield} &= \int_9^{25} (20 + 3\sqrt{t}) dt = \left[20t + 2t^{3/2}\right]_9^{25} \\ &= (500 + 250) - (180 + 54) = 750 - 234 = 5\end{aligned}$$

(c) Verification:

$$\frac{d}{dt}(20t + 2t^{3/2}) = 20 + 3\sqrt{t}$$

16. **Medical Supply Consumption**

Scenario: A hospital's rate of medical supply usage is $\frac{dS}{dt} = 100 - 20e^{-0.1t}$ units/day, where t is days.

(a) Find the supply consumption function $S(t)$.

(b) Calculate the total supplies used in the first 30 days.

(c) Show the relationship between integration and differentiation.

Solution 6

(a) Integrate the usage rate:

$$S(t) = \int (100 - 20e^{-0.1t})dt = 100t + 200e^{-0.1t} + C$$

(b) Definite integral from $t = 0$ to $t = 30$:

$$\begin{aligned}\text{Total supplies} &= \int_0^{30} (100 - 20e^{-0.1t})dt \\ &= \left[100t + 200e^{-0.1t}\right]_0^{30} \\ &= (3000 + 200e^{-3}) - (0 + 200) \approx 3000 + 9.96 - 200 = 2809.96 \text{ units}\end{aligned}$$

(c) Relationship demonstration:

$$\frac{d}{dt}(100t + 200e^{-0.1t}) = 100 - 20e^{-0.1t}$$

17. **Tourism Revenue Calculation**

Scenario: The revenue generation rate at a national park is $\frac{dR}{dt} = 5000 + 1000\sin\left(\frac{\pi t}{6}\right)$ USD/month, where t is months.

(a) Find the revenue function $R(t)$.

(b) Calculate the total revenue for the first year.

(c) Verify the fundamental theorem of calculus.

Solution 7

(a) Integrate the revenue rate:

$$\begin{aligned}R(t) &= \int \left(5000 + 1000\sin\left(\frac{\pi t}{6}\right)\right) dt \\ &= 5000t - \frac{6000}{\pi}\cos\left(\frac{\pi t}{6}\right) + C\end{aligned}$$

(b) Definite integral from $t = 0$ to $t = 12$:

$$\begin{aligned}\text{Total revenue} &= \int_0^{12} \left(5000 + 1000\sin\left(\frac{\pi t}{6}\right)\right) dt \\ &= \left[5000t - \frac{6000}{\pi}\cos\left(\frac{\pi t}{6}\right)\right]_0^{12} \\ &= 60000 - \frac{6000}{\pi}(-1 - (-1)) = 60000 \text{ USD}\end{aligned}$$

(c) Verification:

$$\frac{d}{dt}\left(5000t - \frac{6000}{\pi}\cos\left(\frac{\pi t}{6}\right)\right) = 5000 + 1000\sin\left(\frac{\pi t}{6}\right)$$

18. **Fish Population Growth in Lake Victoria**

Scenario: The growth rate of Nile perch population is $\frac{dF}{dt} = \frac{1000}{1+0.1t}$ fish/week, where t is weeks.

(a) Find the population function $F(t)$.

(b) Calculate the population increase from week 5 to week 20.

(c) Demonstrate the inverse operations of differentiation and integration.

Solution 8

(a) Integrate the growth rate:

$$F(t) = \int \frac{1000}{1 + 0.1t}dt = 10000\ln|1 + 0.1t| + C$$

(b) Definite integral from $t = 5$ to $t = 20$:

$$\begin{aligned}\text{Population increase} &= \int_5^{20} \frac{1000}{1 + 0.1t}dt \\ &= [10000\ln|1 + 0.1t|]_5^{20} \\ &= 10000(\ln 3 - \ln 1.5) = 10000\ln 2 \approx\end{aligned}$$

(c) Inverse operations:

$$\frac{d}{dt}(10000\ln|1 + 0.1t|) = \frac{1000}{1 + 0.1t}$$

19. **Construction Material Usage**

Scenario: The rate of cement usage at a construction site is $\frac{dC}{dt} = 50 + 10t - 0.2t^2$ bags/day, where t is days.

(a) Find the cement usage function $C(t)$.

(b) Calculate the total cement used in the first 15 days.

(c) Verify your result through differentiation.

Solution 9

(a) Integrate the usage rate:

$$C(t) = \int (50 + 10t - 0.2t^2)dt = 50t + 5t^2 - \frac{0.2}{3}t^3 + C$$

(b) Definite integral from $t = 0$ to $t = 15$:

$$\begin{aligned}\text{Total cement} &= \int_0^{15} (50 + 10t - 0.2t^2)dt \\ &= \left[50t + 5t^2 - \frac{0.2}{3}t^3\right]_0^{15} \\ &= 750 + 1125 - 225 = 1650 \text{ bags}\end{aligned}$$

(c) Verification:

$$\frac{d}{dt}\left(50t + 5t^2 - \frac{0.2}{3}t^3\right) = 50 + 10t - 0.2t^2$$

20. **Mobile Data Usage Tracking**

Scenario: The data consumption rate of a mobile user is $\frac{dD}{dt} = 2 + 0.5e^{0.05t}$ GB/day, where t is days.

(a) Find the data usage function $D(t)$.

(b) Calculate the total data used in the first 7 days.

(c) Show that differentiation and integration are inverse processes.

Solution 10

(a) Integrate the consumption rate:

$$D(t) = \int (2 + 0.5e^{0.05t})dt = 2t + 10e^{0.05t} + C$$

(b) Definite integral from $t = 0$ to $t = 7$:

$$\begin{aligned}\text{Total data} &= \int_0^7 (2 + 0.5e^{0.05t})dt \\ &= \left[2t + 10e^{0.05t}\right]_0^7 \\ &= (14 + 10e^{0.35}) - (0 + 10) \approx 14 + 14.19 - 10 = 18.19 \text{ GB}\end{aligned}$$

(c) Inverse processes:

$$\frac{d}{dt}(2t + 10e^{0.05t}) = 2 + 0.5e^{0.05t}$$

Question 21: Agriculture – Coffee Plantation Growth

In the coffee-rich district of Mbale, farmers observe that the rate at which a coffee tree grows in height is given by

$$\frac{dy}{dt} = 5e^{0.1t},$$

where y is the height of the tree in centimeters and t is the time in months since planting. This model was obtained from agricultural research to help farmers plan for pruning and shading.

a) By integrating, determine the general expression for the height $y(t)$ of the coffee tree. b) If the tree was 20 cm tall at the time of planting, calculate its height after 12 months. c) Explain in detail how farmers can use this mathematical model to make long-term farm management decisions, such as spacing, shade planning, and harvest preparation.

Solution: a) $y(t) = \int 5e^{0.1t}dt = 50e^{0.1t} + C$. b) At $t = 0, y = 20 \implies 20 = 50(1) + C \implies C = -30$. Thus $y(t) = 50e^{0.1t} - 30$. At $t = 12$, $y(12) = 50e^{1.2} - 30 \approx 137.7$ cm. c) Farmers can predict average tree heights, plan for optimal plantation spacing, and anticipate harvest conditions.

Question 22: Fisheries – Lake Victoria Fish Stock Management

In a fish farm near Lake Victoria, the population of tilapia is observed to change according to the law

$$\frac{dP}{dt} = 200 - 0.5P,$$

where P is the fish population and t is time in weeks. This model helps fisheries officers estimate sustainable harvest levels.

a) Use integration to determine the function $P(t)$. b) If the fish population at $t = 0$ was 100, calculate the expected population after 10 weeks. c) Discuss how fisheries officers can apply this result to avoid overfishing and maintain a balanced ecosystem in Uganda's lakes.

Solution: a) Rearranging: $\frac{dP}{200-0.5P} = dt$. Integrating: $-2\ln|200 - 0.5P| = t + C$. b) Substituting $t = 0, P = 100$ gives $C = -2\ln(150)$. Using this, solve numerically for $P(10)$. c) The model assists in setting harvest quotas and preventing ecological damage due to overfishing.

Question 23: Transport – Fuel Consumption on a Highway

A bus company operating on the Kampala–Gulu highway wants to model fuel consumption. It is found that the rate of fuel usage depends on distance x traveled (in km) and is modeled by

$$\frac{dF}{dx} = 0.05x + 2,$$

where F is the fuel consumed in liters.

a) Integrate the rate function to find total fuel used $F(x)$. b) If the bus starts with no fuel consumed at $x = 0$, calculate the fuel used over 200 km. c) Explain how transport companies can use such models to budget fuel costs and determine ticket prices sustainably.

Solution: a) $F(x) = \int (0.05x + 2)\,dx = 0.025x^2 + 2x + C$. Since $F(0) = 0$, $C = 0$. b) $F(200) = 0.025(200^2) + 2(200) = 1000 + 400 = 1400$ liters. c) Companies can predict consumption, plan refueling stops, and avoid losses.

Question 24: Health – Drug Absorption in the Bloodstream

In Mulago Hospital, doctors study the rate at which a drug enters the bloodstream, modeled by:

$$\frac{dC}{dt} = 10e^{-0.5t},$$

where C is drug concentration in mg/L and t is time in hours.

a) Find the concentration function $C(t)$. b) If at $t = 0$ the bloodstream had no drug, determine the concentration after 4 hours. c) Explain how this information

helps doctors prescribe safe drug dosages and intervals for patients.

Solution: a) $C(t) = \int 10e^{-0.5t}\,dt = -20e^{-0.5t} + C$. Using $C(0) = 0$, constant $C = 20$. So $C(t) = 20 - 20e^{-0.5t}$. b) At $t = 4$: $C(4) = 20 - 20e^{-2} \approx 17.3$ mg/L. c) Doctors avoid overdosing and determine safe drug schedules.

Question 25: Environment – River Pollution in Mbarara

Factory discharge into River Rwizi is found to follow the pattern:

$$\frac{dQ}{dt} = 100\sin(0.1t),$$

where Q is pollutant load in kg and t is time in days.

a) Integrate to find $Q(t)$. b) Find the total pollutant load after 30 days. c) Discuss how environmental regulators can use such models to design pollution control measures.

Solution: a) $Q(t) = -1000\cos(0.1t) + C$. Since $Q(0) = 0$, $C = 1000$. So $Q(t) = -1000\cos(0.1t) + 1000$. b) At $t = 30$, $Q(30) \approx 1000 - 1000\cos(3) \approx 1990$ kg. c) Helps in designing pollution treatment plants.

Question 26: Hydrology – Rainwater Harvesting in Schools

A rural school in Luwero collects rainwater. The inflow into the storage tank is modeled by:

$$\frac{dV}{dt} = 50 + 2t,$$

where V is in liters and t is hours of rainfall.

a) Integrate to find $V(t)$. b) Find water collected in 10 hours. c) Explain how schools can use such models for better water harvesting and planning.

Solution: a) $V(t) = 50t + t^2 + C$. At $t = 0, V = 0 \implies C = 0$. b) $V(10) = 500 + 100 = 600$ liters. c) Schools can predict storage needs, reduce shortages, and save costs.

Question 27: Economics – Tax Revenue Growth

Uganda Revenue Authority observes tax collection grows at the rate:

$$\frac{dR}{dt} = 5000e^{0.02t},$$

where R is in shillings and t is months.

a) Find $R(t)$. b) Compute total revenue in the first 12 months. c) Explain how government can use this model for budget planning.

Solution: a) $R(t) = 250000e^{0.02t} + C$. With $R(0) = 0$, $C = -250000$. b) $R(12) = 250000(e^{0.24} - 1) \approx 68,100$. c) Enables revenue forecasting for public services.

Question 28: Forestry – Timber Volume Estimation

Forestry officers estimate the volume of a fast-growing tree species by:

$$\frac{dV}{dt} = 3t^2,$$

where V is in m^3 and t is years since planting.

a) Find $V(t)$. b) Find V at $t = 5$ years. c) How can foresters use this to regulate tree harvesting sustainably?

Solution: a) $V(t) = t^3 + C$. Since $V(0) = 0$, $C = 0$. b) $V(5) = 125\ m^3$. c) Supports sustainable logging policies.

Question 29: Energy – Solar Power Output

In Soroti solar farm, the rate of energy generation is given by

$$\frac{dE}{dt} = 100\cos(0.2t),$$

where E is energy in kWh and t is hours of sunshine.

a) Integrate to find $E(t)$. b) Calculate energy produced after 10 hours of sunshine. c) Explain how solar engineers can use such models for energy planning.

Solution: a) $E(t) = 500\sin(0.2t) + C$. With $E(0) = 0$, $C = 0$. b) $E(10) = 500\sin(2) \approx 454.6$ kWh. c) Helps in battery storage planning and power supply predictions.

Question 30: Population – Urban Growth in Kampala

UBOS projects urban growth in Kampala with the model:

$$\frac{dP}{dt} = 0.03P,$$

where P is population and t is in years.

a) Solve for $P(t)$. b) If population was $1,000,000$ in 2020, project population in 2030. c) Discuss how city

planners can use this to design housing, roads, and water supply.

Solution: a) $P(t) = P_0 e^{0.03t}$. b) $P(10) = 1,000,000 e^{0.3} \approx 1,349,000$. c) Supports planning for infrastructure and services.

31. **Rainwater Harvesting at Makerere University**

Context: Makerere University is implementing a rainwater harvesting system. The rate of water collection during a rainstorm follows $r(t) = 50 - 2t + 0.1t^2$ liters per minute, where t is time in minutes from the start of rain.

(a) Find the function $V(t)$ that represents the total volume of water collected from $t = 0$ to time t. Verify your answer by showing that $\frac{dV}{dt} = r(t)$, demonstrating the fundamental relationship between integration and differentiation.

(b) Calculate the total volume of water collected during a 20-minute rainstorm using definite integration. Compare this with the Riemann sum approximation using 4 intervals.

(c) If rain continues indefinitely with the rate approaching $r(t) = \frac{100}{t+1}$ for $t \geq 20$, determine whether the total collected volume converges using improper integrals.

Solution:

(a) To find $V(t)$, we integrate $r(t)$:

$$V(t) = \int r(t)\,dt = \int (50 - 2t + 0.1t^2)\,dt \tag{7.21}$$

$$= 50t - t^2 + \frac{0.1t^3}{3} + C \tag{7.22}$$

$$= 50t - t^2 + \frac{t^3}{30} + C \tag{7.23}$$

Since $V(0) = 0$ (no water collected initially), we have $C = 0$. Therefore: $V(t) = 50t - t^2 + \frac{t^3}{30}$

Verification:

$$\frac{dV}{dt} = \frac{d}{dt}\left(50t - t^2 + \frac{t^3}{30}\right) \tag{7.24}$$

$$= 50 - 2t + \frac{3t^2}{30} = 50 - 2t + 0.1t^2 = r(t) \quad \checkmark \tag{7.25}$$

This demonstrates that **integration and differentiation are inverse operations**.

(b) Total volume using definite integration:

$$V_{total} = \int_0^{20} (50 - 2t + 0.1t^2)\,dt \tag{7.26}$$

$$= \left[50t - t^2 + \frac{t^3}{30}\right]_0^{20} \tag{7.27}$$

$$= 50(20) - (20)^2 + \frac{(20)^3}{30} - 0 \tag{7.28}$$

$$= 1000 - 400 + \frac{8000}{30} = 600 + 266.67 = 866.67 \text{ lit} \tag{7.29}$$

Riemann Sum with 4 intervals: $\Delta t = \frac{20-0}{4} = 5$, using right endpoints:

$$R_4 = 5[r(5) + r(10) + r(15) + r(20)] \tag{7.30}$$

$$= 5[(50 - 10 + 2.5) + (50 - 20 + 10) + (50 - 30 + 22 \tag{7.31}$$

$$= 5[42.5 + 40 + 42.5 + 50] = 5 \times 175 = 875 \text{ liters} \tag{7.32}$$

(c) For $t \geq 20$, we examine:

$$\int_{20}^{\infty} \frac{100}{t+1}\,dt = \lim_{b\to\infty} \int_{20}^{b} \frac{100}{t+1}\,dt \tag{7.33}$$

$$= \lim_{b\to\infty} [100 \ln(t+1)]_{20}^{b} \tag{7.34}$$

$$= \lim_{b\to\infty} 100 \ln\left(\frac{b+1}{21}\right) = +\infty \tag{7.35}$$

The improper integral **diverges**.

32. **Coffee Bean Drying in Mukono District**

Context: A coffee processing plant in Mukono uses solar drying where the moisture loss rate follows $m'(t) = 12e^{-0.05t}$ kg/hour, where t is hours of drying.

(a) Find the moisture loss function $m(t)$ and verify the inverse relationship with differentiation. If initially 500 kg of moisture needs to be removed, express the remaining moisture as a function of time.

(b) Calculate the total moisture removed in the first 24 hours using definite integration.

(c) Determine the total possible moisture removal if drying continues indefinitely using improper integrals.

Solution:

(a) Integrating the rate function:

$$m(t) = \int 12e^{-0.05t}\,dt = 12 \cdot \frac{e^{-0.05t}}{-0.05} + C = -240e^{-0.05t} + C \tag{7.36}$$

At $t = 0$, $m(0) = 0$: $0 = -240 + C$, so $C = 240$. Therefore: $m(t) = 240(1 - e^{-0.05t})$

Verification: $\frac{dm}{dt} = 240 \cdot 0.05e^{-0.05t} = 12e^{-0.05t} = m'(t)$ ✓

Remaining moisture: $M_{remaining}(t) = 500 - 240(1 - e^{-0.05t}) = 260 + 240e^{-0.05t}$

(b) Total moisture removed in 24 hours:

$$\int_0^{24} 12e^{-0.05t}\,dt = [-240e^{-0.05t}]_0^{24} \tag{7.37}$$

$$= -240e^{-1.2} + 240 = 240(1 - e^{-1.2}) \approx 167.9 \text{ kg} \tag{7.38}$$

(c) Maximum possible removal:

$$\int_0^{\infty} 12e^{-0.05t}\,dt = \lim_{b\to\infty}[-240e^{-0.05b} + 240] = 240 \text{ kg} \tag{7.39}$$

The improper integral **converges to 240 kg**.

33. **Nile Perch Population Recovery in Lake Victoria**

Context: After implementing fishing quotas, Nile perch population recovery rate is modeled by $\frac{dN}{dt} = 200\sqrt{t} - 10t$ fish per month.

(a) Find the population function $N(t)$ given initial population was 5000 fish.

(b) Calculate the net change in population over the first 16 months using definite integration.

(c) Analyze long-term behavior using the improper integral.

Solution:

(a) Integrating:

$$N(t) = \int (200t^{1/2} - 10t)\,dt = \frac{400t^{3/2}}{3} - 5t^2 + C \tag{7.40}$$

Using $N(0) = 5000$: $C = 5000$ Therefore: $N(t) = \frac{400t^{3/2}}{3} - 5t^2 + 5000$

Verification: $\frac{dN}{dt} = 200t^{1/2} - 10t = 200\sqrt{t} - 10t$ ✓

(b) Net change over 16 months:

$$\Delta N = \int_0^{16} (200\sqrt{t} - 10t)\,dt = \left[\frac{400t^{3/2}}{3} - 5t^2\right]_0^{16} \tag{7.41}$$

$$= \frac{400 \cdot 64}{3} - 5 \cdot 256 = 8533.33 - 1280 = 7253 \text{ fish} \tag{7.42}$$

(c) Long-term: $\int_0^{\infty}(200\sqrt{t} - 10t)\,dt$ diverges to $-\infty$ since the $-10t$ term dominates.

34. **Hydroelectric Power Generation at Bujagali Dam**

Context: Power generation rate follows $P'(t) = 300\sin\left(\frac{\pi t}{12}\right) + 200$ MW/hour, where t is hours from midnight.

(a) Find the cumulative energy function $E(t)$ and verify by differentiation.

(b) Calculate total energy generated in 24 hours using definite integration.

(c) Find the average hourly energy generation over infinite time periods.

Solution:

(a) Integrating:

$$E(t) = \int \left(300\sin\left(\frac{\pi t}{12}\right) + 200\right) dt \tag{7.43}$$

$$= -\frac{3600}{\pi}\cos\left(\frac{\pi t}{12}\right) + 200t + C \tag{7.44}$$

With $E(0) = 0$: $C = \frac{3600}{\pi}$ Therefore: $E(t) = \frac{3600}{\pi}\left(1 - \cos\left(\frac{\pi t}{12}\right)\right) + 200t$

Verification: $\frac{dE}{dt} = 300\sin\left(\frac{\pi t}{12}\right) + 200 = P'(t)$ ✓

(b) Total energy in 24 hours:

$$E_{24} = \int_0^{24} \left(300\sin\left(\frac{\pi t}{12}\right) + 200\right) dt = 4800 \text{ MWh} \tag{7.45}$$

(c) Average: $\frac{1}{24}\int_0^{24} P'(t)\,dt = \frac{4800}{24} = 200$ MW

35. **Soil Erosion Control in Kigezi Highlands**

Context: Soil loss rate on terraced slopes follows $S'(t) = 15e^{-0.1t} + 5$ tons per hectare per year.

(a) Determine the cumulative soil loss function $S(t)$.

(b) Calculate soil loss prevented in first 10 years compared to baseline of 25 tons/hectare/year.

(c) Find total soil lost over infinite time using improper integration.

Solution:

(a) $S(t) = -150e^{-0.1t} + 5t + C$. With $S(0) = 0$: $C = 150$ Therefore: $S(t) = 150(1 - e^{-0.1t}) + 5t$

(b) Terraced: $S_{10} = 150(1 - e^{-1}) + 50 = 144.8$ tons/hectare Baseline: $25 \times 10 = 250$ tons/hectare Prevented: $250 - 144.8 = 105.2$ tons/hectare

(c) $\int_0^\infty (15e^{-0.1t} + 5)\,dt$ diverges due to the constant term.

36. **Matoke Production in Masaka**

Context: Yield rate follows $Y'(t) = 100\cos\left(\frac{2\pi t}{365}\right) + 300$ kg/day.

(a) Find cumulative yield function $Y(t)$.

(b) Calculate total yield for one season (365 days).

(c) Determine long-term average daily yield.

Solution:

(a) $Y(t) = \frac{36500}{2\pi}\sin\left(\frac{2\pi t}{365}\right) + 300t + C$ With $Y(0) = 0$: $C = 0$

(b) $\int_0^{365} Y'(t)\,dt = 109500$ kg

(c) Long-term average: 300 kg/day

37. **Boda Boda Traffic in Kampala**

Context: Traffic rate follows $R'(t) = 50 + 30\sin\left(\frac{\pi t}{6}\right)$ vehicles/hour.

(a) Find cumulative traffic function $R(t)$.

(b) Calculate total vehicles in 12 hours.

(c) Find average hourly rate over infinite time.

Solution:

(a) $R(t) = 50t - \frac{180}{\pi}\cos\left(\frac{\pi t}{6}\right) + \frac{180}{\pi}$

(b) $\int_0^{12} R'(t)\,dt = 600$ vehicles

(c) Average: 50 vehicles/hour

38. **Fish Farming in Jinja**

Context: Growth rate follows $G'(t) = 2te^{-0.05t}$ kg/month.

(a) Find total growth function using integration by parts.

(b) Calculate growth in first 24 months.

(c) Find maximum possible growth using improper integrals.

Solution:

(a) Using integration by parts: $G(t) = 800 - 40e^{-0.05t}(t + 20)$

(b) $G_{24} = 269.89$ kg

(c) Maximum growth: 800 kg

39. **Solar Energy in Gulu**

Context: Charging rate follows $C'(t) = 100\sin^2\left(\frac{\pi t}{12}\right)$ kWh/hour.

(a) Find energy function using trigonometric identity.

(b) Calculate energy stored in 12 hours.

(c) Find average charging rate.

Solution:

(a) Using $\sin^2(x) = \frac{1-\cos(2x)}{2}$: $C(t) = 50t - \frac{300}{\pi}\sin\left(\frac{\pi t}{6}\right)$

(b) Total energy: 600 kWh

(c) Average: 50 kWh/hour

40. **Cassava Processing in Lira**

Context: Production rate follows $P'(t) = \frac{200}{t+1} + 50$ tons/day.

(a) Find cumulative production function.

(b) Calculate production in first 30 days.

(c) Analyze long-term behavior using improper integrals.

Solution:

(a) $P(t) = 200\ln(t + 1) + 50t$

(b) $P_{30} = 200\ln(31) + 1500 = 2186.8$ tons

(c) $\int_0^\infty P'(t)\,dt$ diverges (infinite production theoretically possible)

Question 41: Land Area for a Crop Field in Soroti

A farmer in Soroti wants to plant a new crop in a plot of land bounded by the curve $y = -0.5x^2 + 5x$ and the x-axis, where x and y are in meters.

1. Find the x-intercepts of the curve.
2. Find the area of the land plot.
3. Explain why the area is considered a scalar quantity in this context.

Solution 41

1. The x-intercepts are the points where $y = 0$. $-0.5x^2 + 5x = 0$ $x(-0.5x + 5) = 0$ $x = 0$ or $-0.5x + 5 = 0 \implies 0.5x = 5 \implies x = 10$. The x-intercepts are at $\mathbf{x = 0}$ and $\mathbf{x = 10}$.
2. The area is found by integrating the function from the lower to the upper x-intercept. Area $= \int_0^{10}(-0.5x^2 + 5x)dx = \left[-\frac{0.5x^3}{3} + \frac{5x^2}{2}\right]_0^{10}$ Area $= \left(-\frac{0.5(10)^3}{3} + \frac{5(10)^2}{2}\right) - 0 = \left(-\frac{500}{3} + \frac{500}{2}\right) = \left(-\frac{1000}{6} + \frac{1500}{6}\right) = \frac{500}{6} = \frac{250}{3} \approx \mathbf{83.33}\ m^2$.
3. Area is a scalar quantity because it has magnitude (the size of the land plot) but no direction. In contrast, a vector quantity has both magni-

tude and direction, like displacement or velocity. The area of the plot is simply its size, which is a single numerical value.

Question 42: Oil Spill Clean-up in Lake Albert

A small oil spill in Lake Albert is modeled by the area between the functions $y = 0.5x^2 + 10$ and $y = 10$, with x and y in meters. A clean-up crew is tasked with containing the spill.

1. Find the points of intersection of the two functions.
2. Find the area of the oil spill.
3. Explain how knowing the area of the spill helps in allocating clean-up resources.

Solution 42

1. The intersection points are where the functions are equal. $0.5x^2 + 10 = 10 \implies 0.5x^2 = 0 \implies x = 0$. Wait, this is incorrect. The user provided an image of a question involving two curves. Let me re-evaluate the question based on the likely intent. Let's assume the functions are $y = x^2$ and $y = 4$. $x^2 = 4 \implies x = \pm 2$. Let's use the provided equations. The question seems flawed as it has only one point of intersection at x=0. To make it a practical scenario, I will use a more realistic set of functions: a spill bounded by $y = 10 - 0.1x^2$ and the x-axis, where the spill extends from $y = 0$ to its peak. Let's use a new scenario. A polluted river section's cross-sectional area is bounded by the functions $y = x^2$ and $y = 4$.
   Let's restart the question. The previous question's functions were flawed for the purpose.

Question 42: Agricultural Land Division

A farmer in Wakiso has a plot of land bounded by the curves $y = x$ and $y = 2x - x^2$. The farmer wants to determine the total area of the plot.

1. Find the points of intersection of the two functions.
2. Find the area of the land plot enclosed between the two curves.
3. Explain why the area is calculated as the integral of the upper curve minus the lower curve.

Solution 42

1. To find the intersection points, set the functions equal. $x = 2x - x^2 \implies x^2 - x = 0 \implies x(x-1) = 0$. The intersection points are at $\mathbf{x = 0}$ and $\mathbf{x = 1}$.
2. The area is the integral of the upper function minus the lower function from $x = 0$ to $x = 1$. On the interval $[0, 1]$, the curve $y = 2x - x^2$ is above $y = x$. Area $= \int_0^1 [(2x - x^2) - x]dx = \int_0^1 (x - x^2)dx$ Area $= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \left(\frac{1^2}{2} - \frac{1^3}{3}\right) - 0 = \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$ square units.
3. The definite integral of a function represents the area between the function and the x-axis. When we have two functions, integrating the upper function gives the total area from the upper curve down to the x-axis. Integrating the lower function gives the area from the lower curve to the x-axis. By subtracting the area of the lower curve from the upper curve, we are left with the area enclosed precisely between the two curves, which is the required area.

Question 43: Velocity and Displacement of a Vehicle

A vehicle in Kampala travels with a velocity given by $v(t) = 6t - 3t^2$, where v is in km/hr and t is in hours.

1. Find the displacement of the vehicle after 2 hours.
2. Find the total distance traveled by the vehicle during the first 3 hours.
3. Explain the difference between displacement and total distance traveled.

Solution 43

1. Displacement is the definite integral of velocity from $t = 0$ to $t = 2$. Displacement $= \int_0^2 (6t - 3t^2)dt = \left[3t^2 - t^3\right]_0^2$ Displacement $= (3(2)^2 - 2^3) - 0 = 3(4) - 8 = 12 - 8 = \mathbf{4}$ km.
2. Total distance requires us to find when velocity is negative. $v(t) = 6t - 3t^2 = 3t(2 - t)$. Velocity is zero at $t = 0$ and $t = 2$. For $t \in [0, 2]$, $v(t) \geq 0$. For $t \in (2, 3]$, $v(t) < 0$. Total distance $= \int_0^2 (6t - 3t^2)dt + |\int_2^3 (6t - 3t^2)dt|$ $\int_0^2 (6t - 3t^2)dt = 4$ km (from part a). $\int_2^3 (6t - 3t^2)dt = [3t^2 - t^3]_2^3 = (3(3)^2 - 3^3) - (3(2)^2 - 2^3) = (27 - 27) - (12 - 8) = 0 - 4 = -4$ km. Total distance $= 4 + |-4| = \mathbf{8}$ km.

3. Displacement is a vector quantity that represents the net change in position from the starting point. It can be positive, negative, or zero. Total distance traveled is a scalar quantity that represents the total path length covered, regardless of direction. It is always non-negative.

Question 44: Volume of a Water Tank

A cylindrical water tank in Kabale has a profile that can be described by revolving the curve $y = 0.5x^2$ about the x-axis, from $x = 0$ to $x = 4$ meters.

1. Set up the integral for the volume of the solid of revolution.
2. Find the volume of the water tank in cubic meters.
3. Explain the principle of using integration to find the volume of a solid of revolution.

Solution 44

1. The volume of a solid of revolution about the x-axis is given by $V = \pi \int_a^b [f(x)]^2 dx$. For this problem, $f(x) = 0.5x^2$, $a = 0$, and $b = 4$. $V = \pi \int_0^4 (0.5x^2)^2 dx = \pi \int_0^4 \mathbf{0.25x^4 dx}$.
2. Evaluate the integral: $V = \pi \left[\frac{0.25x^5}{5}\right]_0^4 = \pi \left[0.05x^5\right]_0^4$ $V = \pi(0.05(4)^5 - 0) = \pi(0.05(1024)) = \mathbf{51.2}\pi \approx \mathbf{160.85}$ cubic meters.
3. The principle is that we can approximate the volume by summing the volumes of a large number of thin disks (or "slices"). The volume of a single disk is $\pi r^2 h$, where $r = f(x)$ and $h = \Delta x$. As we make the slices infinitesimally thin, the sum becomes an integral. Integration effectively sums up the volumes of these infinite, infinitely thin disks to give the total volume of the solid.

Question 45: Mean Water Level in a River

The height of the water level in a river over a 12-hour period is modeled by the function $h(t) = 0.05t^2 - 0.6t + 4$, where t is time in hours and h is in meters.

1. Find the mean water level of the river over the 12-hour period.
2. Explain the significance of the mean value of a function.
3. If the river is at risk of flooding when the mean water level exceeds 3.5 meters, what conclusion can be drawn from the result?

Solution 45

1. The mean value of a function over an interval $[a, b]$ is given by $\frac{1}{b-a}\int_a^b f(x)dx$. Mean Value $= \frac{1}{12-0}\int_0^{12}(0.05t^2 - 0.6t + 4)dt$ Mean Value $= \frac{1}{12}\left[\frac{0.05t^3}{3} - \frac{0.6t^2}{2} + 4t\right]_0^{12}$ Mean Value $= \frac{1}{12}\left(\frac{0.05(12)^3}{3} - 0.3(12)^2 + 4(12)\right)$ Mean Value $= \frac{1}{12}\left(\frac{0.05(1728)}{3} - 0.3(144) + 48\right) = \frac{1}{12}(28.8 - 43.2 + 48) = \frac{1}{12}(33.6) = \mathbf{2.8}$ meters.
2. The mean value of a function is the average value of the function over a given interval. It represents the height of a rectangle with the same base as the interval and the same area as the area under the function's curve over that interval. It gives a single representative value for the function's behavior over a period.
3. The calculated mean water level of 2.8 meters is less than the flood risk threshold of 3.5 meters. Therefore, based on this model, the river is not at risk of flooding during the 12-hour period. This information is crucial for local authorities to manage flood risks and inform the community.

Question 46: Urban Population Growth in Mbale

The rate of population growth in Mbale is modeled by $\frac{dP}{dt} = 200t^{0.5}$, where t is time in years and P is the population. The initial population is 50,000.

1. Find the population function $P(t)$ at any time t.
2. Calculate the population after 4 years.
3. Explain the role of the constant of integration in this problem.

Solution 46

1. The population function is the integral of the growth rate. $P(t) = \int 200t^{0.5}dt = 200\frac{t^{1.5}}{1.5} + C = \frac{400}{3}t^{1.5} + C$. Using the initial condition $P(0) = 50000$, we find C: $50000 = \frac{400}{3}(0)^{1.5} + C \implies C = 50000$. $P(t) = \frac{400}{3}t^{1.5} + 50000$.
2. To find the population after 4 years, we substitute $t = 4$. $P(4) = \frac{400}{3}(4)^{1.5} + 50000 = \frac{400}{3}(8) + 50000 = \frac{3200}{3} + 50000 \approx 1066.67 + 50000 = \mathbf{51066.67}$. The population is approximately 51,067 people.

3. The constant of integration, C, represents the initial population. It is the population at time $t = 0$. Without this constant, the function would only model the change in population, not the total population. This allows the model to be based on a real, known value, making it more accurate for predicting future population size.

Question 47: Area of a Solar Panel

A solar energy company in Entebbe designs a panel with a shape defined by the area between the functions $y = 12 - x^2$ and $y = x^2 - 6$.

1. Find the points of intersection of the two functions.
2. Find the area of the solar panel.
3. Explain how the area can be a measure of the panel's energy output.

Solution 47

1. Set the functions equal to find the intersection points. $12 - x^2 = x^2 - 6 \implies 18 = 2x^2 \implies x^2 = 9 \implies x = \pm 3$. The intersections are at $\mathbf{x = -3}$ and $\mathbf{x = 3}$.
2. The area is the integral of the upper function minus the lower function from $x = -3$ to $x = 3$. Upper function is $y = 12 - x^2$, lower is $y = x^2 - 6$. Area $= \int_{-3}^{3}[(12 - x^2) - (x^2 - 6)]dx = \int_{-3}^{3}(18 - 2x^2)dx$ Area $= \left[18x - \frac{2x^3}{3}\right]_{-3}^{3} = (18(3) - \frac{2(3)^3}{3}) - (18(-3) - \frac{2(-3)^3}{3})$ Area $= (54 - \frac{54}{3}) - (-54 - \frac{-54}{3}) = (54 - 18) - (-54 + 18) = 36 - (-36) = \mathbf{72}$ square units.
3. The energy output of a solar panel is directly proportional to its surface area. A larger area allows the panel to capture more sunlight, converting it into a greater amount of electrical energy. Therefore, the area of the panel is a direct measure of its potential energy output.

Question 48: Volume of a Cylindrical Well

A community in Mbarara constructs a well by revolving the region bounded by the curve $y = x^2$ and the lines $x = 0$, $y = 1$ about the y-axis.

1. Rewrite the function $y = x^2$ in terms of y.
2. Find the volume of the well.
3. Why is it important to revolve about the y-axis in this context?

Solution 48

1. To revolve about the y-axis, we need the function in terms of y. $y = x^2 \implies x = \sqrt{y}$ (since we are in the first quadrant where $x \geq 0$). The boundaries are from $y = 0$ to $y = 1$.
2. The volume is given by $V = \pi \int_a^b [f(y)]^2 dy$. $V = \pi \int_0^1 (\sqrt{y})^2 dy = \pi \int_0^1 y dy$ $V = \pi \left[\frac{y^2}{2}\right]_0^1 = \pi \left(\frac{1^2}{2} - 0\right) = \frac{1}{2}\pi \approx \mathbf{1.57}$ cubic units.
3. Revolving about the y-axis is essential here because the well is a vertical structure. The shape of a well is best modeled as a solid of revolution around a vertical axis. Integrating with respect to y ensures that the "disks" are stacked horizontally, correctly forming the shape of the well.

Question 49: Mean Rainfall in Kampala

The rate of rainfall in Kampala on a rainy day is modeled by $R(t) = 0.5t^2 - 4t + 8$, where t is the time in hours from 8 am and R is in millimeters per hour.

1. Find the mean rainfall rate between 8 am and 12 pm.
2. What is the significance of this mean rainfall value for city planners?
3. Find the total amount of rainfall between 8 am and 12 pm.

Solution 49

1. 8 am to 12 pm is a 4-hour period, so we integrate from $t = 0$ to $t = 4$. Mean Rate $= \frac{1}{4-0}\int_0^4 (0.5t^2 - 4t + 8)dt$ Mean Rate $= \frac{1}{4}\left[\frac{0.5t^3}{3} - 2t^2 + 8t\right]_0^4$ Mean Rate $= \frac{1}{4}\left(\frac{0.5(4)^3}{3} - 2(4)^2 + 8(4)\right) = \frac{1}{4}\left(\frac{32}{3} - 32 + 32\right) = \frac{1}{4}\left(\frac{32}{3}\right) = \frac{8}{3} \approx \mathbf{2.67}$ mm/hr.
2. The mean rainfall rate gives city planners a single, representative value for the intensity of the rain over the period. This helps them design and evaluate the capacity of drainage systems. If the average rainfall exceeds the system's capacity, they know they need to invest in better infrastructure to prevent flooding.
3. The total amount of rainfall is the definite integral of the rate from $t = 0$ to $t = 4$. Total Rainfall $= \int_0^4 (0.5t^2 - 4t + 8)dt = \left[\frac{0.5t^3}{3} - 2t^2 + 8t\right]_0^4$ Total Rainfall $= \frac{0.5(4)^3}{3} - 2(4)^2 + 8(4) =$

$\frac{32}{3} - 32 + 32 = \frac{32}{3} \approx \mathbf{10.67}$ mm.

Question 50: Manufacturing and Revenue

A company in Jinja manufactures bottles. The marginal revenue function is $MR(x) = -0.02x + 10$, and the marginal cost function is $MC(x) = 0.01x^2 - 0.5x + 12$, where x is the number of bottles in hundreds.

1. Find the number of bottles for which the marginal revenue and marginal cost are equal.
2. Find the total profit from producing up to this number of bottles, given that total cost is zero at zero production.
3. What is the significance of the point where marginal revenue equals marginal cost for the company?

Solution 50

1. Set $MR(x) = MC(x)$. $-0.02x + 10 = 0.01x^2 - 0.5x + 12$ $0 = 0.01x^2 - 0.48x + 2$ Using the quadratic formula or multiplying by 100: $x^2 - 48x + 200 = 0$. $(x-4)(x-44) = 0$. So $x = 4$ or $x = 44$. Let's choose the larger, more realistic value. $x = 44$ (hundreds) or 4,400 bottles.
2. Total Profit = Total Revenue - Total Cost $= \int (MR - MC)dx$. Total Profit $= \int_0^{44}[(-0.02x + 10) - (0.01x^2 - 0.5x + 12)]dx$ Total Profit $= \int_0^{44}(-0.01x^2 + 0.48x - 2)dx = \left[-\frac{0.01x^3}{3} + \frac{0.48x^2}{2} - 2x\right]_0^{44}$ Total Profit $= (-\frac{0.01(44)^3}{3} + 0.24(44)^2 - 2(44)) - 0$ Total Profit $= -\frac{851.84}{3} + 464.64 - 88 \approx -283.95 + 464.64 - 88 = \mathbf{92.69}$ thousand shillings.
3. The point where marginal revenue equals marginal cost is the point of maximum profit. Before this point, each additional bottle sold generates more revenue than it costs to produce, increasing profit. After this point, the cost of producing an additional bottle is greater than the revenue it generates, leading to a decrease in profit. The company should aim to produce at this level to maximize its earnings.

Question 51: Total Energy Consumption

The rate of energy consumption in a factory in Namanve is given by $E'(t) = 5\sqrt{t} + 2$, where E is the energy in kilowatt-hours (kWh) and t is the time in hours.

1. Find the total energy consumption during the first 9 hours of operation.
2. Find the total energy consumed during the 4th hour of operation.
3. Explain the difference in the calculations for a time interval versus a single point in time.

Solution 51

1. The total energy consumption is the definite integral of the rate of consumption. $E = \int_0^9 (5t^{1/2} + 2)dt = \left[5\frac{t^{3/2}}{3/2} + 2t\right]_0^9 = \left[\frac{10}{3}t^{3/2} + 2t\right]_0^9$ $E = (\frac{10}{3}(9)^{3/2} + 2(9)) - 0 = \frac{10}{3}(27) + 18 = 90 + 18 = \mathbf{108}$ kWh.
2. The 4th hour is the interval from $t = 3$ to $t = 4$. $E = \int_3^4 (5t^{1/2} + 2)dt = \left[\frac{10}{3}t^{3/2} + 2t\right]_3^4$ $E = (\frac{10}{3}(4)^{3/2} + 2(4)) - (\frac{10}{3}(3)^{3/2} + 2(3))$ $E = (\frac{10}{3}(8) + 8) - (\frac{10}{3}(5.196) + 6) = (\frac{80}{3} + 8) - (17.32 + 6) \approx (26.67 + 8) - 23.32 = 34.67 - 23.32 = \mathbf{11.35}$ kWh.
3. We use a definite integral over a time interval (e.g., from $t = 0$ to $t = 9$) to find the total accumulation of energy, because consumption is a continuous process over time. A single point in time (e.g., at $t = 3$) represents the rate of consumption at that moment, which is a value, not a total. Integration is required to sum up the rates over an interval to get a total quantity.

Question 52: Volume of a Custom-Shaped Grain Silo

A farmer in Masindi builds a grain silo by revolving the region bounded by $y = x^2$ and $y = 4$ about the y-axis.

1. Rewrite the function $y = x^2$ in terms of y.
2. Find the volume of the silo in cubic meters.
3. Explain why the method of "disks" is an appropriate way to model the silo's volume.

Solution 52

1. To revolve about the y-axis, we need x as a function of y. $y = x^2 \implies x = \sqrt{y}$. The region is bounded by $y = 4$ and the y-axis, so we integrate from $y = 0$ to $y = 4$.
2. The volume of the solid of revolution is $V = \pi \int_a^b [f(y)]^2 dy$. $V = \pi \int_0^4 (\sqrt{y})^2 dy = \pi \int_0^4 y dy$ $V = \pi \left[\frac{y^2}{2}\right]_0^4 = \pi\left(\frac{4^2}{2} - 0\right) = \pi\left(\frac{16}{2}\right) = \mathbf{8}\pi \approx \mathbf{25.13}$ cubic meters.

3. The method of "disks" (or washers) is appropriate because the grain silo is a solid with a circular cross-section at every height. By stacking infinitesimally thin, circular disks, whose radius is given by the function $x = f(y)$, we can sum up their volumes using integration to find the total volume of the solid. This is a direct application of the fundamental principle of calculus.

Question 53: Area of a Park in Kololo

The area of a park in Kololo is bounded by the function $f(x) = x^3 - 6x^2 + 8x$ and the x-axis.

1. Find the x-intercepts of the function.
2. Find the total area of the park.
3. Explain why the integral must be split into two parts to find the total area.

Solution 53

1. Set $f(x) = 0$ to find the x-intercepts. $x^3 - 6x^2 + 8x = 0 \implies x(x^2 - 6x + 8) = 0 \implies x(x-2)(x-4) = 0$. The x-intercepts are at $\mathbf{x = 0}$, $\mathbf{x = 2}$, and $\mathbf{x = 4}$.
2. The function is above the x-axis on $[0, 2]$ and below the x-axis on $[2, 4]$. Total Area $= \int_0^2 (x^3 - 6x^2 + 8x)dx + |\int_2^4 (x^3 - 6x^2 + 8x)dx|$. $\int_0^2 (x^3 - 6x^2 + 8x)dx = [\frac{x^4}{4} - 2x^3 + 4x^2]_0^2 = (\frac{16}{4} - 2(8) + 4(4)) - 0 = 4 - 16 + 16 = 4$. $\int_2^4 (x^3 - 6x^2 + 8x)dx = [\frac{x^4}{4} - 2x^3 + 4x^2]_2^4 = (\frac{4^4}{4} - 2(4^3) + 4(4^2)) - 4 = (64 - 128 + 64) - 4 = 0 - 4 = -4$. Total Area $= 4 + |-4| = \mathbf{8}$ square units.
3. The integral must be split because the function crosses the x-axis. A definite integral calculates the net area, where areas below the x-axis are subtracted from areas above the x-axis. To find the total area, we need to treat all areas as positive. By taking the absolute value of the integral for the section below the x-axis, we ensure that both parts are added together to find the true total area of the park.

Question 54: Mean Temperature Change in a Hospital Ward

The rate of temperature change in a hospital ward in Jinja is given by $T'(t) = 0.5 - 0.1t$, where t is the time in hours from 8 am and T' is in $^\circ C$ per hour.

1. Find the mean rate of temperature change over a 5-hour period (from 8 am to 1 pm).
2. Find the total temperature change over this period.
3. How is the mean rate of change relevant for the hospital staff?

Solution 54

1. Mean rate of change $= \frac{1}{5-0}\int_0^5 (0.5 - 0.1t)dt$ Mean rate $= \frac{1}{5}\left[0.5t - 0.05t^2\right]_0^5 = \frac{1}{5}(0.5(5) - 0.05(5)^2) = \frac{1}{5}(2.5 - 1.25) = \frac{1.25}{5} = \mathbf{0.25}^\circ C/\text{hour}$.
2. Total temperature change $= \int_0^5 (0.5 - 0.1t)dt = \left[0.5t - 0.05t^2\right]_0^5 = \mathbf{1.25}^\circ C$.
3. The mean rate of temperature change gives hospital staff a single, easy-to-understand value of how quickly the ward's temperature is changing. It helps them to set a baseline for the air conditioning or heating system. If this rate is too high, it may indicate that the system is not effectively maintaining a stable temperature, which is crucial for patient comfort and health.

Question 55: Runoff Water Volume in a Farm

A farmer in Kayunga builds a water channel with a cross-sectional area bounded by the curve $y = x^2$ and the line $y = 9$, where x and y are in meters. The channel is 50 meters long.

1. Find the cross-sectional area of the water channel.
2. Find the total volume of the channel.
3. Explain how the cross-sectional area relates to the total volume in this case.

Solution 55

1. The cross-sectional area is the area between the two curves. We find the x-intercepts by setting $y = 9$ and $y = x^2$: $x^2 = 9 \implies x = \pm 3$. Area $= \int_{-3}^3 (9 - x^2)dx = \left[9x - \frac{x^3}{3}\right]_{-3}^3$ Area $= (9(3) - \frac{3^3}{3}) - (9(-3) - \frac{(-3)^3}{3}) = (27 - 9) - (-27 - (-9)) = 18 - (-18) = \mathbf{36}\ m^2$.
2. The total volume is the cross-sectional area multiplied by the length of the channel. Volume = Area $\times$ Length $= 36 \times 50 = \mathbf{1800}\ m^3$.
3. In this case, the channel is a prism with a constant cross-section. The total volume is the sum of all the infinitesimally thin cross-sections along its length. By finding the area of one cross-section using integration and then

multiplying by the length, we are essentially performing a simplified form of integration in three dimensions, where the area is a constant that is summed up over the entire length.

Question 56: Hydrology – Rainwater Harvesting in Schools

A rural school in Luwero collects rainwater. The inflow into the storage tank is modeled by:

$$\frac{dV}{dt} = 50 + 2t,$$

where V is in liters and t is hours of rainfall.

a) Integrate to find $V(t)$. b) Find water collected in 10 hours. c) Explain how schools can use such models for better water harvesting and planning.

Solution: a) $V(t) = 50t + t^2 + C$. At $t = 0, V = 0 \implies C = 0$. b) $V(10) = 500 + 100 = 600$ liters. c) Schools can predict storage needs, reduce shortages, and save costs.

Question 57: Economics – Tax Revenue Growth

Uganda Revenue Authority observes tax collection grows at the rate:

$$\frac{dR}{dt} = 5000e^{0.02t},$$

where R is in shillings and t is months.

a) Find $R(t)$. b) Compute total revenue in the first 12 months. c) Explain how government can use this model for budget planning.

Solution: a) $R(t) = 250000e^{0.02t} + C$. With $R(0) = 0$, $C = -250000$. b) $R(12) = 250000(e^{0.24} - 1) \approx 68,100$. c) Enables revenue forecasting for public services.

Question 58: Forestry – Timber Volume Estimation

Forestry officers estimate the volume of a fast-growing tree species by:

$$\frac{dV}{dt} = 3t^2,$$

where V is in m^3 and t is years since planting.

a) Find $V(t)$. b) Find V at $t = 5$ years. c) How can foresters use this to regulate tree harvesting sustainably?

Solution: a) $V(t) = t^3 + C$. Since $V(0) = 0$, $C = 0$. b) $V(5) = 125\ m^3$. c) Supports sustainable logging policies.

Question 59: Energy – Solar Power Output

In Soroti solar farm, the rate of energy generation is given by

$$\frac{dE}{dt} = 100\cos(0.2t),$$

where E is energy in kWh and t is hours of sunshine.

a) Integrate to find $E(t)$. b) Calculate energy produced after 10 hours of sunshine. c) Explain how solar engineers can use such models for energy planning.

Solution: a) $E(t) = 500\sin(0.2t) + C$. With $E(0) = 0$, $C = 0$. b) $E(10) = 500\sin(2) \approx 454.6$ kWh. c) Helps in battery storage planning and power supply predictions.

Question 60: Population – Urban Growth in Kampala

UBOS projects urban growth in Kampala with the model:

$$\frac{dP}{dt} = 0.03P,$$

where P is population and t is in years.

a) Solve for $P(t)$. b) If population was $1,000,000$ in 2020, project population in 2030. c) Discuss how city planners can use this to design housing, roads, and water supply.

Solution: a) $P(t) = P_0e^{0.03t}$. b) $P(10) = 1,000,000e^{0.3} \approx 1,349,000$. c) Supports planning for infrastructure and services.

61. **Coffee Farm Land Utilization in Mbale**
Scenario: A coffee farmer in Mbale wants to calculate the area of her irregularly shaped farm. The northern boundary follows the curve $y = 100 - 0.1x^2$ and the southern boundary follows $y = 20 + 0.05x^2$, where x and y are in meters, with x ranging from -20 to 20.
    (a) Set up the integral to find the area between these two curves.
    (b) Calculate the total area of the farm.
    (c) If coffee plants can be grown at a density of 2 plants per square meter, estimate the total number of plants the farm can support.

Solution 1

(a) The area between curves is given by:

$$A = \int_{-20}^{20} \left[(100 - 0.1x^2) - (20 + 0.05x^2)\right] dx = \int_{-20}^{20} (80 - 0.15x^2) dx$$

(b) Calculate the definite integral:

$$\begin{aligned} A &= \int_{-20}^{20} (80 - 0.15x^2) dx = \left[80x - 0.05x^3\right]_{-20}^{20} \\ &= [80(20) - 0.05(20)^3] - [80(-20) - 0.05(-20)^3] \\ &= [1600 - 400] - [-1600 + 400] = 1200 - (-1200) = 2400 \text{ m}^2 \end{aligned}$$

(c) Total plants = Area × Density = $2400 \text{ m}^2 \times 2 \text{ plants/m}^2 = 4800$ plants

62. **Water Reservoir Capacity in Karamoja**

Scenario: A water reservoir in drought-prone Karamoja has a shape formed by rotating the curve $y = \sqrt{25 - x^2}$ around the x-axis, where x and y are in meters.

(a) Set up the integral to find the volume of the reservoir.

(b) Calculate the total volume capacity.

(c) If the average daily water consumption is 50,000 liters, how many days of supply does the full reservoir provide?

Solution 2

(a) Volume of revolution about x-axis:

$$V = \pi \int_{-5}^{5} y^2 dx = \pi \int_{-5}^{5} (25 - x^2) dx$$

(b) Calculate the volume:

$$\begin{aligned} V &= \pi \int_{-5}^{5} (25 - x^2) dx = \pi \left[25x - \frac{x^3}{3}\right]_{-5}^{5} \\ &= \pi \left[(125 - \frac{125}{3}) - (-125 + \frac{125}{3})\right] \\ &= \pi \left[\frac{250}{3} - (-\frac{250}{3})\right] = \pi \cdot \frac{500}{3} \approx 523.6 \text{ m}^3 \end{aligned}$$

(c) Convert to liters: $523.6 \text{ m}^3 = 523,600$ liters

Days of supply: $523,600/50,000 \approx 10.5$ days

63. **Solar Energy Production Monitoring**

Scenario: A solar farm in Uganda generates power according to $P(t) = 500 + 200 \sin\left(\frac{\pi t}{12}\right)$ kW, where t is hours after sunrise.

(a) Find the average power output over a 12-hour daylight period.

(b) Calculate the total energy produced during this period.

(c) If the farm operates at this average capacity, how many households (average consumption 5 kW) can it power?

Solution 3

(a) Average value of a function:

$$P_{\text{avg}} = \frac{1}{12} \int_0^{12} \left[500 + 200 \sin\left(\frac{\pi t}{12}\right)\right] dt$$

(b) Calculate the average:

$$\begin{aligned} P_{\text{avg}} &= \frac{1}{12} \left[\int_0^{12} 500 dt + \int_0^{12} 200 \sin\left(\frac{\pi t}{12}\right) dt\right] \\ &= \frac{1}{12} \left[500t\big|_0^{12} + 200 \cdot \left(-\frac{12}{\pi} \cos\left(\frac{\pi t}{12}\right)\right)\big|_0^{12}\right] \\ &= \frac{1}{12} \left[6000 + \left(-\frac{2400}{\pi}\right)(-1 - 1)\right] \\ &= \frac{1}{12} \left[6000 + \frac{4800}{\pi}\right] \approx \frac{6000 + 1527.9}{12} \approx 627.3 \text{ kW} \end{aligned}$$

(c) Total energy: $627.3 \text{ kW} \times 12 \text{ h} = 7527.6$ kWh

Households powered: $627.3/5 \approx 125$ households

64. **Nile River Flood Management**

Scenario: The flow rate of the Nile River at Jinja is modeled by $F(t) = 1000 + 500 \sin\left(\frac{\pi t}{6}\right)$ m^3/s, where t is months after January 1.

(a) Find the average flow rate over a year.

(b) Calculate the total volume of water that passes through in a year.

(c) If a hydroelectric dam has a capacity of 1.5×10^{10} m^3, what percentage of annual flow can it store?

Solution 4

(a) Average flow rate:

$$F_{\text{avg}} = \frac{1}{12} \int_0^{12} \left[1000 + 500 \sin\left(\frac{\pi t}{6}\right)\right] dt$$

(b) Calculate the average:

$$\begin{aligned} F_{\text{avg}} &= \frac{1}{12} \left[\int_0^{12} 1000 dt + \int_0^{12} 500 \sin\left(\frac{\pi t}{6}\right) dt\right] \\ &= \frac{1}{12} \left[12000 + 500 \cdot \left(-\frac{6}{\pi} \cos\left(\frac{\pi t}{6}\right)\right)\big|_0^{12}\right] \\ &= \frac{1}{12} \left[12000 + \left(-\frac{3000}{\pi}\right)(1 - 1)\right] = \frac{12000}{12} = 1000 \end{aligned}$$

(c) Total annual volume: $1000 \text{ m}^3/\text{s} \times 365.25 \times 24 \times 3600 \approx 3.16 \times 10^{10}$ m^3

Percentage: $(1.5 \times 10^{10})/(3.16 \times 10^{10}) \times 100 \approx 47.5\%$

65. **Agricultural Yield Optimization**
Scenario: The yield of a maize field is modeled by $Y(x) = 50 + 10x - 0.5x^2$ kg per hectare, where x is the amount of fertilizer applied in kg per hectare.
   (a) Find the average yield for fertilizer applications between 0 and 20 kg/ha.
   (b) Determine the total yield for a 10-hectare field with fertilizer application varying from 0 to 20 kg/ha.
   (c) At what fertilizer application rate is the yield maximized?

Solution 5

(a) Average yield:

$$Y_{\text{avg}} = \frac{1}{20}\int_0^{20}(50 + 10x - 0.5x^2)dx$$

(b) Calculate the average:

$$\begin{aligned} Y_{\text{avg}} &= \frac{1}{20}\left[50x + 5x^2 - \frac{0.5}{3}x^3\right]_0^{20} \\ &= \frac{1}{20}\left[1000 + 2000 - \frac{4000}{3}\right] \\ &= \frac{1}{20}[3000 - 1333.33] = \frac{1666.67}{20} \approx 83.33 \text{ kg/ha} \end{aligned}$$

(c) Total yield: 83.33 kg/ha×10 ha = 833.3 kg
Maximize yield: $Y'(x) = 10 - x = 0 \Rightarrow x = 10$ kg/ha

66. **Urban Traffic Flow Analysis in Kampala**
Scenario: The rate of vehicles passing through a busy Kampala intersection is $V(t) = 500 + 300\cos\left(\frac{\pi t}{12}\right)$ vehicles/hour, where t is hours after 6:00 AM.
   (a) Find the average traffic flow between 6:00 AM and 6:00 PM.
   (b) Calculate the total number of vehicles passing through during this period.
   (c) At what time does the maximum flow occur?

Solution 6

(a) Average flow rate:

$$V_{\text{avg}} = \frac{1}{12}\int_0^{12}\left[500 + 300\cos\left(\frac{\pi t}{12}\right)\right]dt$$

(b) Calculate the average:

$$\begin{aligned} V_{\text{avg}} &= \frac{1}{12}\left[\int_0^{12} 500dt + \int_0^{12} 300\cos\left(\frac{\pi t}{12}\right)dt\right] \\ &= \frac{1}{12}\left[6000 + 300\cdot\frac{12}{\pi}\sin\left(\frac{\pi t}{12}\right)\Big|_0^{12}\right] \\ &= \frac{1}{12}\left[6000 + \frac{3600}{\pi}(0-0)\right] = \frac{6000}{12} = 500 \text{ vehicles} \end{aligned}$$

(c) Total vehicles: 500 vehicles/hour × 12 hours = 6000 vehicles
Maximum flow when $\cos\left(\frac{\pi t}{12}\right) = 1 \Rightarrow t = 0$ (6:00 AM)

67. **Fish Population Dynamics in Lake Victoria**
Scenario: The population growth rate of Nile perch is modeled by $\frac{dP}{dt} = 1000e^{0.1t}$ fish/year, where t is years.
   (a) Find the total population increase from year 2 to year 5.
   (b) What is the average growth rate during this period?
   (c) If the initial population at t=2 is 10,000 fish, what is the population at t=5?

Solution 7

(a) Total population increase:

$$\Delta P = \int_2^5 1000e^{0.1t}dt$$

(b) Calculate the increase:

$$\begin{aligned} \Delta P &= \left[\frac{1000}{0.1}e^{0.1t}\right]_2^5 = 10000\left(e^{0.5} - e^{0.2}\right) \\ &\approx 10000(1.6487 - 1.2214) = 10000 \times 0.4273 = 4273 \end{aligned}$$

(c) Average growth rate: $\frac{4273}{3} \approx 1424.3$ fish/year
Population at t=5: 10000 + 4273 = 14273 fish

68. **Medical Supply Distribution**
Scenario: The rate of medical supply usage at a Kampala hospital is $U(t) = 200 - 50e^{-0.2t}$ units/day, where t is days after a disease outbreak.
   (a) Find the total supplies used in the first 10 days.
   (b) What is the average daily usage during this period?
   (c) When does the usage rate drop below 150 units/day?

Solution 8

(a) Total supplies used:

$$T = \int_0^{10} (200 - 50e^{-0.2t})dt$$

(b) Calculate the total:

$$\begin{aligned} T &= \left[200t + 250e^{-0.2t}\right]_0^{10} \\ &= [2000 + 250e^{-2}] - [0 + 250] \\ &= 2000 + 33.83 - 250 = 1783.83 \text{ units} \end{aligned}$$

(c) Average daily usage: $1783.83/10 = 178.38$ units/day
Solve $200 - 50e^{-0.2t} = 150 \Rightarrow e^{-0.2t} = 1 \Rightarrow t = 0$ days

69. **Water Tank Design for Rural Communities**
Scenario: A cylindrical water tank is designed by rotating the curve $y = \sqrt{10x}$ about the y-axis, where x and y are in meters.
(a) Set up the integral to find the volume of the tank.
(b) Calculate the capacity of the tank in liters.
(c) If the average daily water requirement is 20 liters per person, how many people can this tank serve?

Solution 9

(a) Volume of revolution about y-axis:

$$V = \pi \int_0^h x^2 dy$$

Since $y = \sqrt{10x}$, then $x = y^2/10$, and $x^2 = y^4/100$

(b) Calculate the volume:

$$\begin{aligned} V &= \pi \int_0^{10} \frac{y^4}{100} dy = \frac{\pi}{100}\left[\frac{y^5}{5}\right]_0^{10} \\ &= \frac{\pi}{100} \cdot \frac{100000}{5} = \frac{\pi}{100} \cdot 20000 = 200\pi \approx 628.3 \text{ m}^3 \end{aligned}$$

(c) Capacity: $628.3 \text{ m}^3 = 628,300$ liters
People served: $628,300/20 = 31,415$ people

70. **Solar Cooker Performance Analysis**
Scenario: The temperature of a solar cooker follows $T(t) = 20 + 80\sin\left(\frac{\pi t}{8}\right)$ °C, where t is hours after sunrise.
(a) Find the average temperature between t=2 and t=6 hours.
(b) Calculate the total thermal energy accumulated (proportional to the integral of temperature).
(c) At what time does the cooker reach its maximum temperature?

Solution 10

(a) Average temperature:

$$T_{\text{avg}} = \frac{1}{4}\int_2^6 \left[20 + 80\sin\left(\frac{\pi t}{8}\right)\right] dt$$

(b) Calculate the average:

$$\begin{aligned} T_{\text{avg}} &= \frac{1}{4}\left[20t - \frac{640}{\pi}\cos\left(\frac{\pi t}{8}\right)\right]_2^6 \\ &= \frac{1}{4}\left[(120 - \frac{640}{\pi}\cos(0.75\pi)) - (40 - \frac{640}{\pi}\cos(0.25\right. \\ &= \frac{1}{4}\left[(120 - \frac{640}{\pi}(-0.7071)) - (40 - \frac{640}{\pi}(0.7071))\right. \\ &= \frac{1}{4}[120 + 144.3 - 40 + 144.3] = \frac{368.6}{4} = 92.15°\text{C} \end{aligned}$$

(c) Total thermal energy proportional to: $\int_2^6 T(t)dt \approx 4 \times 92.15 = 368.6$ °C·hours
Maximum temperature when $\sin\left(\frac{\pi t}{8}\right) = 1 \Rightarrow \frac{\pi t}{8} = \frac{\pi}{2} \Rightarrow t = 4$ hours

Question 71: Agriculture – Coffee Plantation Growth

In the coffee-rich district of Mbale, farmers observe that the rate at which a coffee tree grows in height is given by

$$\frac{dy}{dt} = 5e^{0.1t},$$

where y is the height of the tree in centimeters and t is the time in months since planting. This model was obtained from agricultural research to help farmers plan for pruning and shading.

a) By integrating, determine the general expression for the height $y(t)$ of the coffee tree. b) If the tree was 20 cm tall at the time of planting, calculate its height after 12 months. c) Explain in detail how farmers can use this mathematical model to make long-term farm management decisions, such as spacing, shade planning, and harvest preparation.

Solution: a) $y(t) = \int 5e^{0.1t}dt = 50e^{0.1t} + C$. b) At $t = 0, y = 20 \implies 20 = 50(1) + C \implies C = -30$. Thus $y(t) = 50e^{0.1t} - 30$. At $t = 12$, $y(12) = 50e^{1.2} - 30 \approx 137.7$ cm. c) Farmers can

predict average tree heights, plan for optimal plantation spacing, and anticipate harvest conditions.

Question 72: Fisheries – Lake Victoria Fish Stock Management

In a fish farm near Lake Victoria, the population of tilapia is observed to change according to the law

$$\frac{dP}{dt} = 200 - 0.5P,$$

where P is the fish population and t is time in weeks. This model helps fisheries officers estimate sustainable harvest levels.

a) Use integration to determine the function $P(t)$. b) If the fish population at $t = 0$ was 100, calculate the expected population after 10 weeks. c) Discuss how fisheries officers can apply this result to avoid overfishing and maintain a balanced ecosystem in Uganda's lakes.

Solution: a) Rearranging: $\frac{dP}{200-0.5P} = dt$. Integrating: $-2\ln|200 - 0.5P| = t + C$. b) Substituting $t = 0, P = 100$ gives $C = -2\ln(150)$. Using this, solve numerically for $P(10)$. c) The model assists in setting harvest quotas and preventing ecological damage due to overfishing.

Question 73: Transport – Fuel Consumption on a Highway

A bus company operating on the Kampala–Gulu highway wants to model fuel consumption. It is found that the rate of fuel usage depends on distance x traveled (in km) and is modeled by

$$\frac{dF}{dx} = 0.05x + 2,$$

where F is the fuel consumed in liters.

a) Integrate the rate function to find total fuel used $F(x)$. b) If the bus starts with no fuel consumed at $x = 0$, calculate the fuel used over 200 km. c) Explain how transport companies can use such models to budget fuel costs and determine ticket prices sustainably.

Solution: a) $F(x) = \int (0.05x + 2)\,dx = 0.025x^2 + 2x + C$. Since $F(0) = 0$, $C = 0$. b) $F(200) = 0.025(200^2) + 2(200) = 1000 + 400 = 1400$ liters. c) Companies can predict consumption, plan refueling stops, and avoid losses.

Question 74: Health – Drug Absorption in the Bloodstream

In Mulago Hospital, doctors study the rate at which a drug enters the bloodstream, modeled by:

$$\frac{dC}{dt} = 10e^{-0.5t},$$

where C is drug concentration in mg/L and t is time in hours.

a) Find the concentration function $C(t)$. b) If at $t = 0$ the bloodstream had no drug, determine the concentration after 4 hours. c) Explain how this information helps doctors prescribe safe drug dosages and intervals for patients.

Solution: a) $C(t) = \int 10e^{-0.5t}\,dt = -20e^{-0.5t} + C$. Using $C(0) = 0$, constant $C = 20$. So $C(t) = 20 - 20e^{-0.5t}$. b) At $t = 4$: $C(4) = 20 - 20e^{-2} \approx 17.3$ mg/L. c) Doctors avoid overdosing and determine safe drug schedules.

Question 75: Environment – River Pollution in Mbarara

Factory discharge into River Rwizi is found to follow the pattern:

$$\frac{dQ}{dt} = 100\sin(0.1t),$$

where Q is pollutant load in kg and t is time in days.

a) Integrate to find $Q(t)$. b) Find the total pollutant load after 30 days. c) Discuss how environmental regulators can use such models to design pollution control measures.

Solution: a) $Q(t) = -1000\cos(0.1t) + C$. Since $Q(0) = 0$, $C = 1000$. So $Q(t) = -1000\cos(0.1t) + 1000$. b) At $t = 30$, $Q(30) \approx 1000 - 1000\cos(3) \approx 1990$ kg. c) Helps in designing pollution treatment plants.

76. **Coffee Farm Irrigation Coverage in Mbale District**
Context: A coffee farm in Mbale uses a sprinkler system where water coverage follows the curve $y = 4 - x^2$ (in hundreds of square meters) for $x \in [0, 2]$ representing distance from the main pipe in kilometers.
(a) Find the total irrigated area under the coverage curve bounded by the x-axis.

Demonstrate that area is a scalar quantity (positive value only).

(b) The farm also has a secondary irrigation system following $y = x^2$. Calculate the area between the two irrigation curves to determine the differential coverage zone.

(c) If the sprinkler rotates to create a solid of revolution about the x-axis, find the volume of the water coverage zone in cubic kilometers.

Solution:

(a) Area under curve $y = 4 - x^2$:

$$A = \int_0^2 (4 - x^2)\,dx \tag{7.46}$$

$$= \left[4x - \frac{x^3}{3}\right]_0^2 \tag{7.47}$$

$$= 4(2) - \frac{(2)^3}{3} - 0 \tag{7.48}$$

$$= 8 - \frac{8}{3} = \frac{24-8}{3} = \frac{16}{3} \text{ hundred m}^2 \tag{7.49}$$

Since we integrate a positive function over the given interval, the result is positive, confirming that **area is a scalar quantity** with magnitude $\frac{16}{3} \times 100 = 533.33$ m².

(b) Area between curves $y = 4 - x^2$ and $y = x^2$:

First, find intersection points: $4 - x^2 = x^2 \Rightarrow 4 = 2x^2 \Rightarrow x = \sqrt{2}$

$$A_{between} = \int_0^{\sqrt{2}} [(4 - x^2) - x^2]\,dx \tag{7.50}$$

$$= \int_0^{\sqrt{2}} (4 - 2x^2)\,dx \tag{7.51}$$

$$= \left[4x - \frac{2x^3}{3}\right]_0^{\sqrt{2}} \tag{7.52}$$

$$= 4\sqrt{2} - \frac{2(\sqrt{2})^3}{3} \tag{7.53}$$

$$= 4\sqrt{2} - \frac{2 \cdot 2\sqrt{2}}{3} \tag{7.54}$$

$$= 4\sqrt{2} - \frac{4\sqrt{2}}{3} = \frac{12\sqrt{2} - 4\sqrt{2}}{3} = \frac{8\sqrt{2}}{3} \text{ hundred m}^2 \tag{7.55}$$

Differential coverage: $\frac{8\sqrt{2}}{3} \times 100 = 377.12$ m²

(c) Volume of revolution about x-axis:

$$V = \pi \int_0^2 (4 - x^2)^2\,dx \tag{7.56}$$

$$= \pi \int_0^2 (16 - 8x^2 + x^4)\,dx \tag{7.57}$$

$$= \pi \left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_0^2 \tag{7.58}$$

$$= \pi \left[32 - \frac{64}{3} + \frac{32}{5}\right] \tag{7.59}$$

$$= \pi \left[\frac{480 - 320 + 96}{15}\right] = \frac{256\pi}{15} \text{ km}^3 \tag{7.60}$$

Water coverage volume: $\frac{256\pi}{15} \approx 53.54$ km³

77. **Matoke Transportation Analysis in Kampala**

Context: A matoke truck's velocity while navigating Kampala traffic follows $v(t) = 2t^2 - 6t + 4$ km/h for the first 3 hours of delivery.

(a) Calculate the displacement of the truck over the 3-hour period and interpret the physical meaning of positive/negative areas under the velocity curve.

(b) Find when the truck changes direction and calculate the total distance traveled (sum of absolute displacements).

(c) If acceleration $a(t) = \frac{dv}{dt}$, find the average acceleration over the 3-hour journey and relate it to the mean value theorem.

Solution:

(a) Displacement calculation:

$$s(t) = \int_0^3 (2t^2 - 6t + 4)\,dt \tag{7.61}$$

$$= \left[\frac{2t^3}{3} - 3t^2 + 4t\right]_0^3 \tag{7.62}$$

$$= \frac{2(27)}{3} - 3(9) + 4(3) - 0 \tag{7.63}$$

$$= 18 - 27 + 12 = 3 \text{ km} \tag{7.64}$$

Physical interpretation: The truck's net displacement is 3 km forward. Positive areas under the velocity curve represent forward motion, while negative areas represent backward motion.

(b) Direction changes occur when $v(t) = 0$:

$$2t^2 - 6t + 4 = 0 \quad (7.65)$$
$$t^2 - 3t + 2 = 0 \quad (7.66)$$
$$(t-1)(t-2) = 0 \quad (7.67)$$

The truck stops at $t = 1$ hour and $t = 2$ hours.

Velocity analysis: - $t \in [0,1]$: $v(0) = 4 > 0$ (forward) - $t \in [1,2]$: $v(1.5) = 2(1.5)^2 - 6(1.5) + 4 = -0.5 < 0$ (backward) - $t \in [2,3]$: $v(2.5) = 2(2.5)^2 - 6(2.5) + 4 = 1.5 > 0$ (forward)

Distance calculation:

$$d_1 = \int_0^1 (2t^2 - 6t + 4)\,dt = \left[\frac{2t^3}{3} - 3t^2 + 4t\right]_0^1 = \frac{2}{3} - 3 + 4 = \frac{5}{3} \quad (7.68)$$

$$d_2 = \int_1^2 |2t^2 - 6t + 4|\,dt = -\int_1^2 (2t^2 - 6t + 4)\,dt = \frac{4}{3} \quad (7.69)$$

$$d_3 = \int_2^3 (2t^2 - 6t + 4)\,dt = \frac{5}{3} \quad (7.70)$$

Total distance: $d_1 + d_2 + d_3 = \frac{5}{3} + \frac{4}{3} + \frac{5}{3} = \frac{14}{3} \approx 4.67$ km

(c) Acceleration: $a(t) = \frac{dv}{dt} = 4t - 6$

Average acceleration using mean value theorem:

$$\bar{a} = \frac{1}{3-0}\int_0^3 (4t-6)\,dt \quad (7.71)$$
$$= \frac{1}{3}\left[2t^2 - 6t\right]_0^3 \quad (7.72)$$
$$= \frac{1}{3}[18 - 18] = 0 \text{ km/h}^2 \quad (7.73)$$

The average acceleration is 0, meaning the truck returns to its initial velocity after 3 hours.

78. **Lake Victoria Water Level Management**

Context: Lake Victoria's water level variation follows $h(x) = 3 + \sin(x)$ meters above minimum level, where x represents months (0 to 2π).

(a) Calculate the area under the water level curve over one complete year (12 months) to determine total water volume indicator.

(b) Compare this with a constant water level of 3 meters by finding the area between $h(x) = 3 + \sin(x)$ and $h(x) = 3$.

(c) Model the lake as a solid of revolution about the x-axis and calculate the volume of water above the 3-meter baseline.

Solution:

(a) Area under water level curve (scaling 2π to 12 months):

$$A = \int_0^{2\pi} (3 + \sin(x))\,dx \quad (7.74)$$
$$= [3x - \cos(x)]_0^{2\pi} \quad (7.75)$$
$$= [6\pi - \cos(2\pi)] - [0 - \cos(0)] \quad (7.76)$$
$$= 6\pi - 1 + 1 = 6\pi \text{ meter-months} \quad (7.77)$$

Total water volume indicator: $6\pi \approx 18.85$ meter-months

(b) Area between variable and constant levels:

$$A_{diff} = \int_0^{2\pi} [(3 + \sin(x)) - 3]\,dx \quad (7.78)$$
$$= \int_0^{2\pi} \sin(x)\,dx \quad (7.79)$$
$$= [-\cos(x)]_0^{2\pi} \quad (7.80)$$
$$= -\cos(2\pi) + \cos(0) = -1 + 1 = 0 \quad (7.81)$$

The net difference is 0, confirming that seasonal variations average out over a complete cycle.

(c) Volume above 3-meter baseline (revolution about x-axis):

$$V = \pi \int_0^{2\pi} \sin^2(x)\,dx \quad (7.82)$$

Using identity $\sin^2(x) = \frac{1-\cos(2x)}{2}$:

$$V = \pi \int_0^{2\pi} \frac{1 - \cos(2x)}{2}\,dx \quad (7.83)$$
$$= \frac{\pi}{2}\int_0^{2\pi} (1 - \cos(2x))\,dx \quad (7.84)$$
$$= \frac{\pi}{2}\left[x - \frac{\sin(2x)}{2}\right]_0^{2\pi} \quad (7.85)$$
$$= \frac{\pi}{2}[2\pi - 0] = \pi^2 \text{ cubic meter-months} \quad (7.86)$$

Volume above baseline: $\pi^2 \approx 9.87$ cubic meter-months

79. **Banana Plantation Yield Distribution in Masaka**

Context: A banana plantation's yield density follows $f(x) = xe^{-x}$ tons per hectare, where x represents distance from the irrigation center (0 to 4 km).

(a) Calculate the total yield area under the density curve from 0 to 4 km using integration by parts.

(b) Find the mean yield per hectare over the 4 km plantation area using the mean value theorem.

(c) Model the plantation as a solid of revolution about the y-axis and calculate the volume representing yield distribution.

Solution:

(a) Total yield using integration by parts: Let $u = x$, $dv = e^{-x}dx$, then $du = dx$, $v = -e^{-x}$

$$\int xe^{-x}\,dx = -xe^{-x} - \int (-e^{-x})\,dx \tag{7.87}$$

$$= -xe^{-x} - e^{-x} = -e^{-x}(x+1) \tag{7.88}$$

$$\text{Total yield} = \int_0^4 xe^{-x}\,dx \tag{7.89}$$

$$= [-e^{-x}(x+1)]_0^4 \tag{7.90}$$

$$= -e^{-4}(5) - (-e^0(1)) \tag{7.91}$$

$$= -5e^{-4} + 1 \tag{7.92}$$

$$= 1 - \frac{5}{e^4} \approx 1 - 0.091 = 0.909 \text{ tons} \tag{7.93}$$

(b) Mean yield per hectare:

$$\bar{f} = \frac{1}{4-0}\int_0^4 xe^{-x}\,dx \tag{7.94}$$

$$= \frac{1}{4}(0.909) = 0.227 \text{ tons per hectare} \tag{7.95}$$

(c) Volume of revolution about y-axis using shell method:

$$V = 2\pi \int_0^4 x \cdot xe^{-x}\,dx = 2\pi \int_0^4 x^2e^{-x}\,dx \tag{7.96}$$

Using integration by parts twice:

$$\int x^2e^{-x}\,dx = -x^2e^{-x} + 2\int xe^{-x}\,dx \tag{7.97}$$

$$= -x^2e^{-x} - 2e^{-x}(x+1) \tag{7.98}$$

$$= -e^{-x}(x^2 + 2x + 2) \tag{7.99}$$

$$V = 2\pi[-e^{-x}(x^2+2x+2)]_0^4 \tag{7.100}$$

$$= 2\pi[-e^{-4}(16+8+2) + e^0(2)] \tag{7.101}$$

$$= 2\pi[2 - 26e^{-4}] \approx 2\pi(1.526) = 3.05\pi \text{ cubic km} \tag{7.102}$$

80. **Solar Panel Efficiency in Gulu District**

Context: Solar panel efficiency varies throughout the day following $E(t) = 0.5t(6-t)$ percent for $t \in [0, 6]$ hours after sunrise.

(a) Calculate the area under the efficiency curve to determine total daily energy collection potential.

(b) Find when efficiency is maximum and compare the area above this maximum level with the area below it.

(c) Calculate the average efficiency over the 6-hour period and interpret its practical significance.

Solution:

(a) Total energy collection potential:

$$A = \int_0^6 0.5t(6-t)\,dt = \int_0^6 (3t - 0.5t^2)\,dt \tag{7.103}$$

$$= \left[\frac{3t^2}{2} - \frac{0.5t^3}{3}\right]_0^6 \tag{7.104}$$

$$= \left[\frac{3(36)}{2} - \frac{0.5(216)}{3}\right] \tag{7.105}$$

$$= 54 - 36 = 18 \text{ percent-hours} \tag{7.106}$$

(b) Maximum efficiency occurs when $\frac{dE}{dt} = 0$:

$$\frac{d}{dt}[0.5t(6-t)] = \frac{d}{dt}[3t - 0.5t^2] = 3 - t = 0 \tag{7.107}$$

Maximum at $t = 3$ hours with $E(3) = 0.5(3)(3) = 4.5\%$

Area above maximum level: Since $E(t) = 0.5t(6-t)$ is a parabola opening

downward with maximum at $t = 3$, we compare with constant level $E = 4.5$:

$$A_{above} = \int_0^6 \max(0, E(t) - 4.5)\, dt \tag{7.108}$$

$$= \int_1^5 (3t - 0.5t^2 - 4.5)\, dt \text{ (after finding intersection points)} \tag{7.109}$$

This requires solving $3t - 0.5t^2 = 4.5$, giving $t = 1.5$ and $t = 4.5$.

(c) Average efficiency:

$$\bar{E} = \frac{1}{6}\int_0^6 0.5t(6 - t)\, dt = \frac{18}{6} = 3\% \text{ average efficiency} \tag{7.110}$$

Practical significance: The solar panels operate at 3

81. **Nile River Flow Analysis at Jinja**

Context: Water flow rate at Jinja follows $Q(t) = 100 + 50\cos\left(\frac{\pi t}{6}\right)$ m³/s over 12 hours.

(a) Calculate the total water volume flowing past Jinja in 12 hours by finding the area under the flow rate curve.

(b) Determine the area between the actual flow rate and the average flow rate to assess flow variability.

(c) Model the river cross-section as rotating about the time axis to find a theoretical volume measure.

Solution:

(a) Total water volume:

$$V = \int_0^{12} \left(100 + 50\cos\left(\frac{\pi t}{6}\right)\right) dt \tag{7.111}$$

$$= \left[100t + 50 \cdot \frac{\sin\left(\frac{\pi t}{6}\right)}{\frac{\pi}{6}}\right]_0^{12} \tag{7.112}$$

$$= \left[100t + \frac{300}{\pi}\sin\left(\frac{\pi t}{6}\right)\right]_0^{12} \tag{7.113}$$

$$= 1200 + \frac{300}{\pi}\sin(2\pi) - 0 = 1200 \text{ m}^3 \tag{7.114}$$

(b) Average flow rate: $\bar{Q} = \frac{1200}{12} = 100$ m³/s

Area between actual and average flow:

$$A_{var} = \int_0^{12} |Q(t) - 100|\, dt = \int_0^{12} \left|50\cos\left(\frac{\pi t}{6}\right)\right| dt \tag{7.115}$$

Since $\cos\left(\frac{\pi t}{6}\right)$ changes sign, we integrate over intervals:

$$A_{var} = 2\int_0^3 50\cos\left(\frac{\pi t}{6}\right) dt + 2\int_6^9 50\cos\left(\frac{\pi t}{6}\right) dt \tag{7.116}$$

$$= 4 \times \frac{300}{\pi} = \frac{1200}{\pi} \approx 382 \text{ m}^3 \tag{7.117}$$

(c) Volume of revolution about t-axis:

$$V_{rev} = \pi\int_0^{12} \left(100 + 50\cos\left(\frac{\pi t}{6}\right)\right)^2 dt \tag{7.118}$$

$$= \pi\int_0^{12} \left(10000 + 10000\cos\left(\frac{\pi t}{6}\right) + 2500\cos^2\left(\frac{\pi t}{6}\right)\right. \tag{7.119}$$

Using $\cos^2(x) = \frac{1+\cos(2x)}{2}$:

$$V_{rev} = \pi[120000 + 0 + 15000] = 135000\pi \text{ m}^3 \tag{7.120}$$

82. **Tea Leaf Processing Temperature in Fort Portal**

Context: Tea processing temperature follows $T(t) = 80 + 20e^{-0.1t}$ °C over 10 hours of processing.

(a) Find the area under the temperature curve representing total thermal energy input over 10 hours.

(b) Calculate the mean processing temperature and compare it with the initial temperature.

(c) Model thermal distribution as a solid of revolution about the temperature axis and find the volume.

Solution:

(a) Total thermal energy representation:

$$A = \int_0^{10} (80 + 20e^{-0.1t})\, dt \tag{7.121}$$

$$= \left[80t + 20 \cdot \frac{e^{-0.1t}}{-0.1}\right]_0^{10} \tag{7.122}$$

$$= \left[80t - 200e^{-0.1t}\right]_0^{10} \tag{7.123}$$

$$= [800 - 200e^{-1}] - [0 - 200] \tag{7.124}$$

$$= 800 - 200e^{-1} + 200 = 1000 - \frac{200}{e} \approx 926.4 \text{ °C-ho} \tag{7.125}$$

(b) Mean processing temperature:

$$\bar{T} = \frac{1}{10}\int_0^{10}(80 + 20e^{-0.1t})\,dt \tag{7.126}$$

$$= \frac{926.4}{10} = 92.64\ ^\circ\text{C} \tag{7.127}$$

Initial temperature: $T(0) = 80 + 20 = 100\ ^\circ$C

The mean temperature (92.64°C) is lower than initial temperature due to exponential decay.

(c) Volume of revolution about temperature axis (using shell method):

$$V = 2\pi\int_0^{10} t(80 + 20e^{-0.1t})\,dt \tag{7.128}$$

$$= 2\pi\left[80\int_0^{10} t\,dt + 20\int_0^{10} te^{-0.1t}\,dt\right] \tag{7.129}$$

For the exponential integral, using integration by parts:

$$\int te^{-0.1t}\,dt = -10te^{-0.1t} + 100e^{-0.1t} \tag{7.130}$$

$$V = 2\pi[40(100) + 20(100 - 110e^{-1})] \tag{7.131}$$

$$= 2\pi[4000 + 2000 - 2200e^{-1}] \approx 11191\pi \text{ cubic units} \tag{7.132}$$

83. **Boda Boda Acceleration Pattern in Kampala**

Context: A boda boda's acceleration in city traffic follows $a(t) = 6 - 2t$ m/s² for the first 4 seconds, starting from rest.

(a) Find the velocity function by integrating acceleration, then find displacement by integrating velocity.

(b) Calculate the total distance traveled and compare with net displacement over 4 seconds.

(c) Find the average velocity over the journey and verify using the mean value theorem.

Solution:

(a) Finding velocity and displacement:

Velocity: $v(t) = \int a(t)\,dt = \int(6 - 2t)\,dt = 6t - t^2 + C$

Since starting from rest, $v(0) = 0$, so $C = 0$. Therefore: $v(t) = 6t - t^2$

Displacement: $s(t) = \int v(t)\,dt = \int(6t - t^2)\,dt = 3t^2 - \frac{t^3}{3} + C$

With $s(0) = 0$, we have $C = 0$. Therefore: $s(t) = 3t^2 - \frac{t^3}{3}$

Displacement at $t = 4$: $s(4) = 3(16) - \frac{64}{3} = 48 - 21.33 = 26.67$ m

(b) The boda boda changes direction when $v(t) = 0$: $6t - t^2 = 0 \Rightarrow t(6 - t) = 0 \Rightarrow t = 0$ or $t = 6$

Since we only consider $t \in [0, 4]$, the boda boda doesn't change direction.

Total distance = Net displacement = 26.67 m

(c) Average velocity:

$$\bar{v} = \frac{1}{4}\int_0^4 (6t - t^2)\,dt \tag{7.133}$$

$$= \frac{1}{4}[s(4) - s(0)] = \frac{26.67}{4} = 6.67\ \text{m/s} \tag{7.134}$$

By mean value theorem: $\bar{v} = \frac{s(4)-s(0)}{4-0} = 6.67$ m/s

84. **Maize Yield Distribution in Soroti District**

Context: Maize yield across a field follows $y = 4x - x^2$ tons per hectare, where x represents distance (0 to 4 km) from the storage facility.

(a) Calculate the total yield area under the curve from 0 to 4 km.

(b) Find where yield is maximum and calculate the area of high-yield zones (above mean yield level).

(c) Model the field as a solid of revolution about the x-axis to determine storage volume requirements.

Solution:

(a) Total yield:

$$A = \int_0^4 (4x - x^2)\,dx \tag{7.135}$$

$$= \left[2x^2 - \frac{x^3}{3}\right]_0^4 \tag{7.136}$$

$$= 32 - \frac{64}{3} = \frac{96 - 64}{3} = \frac{32}{3} \approx 10.67\ \text{tons} \tag{7.137}$$

(b) Maximum yield when $\frac{dy}{dx} = 4 - 2x = 0 \Rightarrow x = 2$ km

Maximum yield: $y(2) = 4(2) - (2)^2 = 4$ tons per hectare

Mean yield: $\bar{y} = \frac{10.67}{4} = 2.67$ tons per hectare

High-yield zones where $y > 2.67$: $4x - x^2 > 2.67 \Rightarrow x^2 - 4x + 2.67 < 0$
Solving: $x = 2 \pm \sqrt{4 - 2.67} = 2 \pm 1.15$
High-yield zone: $x \in [0.85, 3.15]$ km

85. **Maize Yield Optimization in Kakuto Plains**
Scenario: The yield of a maize field in Kakuto Plains is modeled by $Y(x) = 40 + 8x - 0.4x^2$ kg per hectare, where x is the amount of fertilizer applied in kg per hectare.
   (a) Find the average yield for fertilizer applications between 0 and 15 kg/ha.
   (b) Determine the total yield for a 5-hectare field with fertilizer application varying from 0 to 15 kg/ha.
   (c) At what fertilizer application rate is the yield maximized?

Solution 5

(a) Average yield:

$$Y_{\text{avg}} = \frac{1}{15}\int_0^{15}(40 + 8x - 0.4x^2)dx$$

(b) Calculate the average:

$$\begin{aligned} Y_{\text{avg}} &= \frac{1}{15}\left[40x + 4x^2 - \frac{0.4}{3}x^3\right]_0^{15} \\ &= \frac{1}{15}\left[600 + 900 - \frac{0.4}{3}(3375)\right] \\ &= \frac{1}{15}[1500 - 450] = \frac{1050}{15} = 70 \text{ kg/ha} \end{aligned}$$

(c) Total yield: 70 kg/ha $\times$ 5 ha = 350 kg
Maximize yield: $Y'(x) = 8 - 0.8x = 0 \Rightarrow x = 10$ kg/ha

86. **Water Flow in Kakuto Irrigation Channel**
Scenario: The flow rate in a Kakuto irrigation channel is modeled by $F(t) = 50 + 20\cos\left(\frac{\pi t}{12}\right)$ liters/second, where t is hours after 6:00 AM.
   (a) Find the average flow rate between 6:00 AM and 6:00 PM.
   (b) Calculate the total volume of water that flows through during this period.
   (c) If a typical farm requires 50,000 liters daily for irrigation, what percentage of this need does this channel provide?

Solution 6

(a) Average flow rate:

$$F_{\text{avg}} = \frac{1}{12}\int_0^{12}\left[50 + 20\cos\left(\frac{\pi t}{12}\right)\right]dt$$

(b) Calculate the average:

$$\begin{aligned} F_{\text{avg}} &= \frac{1}{12}\left[\int_0^{12}50dt + \int_0^{12}20\cos\left(\frac{\pi t}{12}\right)dt\right] \\ &= \frac{1}{12}\left[600 + 20\cdot\frac{12}{\pi}\sin\left(\frac{\pi t}{12}\right)\Big|_0^{12}\right] \\ &= \frac{1}{12}\left[600 + \frac{240}{\pi}(0 - 0)\right] = \frac{600}{12} = 50 \text{ liters/second} \end{aligned}$$

(c) Total volume: 50 l/s $\times$ 12 $\times$ 3600 s = 2, 160, 000 liters
Percentage of farm need: $(2,160,000/50,000) \times 100 = 4320\%$ (serves many farms)

87. **Soil Erosion Analysis in Kakuto Slopes**
Scenario: The rate of soil erosion on a Kakuto hillside is modeled by $E(t) = 5 + 2e^{-0.1t}$ kg/day, where t is days after the start of rainy season.
   (a) Find the total soil loss in the first 30 days of rainy season.
   (b) What is the average daily erosion during this period?
   (c) If the initial soil depth was 50 cm and soil density is 1200 kg/m^3, what percentage of topsoil is lost?

Solution 7

(a) Total soil loss:

$$T = \int_0^{30}(5 + 2e^{-0.1t})dt$$

(b) Calculate the total:

$$\begin{aligned} T &= \left[5t - 20e^{-0.1t}\right]_0^{30} \\ &= [150 - 20e^{-3}] - [0 - 20] \\ &= 150 - 0.996 + 20 = 169.004 \text{ kg} \end{aligned}$$

(c) Average daily erosion: $169.004/30 = 5.63$ kg/day
For a 100 m^2 area: Volume lost = $169.004/1200 = 0.1408$ m^3
Depth lost = $0.1408/100 = 0.001408$ m = 0.1408 cm
Percentage lost: $(0.1408/50) \times 100 = 0.28\%$

88. **Medical Supply Usage in Kakuto Clinic**
Scenario: The rate of medical supply usage at Kakuto Clinic is $U(t) = 30 - 10e^{-0.2t}$ units/day, where t is days after a disease outbreak.
   (a) Find the total supplies used in the first 14 days.

(b) What is the average daily usage during this period?

(c) When does the usage rate drop to 25 units/day?

Solution 8

(a) Total supplies used:

$$T = \int_0^{14} (30 - 10e^{-0.2t})dt$$

(b) Calculate the total:

$$\begin{aligned} T &= \left[30t + 50e^{-0.2t}\right]_0^{14} \\ &= [420 + 50e^{-2.8}] - [0 + 50] \\ &= 420 + 3.72 - 50 = 373.72 \text{ units} \end{aligned}$$

(c) Average daily usage: $373.72/14 = 26.69$ units/day
Solve $30 - 10e^{-0.2t} = 25 \Rightarrow e^{-0.2t} = 0.5 \Rightarrow t = -\ln(0.5)/0.2 \approx 3.47$ days

89. **Traditional Grain Storage Silo Design**
Scenario: A cylindrical grain silo is designed by rotating the curve $y = \sqrt{8x}$ about the y-axis, where x and y are in meters.

(a) Set up the integral to find the volume of the silo.

(b) Calculate the capacity of the silo in cubic meters.

(c) If sorghum has a density of 700 kg/m^3, how many kilograms of grain can the silo hold?

Solution 9

(a) Volume of revolution about y-axis:

$$V = \pi \int_0^h x^2 dy$$

Since $y = \sqrt{8x}$, then $x = y^2/8$, and $x^2 = y^4/64$

(b) Calculate the volume (for y from 0 to 8):

$$\begin{aligned} V &= \pi \int_0^8 \frac{y^4}{64} dy = \frac{\pi}{64}\left[\frac{y^5}{5}\right]_0^8 \\ &= \frac{\pi}{64} \cdot \frac{32768}{5} = \frac{\pi}{64} \cdot 6553.6 = 102.4\pi \approx 321.7 \text{ m}^3 \end{aligned}$$

(c) Grain capacity: 321.7 m^3×700 kg/m^3 = 225, 190 kg

90. **Solar Dryer Performance Analysis**
Scenario: The temperature of a solar food dryer in Kakuto follows $T(t) = 25 + 60\sin\left(\frac{\pi t}{9}\right)$ °C, where t is hours after sunrise.

(a) Find the average temperature between t=3 and t=6 hours.

(b) Calculate the total thermal energy accumulated (proportional to the integral of temperature).

(c) At what time does the dryer reach its maximum temperature?

Solution 10

(a) Average temperature:

$$T_{\text{avg}} = \frac{1}{3}\int_3^6 \left[25 + 60\sin\left(\frac{\pi t}{9}\right)\right] dt$$

(b) Calculate the average:

$$\begin{aligned} T_{\text{avg}} &= \frac{1}{3}\left[25t - \frac{540}{\pi}\cos\left(\frac{\pi t}{9}\right)\right]_3^6 \\ &= \frac{1}{3}\left[(150 - \frac{540}{\pi}\cos(\frac{2\pi}{3})) - (75 - \frac{540}{\pi}\cos(\frac{\pi}{3}))\right] \\ &= \frac{1}{3}\left[(150 - \frac{540}{\pi}(-0.5)) - (75 - \frac{540}{\pi}(0.5))\right] \\ &= \frac{1}{3}\left[150 + 85.95 - 75 + 85.95\right] = \frac{246.9}{3} = 82.3°\text{C} \end{aligned}$$

(c) Total thermal energy proportional to: $\int_3^6 T(t)dt \approx 3 \times 82.3 = 246.9$ °C·hours
Maximum temperature when $\sin\left(\frac{\pi t}{9}\right) = 1 \Rightarrow \frac{\pi t}{9} = \frac{\pi}{2} \Rightarrow t = 4.5$ hours

Question 91: Agriculture – Coffee Plantation Growth

In the coffee-rich district of Mbale, farmers observe that the rate at which a coffee tree grows in height is given by

$$\frac{dy}{dt} = 5e^{0.1t},$$

where y is the height of the tree in centimeters and t is the time in months since planting. This model was obtained from agricultural research to help farmers plan for pruning and shading.

a) By integrating, determine the general expression for the height $y(t)$ of the coffee tree. b) If the tree was 20 cm tall at the time of planting, calculate its height after 12 months. c) Explain in detail how farmers can use this mathematical model to make long-term farm management decisions, such as spacing, shade planning, and harvest preparation.

Solution: a) $y(t) = \int 5e^{0.1t}dt = 50e^{0.1t} + C$. b) At $t = 0, y = 20 \implies 20 = 50(1) + C \implies C = -30$. Thus $y(t) = 50e^{0.1t} - 30$. At $t = 12$, $y(12) = 50e^{1.2} - 30 \approx 137.7$ cm. c) Farmers can

predict average tree heights, plan for optimal plantation spacing, and anticipate harvest conditions.

Question 92: Fisheries – Lake Victoria Fish Stock Management

In a fish farm near Lake Victoria, the population of tilapia is observed to change according to the law

$$\frac{dP}{dt} = 200 - 0.5P,$$

where P is the fish population and t is time in weeks. This model helps fisheries officers estimate sustainable harvest levels.

a) Use integration to determine the function $P(t)$. b) If the fish population at $t = 0$ was 100, calculate the expected population after 10 weeks. c) Discuss how fisheries officers can apply this result to avoid overfishing and maintain a balanced ecosystem in Uganda's lakes.

Solution: a) Rearranging: $\frac{dP}{200-0.5P} = dt$. Integrating: $-2\ln|200 - 0.5P| = t + C$. b) Substituting $t = 0, P = 100$ gives $C = -2\ln(150)$. Using this, solve numerically for $P(10)$. c) The model assists in setting harvest quotas and preventing ecological damage due to overfishing.

Question 93: Transport – Fuel Consumption on a Highway

A bus company operating on the Kampala–Gulu highway wants to model fuel consumption. It is found that the rate of fuel usage depends on distance x traveled (in km) and is modeled by

$$\frac{dF}{dx} = 0.05x + 2,$$

where F is the fuel consumed in liters.

a) Integrate the rate function to find total fuel used $F(x)$. b) If the bus starts with no fuel consumed at $x = 0$, calculate the fuel used over 200 km. c) Explain how transport companies can use such models to budget fuel costs and determine ticket prices sustainably.

Solution: a) $F(x) = \int (0.05x + 2)\,dx = 0.025x^2 + 2x + C$. Since $F(0) = 0$, $C = 0$. b) $F(200) = 0.025(200^2) + 2(200) = 1000 + 400 = 1400$ liters. c) Companies can predict consumption, plan refueling stops, and avoid losses.

Question 94: Health – Drug Absorption in the Bloodstream

In Mulago Hospital, doctors study the rate at which a drug enters the bloodstream, modeled by:

$$\frac{dC}{dt} = 10e^{-0.5t},$$

where C is drug concentration in mg/L and t is time in hours.

a) Find the concentration function $C(t)$. b) If at $t = 0$ the bloodstream had no drug, determine the concentration after 4 hours. c) Explain how this information helps doctors prescribe safe drug dosages and intervals for patients.

Solution: a) $C(t) = \int 10e^{-0.5t}\,dt = -20e^{-0.5t} + C$. Using $C(0) = 0$, constant $C = 20$. So $C(t) = 20 - 20e^{-0.5t}$. b) At $t = 4$: $C(4) = 20 - 20e^{-2} \approx 17.3$ mg/L. c) Doctors avoid overdosing and determine safe drug schedules.

Question 95: Environment – River Pollution in Mbarara

Factory discharge into River Rwizi is found to follow the pattern:

$$\frac{dQ}{dt} = 100\sin(0.1t),$$

where Q is pollutant load in kg and t is time in days.

a) Integrate to find $Q(t)$. b) Find the total pollutant load after 30 days. c) Discuss how environmental regulators can use such models to design pollution control measures.

Solution: a) $Q(t) = -1000\cos(0.1t) + C$. Since $Q(0) = 0$, $C = 1000$. So $Q(t) = -1000\cos(0.1t) + 1000$. b) At $t = 30$, $Q(30) \approx 1000 - 1000\cos(3) \approx 1990$ kg. c) Helps in designing pollution treatment plants.

Question 96: Hydrology – Rainwater Harvesting in Schools

A rural school in Luwero collects rainwater. The inflow into the storage tank is modeled by:

$$\frac{dV}{dt} = 50 + 2t,$$

where V is in liters and t is hours of rainfall.

a) Integrate to find $V(t)$. b) Find water collected in 10 hours. c) Explain how schools can use such models for better water harvesting and planning.

Solution: a) $V(t) = 50t + t^2 + C$. At $t = 0, V = 0 \implies C = 0$. b) $V(10) = 500 + 100 = 600$ liters. c) Schools can predict storage needs, reduce shortages, and save costs.

Question 97: Economics – Tax Revenue Growth

Uganda Revenue Authority observes tax collection grows at the rate:

$$\frac{dR}{dt} = 5000e^{0.02t},$$

where R is in shillings and t is months.

a) Find $R(t)$. b) Compute total revenue in the first 12 months. c) Explain how government can use this model for budget planning.

Solution: a) $R(t) = 250000e^{0.02t} + C$. With $R(0) = 0$, $C = -250000$. b) $R(12) = 250000(e^{0.24} - 1) \approx 68,100$. c) Enables revenue forecasting for public services.

Question 98: Forestry – Timber Volume Estimation

Forestry officers estimate the volume of a fast-growing tree species by:

$$\frac{dV}{dt} = 3t^2,$$

where V is in m^3 and t is years since planting.

a) Find $V(t)$. b) Find V at $t = 5$ years. c) How can foresters use this to regulate tree harvesting sustainably?

Solution: a) $V(t) = t^3 + C$. Since $V(0) = 0$, $C = 0$. b) $V(5) = 125\ m^3$. c) Supports sustainable logging policies.

Question 99: Energy – Solar Power Output

In Soroti solar farm, the rate of energy generation is given by

$$\frac{dE}{dt} = 100\cos(0.2t),$$

where E is energy in kWh and t is hours of sunshine.

a) Integrate to find $E(t)$. b) Calculate energy produced after 10 hours of sunshine. c) Explain how solar engineers can use such models for energy planning.

Solution: a) $E(t) = 500\sin(0.2t) + C$. With $E(0) = 0$, $C = 0$. b) $E(10) = 500\sin(2) \approx 454.6$ kWh. c) Helps in battery storage planning and power supply predictions.

Question 100: Population – Urban Growth in Kampala

UBOS projects urban growth in Kampala with the model:

$$\frac{dP}{dt} = 0.03P,$$

where P is population and t is in years.

a) Solve for $P(t)$. b) If population was $1,000,000$ in 2020, project population in 2030. c) Discuss how city planners can use this to design housing, roads, and water supply.

Solution: a) $P(t) = P_0e^{0.03t}$. b) $P(10) = 1,000,000e^{0.3} \approx 1,349,000$. c) Supports planning for infrastructure and services.

101. **Coffee Farm Land Utilization in Kakuto Hills**

Scenario: A coffee farmer in Kakuto Hills wants to calculate the area of his irregularly shaped farm. The eastern boundary follows the curve $y = 80 - 0.08x^2$ and the western boundary follows $y = 15 + 0.03x^2$, where x and y are in meters, with x ranging from -25 to 25.

(a) Set up the integral to find the area between these two curves.

(b) Calculate the total area of the farm in hectares.

(c) If coffee plants can be grown at a density of 3 plants per square meter, estimate the total number of plants the farm can support.

Solution 101

(a) The area between curves is given by:

$$A = \int_{-25}^{25} \left[(80 - 0.08x^2) - (15 + 0.03x^2)\right] dx = \int_{-25}^{25} (65$$

(b) Calculate the definite integral:

$$A = \int_{-25}^{25} (65 - 0.11x^2)dx = \left[65x - \frac{0.11}{3}x^3\right]_{-25}^{25}$$
$$= [1625 - 572.92] - [-1625 + 572.92] = 1052.08 - ($$

In hectares: $2104.16/10,000 = 0.2104$ hectares

(c) Total plants = Area × Density = $2104.16\ \text{m}^2 \times 3\ \text{plants/m}^2 = 6312$ plants

102. **Water Reservoir Capacity in Kakuto Valley**
Scenario: A water reservoir in Kakuto Valley has a shape formed by rotating the curve $y = \sqrt{16 - x^2}$ around the x-axis, where x and y are in meters.
   (a) Set up the integral to find the volume of the reservoir.
   (b) Calculate the total volume capacity in cubic meters.
   (c) If the average daily water consumption in Kakuto is 30,000 liters, how many days of supply does the full reservoir provide?

Solution 102

(a) Volume of revolution about x-axis:

$$V = \pi \int_{-4}^{4} y^2 dx = \pi \int_{-4}^{4} (16 - x^2) dx$$

(b) Calculate the volume:

$$\begin{aligned} V &= \pi \int_{-4}^{4} (16 - x^2) dx = \pi \left[16x - \frac{x^3}{3} \right]_{-4}^{4} \\ &= \pi \left[(64 - \frac{64}{3}) - (-64 + \frac{64}{3}) \right] \\ &= \pi \left[\frac{128}{3} - (-\frac{128}{3}) \right] = \pi \cdot \frac{256}{3} \approx 268.08\ \text{m}^3 \end{aligned}$$

(c) Convert to liters: $268.08\ \text{m}^3 = 268,080$ liters
Days of supply: $268,080/30,000 \approx 8.94$ days

103. **Fish Population Dynamics in Kakuto Lake**
Scenario: The population growth rate of Nile tilapia in Kakuto Lake is modeled by $\frac{dP}{dt} = 800e^{0.08t}$ fish/year, where t is years.
   (a) Find the total population increase from year 3 to year 8.
   (b) What is the average growth rate during this period?
   (c) If the initial population at t=3 is 8,000 fish, what is the population at t=8?

Solution 103

(a) Total population increase:

$$\Delta P = \int_{3}^{8} 800e^{0.08t} dt$$

(b) Calculate the increase:

$$\begin{aligned} \Delta P &= \left[\frac{800}{0.08} e^{0.08t} \right]_{3}^{8} = 10000 \left(e^{0.64} - e^{0.24} \right) \\ &\approx 10000(1.8965 - 1.2712) = 10000 \times 0.6253 = 6253 \end{aligned}$$

(c) Average growth rate: $\frac{6253}{5} \approx 1250.6$ fish/year
Population at t=8: $8000 + 6253 = 14253$ fish

104. **Solar Energy Production in Kakuto Village**
Scenario: A solar installation in Kakuto Village generates power according to $P(t) = 300 + 150\sin\left(\frac{\pi t}{10}\right)$ W, where t is hours after sunrise.
   (a) Find the average power output over a 10-hour daylight period.
   (b) Calculate the total energy produced during this period.
   (c) If the average household consumes 1.5 kWh daily, how many households can this installation support?

Solution 104

(a) Average value of a function:

$$P_{\text{avg}} = \frac{1}{10} \int_{0}^{10} \left[300 + 150\sin\left(\frac{\pi t}{10}\right) \right] dt$$

(b) Calculate the average:

$$\begin{aligned} P_{\text{avg}} &= \frac{1}{10} \left[\int_{0}^{10} 300 dt + \int_{0}^{10} 150\sin\left(\frac{\pi t}{10}\right) dt \right] \\ &= \frac{1}{10} \left[3000 + 150 \cdot \left(-\frac{10}{\pi} \cos\left(\frac{\pi t}{10}\right) \right) \Big|_0^{10} \right] \\ &= \frac{1}{10} \left[3000 + \left(-\frac{1500}{\pi} \right) (-1 - 1) \right] \\ &= \frac{1}{10} \left[3000 + \frac{3000}{\pi} \right] \approx \frac{3000 + 954.9}{10} = 395.5\ \text{W} \end{aligned}$$

(c) Total energy: $395.5\ \text{W} \times 10\ \text{h} = 3955\ \text{Wh} = 3.955\ \text{kWh}$
Households supported: $3.955/1.5 \approx 2.64$ households (serves 2 households completely)

Chapter 8

Permutations and Combinations

8.1 Arranging sets of different objects in a row to distinguish between different arrangements

8.2 Arranging a number of different objects in turn to derive the relationship between the factorial notation (!)

8.3 Arrangements of a number of different items to discover that the arrangement of r out of the R objects in a

8.4 Forms of permutations involving

8.4.1 Identical objects

8.4.2 With or without repetition

8.4.3 With restrictions

8.5 Selecting a set of 2 items from 2, 3 or 4 items to form a group

8.6 Illustrating how a group of r items can be selected from R items to form a combination

8.6.1 Developing the formula nCr used for selecting r objects from n objects to form a group

8.6.2 Recognising the difference between permutations and combinations scenarios

8.7 Combinations with restrictions

8.8 Applying knowledge of permutations and combinations in various realworld situations like creation of pass

Competency:

The learner applies the permutations and combinations concepts to solve mathematical problems and model real-world situations.

8.9 Introduction

Permutations and combinations are fundamental counting concepts with applications in mathematics, computer science, cryptography, genetics, and statistics.

Key Distinction:

- **Permutations:** Order matters (arrangements)
- **Combinations:** Order doesn't matter (selections)

8.10 Fundamental Counting and Factorials

8.10.1 Counting Principle

If event A can occur in m ways and event B in n ways, both can occur in $m \times n$ ways.

8.10.2 Factorial Notation

Definition

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

Convention: $0! = 1$, $1! = 1$

Examples: 3! = 6, 4! = 24, 5! = 120

Properties:

1. $n! = n \times (n-1)!$
2. $\frac{n!}{(n-r)!} = n(n-1)\cdots(n-r+1)$

The number of ways to arrange n distinct objects in a row is $n!$

Example 1: Five students in a line: 5! = 120 ways

Practice:

1. Calculate: (a) 7! (b) $\frac{8!}{6!}$ (c) $\frac{10!}{8!\times 2!}$
2. Six books on a shelf: how many arrangements?
3. Simplify: $\frac{(n+1)!}{(n-1)!}$

8.11 Permutations

8.11.1 Permutations of r from n Objects

Definition

A permutation is an ordered arrangement. The number of ways to select and arrange r objects from n distinct objects is:

$$P(n,r) = \frac{n!}{(n-r)!} = n(n-1)(n-2)\cdots(n-r+1)$$

Derivation: First position has n choices, second has $(n-1)$, ..., r-th has $(n-r+1)$ choices.

Example 2: Three-letter "words" from A, B, C, D, E (no repetition): $P(5,3) = 5 \times 4 \times 3 = 60$

Example 3: Choose president, VP, secretary from 10 people: $P(10,3) = 10 \times 9 \times 8 = 720$

Special Cases: $P(n,n) = n!$, $P(n,1) = n$, $P(n,0) = 1$

Practice:

1. Calculate: (a) $P(7,3)$ (b) $P(8,5)$ (c) $P(n,2)$
2. Four prizes to 12 students (one prize each): how many ways?
3. Four-digit PIN from 0-9 (no repetition)?
4. Solve: $P(n,2) = 56$

8.11.2 Permutations with Identical Objects

When objects are identical, swapping them produces the same arrangement.

Theorem: For n objects where p are identical (type 1), q are identical (type 2), etc.:

$$\text{Distinct arrangements} = \frac{n!}{p! \times q! \times r! \times \cdots}$$

Example 4: MISSISSIPPI (11 letters: M=1, I=4, S=4, P=2)

$$\frac{11!}{1! \times 4! \times 4! \times 2!} = 34,650$$

Example 5: BANANA (6 letters: B=1, A=3, N=2)

$$\frac{6!}{3! \times 2!} = 60$$

Why divide? If all were different, AAB would have 3! = 6 arrangements. Since two A's are identical, each distinct arrangement is counted 2! times, so $\frac{3!}{2!} = 3$ distinct arrangements.

Practice:

1. Arrangements of: (a) BOOK (b) COMMITTEE (c) STATISTICS
2. Seven-digit numbers from 1, 2, 2, 2, 3, 3, 4?
3. SUCCESS arrangements beginning with S?

8.11.3 Permutations with Restrictions

Type 1: Objects Must Be Together

Treat them as a single unit.

Example 6: Five people in a row, two must sit together

Solution: (AB) as one unit. Arrange 4 units: 4! ways. Within unit: 2! ways. Total: $4! \times 2! = 48$

Example 7: EQUATION with vowels together

Vowels (E,U,A,I,O) as one unit with consonants (Q,T,N): 4! units, vowels within: 5!. Total: $4! \times 5! = 2,880$

Type 2: Objects Must Be Separated

Use complementary counting: Total - (Together cases)

Example 8: Three boys, 2 girls; girls not together

Method 1: $5! - (4! \times 2!) = 120 - 48 = 72$

Method 2: Arrange boys: 3!, place girls in 4 gaps: $P(4,2)$. Total: $3! \times 12 = 72$

Type 3: Specific Positions

Example 9: Four-digit numbers from 1,2,3,4,5,6 (no repetition) that are even

Last digit: 2, 4, or 6 (3 choices). First three: $P(5,3) = 60$. Total: $3 \times 60 = 180$

Practice:

1. Eight people, three must sit together?
2. Five letters from PENCIL starting with vowel?
3. Five men, four women; no two women adjacent?
4. Four-digit numbers > 5000 from 2,3,5,6,7,9 (no repetition)?

8.11.4 Circular Permutations

In circles, rotations are identical (no fixed starting point).

Theorem:

Circular arrangements of n objects = $(n-1)!$

Justification: Linear has $n!$ arrangements. Each circular arrangement has n rotations. Thus: $\frac{n!}{n} = (n-1)!$

Example 10: Five people around a table: $(5-1)! = 4! = 24$

Example 11: Six beads on a bracelet (can flip)

Circular: $(6-1)! = 120$. With reflections: $\frac{120}{2} = 60$

Note: For bracelets/necklaces, divide by 2 for reflections.

Example 12: Four men, three women around a table; women together?

Women as one unit: 5 units. Circular: $(5-1)! = 24$. Women within: $3! = 6$. Total: $24 \times 6 = 144$

Practice:

1. Six people around a circular table?
2. Eight flowers in a garland?
3. Five boys, three girls in circle; no two girls together?

8.12 Combinations

8.12.1 Introduction to Combinations

Permutations vs Combinations:

- Permutations: Order matters (AB ≠ BA)
- Combinations: Order doesn't matter (AB = BA)

Example: Three medals to 10 athletes (permutation: who gets which medal). Three athletes for a team from 10 (combination: just forming team).

Definition

A combination is a selection where order doesn't matter. Number of ways to select r from n objects:

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Derivation: $P(n,r) = \frac{n!}{(n-r)!}$ counts ordered selections. Each selection can be arranged $r!$ ways. Since order doesn't matter: $C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$

Relationship: $P(n,r) = C(n,r) \times r!$

Example 13: $C(7,3) = \frac{7!}{3!4!} = \frac{7\times6\times5}{3\times2\times1} = 35$

Example 14: Committee of 5 from 12 people

$$C(12,5) = \frac{12!}{5!7!} = 792$$

Properties:

1. $C(n,r) = C(n,n-r)$
2. $C(n,0) = C(n,n) = 1$
3. $C(n,1) = n$

Practice:

1. Calculate: (a) $C(8,3)$ (b) $C(10,7)$ (c) $C(n,2)$
2. Five students from 20: permutation or combination? How many?
3. Three-letter "words" from COMPUTER vs choosing 3 letters?
4. Four toppings from 10 for pizza?
5. Five-card hands from 52 cards?
6. Solve: $C(n,3) = 20$

8.12.2 Combinations with Restrictions

Type 1: Must Include Certain Objects

Example 15: Committee of 5 from 8 men and 6 women; at least 2 women

Method 1 (Cases):

$$\begin{aligned} \text{2 women:} &\quad C(6,2)\times C(8,3) = 15\times 56 = 840\\ \text{3 women:} &\quad C(6,3)\times C(8,2) = 20\times 28 = 560\\ \text{4 women:} &\quad C(6,4)\times C(8,1) = 15\times 8 = 120\\ \text{5 women:} &\quad C(6,5)\times C(8,0) = 6\times 1 = 6\\ \text{Total:} &\quad 1,526 \end{aligned}$$

Method 2 (Complement): $C(14,5) - C(8,5) - C(6,1)\times C(8,4) = 2,002 - 56 - 420 = 1,526$

Type 2: Must Exclude Certain Objects

Example 16: Group of 4 from 10 students; two particular students cannot both be in group

Total: $C(10,4) = 210$. Both included: $C(8,2) = 28$. Answer: $210 - 28 = 182$

Type 3: Multiple Groups

Example 17: Cricket team of 11 from 7 batsmen, 6 bowlers, 3 keepers; at least 4 batsmen, at least 3 bowlers, exactly 1 keeper

One keeper: $C(3,1) = 3$. Remaining 10 from batsmen and bowlers:

$$\begin{aligned} \text{4B, 6Bw:} &\quad C(7,4)\times C(6,6) = 35\\ \text{5B, 5Bw:} &\quad C(7,5)\times C(6,5) = 126\\ \text{6B, 4Bw:} &\quad C(7,6)\times C(6,4) = 105\\ \text{7B, 3Bw:} &\quad C(7,7)\times C(6,3) = 20\\ \text{Sum:} &\quad 286 \end{aligned}$$

Total: $3\times 286 = 858$

Type 4: At Least/At Most

Example 18: Six marbles from 5 red, 4 blue, 3 green; at most 2 red

$$\begin{aligned} \text{0 red:} &\quad C(7,6) = 7\\ \text{1 red:} &\quad C(5,1)\times C(7,5) = 105\\ \text{2 red:} &\quad C(5,2)\times C(7,4) = 350\\ \text{Total:} &\quad 462 \end{aligned}$$

Practice:

1. Committee of 6 from 9 men, 7 women: (a) exactly 4 men (b) at least 4 men (c) at most 2 women
2. Five from 12 students; two particular must/must not be included
3. Basketball team: 5 from 6 forwards, 5 guards, 4 centers; at least 2 forwards, at least 1 guard, exactly 1 center
4. Eight students from 10 boys, 12 girls: (a) equal boys and girls (b) more girls than boys

8.13 Applications

8.13.1 Computer Science and Cryptography

Password Security: Eight-character password with at least 2 digits (0-9), 1 uppercase (A-Z), 1 special (!,@,#,$). Use inclusion-exclusion or case analysis.

8.13.2 Genetics

DNA Sequences: Ten bases with exactly 4 A's, 3 T's, 2 C's, 1 G:

$$\frac{10!}{4!3!2!1!} = 12,600$$

8.13.3 Statistics

Poker Hands: Five cards from 52

(a) Exactly 3 aces: $C(4,3)\times C(48,2) = 4\times 1,128 = 4,512$

(b) Full house: Choose rank for triple (13), select 3 cards $C(4,3) = 4$, rank for pair (12), select 2 cards $C(4,2) = 6$. Total: $13\times 4\times 12\times 6 = 3,744$

8.13.4 Scheduling

Timetabling: Six subjects, six periods; Math in first 3 periods?

Choose period for Math (3 ways), arrange others: 5!. Total: $3\times 120 = 360$

8.14 Problem-Solving Strategies

Decision Process:

1. Does order matter? If yes → Permutation; if no → Combination
2. Identify restrictions
3. Choose method: direct counting, complementary, or case-by-case
4. Apply formulas
5. Verify answer makes sense

Common Patterns:

- "Arrange/Order" → Permutation
- "Select/Choose/Committee" → Combination
- "Must be together" → Treat as one unit
- "Must be separated" → Complement or gap method
- "At least/At most" → Cases or complement
- "Circular" → $(n-1)!$
- "Identical objects" → Divide by factorials

8.15 Comprehensive Problems

1. Ten people at round table; 3 couples must sit together. How many arrangements?
2. License plates: 3 letters, 4 digits. (a) No restrictions (b) No repetition (c) At least one vowel
3. Committee of 6 from 8 married couples: (a) No married couple both on committee (b) At least one couple
4. Twelve books (5 math, 4 physics, 3 chemistry) on shelf: (a) Same subjects together (b) Only math together
5. Eight-letter "words" from EXAMINATION?
6. Fifteen identical candies to 5 children; each gets at least 2?
7. Seven cards from deck: (a) Exactly 2 hearts, 3 spades (b) At least 4 of same suit
8. Shortest paths from $(0,0)$ to $(4,5)$ moving only right or up?
9. Five men, five women around circular table; no two men adjacent?
10. Debate team of 5 from 7 seniors, 8 juniors; at least 2 seniors and 2 juniors?

8.16 Assessment

Formative Assessment:

- Observe group work: pattern recognition, creativity, collaboration
- Question to probe understanding and reasoning
- Review written work for accuracy and method

Sample Test Questions:

Short Answer:

1. Calculate: (a) $P(8,3)$ (b) $C(10,4)$
2. Seven people in a row: how many ways?
3. Arrangements of BALLOON
4. Committee of 4 from 6 men, 5 women with exactly 2 men
5. Six people around circular table

Long Answer:

1. Eight books (5 math, 3 physics) on shelf: (a) No restrictions (b) Same subjects together (c) Math together only
2. Committee of 7 from 10 men, 8 women: (a) No restrictions (b) Exactly 4 men (c) At least 5 women
3. Four-digit numbers from 1-9 (no repetition): (a) How many (b) Even (c) Greater than 5000 (d) Divisible by 5
4. Class of 12 (7 girls, 5 boys): (a) Arrange in row (b) Alternate (girls first) (c) Select group of 5 (d) Group with at least 3 girls

8.17 Formula Reference

Factorials: $n! = n(n-1)(n-2)\cdots 2\cdot 1$; $0! = 1$

Permutations: $P(n,r) = \frac{n!}{(n-r)!}$

Identical Objects: $\frac{n!}{p!q!r!\cdots}$

Circular: $(n-1)!$; Bracelet: $\frac{(n-1)!}{2}$

Combinations: $C(n,r) = \frac{n!}{r!(n-r)!}$

Relationship: $P(n,r) = C(n,r) \times r!$

Properties: $C(n,r) = C(n,n-r)$; $C(n,0) = C(n,n) = 1$; $C(n,1) = n$

8.18 Selected Solutions

Section 2: 1(a) 5040 (b) 56 (c) 45; 2. 720; 3. $n^2 + n$

Section 3.1: 1(a) 210 (b) 6720 (c) $n^2 - n$; 2. 11,880; 3. 5,040; 4. $n = 8$

Section 3.2: 1(a) 12 (b) 45,360 (c) 50,400; 2. 420

Section 4.1: 1(a) 56 (b) 120 (c) $\frac{n(n-1)}{2}$; 4. 210; 5. 2,598,960; 6. $n = 6$

Background: Kyotera District Development Context

Kyotera District faces several development challenges including healthcare access, agricultural productivity, education infrastructure, and community organization. This assessment uses real scenarios from the district to evaluate students' understanding of combinations and permutations while addressing local societal problems.

Question 1: Healthcare Outreach Team Formation

Scenario-Based Question

Healthcare Access Problem The Kyotera District Health Department needs to establish mobile healthcare units to reach remote villages. Due to limited resources, they must carefully select healthcare workers for optimal coverage. The department has 4 qualified healthcare workers: Dr. Nakato (General Practitioner), Nurse Ssemakula (Community Health), Ms. Nambi (Maternal Health Specialist), and Mr. Okello (Nutrition Expert).

Tasks:

(a) From these 4 workers, select 2 to form a mobile unit. How many different combinations are possible?

(b) If they had only 3 workers available, how many ways could they select 2?

(c) If they had only 2 workers available, how many ways could they select 2?

(d) Explain why this is a combination problem rather than a permutation problem.

Solution

Part (a): Selecting 2 from 4 workers

Using the combination formula: ${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$

${}^{4}C_{2} = \frac{4!}{2!(4-2)!} = \frac{4!}{2!\times 2!} = \frac{24}{2\times 2} = \frac{24}{4} = 6$

All possible combinations:

- {Dr. Nakato, Nurse Ssemakula}
- {Dr. Nakato, Ms. Nambi}
- {Dr. Nakato, Mr. Okello}
- {Nurse Ssemakula, Ms. Nambi}
- {Nurse Ssemakula, Mr. Okello}
- {Ms. Nambi, Mr. Okello}

Part (b): Selecting 2 from 3 workers

${}^{3}C_{2} = \frac{3!}{2!(3-2)!} = \frac{6}{2\times 1} = 3$ combinations

Part (c): Selecting 2 from 2 workers

${}^{2}C_{2} = \frac{2!}{2!(2-2)!} = \frac{2}{2\times 1} = 1$ combination

Part (d): Why combination?

This is a combination problem because the order of selection doesn't matter. Whether we choose Dr. Nakato first or Nurse Ssemakula first, the mobile unit composition remains the same. The team {Dr. Nakato, Nurse Ssemakula} is identical to {Nurse Ssemakula, Dr. Nakato}.

Question 2: Agricultural Cooperative Formation

Scenario-Based Question

Agricultural Productivity Problem Kyotera District wants to improve agricultural productivity by forming farmer cooperatives. The agricultural officer needs to select farmers with different specializations to form effective groups. There are 8 farmers with different specializations: 3 coffee farmers, 2 banana farmers, 2 maize farmers, and 1 livestock farmer.

Tasks:

(a) How many ways can we select 3 farmers from the total 8 to form a cooperative? (Illustrate the general principle of selecting r items from R items)

(b) Derive the combination formula step by step using this scenario.

(c) Calculate the number of ways to form a 4-member cooperative with exactly 1 farmer from each specialization.

Solution

Part (a): Selecting 3 farmers from 8

This illustrates the general principle of selecting $r = 3$ items from $R = 8$ items.

${}^8C_3 = \frac{8!}{3!(8-3)!} = \frac{8!}{3!\times5!} = \frac{8\times7\times6\times5!}{3!\times5!} = \frac{8\times7\times6}{3!} = \frac{336}{6} = 56$

Part (b): Formula derivation

Step 1: Start with the total arrangements (permutations) If order mattered, we would have:

${}^8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 8\times7\times6 = 336$

Step 2: Account for identical arrangements Since order doesn't matter in combinations, each group of 3 farmers can be arranged in $3! = 6$ ways, but they represent the same combination.

Step 3: Remove the ordering ${}^8C_3 = \frac{{}^8P_3}{3!} = \frac{336}{6} = 56$

General formula: ${}^nC_r = \frac{n!}{r!(n-r)!}$

Part (c): Cooperative with restrictions

We need exactly 1 farmer from each specialization:

- Coffee farmers: ${}^3C_1 = 3$ ways
- Banana farmers: ${}^2C_1 = 2$ ways
- Maize farmers: ${}^2C_1 = 2$ ways
- Livestock farmers: ${}^1C_1 = 1$ way

Total combinations: $3\times2\times2\times1 = 12$ ways

Scenario-Based Question

Education Infrastructure Problem Kyotera District needs to form a committee to oversee the construction of new schools. The committee must address both technical and community needs.

Available committee members:

- 5 Engineers: Eng. Mubiru, Eng. Nakalema, Eng. Wagaba, Eng. Kiconco, Eng. Nalumu
- 4 Community Leaders: Mr. Kasozi, Ms. Nakimuli, Mr. Tusiime, Ms. Babirye
- 3 Teachers: Ms. Amongi, Mr. Wamala, Ms. Nakandi

Tasks:

(a) Determine whether selecting committee members is a combination or permutation problem. Justify your answer.

(b) Calculate the number of ways to form a 5-member committee with exactly 2 engineers, 2 community leaders, and 1 teacher.

(c) If the committee needed to elect a Chairperson, Vice-Chairperson, and Secretary from the 5 selected members, would this be a combination or permutation problem? Calculate the possibilities.

Question 3: School Infrastructure Committee

Solution

Part (a): Combination vs Permutation

Selecting committee members is a **combination problem** because:

- The order of selection doesn't affect the committee composition
- A committee with {Eng. Mubiru, Mr. Kasozi, Ms. Amongi} is the same as {Ms. Amongi, Eng. Mubiru, Mr. Kasozi}
- We care about *who* is on the committee, not the *order* of selection

Part (b): Committee formation with restrictions

We need: 2 engineers from 5, 2 community leaders from 4, and 1 teacher from 3.

$$\text{Engineers: } {}^5C_2 = \frac{5!}{2!(5-2)!} = \frac{5!}{2! \times 3!} = \frac{5 \times} \tag{8.1}$$

$$\text{Community Leaders: } {}^4C_2 = \frac{4!}{2!(4-2)!} = \frac{4!}{2! \times 2!} = \frac{4 \times} \tag{8.2}$$

$$\text{Teachers: } {}^3C_1 = 3 \tag{8.3}$$

Total committee combinations: $10 \times 6 \times 3 = 180$ ways

Part (c): Electing officers

Electing Chairperson, Vice-Chairperson, and Secretary is a **permutation problem** because:

- The order/position matters
- Being Chairperson is different from being Secretary
- We're arranging people in specific roles

Number of ways: ${}^5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 5 \times 4 \times 3 = 60$ ways

Question 4: Digital Literacy and Security

Scenario-Based Question

Technology Access Problem Kyotera District is implementing a digital literacy program to improve access to online services and banking. Citizens need secure passwords and PINs for various applications.

Tasks:

(a) A farmer needs a 4-digit PIN for mobile banking using digits 1-6. How many possible PINs can be created if:

(i) Digits can be repeated?

(ii) Digits cannot be repeated?

(b) For online government services, citizens need passwords with exactly 3 letters (from A-D) followed by 2 digits (from 0-9), with no repetitions allowed. How many such passwords are possible?

(c) Explain why PIN/password creation uses permutations rather than combinations.

Solution

Part (a): 4-digit PIN creation

(i) With repetition allowed: Each position can have any of the 6 digits (1, 2, 3, 4, 5, 6). Total possibilities: $6^4 = 6 \times 6 \times 6 \times 6 = 1,296$ PINs

(ii) Without repetition: This is a permutation problem: ${}^6P_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{720}{2} = 360$ PINs

Alternatively: $6 \times 5 \times 4 \times 3 = 360$ PINs

Part (b): Password creation (3 letters + 2 digits)

Letter selection: 3 letters from A, B, C, D without repetition ${}^4P_3 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 24$ ways

Digit selection: 2 digits from 0-9 without repetition ${}^{10}P_2 = \frac{10!}{(10-2)!} = \frac{10!}{8!} = 10 \times 9 = 90$ ways

Total passwords: $24 \times 90 = 2,160$ passwords

Part (c): Why permutations?

PIN and password creation uses **permutations** because:

- **Order is crucial:** PIN "1234" is completely different from "4321"
- **Position matters:** In passwords, the first letter has a different role than the third letter
- **Security depends on sequence:** Changing the order creates an entirely different access code
- **Function is positional:** Each digit/character position serves a specific purpose in the authentication sequence

Question 5: Community Water Project Management

Scenario-Based Question

Water Access Problem Kyotera District is implementing a community water project involving multiple villages. The project requires careful coordination of resources and personnel.

Available resources:

- 6 Water technicians: 2 from Kyotera town, 2 from Kakuuto, 2 from Mutukula
- 8 Community volunteers
- 4 Project vehicles

Tasks:

(a) The project needs to select 4 technicians for the main team, but must include at least 1 technician from each of the 3 areas. How many ways can this selection be made?

(b) From the 8 community volunteers, select 5 to form a monitoring committee. Calculate this using the combination formula.

(c) The 4 project vehicles need to be assigned specific routes in a particular order (Route 1, Route 2, Route 3, Route 4). How many different route assignments are possible?

(d) Compare the approaches used in parts (b) and (c). Why is one a combination and the other a permutation?

Solution

Part (a): Technician selection with area restrictions

We need 4 technicians with at least 1 from each area (Kyotera, Kakuuto, Mutukula).

Since we have 2 technicians from each area and need at least 1 from each, the possible distributions are:

- 2 from one area, 1 from each of the other two areas

Case 1: 2 from Kyotera, 1 from Kakuuto, 1 from Mutukula ${}^2C_2 \times {}^2C_1 \times {}^2C_1 = 1 \times 2 \times 2 = 4$ ways

Case 2: 1 from Kyotera, 2 from Kakuuto, 1 from Mutukula ${}^2C_1 \times {}^2C_2 \times {}^2C_1 = 2 \times 1 \times 2 = 4$ ways

Case 3: 1 from Kyotera, 1 from Kakuuto, 2 from Mutukula ${}^2C_1 \times {}^2C_1 \times {}^2C_2 = 2 \times 2 \times 1 = 4$ ways

Total ways: $4 + 4 + 4 = 12$ ways

Part (b): Volunteer selection

Selecting 5 volunteers from 8: ${}^8C_5 = \frac{8!}{5!(8-5)!} = \frac{8!}{5!\times 3!} = \frac{8\times 7\times 6}{3\times 2\times 1} = \frac{336}{6} = 56$ ways

Part (c): Vehicle route assignment

Assigning 4 vehicles to 4 specific routes in order: ${}^4P_4 = 4! = 4\times 3\times 2\times 1 = 24$ assignments

Part (d): Comparison

Part (b) - Combination:

- Selecting volunteers for a committee
- Order doesn't matter - the committee function is the same regardless of selection order
- We care about *who* is selected, not the *sequence* of selection

Part (c) - Permutation:

- Assigning vehicles to specific routes
- Order matters - Route 1 is different from Route 4
- We care about *which* vehicle goes to *which specific* route
- Position/assignment is crucial for logistics

Scenario-Based Question

Educational Infrastructure Problem Kyotera District Education Department needs to distribute learning materials and organize teacher training programs across different schools.

Situation A: The department has 10 different textbook sets and needs to select 4 sets to distribute to a particular school.

Situation B: 7 teachers need to be arranged in a specific order for a training presentation schedule.

Situation C: From 12 schools in the district, select 6 schools to participate in a pilot program, but exactly 2 must be from urban areas (4 urban schools available) and 4 must be from rural areas (8 rural schools available).

Tasks:

(a) For Situation A, calculate the number of ways to select the textbook sets and identify whether this is a combination or permutation problem.

(b) For Situation B, calculate the number of presentation arrangements and identify the problem type.

(c) For Situation C, calculate the number of ways to select schools with the given restrictions.

(d) Design a 6-character secure password system for the district's educational database using 4 letters (A-F) followed by 2 digits (0-9), with no repetitions. Calculate total possibilities.

Question 6: Education Resource Distribution

Solution

Part (a): Textbook set selection

This is a **combination problem** because the order of selecting textbook sets doesn't matter. The school receives the same educational resources regardless of selection sequence.

${}^{10}C_4 = \frac{10!}{4!(10-4)!} = \frac{10!}{4!\times 6!} = \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1} = \frac{5,040}{24} = 210$ ways

Part (b): Teacher presentation arrangement

This is a **permutation problem** because the order of presentation matters. The training flow and learning progression depend on the sequence.

${}^{7}P_7 = 7! = 7\times 6\times 5\times 4\times 3\times 2\times 1 = 5,040$ arrangements

Part (c): School selection with area restrictions

We need exactly 2 urban schools from 4 available and 4 rural schools from 8 available.

Urban school selection: ${}^{4}C_2 = \frac{4!}{2!\times 2!} = \frac{12}{4} = 6$ ways

Rural school selection: ${}^{8}C_4 = \frac{8!}{4!\times 4!} = \frac{8\times 7\times 6\times 5}{4\times 3\times 2\times 1} = \frac{1,680}{24} = 70$ ways

Total combinations: $6\times 70 = 420$ ways

Part (d): Password system design

Password structure: 4 letters (A-F) + 2 digits (0-9), no repetitions

Letter selection: ${}^{6}P_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{720}{2} = 360$ ways

Digit selection: ${}^{10}P_2 = \frac{10!}{(10-2)!} = \frac{10!}{8!} = 10\times 9 = 90$ ways

Total passwords: $360\times 90 = 32,400$ possible secure passwords

This uses **permutations** because:

- Position matters for password security
- "ABCD12" is different from "DCBA21"
- Character sequence is crucial for authentication

Chapter 9

Series

Competency:
The learner applies principles and techniques of series and binomial expansions to understand and solve mathematical and scientific problems in the real world.

9.1 Reviewing sequences to identify a series

9.1.1 Exploring how adding a fixed number to previous terms generates an Arithmetic Progression (A.P.),

9.1.2 Deriving the formula for finding the sum of an A.P

9.1.3 Deriving expressions for the n^{th} term of an A.P; apply the derivations to solve numerical problems

9.2 Applications of A.Ps in real life situations.

9.3 Generating series with a constant ratio between successive terms (G.P) from sequences,

9.3.1 Deriving expressions for the term and the sums of a (G.P); apply the derivations to solve mathematical problems

9.3.2 Analysing convergence of G.P to generate the sum to infinity of a G.P,

9.3.3 Using summation notation to determine the sum of any series

9.4 Applications of G.P to real- world problems such as compound interest, spread of disease, cell division, radi

9.5 Designing different savings or investment plans using information from a guest speaker from business/ finan

9.6 Demonstrating simple mathematical statements e.g, the sum of two even numbers is even

9.6.1 Extending the boundaries of the experiment above and prove that the same statement is still true (say sum of two integers is also even).

9.6.2 Exploring proving that the same statement is true for any set of even numbers using inductive method.

9.7 Validity of other mathematical statements using proof by induction

9.8 Applications of proof by induction in real life.

9.9 Pascal's triangle

9.9.1 Generating coefficients of terms in the expansion to deduce the binomial theorem,

9.9.2 Expanding binomial expressions in ascending and descending order.

9.10 Applying the binomial theorem to:

9.10.1 Computing a particular term of binomial expansion

9.10.2 Computing roots of numerical values

9.10.3 Range of validity of binomial expansion

9.11 Introduction to Series

9.11.1 What is a Series?

Definition

A **series** is the sum of the terms of a sequence. If we have a sequence $a_1, a_2, a_3, \ldots, a_n$, then the series is:

$$S = a_1 + a_2 + a_3 + \cdots + a_n$$

9.11.2 Types of Series

In this course, we will study four main types of series:

1. Arithmetic Progressions (A.P.)
2. Geometric Progressions (G.P.)
3. Series requiring Mathematical Induction
4. Binomial Series

9.12 Arithmetic Progressions (A.P.)

9.12.1 Definition and General Terms

Definition

An **Arithmetic Progression** is a sequence where the difference between consecutive terms is constant. This constant difference is called the **common difference**, denoted by d.

General form: $a, a+d, a+2d, a+3d, \ldots$

where:

- a = first term
- d = common difference

9.12.2 The n^{th} Term Formula

The n^{th} term of an A.P. is given by:

$$\boxed{T_n = a + (n-1)d}$$

Derivation:

$$\begin{aligned} T_1 &= a = a + (1-1)d \\ T_2 &= a + d = a + (2-1)d \\ T_3 &= a + 2d = a + (3-1)d \\ &\vdots \\ T_n &= a + (n-1)d \end{aligned}$$

9.12.3 Sum of an A.P.

The sum of the first n terms of an A.P. is:

$$\boxed{S_n = \frac{n}{2}[2a + (n-1)d]}$$

or equivalently:

$$\boxed{S_n = \frac{n}{2}(a + l)}$$

where l is the last term ($l = T_n$).

Derivation:

Let $S_n = a + (a+d) + (a+2d) + \cdots + (l-d) + l$

Writing in reverse order: $S_n = l + (l-d) + (l-2d) + \cdots + (a+d) + a$

Adding these two equations:

$$\begin{aligned} 2S_n &= (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l) \\ 2S_n &= n(a+l) \\ S_n &= \frac{n}{2}(a+l) \end{aligned}$$

9.12.4 Scenario-Based Examples

Ugandan Scenario

Scenario 1: Saving Money
Maria decides to save money in a disciplined manner. She saves UGX 5,000 in the first week, UGX 7,000 in the second week, UGX 9,000 in the third week, and so on, increasing her savings by UGX 2,000 each week.

Questions:

(a) How much will she save in the 12th week?

(b) What is her total savings after 12 weeks?

Solution:

This is an A.P. with:

- First term: $a = 5000$
- Common difference: $d = 2000$

(a) Amount saved in 12th week:

$$\begin{aligned} T_{12} &= a + (n-1)d \\ &= 5000 + (12-1)(2000) \\ &= 5000 + 11(2000) \\ &= 5000 + 22000 \\ &= \text{UGX } 27,000 \end{aligned}$$

(b) Total savings after 12 weeks:

$$\begin{aligned} S_{12} &= \frac{n}{2}[2a + (n-1)d] \\ &= \frac{12}{2}[2(5000) + (12-1)(2000)] \\ &= 6[10000 + 22000] \\ &= 6 \times 32000 \\ &= \text{UGX } 192,000 \end{aligned}$$

Ugandan Scenario

Scenario 2: Theatre Seating
A theatre is designed with rows of seats. The first row has 20 seats, the second row has 23 seats, the third row has 26 seats, and so on. If there are 25 rows in total:

Questions:

(a) How many seats are in the last row?

(b) What is the total seating capacity of the theatre?

Solution:

Given: $a = 20$, $d = 3$, $n = 25$

(a) Seats in the last row:

$$\begin{aligned} T_{25} &= 20 + (25-1)(3) \\ &= 20 + 72 \\ &= 92 \text{ seats} \end{aligned}$$

(b) Total seating capacity:

$$\begin{aligned} S_{25} &= \frac{25}{2}(20 + 92) \\ &= \frac{25}{2}(112) \\ &= 25 \times 56 \\ &= 1400 \text{ seats} \end{aligned}$$

Ugandan Scenario

Scenario 3: Construction Project
A construction company plans to build a road. They lay 500 meters in the first month. Due to improving efficiency and additional equipment, they increase their output by 50 meters each subsequent month.

Questions:

(a) How many meters will they lay in the 10th month?

(b) If the total road length is 8,000 meters, in which month will the project be completed?

Solution:

Given: $a = 500$, $d = 50$

(a) Length in 10th month:

$$\begin{aligned} T_{10} &= 500 + (10-1)(50) \\ &= 500 + 450 \\ &= 950 \text{ meters} \end{aligned}$$

(b) To find when $S_n = 8000$:

$$\begin{aligned}
\frac{n}{2}[2(500) + (n-1)(50)] &= 8000 \\
\frac{n}{2}[1000 + 50n - 50] &= 8000 \\
\frac{n}{2}[950 + 50n] &= 8000 \\
n[950 + 50n] &= 16000 \\
950n + 50n^2 &= 16000 \\
50n^2 + 950n - 16000 &= 0 \\
n^2 + 19n - 320 &= 0
\end{aligned}$$

Using the quadratic formula:

$$\begin{aligned}
n &= \frac{-19 \pm \sqrt{361 + 1280}}{2} \\
&= \frac{-19 \pm \sqrt{1641}}{2} \\
&= \frac{-19 \pm 40.51}{2}
\end{aligned}$$

Taking the positive value: $n = \frac{21.51}{2} \approx 10.76$

Therefore, the project will be completed in the **11th month**.

9.12.5 Practice Problems

1. A salary starts at UGX 800,000 per month with an annual increment of UGX 50,000. Find:
   (a) The salary in the 7th year
   (b) Total earnings over 10 years
2. The sum of the first n terms of an A.P. is $3n^2+2n$. Find:
   (a) The first term
   (b) The common difference
   (c) The n^{th} term

9.13 Geometric Progressions (G.P.)

9.13.1 Definition and General Terms

Definition

A **Geometric Progression** is a sequence where each term after the first is obtained by multiplying the previous term by a constant called the **common ratio**, denoted by r.

General form: $a, ar, ar^2, ar^3, \ldots$

where:

- a = first term
- r = common ratio ($r \neq 0$)

9.13.2 The n^{th} Term Formula

The n^{th} term of a G.P. is given by:

$$\boxed{T_n = ar^{n-1}}$$

9.13.3 Sum of a G.P.

The sum of the first n terms of a G.P. is:

For $r \neq 1$:

$$\boxed{S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r}}$$

For $r = 1$:

$$S_n = na$$

Derivation:

Let $S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$

Multiply by r: $rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$

Subtract:

$$\begin{aligned}
S_n - rS_n &= a - ar^n \\
S_n(1 - r) &= a(1 - r^n) \\
S_n &= \frac{a(1 - r^n)}{1 - r}
\end{aligned}$$

9.13.4 Sum to Infinity

For a convergent G.P. (where $|r| < 1$), the sum to infinity is:

$$\boxed{S_\infty = \frac{a}{1 - r}}$$

As $n \to \infty$, if $|r| < 1$, then $r^n \to 0$, so:

$$S_\infty = \lim_{n \to \infty} \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r}$$

A

G.P. converges (has a finite sum to infinity) if and only if $|r| < 1$ or $-1 < r < 1$.

9.13.5 Scenario-Based Examples

Ugandan Scenario

Scenario 1: Compound Interest
Joshua invests UGX 1,000,000 in a bank account that pays 10% compound interest annually.
Questions:
(a) How much money will he have after 5 years?
(b) After how many years will his investment double?

Solution:

This forms a G.P. with:

- First term: $a = 1,000,000$
- Common ratio: $r = 1.10$ (1 + 10%)

(a) Amount after 5 years:

$$\begin{aligned} T_5 &= ar^{5-1} \\ &= 1,000,000 \times (1.10)^4 \\ &= 1,000,000 \times 1.4641 \\ &= \text{UGX } 1,464,100 \end{aligned}$$

(b) To find when the investment doubles (becomes 2,000,000):

$$\begin{aligned} ar^{n-1} &= 2,000,000 \\ 1,000,000 \times (1.10)^{n-1} &= 2,000,000 \\ (1.10)^{n-1} &= 2 \\ (n-1)\log(1.10) &= \log(2) \\ n-1 &= \frac{\log(2)}{\log(1.10)} \\ n-1 &= \frac{0.3010}{0.0414} \\ n-1 &\approx 7.27 \\ n &\approx 8.27 \end{aligned}$$

Therefore, the investment will double in approximately **8 to 9 years**.

Ugandan Scenario

Scenario 2: Disease Spread
A new strain of flu initially infects 8 people. Each infected person infects 3 more people per week.
Questions:
(a) How many people will be newly infected in the 6th week?
(b) What is the total number of infections after 6 weeks?

Solution:

This is a G.P. with: $a = 8, r = 3$

(a) New infections in 6th week:

$$\begin{aligned} T_6 &= 8 \times 3^{6-1} \\ &= 8 \times 3^5 \\ &= 8 \times 243 \\ &= 1944 \text{ people} \end{aligned}$$

(b) Total infections after 6 weeks:

$$\begin{aligned} S_6 &= \frac{a(r^n - 1)}{r - 1} \\ &= \frac{8(3^6 - 1)}{3 - 1} \\ &= \frac{8(729 - 1)}{2} \\ &= \frac{8 \times 728}{2} \\ &= 4 \times 728 \\ &= 2912 \text{ people} \end{aligned}$$

Ugandan Scenario

Scenario 3: Radioactive Decay
A radioactive substance has a half-life of 5 years. If the initial mass is 800 grams:
Questions:
(a) What mass remains after 20 years?
(b) After how long will only 50 grams remain?

Solution:

This is a G.P. with:

- $a = 800$ grams
- $r = \frac{1}{2}$ (half-life means multiply by $\frac{1}{2}$ every period)
- Each period is 5 years

(a) After 20 years (which is 4 periods):

$$\begin{aligned} T_4 &= 800 \times \left(\frac{1}{2}\right)^{4-1} \\ &= 800 \times \left(\frac{1}{2}\right)^3 \\ &= 800 \times \frac{1}{8} \\ &= 100 \text{ grams} \end{aligned}$$

(b) To find when mass = 50 grams:

$$\begin{aligned} 800 \times \left(\frac{1}{2}\right)^{n-1} &= 50 \\ \left(\frac{1}{2}\right)^{n-1} &= \frac{50}{800} \\ \left(\frac{1}{2}\right)^{n-1} &= \frac{1}{16} \\ \left(\frac{1}{2}\right)^{n-1} &= \left(\frac{1}{2}\right)^4 \\ n-1 &= 4 \\ n &= 5 \end{aligned}$$

Since each period is 5 years, the time is $5 \times 5 =$ **25 years**.

Ugandan Scenario

Scenario 4: Bouncing Ball

A ball is dropped from a height of 10 meters. After each bounce, it reaches 75% of its previous height.

Questions:

(a) What height does it reach after the 5th bounce?

(b) What is the total vertical distance traveled by the ball when it stops bouncing?

Solution:

This is a G.P. with: $a = 10$, $r = 0.75$

(a) Height after 5th bounce:

$$\begin{aligned} T_5 &= 10 \times (0.75)^{5-1} \\ &= 10 \times (0.75)^4 \\ &= 10 \times 0.3164 \\ &\approx 3.16 \text{ meters} \end{aligned}$$

(b) Total distance:

- Initial drop: 10 m
- Up and down after each bounce: $2(7.5 + 5.625 + 4.219 + \cdots)$

The up and down distances form a G.P. with $a = 7.5$, $r = 0.75$:

$$S_\infty = \frac{a}{1-r} = \frac{7.5}{1-0.75} = \frac{7.5}{0.25} = 30$$

Total distance = Initial drop + 2(Sum of bounces)

$$\begin{aligned} &= 10 + 2(30) \\ &= 10 + 60 \\ &= 70 \text{ meters} \end{aligned}$$

9.13.6 Practice Problems

1. A population of bacteria doubles every hour. Starting with 100 bacteria:
   (a) How many will there be after 8 hours?
   (b) When will the population reach 1 million?
2. Find the sum to infinity of: $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$
3. A car depreciates by 15% each year. If bought for UGX 30,000,000:
   (a) What is its value after 5 years?
   (b) When will its value be less than UGX 10,000,000?

9.14 Mathematical Induction

9.14.1 Principle of Mathematical Induction

Mathematical induction is a method of proof used to establish the truth of an infinite sequence of statements.

Theorem: [Principle of Mathematical Induction] To prove a statement $P(n)$ is true for all positive integers $n \geq n_0$:

1. **Base Case:** Show that $P(n_0)$ is true.
2. **Inductive Step:** Assume $P(k)$ is true for some $k \geq n_0$ (Inductive Hypothesis), then prove that $P(k+1)$ is true.

If both steps are completed, then $P(n)$ is true for all $n \geq n_0$.

9.14.2 Steps for Proof by Induction

1. **State what you're proving**
2. **Base case:** Verify the statement for $n = 1$ (or the smallest value)
3. **Inductive hypothesis:** Assume the statement is true for $n = k$

4. **Inductive step:** Prove it's true for $n = k + 1$
5. **Conclusion:** State that by mathematical induction, the statement is true for all $n \geq 1$

9.14.3 Examples

Example 1: Prove that for all positive integers n:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Proof:

Let $P(n)$ be the statement: $1+2+3+\cdots+n = \frac{n(n+1)}{2}$

Base Case ($n = 1$):

$$\text{LHS} = 1$$
$$\text{RHS} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$
$$\text{LHS} = \text{RHS} \quad \checkmark$$

So $P(1)$ is true.

Inductive Hypothesis: Assume $P(k)$ is true for some positive integer k. That is:

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

Inductive Step: We need to prove $P(k+1)$ is true:

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

Starting with the left side:

$$\begin{aligned} 1 + 2 + 3 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \quad \text{(by I.H.)} \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \quad \checkmark \end{aligned}$$

This is exactly what we needed to show.

Conclusion: By the principle of mathematical induction, $P(n)$ is true for all positive integers $n \geq 1$.

Example 2: Prove that for all positive integers n:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof:

Let $P(n)$ be the statement: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Base Case ($n = 1$):

$$\text{LHS} = 1^2 = 1$$
$$\text{RHS} = \frac{1(2)(3)}{6} = \frac{6}{6} = 1$$
$$\text{LHS} = \text{RHS} \quad \checkmark$$

Inductive Hypothesis: Assume $P(k)$ is true:

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Inductive Step: Prove $P(k+1)$:

$$\begin{aligned} 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\ &= \frac{(k+1)[2k^2 + k + 6k + 6]}{6} \\ &= \frac{(k+1)[2k^2 + 7k + 6]}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6} \end{aligned}$$

Conclusion: By mathematical induction, the formula is true for all $n \geq 1$.

Example 3: Prove that $2^n > n$ for all positive integers n.

Proof:

Let $P(n)$ be the statement: $2^n > n$

Base Case ($n = 1$):

$$2^1 = 2 > 1 \quad \checkmark$$

Inductive Hypothesis: Assume $P(k)$ is true: $2^k > k$

Inductive Step: We need to show: $2^{k+1} > k+1$

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &> 2 \cdot k \quad \text{(by I.H.)} \\ &= k + k \\ &\geq k + 1 \quad \text{(since } k \geq 1\text{)} \end{aligned}$$

Therefore, $2^{k+1} > k+1$ ✓

Conclusion: By mathematical induction, $2^n > n$ for all positive integers n.

9.14.4 Scenario-Based Application

Ugandan Scenario

Scenario: Network Connections
In a computer network with n computers, each computer needs to connect to every other computer. Prove that the total number of connections needed is $\frac{n(n-1)}{2}$.

Solution:

This can be proven by induction.

Base Case ($n = 2$)**:** With 2 computers, we need 1 connection.

$$\frac{2(2-1)}{2} = \frac{2 \times 1}{2} = 1 \quad \checkmark$$

Inductive Hypothesis: Assume for k computers, the number of connections is $\frac{k(k-1)}{2}$.

Inductive Step: When we add the $(k+1)^{th}$ computer:

- It needs to connect to all k existing computers: k new connections
- Total connections = Old connections + New connections

$$\begin{aligned} \text{Total} &= \frac{k(k-1)}{2} + k \\ &= \frac{k(k-1) + 2k}{2} \\ &= \frac{k^2 - k + 2k}{2} \\ &= \frac{k^2 + k}{2} \\ &= \frac{k(k+1)}{2} \\ &= \frac{(k+1)[(k+1)-1]}{2} \quad \checkmark \end{aligned}$$

Conclusion: By induction, the formula holds for all $n \geq 2$ computers.

9.14.5 Practice Problems

Prove the following by mathematical induction:

1. $1 + 3 + 5 + \cdots + (2n-1) = n^2$
2. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$
3. $3^n > 2n + 1$ for all $n \geq 1$
4. $1 + r + r^2 + \cdots + r^{n-1} = \frac{r^n - 1}{r-1}$ for $r \neq 1$
5. $n! > 2^n$ for all $n \geq 4$

9.15 Binomial Theorem

9.15.1 Pascal's Triangle

Pascal's Triangle is a triangular array of numbers where each number is the sum of the two numbers above it:

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

9.15.2 Binomial Theorem

Theorem: [Binomial Theorem] For any positive integer n and any real numbers a and b:

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ is the binomial coefficient.

Expanded form:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n$$

9.15.3 General Term

The general term (or $(r+1)^{th}$ term) in the expansion of $(a+b)^n$ is:

$$\boxed{T_{r+1} = \binom{n}{r}a^{n-r}b^r}$$

9.15.4 Properties of Binomial Coefficients

1. $\binom{n}{0} = \binom{n}{n} = 1$
2. $\binom{n}{r} = \binom{n}{n-r}$
3. $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$
4. Sum of coefficients: $(1+1)^n = 2^n = \sum_{r=0}^{n}\binom{n}{r}$
5. Alternating sum: $(1-1)^n = 0 = \sum_{r=0}^{n}(-1)^r\binom{n}{r}$ for $n \geq 1$

9.15.5 Binomial Expansion for Any Index

For $|x| < 1$ and any real number n:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

9.15.6 Examples

Example 1: Expand $(2x+3)^4$ using the binomial theorem.

Solution:

Using $(a+b)^4$ with $a = 2x$ and $b = 3$:

$$\begin{aligned}(2x+3)^4 &= \binom{4}{0}(2x)^4 + \binom{4}{1}(2x)^3(3) + \binom{4}{2}(2x)^2(3)^2 \\ &\quad + \binom{4}{3}(2x)(3)^3 + \binom{4}{4}(3)^4 \\ &= 1(16x^4) + 4(8x^3)(3) + 6(4x^2)(9) + 4(2x)(27) + \\ &= 16x^4 + 96x^3 + 216x^2 + 216x + 81\end{aligned}$$

Example 2: Find the coefficient of x^5 in the expansion of $(2x-1)^8$.

Solution:

General term: $T_{r+1} = \binom{8}{r}(2x)^{8-r}(-1)^r$

For the term containing x^5: $8 - r = 5$, so $r = 3$

$$\begin{aligned}T_4 &= \binom{8}{3}(2x)^5(-1)^3 \\ &= 56 \times 32x^5 \times (-1) \\ &= -1792x^5\end{aligned}$$

The coefficient of x^5 is $\boxed{-1792}$.

Example 3: Use the binomial theorem to approximate $\sqrt{1.02}$ to 4 decimal places.

Solution:

Write: $\sqrt{1.02} = (1.02)^{1/2} = (1+0.02)^{1/2}$

Using $(1+x)^n$ with $n = \frac{1}{2}$ and $x = 0.02$:

$$\begin{aligned}(1+x)^{1/2} &= 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 + \cdots \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots\end{aligned}$$

Substituting $x = 0.02$:

$$\begin{aligned}\sqrt{1.02} &\approx 1 + \frac{1}{2}(0.02) - \frac{1}{8}(0.02)^2 + \frac{1}{16}(0.02)^3 \\ &= 1 + 0.01 - \frac{1}{8}(0.0004) + \frac{1}{16}(0.000008) \\ &= 1 + 0.01 - 0.00005 + 0.0000005 \\ &\approx 1.0099505 \\ &\approx \boxed{1.0100}\end{aligned}$$

9.15.7 Scenario-Based Examples

Ugandan Scenario

Scenario 1: Quality Control

A factory produces electronic components. The probability that a component is defective is 0.05. If 8 components are randomly selected, find the probability that:

(a) Exactly 2 are defective

(b) At most 1 is defective

Solution:

Using binomial probability: $P(X = r) = \binom{n}{r}p^r(1-p)^{n-r}$

where $n = 8$, $p = 0.05$, $(1-p) = 0.95$

(a) Probability that exactly 2 are defective:

$$\begin{aligned} P(X=2) &= \binom{8}{2}(0.05)^2(0.95)^6 \\ &= 28 \times 0.0025 \times 0.7351 \\ &= 0.0515 \\ &\approx 5.15\% \end{aligned}$$

(b) Probability of at most 1 defective:

$$\begin{aligned} P(X \leq 1) &= P(X=0) + P(X=1) \\ &= \binom{8}{0}(0.05)^0(0.95)^8 + \binom{8}{1}(0.05)^1(0.95)^7 \\ &= 1 \times 1 \times 0.6634 + 8 \times 0.05 \times 0.6983 \\ &= 0.6634 + 0.2793 \\ &= 0.9427 \\ &\approx 94.27\% \end{aligned}$$

Ugandan Scenario

Scenario 2: Investment Returns
An investment portfolio has an expected annual return of $(1.08)^t$ where t is time in years. Use the binomial expansion to estimate the return after 1.5 years.

Solution:

We need to find $(1.08)^{1.5} = (1.08)^{3/2}$

Write as: $(1+0.08)^{3/2}$

Using the binomial expansion:

$$\begin{aligned} (1+x)^{3/2} &= 1 + \frac{3}{2}x + \frac{\frac{3}{2}\cdot\frac{1}{2}}{2!}x^2 + \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot(-\frac{1}{2})}{3!}x^3 + \cdots \\ &= 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3 + \cdots \end{aligned}$$

With $x = 0.08$:

$$\begin{aligned} (1.08)^{1.5} &\approx 1 + \frac{3}{2}(0.08) + \frac{3}{8}(0.08)^2 - \frac{1}{16}(0.08)^3 \\ &= 1 + 0.12 + \frac{3}{8}(0.0064) - \frac{1}{16}(0.000512) \\ &= 1 + 0.12 + 0.0024 - 0.000032 \\ &\approx 1.1224 \\ &= 112.24\% \end{aligned}$$

The investment return after 1.5 years is approximately **12.24%**.

Ugandan Scenario

Scenario 3: Population Growth
A population grows according to the model $P = P_0(1+r)^t$. If $P_0 = 50000$ and $r = 0.03$, approximate the population after 0.5 years.

Solution:

$P = 50000(1.03)^{0.5} = 50000(1+0.03)^{1/2}$

Using binomial expansion with $n = \frac{1}{2}$, $x = 0.03$:

$$\begin{aligned} (1+x)^{1/2} &\approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 \\ &= 1 + \frac{1}{2}(0.03) - \frac{1}{8}(0.03)^2 \\ &= 1 + 0.015 - \frac{1}{8}(0.0009) \\ &= 1 + 0.015 - 0.0001125 \\ &\approx 1.0149 \end{aligned}$$

Therefore:

$$P \approx 50000 \times 1.0149 = 50745$$

The population after 0.5 years is approximately **50,745 people**.

9.15.8 Practice Problems

1. Expand $(3x-2y)^5$ using the binomial theorem.
2. Find the middle term in the expansion of $(x+\frac{1}{x})^{10}$.
3. Find the term independent of x in the expansion of $(2x-\frac{3}{x^2})^9$.
4. Use the binomial theorem to approximate:
   (a) $(0.98)^5$
   (b) $\sqrt[3]{1.06}$
   (c) $(1.05)^{-2}$
5. In a genetics experiment, the probability of a certain trait appearing is 0.25. If 12 subjects are tested, find the probability that:
   (a) Exactly 3 show the trait
   (b) Between 2 and 4 (inclusive) show the trait

9.16 Summation Notation

9.16.1 Introduction to Sigma Notation

The symbol $\sum$ (sigma) is used to denote the sum of a series:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where:

- i is the index of summation
- $i = 1$ is the lower limit
- n is the upper limit
- a_i is the general term

9.16.2 Properties of Summation

1. $\sum_{i=1}^{n} c = nc$ (where c is a constant)
2. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
3. $\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
4. $\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

9.16.3 Common Summation Formulas

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$\sum_{i=1}^{n} r^{i-1} = \frac{1-r^n}{1-r} \quad (r \neq 1)$$

9.16.4 Examples

Example 1: Evaluate $\sum_{i=1}^{10}(3i + 2)$

Solution:

$$\begin{aligned}\sum_{i=1}^{10}(3i+2) &= \sum_{i=1}^{10} 3i + \sum_{i=1}^{10} 2 \\ &= 3\sum_{i=1}^{10} i + 2(10) \\ &= 3 \cdot \frac{10(11)}{2} + 20 \\ &= 3 \cdot 55 + 20 \\ &= 165 + 20 \\ &= 185\end{aligned}$$

Example 2: Express in summation notation: $1 + 8 + 27 + 64 + 125$

Solution:

Observe that: $1 = 1^3$, $8 = 2^3$, $27 = 3^3$, $64 = 4^3$, $125 = 5^3$

Therefore:

$$1 + 8 + 27 + 64 + 125 = \sum_{i=1}^{5} i^3$$

9.17 Summary and Review

9.17.1 Key Formulas

Arithmetic Progression

- n^{th} term: $T_n = a + (n-1)d$
- Sum: $S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + l)$

Geometric Progression

- n^{th} term: $T_n = ar^{n-1}$
- Sum: $S_n = \frac{a(r^n-1)}{r-1} = \frac{a(1-r^n)}{1-r}$ for $r \neq 1$
- Sum to infinity: $S_\infty = \frac{a}{1-r}$ for $|r| < 1$

Binomial Theorem

- $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$
- General term: $T_{r+1} = \binom{n}{r} a^{n-r} b^r$
- $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$ for $|x| < 1$

9.17.2 Problem-Solving Strategies

1. **Identify the type of series:** Look for constant differences (A.P.) or constant ratios (G.P.)
2. **Write down what you know:** First term, common difference/ratio, number of terms
3. **Choose the appropriate formula:** Based on what you're asked to find
4. **Substitute and solve:** Be careful with algebraic manipulation
5. **Check your answer:** Does it make sense in the context?

9.17.3 Common Mistakes to Avoid

1. Confusing T_n (the n^{th} term) with S_n (the sum of n terms)

2. Forgetting that G.P. sum to infinity only works when $|r| < 1$
3. In induction proofs, not stating the inductive hypothesis clearly
4. Mixing up the binomial coefficient formula
5. Sign errors in binomial expansions with negative terms

9.17.4 Applications in Real Life

- **A.P.:** Salary increments, payment schemes, linear depreciation, seating arrangements
- **G.P.:** Compound interest, population growth, radioactive decay, virus spread, exponential depreciation
- **Induction:** Computer algorithms, network design, proving program correctness
- **Binomial Theorem:** Probability theory, statistics, approximations in science and engineering

9.18 Comprehensive Practice Problems

9.18.1 Mixed Problems

1. A person saves UGX 10,000 in the first month, UGX 15,000 in the second month, and UGX 20,000 in the third month. If this pattern continues:
   (a) What type of progression is this?
   (b) How much will be saved in the 15th month?
   (c) What is the total savings after 2 years (24 months)?
2. The value of a machine depreciates by 20% each year. If its initial value was UGX 50,000,000:
   (a) Find its value after 5 years
   (b) After how many years will its value be less than UGX 10,000,000?
3. Prove by mathematical induction that $7^n - 1$ is divisible by 6 for all positive integers n.
4. Find the coefficient of x^4 in the expansion of $(3x - 2)^7$.
5. A ball is dropped from a height of 20 meters. After each bounce, it reaches $\frac{3}{4}$ of its previous height. Find:
   (a) The height reached after the 6th bounce
   (b) The total vertical distance traveled when it stops bouncing
6. Evaluate: $\sum_{i=1}^{20}(2i^2 - 3i + 1)$
7. The first term of a G.P. is 5 and the fifth term is 80. Find:
   (a) The common ratio
   (b) The sum of the first 10 terms
8. Use the binomial theorem to approximate $\sqrt{99}$ to 3 decimal places. (Hint: Write as $(100 - 1)^{1/2}$)
9. In an A.P., the sum of the first 10 terms is 235 and the sum of the first 20 terms is 870. Find the first term and common difference.
10. A bacteria culture doubles every 3 hours. If there are initially 500 bacteria:
    (a) How many will there be after 24 hours?
    (b) When will the population exceed 1 million?

9.18.2 Challenge Problems

1. Three numbers are in A.P. and their sum is 21. If 2, 3, and 14 are added to them respectively, the results are in G.P. Find the numbers.
2. Prove by induction that $\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$
3. Find the sum to infinity of: $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \cdots$
4. In the expansion of $\left(x + \frac{a}{x^2}\right)^{12}$, the coefficient of x^6 is equal to the coefficient of x^0. Find the value of a.
5. A person deposits UGX 100,000 at the end of each year in an account that pays 8% compound interest annually. What will be the total amount after 10 years?

9.19 Conclusion

This comprehensive study of series provides essential mathematical tools for:

- Understanding patterns in sequences
- Solving real-world financial problems
- Modeling growth and decay processes
- Developing rigorous mathematical proofs
- Making approximations in various applications

Key Takeaways:

1. A.P. and G.P. model many real-world situations
2. Mathematical induction is a powerful proof technique
3. The binomial theorem has wide applications in probability and approximations
4. Understanding the convergence of series is cru-

cial for infinite processes

"Mathematics is not about numbers, equations, computations, or algorithms: it is about understanding."
William Paul Thurston

9.20 More scenario based questions with solutions

Question 1: Saving for a Land Plot in Luwero (A.P.)

A farmer in Luwero wants to buy a plot of land that costs 5,000,000 shillings. He starts by saving 500,000 shillings in the first month and plans to increase his savings by 100,000 shillings each subsequent month.

1. Find the amount the farmer will save in the 10th month.
2. How much money will the farmer have saved in total after 12 months?
3. Explain why this saving plan can be modeled as an Arithmetic Progression (A.P.) and its practical importance for financial planning.

Solution 1

1. The savings form an A.P. with the first term $a = 500,000$ and common difference $d = 100,000$. The formula for the n^{th} term is $a_n = a + (n-1)d$. $a_{10} = 500,000 + (10-1)100,000 = 500,000 + 9(100,000) = 500,000 + 900,000 = \mathbf{1,400,000}$ shillings.
2. The total savings after n months is the sum of the A.P., given by the formula $S_n = \frac{n}{2}(2a+(n-1)d)$. $S_{12} = \frac{12}{2}(2(500,000)+(12-1)100,000) = 6(1,000,000 + 11(100,000)) = 6(1,000,000+1,100,000) = 6(2,100,000) = \mathbf{12,600,000}$ shillings.
3. This is an A.P. because the amount added to the savings each month is a fixed number (100,000 shillings). This predictable, linear growth is the defining characteristic of an A.P. For financial planning, this model is practically important because it allows the farmer to project his savings over time, determine when he can afford the land, and adjust his plan if necessary. It provides a clear and straightforward way to track financial progress.

Question 2: Plant Growth in a Greenhouse (A.P.)

A farmer is tracking the height of a maize plant in his greenhouse. In the first week, the plant grows by 5 cm. In the second week, it grows by 7 cm, and in the third week, by 9 cm. This pattern continues.

1. How much will the plant grow in the 8th week?
2. What is the total height of the plant after 10 weeks, assuming its initial height was 20 cm?
3. Discuss how this type of modeling can help a farmer optimize plant growth and resource use.

Solution 2

1. The weekly growth forms an A.P. with $a = 5$ and $d = 2$. Growth in the 8^{th} week: $a_8 = a + (8-1)d = 5 + 7(2) = 5 + 14 = \mathbf{19}$ cm.
2. The total growth after 10 weeks is the sum of the A.P. $S_{10} = \frac{10}{2}(2a + (10-1)d) = 5(2(5) + 9(2)) = 5(10 + 18) = 5(28) = 140$ cm. Total height = Initial height + Total growth = 20 + 140 = **160** cm.
3. By modeling plant growth with an A.P., the farmer can predict future growth rates. If the actual growth deviates significantly from the model, it could signal a problem like nutrient deficiency or disease. This allows the farmer to take prompt action, optimizing water and fertilizer use and ensuring healthy crop development, ultimately maximizing yield.

Question 3: Microfinance Repayments (A.P.)

A microfinance institution in Uganda gives a loan of 2,000,000 shillings to a small business owner. The repayment schedule requires the owner to pay 100,000 shillings in the first month and decrease the payment by 5,000 shillings each subsequent month.

1. What is the repayment amount in the 15th month?
2. In how many months will the loan be fully repaid?
3. What are the benefits of this type of repayment scheme for a small business owner?

Solution 3

1. The payments form an A.P. with $a = 100,000$ and $d = -5,000$. Payment in the 15^{th} month: $a_{15} = a+(15-1)d = 100,000+14(-5,000) = 100,000 - 70,000 =$ **30, 000** shillings.
2. The sum of the repayments, S_n, must equal the total loan amount of 2,000,000. $S_n = \frac{n}{2}(2a+(n-1)d)$ $2,000,000 = \frac{n}{2}(2(100,000)+(n-1)(-5,000))$ $4,000,000 = n(200,000 - 5000n + 5000) = n(205,000 - 5000n)$ $4,000,000 = 205,000n - 5000n^2$ Divide by 5000: $800 = 41n - n^2 \implies n^2 - 41n + 800 = 0$. Using the quadratic formula: $n = \frac{-(-41)\pm\sqrt{(-41)^2-4(1)(800)}}{2(1)} = \frac{41\pm\sqrt{1681-3200}}{2}$. Since the discriminant is negative, this suggests there is no real solution, which is unusual. Let me re-examine the problem. Perhaps the assumption that the payment is a single amount is flawed. Let me try a different approach. $S_n = 2,000,000$. $a_n = 100,000 + (n-1)(-5000) > 0$. $105000 > 5000n \implies n < 21$. Let's use the sum formula again. $2,000,000 = \frac{n}{2}[200000 + (n-1)(-5000)] \implies 4,000,000 = n[205000 - 5000n] \implies 4000000 = 205000n - 5000n^2 \implies n^2 - 41n + 800 = 0$. Let me try the quadratic formula one more time. The discriminant is negative. Let me revise the question to yield a more practical answer.

Revised Question 3: Microfinance Repayments (A.P.)

A microfinance institution in Uganda gives a loan of 1,000,000 shillings to a small business owner. The repayment schedule requires the owner to pay 100,000 shillings in the first month and decrease the payment by 5,000 shillings each subsequent month.

1. What is the repayment amount in the 10th month?
2. In how many months will the loan be fully repaid?
3. What are the benefits of this type of repayment scheme for a small business owner?

Solution 3

1. The payments form an A.P. with $a = 100,000$ and $d = -5,000$. Payment in the 10^{th} month: $a_{10} = a+(10-1)d = 100,000+9(-5,000) = 100,000 - 45,000 =$ **55, 000** shillings.
2. The sum of the repayments, S_n, must equal 1,000,000. $S_n = \frac{n}{2}(2a + (n-1)d)$ $1,000,000 = \frac{n}{2}(2(100,000) + (n-1)(-5,000))$ $2,000,000 = n(200,000 - 5,000n + 5,000) = n(205,000 - 5,000n)$ $2,000,000 = 205,000n - 5,000n^2$ Divide by 5,000: $400 = 41n-n^2 \implies n^2-41n+400 = 0$. Factor the quadratic: $(n-16)(n-25) = 0$. $n = 16$ or $n = 25$. If $n = 25$, the payment in the 25th month would be $a_{25} = 100,000 + 24(-5,000) = -20,000$, which is not a valid payment. Thus, the solution is **16** months.
3. The benefits of this scheme include a decreasing financial burden for the business owner. In the early stages, when a new business might be generating less revenue, the higher payments are manageable. As the payments decrease, it allows the business to retain more capital for growth. It makes the loan feel more achievable, reducing the stress of repayment.

Question 4: Community Well Maintenance (A.P.)

A community in Nebbi has a well with a known water level. Due to evaporation and usage, the water level drops. On day 1, the level drops by 10 cm. On day 2, it drops by 12 cm, and on day 3, by 14 cm. This pattern continues. The total depth of the well is 20 meters.

1. Find the water level drop on the 15th day.
2. Calculate the total water level drop after 20 days.
3. How can the local water committee use this information to plan for water shortages?

Solution 4

1. The daily drop forms an A.P. with $a = 10$ and $d = 2$. Drop on the 15^{th} day: $a_{15} = a + (15-1)d = 10 + 14(2) = 10 + 28 =$ **38** cm.
2. The total drop after 20 days is the sum of the A.P. $S_{20} = \frac{20}{2}(2a + (20-1)d) = 10(2(10) + 19(2)) = 10(20 + 38) = 10(58) = 580$ cm. 580 cm is equal to **5.8** meters.
3. The water committee can use this model to predict the total drop in water level over a period. By knowing the well's initial depth and the drop rate, they can estimate when the water

will reach a critically low level. This allows them to proactively implement water conservation measures, such as rationing, or to plan for bringing in additional water supplies before a crisis occurs.

Question 5: Agricultural Pest Population (G.P.)

A farmer in Tororo is dealing with a pest infestation. The initial pest population is 100. If left unchecked, the population is expected to double every week.

1. Find the expected pest population after 5 weeks.
2. What is the total number of pests that would have existed in the field over the first 8 weeks (including the initial population)?
3. Explain why an exponential model is more suitable for population growth than a linear one.

Solution 5

1. The population growth forms a G.P. with the first term $a = 100$ and common ratio $r = 2$. The population after 5 weeks is the 6^{th} term (initial week is week 0). So we want a_6. $a_n = ar^{n-1}$. $a_6 = 100(2)^{6-1} = 100(2)^5 = 100(32) = \mathbf{3,200}$ pests.
2. The total number of pests is the sum of the G.P. over the first 8 weeks. $S_n = \frac{a(r^n-1)}{r-1}$. $S_8 = \frac{100(2^8-1)}{2-1} = \frac{100(256-1)}{1} = 100(255) = \mathbf{25,500}$ pests.
3. An exponential model, like a G.P., is more suitable for population growth because populations tend to grow at a rate proportional to their current size. The number of new individuals depends on the number of existing individuals. In contrast, a linear model (A.P.) assumes a constant rate of change, which is not realistic for biological populations that reproduce.

Question 6: Social Media Campaign Reach (G.P.)

A local non-profit in Kampala launches a social media campaign. In the first hour, their post gets 50 shares. In the second hour, it gets 150 shares, and in the third, it gets 450 shares. This pattern continues.

1. Find the number of shares the post is expected to get in the 6th hour.
2. What is the total number of shares the post would have received in the first 7 hours?
3. Explain how understanding this pattern can help the organization's marketing strategy.

Solution 6

1. The shares form a G.P. with $a = 50$ and $r = \frac{150}{50} = 3$. Shares in the 6^{th} hour: $a_6 = ar^{6-1} = 50(3)^5 = 50(243) = \mathbf{12,150}$ shares.
2. The total shares in the first 7 hours is the sum of the G.P. $S_7 = \frac{a(r^7-1)}{r-1} = \frac{50(3^7-1)}{3-1} = \frac{50(2187-1)}{2} = 25(2186) = \mathbf{54,650}$ shares.
3. Understanding this pattern allows the organization to predict the reach of their campaign and the effectiveness of their content. If the ratio (r) is greater than 1, it indicates viral growth. The organization can then strategically use this information to decide when to launch follow-up campaigns or to allocate more resources to content that shows similar growth patterns.

Question 7: Investment with Compound Interest (G.P.)

A businessman in Kampala invests 1,000,000 shillings in a savings account that earns a compound interest of 5% per year.

1. Find the value of the investment after 4 years.
2. Find the total interest earned over the 4 years.
3. Explain the link between compound interest and a Geometric Progression.

Solution 7

1. The value of the investment forms a G.P. with $a = 1,000,000$ and $r = 1 + \frac{5}{100} = 1.05$. The value after 4 years is the 5th term (a_5) since the initial investment is the first term. $a_5 = ar^{5-1} = 1,000,000(1.05)^4 = 1,000,000(1.21550625) = \mathbf{1,215,506.25}$ shillings.
2. Total interest earned = Final value - Initial investment. Interest = $1,215,506.25 - 1,000,000 = \mathbf{215,506.25}$ shillings.
3. Compound interest is a classic example of a G.P. because the value of the investment at the end of each period is found by multiplying the previous value by a constant ratio (1 + the interest rate). The value of the investment grows exponentially, which is the defining characteristic of a Geometric Progression.

Question 8: Total Harvest from a Maize Farm (A.P.)

A farmer in Masindi harvests 100 bags of maize in his first year. Due to improvements in farming techniques, he expects to increase his harvest by 10 bags each subsequent year.

1. How many bags of maize will he harvest in his 15th year?
2. What is the total number of bags he would have harvested over the first 20 years?
3. Explain how this model can be used to set realistic business goals.

Solution 8

1. The annual harvests form an A.P. with $a = 100$ and $d = 10$. Harvest in the 15^{th} year: $a_{15} = a+(15-1)d = 100+14(10) = 100+140 =$ **240** bags.
2. The total harvest over 20 years is the sum of the A.P. $S_{20} = \frac{20}{2}(2a + (20-1)d) = 10(2(100) + 19(10)) = 10(200 + 190) = 10(390) =$ **3, 900** bags.
3. This model helps the farmer set realistic and measurable goals. By projecting the harvest over a long period, he can plan for resource acquisition, storage, and sales. It allows him to track whether his farming techniques are meeting the expected increase in yield, providing a data-driven approach to his business.

Question 9: Water Pump Degradation (G.P.)

A community's solar-powered water pump in Pallisa is experiencing degradation. Its water output decreases by 2% each year. The initial output is 1000 liters per hour.

1. Find the water output of the pump after 5 years.
2. What is the total water output during the first 10 years?
3. How can this information be used to plan for pump replacement?

Solution 9

1. The water output forms a G.P. with $a = 1000$ and $r = 1 - \frac{2}{100} = 0.98$. Output after 5 years is the 6^{th} term. $a_6 = ar^{6-1} = 1000(0.98)^5 \approx 1000(0.9039) =$ **903.9** liters/hour.
2. The total water output over the first 10 years is the sum of the G.P. $S_{10} = \frac{a(1-r^{10})}{1-r} = \frac{1000(1-0.98^{10})}{1-0.98} = \frac{1000(1-0.81707)}{0.02} = \frac{1000(0.18293)}{0.02} = \frac{182.93}{0.02} =$ **9146.5** liters/hour.
3. By modeling the degradation with a G.P., the community can predict when the pump's output will fall below a usable level. This allows them to set a timeline for fundraising and purchasing a new pump before a complete failure occurs, ensuring a continuous supply of water.

Question 10: Building a Community Center (A.P.)

A community in Kabale is raising funds to build a community center. They collect 500,000 shillings in the first week. In each subsequent week, they plan to increase their collection by 75,000 shillings. The target is 10,000,000 shillings.

1. Find the amount of money they will collect in the 12th week.
2. What is the total amount they will have collected after 15 weeks?
3. Based on the total amount collected after 15 weeks, how far are they from their target, and what does this mean for their fundraising efforts?

Solution 10

1. The weekly collections form an A.P. with $a = 500,000$ and $d = 75,000$. Collection in the 12^{th} week: $a_{12} = a + (12-1)d = 500,000+11(75,000) = 500,000+825,000 =$ **1, 325, 000** shillings.
2. The total collected after 15 weeks is the sum of the A.P. $S_{15} = \frac{15}{2}(2a + (15-1)d) = 7.5(2(500,000) + 14(75,000)) = 7.5(1,000,000 + 1,050,000) = 7.5(2,050,000) =$ **15, 375, 000** shillings.
3. After 15 weeks, the community will have collected 15,375,000 shillings, which is more than their 10,000,000 shillings target. They will have a surplus of $15,375,000 - 10,000,000 =$ **5, 375, 000** shillings. This means their fundraising efforts have been highly successful, allowing them to not only meet their goal but also to potentially fund additional projects or unforeseen costs.

11. **Coffee Harvesting Schedule in Mbale District**
Scenario: A coffee farmer in Mbale notices

that her harvest increases by a fixed amount each month during the peak season. In the first month, she harvested 500 kg, and each subsequent month, the harvest increases by 50 kg.

(a) Identify the type of sequence and find the harvest amount in the 6th month.

(b) Calculate the total harvest over the 6-month peak season.

(c) If the coffee sells for 8,000 UGX per kg, what is her total revenue for the season?

Solution 1

(a) This is an Arithmetic Progression (AP) with:

$$a = 500, \quad d = 50$$
$$a_6 = a + 5d = 500 + 5(50) = 500 + 250 = 750 \text{ kg}$$

(b) Sum of AP:

$$S_6 = \frac{6}{2}[2a + (6-1)d] = 3[2(500) + 5(50)]$$
$$= 3[1000 + 250] = 3 \times 1250 = 3750 \text{ kg}$$

(c) Total revenue:

$$\text{Revenue} = 3750 \times 8000 = 30,000,000 \text{ UGX}$$

12. **Loan Repayment Scheme for Small Business**
Scenario: A Kampala market vendor takes a loan where she repays 100,000 UGX in the first month and decreases the payment by 5,000 UGX each subsequent month.

(a) Identify the sequence type and find the payment in the 8th month.

(b) Calculate the total amount paid after 12 months.

(c) How many months will it take for the payment to reduce to zero?

Solution 2

(a) Arithmetic Progression (AP) with:

$$a = 100,000, \quad d = -5,000$$
$$a_8 = a + 7d = 100,000 + 7(-5,000) = 100,000 - 35,000 = 65,000 \text{ UGX}$$

(b) Sum of AP:

$$S_{12} = \frac{12}{2}[2a + (12-1)d] = 6[2(100,000) + 11(-5,000)]$$
$$= 6[200,000 - 55,000] = 6 \times 145,000 = 870,000 \text{ UGX}$$

(c) Solve for n when $a_n = 0$:

$$a + (n-1)d = 0$$
$$100,000 + (n-1)(-5,000) = 0$$
$$-5,000(n-1) = -100,000$$
$$n - 1 = 20 \Rightarrow n = 21 \text{ months}$$

13. **Construction Worker Payment Plan**
Scenario: A construction company in Entebbe pays workers such that each day's pay is 1.5 times the previous day's pay. The first day's pay is 20,000 UGX.

(a) Identify the sequence type and find the payment on the 7th day.

(b) Calculate the total payment for 10 days of work.

(c) If the project lasts 15 days, what will be the payment on the last day?

Solution 3

(a) Geometric Progression (GP) with:

$$a = 20,000, \quad r = 1.5$$
$$a_7 = ar^6 = 20,000 \times (1.5)^6 = 20,000 \times 11.39 = 227,80$$

(b) Sum of GP:

$$S_{10} = a\frac{r^{10}-1}{r-1} = 20,000\frac{(1.5)^{10}-1}{1.5-1}$$
$$= 20,000\frac{57.67-1}{0.5} = 20,000 \times 113.34 = 2,266,800$$

(c) Payment on day 15:

$$a_{15} = ar^{14} = 20,000 \times (1.5)^{14} = 20,000 \times 291.93 = 5,8$$

14. **Fish Population Growth in Lake Victoria**
Scenario: The Nile perch population in a section of Lake Victoria doubles every 3 years. The current population is 5,000 fish.

(a) Identify the sequence type and find the population after 12 years.

(b) Calculate the total population growth over 15 years.

(c) How long will it take for the population to reach 320,000 fish?

Solution 4

(a) Geometric Progression (GP) with:

$$a = 5,000, \quad r = 2 \text{ every 3 years}$$
After 12 years: $n = 12/3 = 4$ periods
$$a_4 = ar^3 = 5,000 \times 2^3 = 5,000 \times 8 = 40,$$

(b) After 15 years: $n = 15/3 = 5$ periods

$$a_5 = ar^4 = 5,000 \times 2^4 = 5,000 \times 16 = 80,000 \text{ fish}$$

(c) Solve for n when $a_n = 320,000$:

$$ar^{n-1} = 320,000$$
$$5,000 \times 2^{n-1} = 320,000$$
$$2^{n-1} = 64 = 2^6$$
$$n - 1 = 6 \Rightarrow n = 7 \text{ periods} = 21 \text{ years}$$

15. **Savings Plan for University Education**
Scenario: A parent in Gulu saves for their child's education by depositing money that increases by a fixed amount each month. The first deposit is 100,000 UGX and increases by 10,000 UGX monthly.
(a) Identify the sequence type and find the deposit amount in the 18th month.
(b) Calculate the total savings after 24 months.
(c) How many months until the deposit reaches 300,000 UGX?

Solution 5

(a) Arithmetic Progression (AP) with:

$$a = 100,000, \quad d = 10,000$$
$$a_{18} = a + 17d = 100,000 + 17(10,000) = 270,000 \text{ UGX}$$

(b) Sum of AP:

$$S_{24} = \frac{24}{2}[2a + (24-1)d] = 12[2(100,000) + 23(10,000)]$$
$$= 12[200,000 + 230,000] = 12 \times 430,000 = 5,160,000 \text{ UGX}$$

(c) Solve for n when $a_n = 300,000$:

$$a + (n-1)d = 300,000$$
$$100,000 + (n-1)(10,000) = 300,000$$
$$(n-1)(10,000) = 200,000$$
$$n - 1 = 20 \Rightarrow n = 21 \text{ months}$$

16. **Mobile Money Adoption Rate**
Scenario: The number of mobile money users in a Kampala suburb increases by 25% each quarter. Currently, there are 2,000 users.
(a) Identify the sequence type and find the number of users after 2 years.
(b) Calculate the total number of new users acquired over 3 years.
(c) How long will it take for the user base to reach 15,000?

Solution 6

(a) Geometric Progression (GP) with:

$$a = 2,000, \quad r = 1.25$$

After 2 years: $n = 8$ quarters

$$a_8 = ar^7 = 2,000 \times (1.25)^7 = 2,000 \times 4.77$$

(b) After 3 years: $n = 12$ quarters

$$a_{12} = ar^{11} = 2,000 \times (1.25)^{11} = 2,000 \times 11.64 =$$

New users $= 23,280 - 2,000 = 21,280$ users

(c) Solve for n when $a_n = 15,000$:

$$ar^{n-1} = 15,000$$
$$2,000 \times (1.25)^{n-1} = 15,000$$
$$(1.25)^{n-1} = 7.5$$
$$(n-1)\log(1.25) = \log(7.5)$$
$$n - 1 = \frac{\log(7.5)}{\log(1.25)} \approx \frac{0.875}{0.097} \approx 9.02$$
$$n \approx 10 \text{ quarters} = 2.5 \text{ years}$$

17. **Agricultural Production Increase**
Scenario: A maize farmer in Lira increases production by a fixed amount each season. The first season's yield was 2,000 kg and increases by 200 kg each season.
(a) Identify the sequence type and find the yield in the 8th season.
(b) Calculate the total production over 10 seasons.
(c) In which season will the yield first exceed 4,000 kg?

Solution 7

(a) Arithmetic Progression (AP) with:

$$a = 2,000, \quad d = 200$$
$$a_8 = a + 7d = 2,000 + 7(200) = 2,000 + 1,400 = 3,400$$

(b) Sum of AP:

$$S_{10} = \frac{10}{2}[2a + (10-1)d] = 5[2(2,000) + 9(200)]$$
$$= 5[4,000 + 1,800] = 5 \times 5,800 = 29,000 \text{ kg}$$

(c) Solve for n when $a_n > 4,000$:

$$a + (n-1)d > 4,000$$
$$2,000 + (n-1)(200) > 4,000$$
$$(n-1)(200) > 2,000$$
$$n - 1 > 10 \Rightarrow n > 11$$

Season 12: $a_{12} = 2,000 + 11(200) = 4,200$ kg

18. **COVID-19 Infection Spread Model**
Scenario: During a outbreak in Jinja, the number of infections triples every 5 days. Initially, there were 10 cases.
(a) Identify the sequence type and find infections after 20 days.
(b) Calculate the total number of people infected over 30 days.
(c) How long until infections reach 10,000?

Solution 8

(a) Geometric Progression (GP) with:

$$a = 10, \quad r = 3 \text{ every 5 days}$$

After 20 days: $n = 20/5 = 4$ periods

$$a_4 = ar^3 = 10 \times 3^3 = 10 \times 27 = 270 \text{ cases}$$

(b) After 30 days: $n = 30/5 = 6$ periods

$$a_6 = ar^5 = 10 \times 3^5 = 10 \times 243 = 2,430 \text{ cases}$$

(c) Solve for n when $a_n = 10,000$:

$$\begin{aligned} ar^{n-1} &= 10,000 \\ 10 \times 3^{n-1} &= 10,000 \\ 3^{n-1} &= 1,000 \\ (n-1)\log(3) &= \log(1,000) \\ n - 1 &= \frac{3}{0.477} \approx 6.29 \\ n &\approx 7.29 \text{ periods} = 36.45 \text{ days} \end{aligned}$$

19. **Construction Material Cost Estimation**
Scenario: The cost of building materials in Kampala increases by a fixed amount each year. In 2020, cement cost 30,000 UGX per bag and increases by 2,000 UGX yearly.
(a) Identify the sequence type and find the cost in 2025.
(b) Calculate the total price increase from 2020 to 2030.
(c) When will the cost reach 50,000 UGX per bag?

Solution 9

(a) Arithmetic Progression (AP) with:

$$a = 30,000, \quad d = 2,000$$

2025 is year 6: $a_6 = a + 5d = 30,000 + 5(2,000) = 40,000$ UGX

(b) 2030 is year 11:

$$a_{11} = a + 10d = 30,000 + 10(2,000) = 50,000 \text{ UGX}$$

Total increase $= 50,000 - 30,000 = 20,000$ UGX

(c) Solve for n when $a_n = 50,000$:

$$\begin{aligned} a + (n-1)d &= 50,000 \\ 30,000 + (n-1)(2,000) &= 50,000 \\ (n-1)(2,000) &= 20,000 \\ n - 1 = 10 \Rightarrow n &= 11 \text{ years} = 2030 \end{aligned}$$

20. **Microfinance Loan Growth**
Scenario: A microfinance institution in Mbale grows its loan portfolio by 20% each year. The initial portfolio was 50,000,000 UGX.
(a) Identify the sequence type and find the portfolio value after 5 years.
(b) Calculate the total growth in value over 7 years.
(c) How long until the portfolio reaches 200,000,000 UGX?

Solution 10

(a) Geometric Progression (GP) with:

$$a = 50,000,000, \quad r = 1.20$$

$$a_5 = ar^4 = 50,000,000 \times (1.20)^4 = 50,000,000 \times 2.073$$

(b) After 7 years:

$$a_7 = ar^6 = 50,000,000 \times (1.20)^6 = 50,000,000 \times$$

Growth $= 149,300,000 - 50,000,000 = 99,300,000$ UG

(c) Solve for n when $a_n = 200,000,000$:

$$\begin{aligned} ar^{n-1} &= 200,000,000 \\ 50,000,000 \times (1.20)^{n-1} &= 200,000,000 \\ (1.20)^{n-1} &= 4 \\ (n-1)\log(1.20) &= \log(4) \\ n - 1 &= \frac{0.602}{0.079} \approx 7.62 \\ n &\approx 8.62 \text{ years} \end{aligned}$$

21. **Smallholder banana plantation — A.P. crop spacing and rows.**
A farmer in Luwero plans banana planting in rows. He starts with a first row of 18 plants. He plans to increase the number of plants per successive row by a constant of 2 plants to allow for staggered planting patterns as soil allows.
(a) Write an expression for the number of plants in the n-th row. Find how many plants are in the 10th row.
(b) If the farmer intends to plant 10 rows, compute the total number of plants.

(c) If each banana mat (plant) costs UGX 5,500, find the total cost for 10 rows.

Solution.

This is an A.P. with first term $a_1 = 18$ and common difference $d = 2$.

(a) $a_n = a_1 + (n-1)d = 18 + (n-1)\cdot 2 = 16 + 2n$. Thus $a_{10} = 18 + (10-1)\cdot 2 = 18 + 18 = 36$ plants.

(b) Sum of first $N = 10$ terms: $S_{10} = \frac{N}{2}(a_1 + a_{10}) = \frac{10}{2}(18 + 36) = 5 \cdot 54 = 270$ plants.

(c) Total cost $= 270 \times 5{,}500 = 1{,}485{,}000$ UGX.

22. **Village savings group (ROSCAs) — A.P. payouts.**

A community SACCO uses a rotating savings scheme where the first member to receive the lump sum in month 1 gets UGX 100,000. Each subsequent month the payout increases by UGX 12,000 to compensate for inflation and incentive.

(a) Formulate the n-th payout amount and find the payout in month 8.

(b) Calculate the total amount paid out over the first 12 months.

(c) If each member contributes equally and there are 12 members, what is the required monthly contribution per member to cover the first 12 payouts (assume the SACCO pays each month from pooled contributions of that same month)?

Solution.

A.P. with $a_1 = 100{,}000$, $d = 12{,}000$.

(a) $a_n = 100{,}000 + (n-1)\cdot 12{,}000$. For $n = 8$: $a_8 = 100{,}000 + 7\cdot 12{,}000 = 100{,}000 + 84{,}000 = 184{,}000$ UGX.

(b) $S_{12} = \frac{12}{2}(a_1 + a_{12})$. First compute $a_{12} = 100{,}000 + 11\cdot 12{,}000 = 100{,}000 + 132{,}000 = 232{,}000$. Then $S_{12} = 6(100{,}000 + 232{,}000) = 6 \cdot 332{,}000 = 1{,}992{,}000$ UGX.

(c) If 12 members share equally: Monthly pooled payment needed = payout that month; but question asks "to cover the first 12 payouts"—interpreting as total contributions over 12 months must equal total payouts. Thus per-member total over 12 months $= \frac{1{,}992{,}000}{12} = 166{,}000$ UGX per member (over the year). So monthly contribution per member $= \frac{166{,}000}{12} \approx 13{,}833.33$ UGX. (If contributions must be integer, round appropriately.)

23. **Rural bus fare increments — A.P. increases.**

A boda-boda cooperative fixes fares along a route: Fare for first kilometer is UGX 1,500; they decide to increase the fare by UGX 200 for every additional km up to 10 km (i.e., an arithmetic increase per km).

(a) Write the fare formula for n kilometers (integer $n \geq 1$). What is the fare for 6 km?

(b) Compute total revenue if in one day a rider travels exactly 1,2,...,6 km each once, paying the respective fares.

(c) If the cooperative wants the average fare for kilometers 1–10 to be less than UGX 3,000, does their scheme meet this target? Show working.

Solution.

A.P.: $a_1 = 1500$, $d = 200$ (per km after first).

(a) Fare for n km: $a_n = 1500 + (n-1)\cdot 200$. For $n = 6$: $a_6 = 1500 + 5\cdot 200 = 1500 + 1000 = 2500$ UGX.

(b) Revenue = sum $S_6 = \frac{6}{2}(a_1 + a_6) = 3(1500 + 2500) = 3 \cdot 4000 = 12{,}000$ UGX.

(c) Average fare for $n = 1\ldots 10$ is $\bar{a} = \frac{S_{10}}{10}$. Compute $a_{10} = 1500 + 9\cdot 200 = 1500 + 1800 = 3300$. Then $S_{10} = \frac{10}{2}(1500 + 3300) = 5 \cdot 4800 = 24{,}000$. Average $\bar{a} = 24000/10 = 2400$ UGX < 3000. So scheme meets the target.

24. **Solar lamp distribution scaling — G.P. adoption.**

An NGO gives solar lamps to households. In the first month they reach 50 households. Due to word-of-mouth the number of new households reached each month multiplies by ratio $r = 1.6$ (i.e., 60% increase month-on-month) for the next 6 months.

(a) Model the number of households reached in month n as a geometric progression. Find households reached in month 5

(take $n = 1$ as first month).

(b) How many households in total are reached over the first 6 months (sum of first 6 terms)?

(c) If the program continues indefinitely but growth slows so that after month 6 the monthly growth ratio becomes $r' = 0.9$ (i.e., the new households per month shrink by 10% each month from the month-7 value), can the NGO approximate the total additional households after month 6 as an infinite GP? If so compute that infinite sum starting from month 7. (Assume month-6 value is used as the starting point for the new geometric tail.)

Solution.

G.P. with $a_1 = 50$, $r = 1.6$.

(a) $a_n = 50(1.6)^{n-1}$. For $n = 5$: $a_5 = 50(1.6)^4$. Compute $(1.6)^2 = 2.56$, $(1.6)^4 = (2.56)^2 = 6.5536$. So $a_5 = 50 \cdot 6.5536 = 327.68 \approx 328$ households (round to whole).

(b) Sum $S_6 = 50\frac{1.6^6-1}{1.6-1} = 50\frac{1.6^6-1}{0.6}$. Compute $1.6^6 = 1.6^4 \cdot 1.6^2 = 6.5536 \cdot 2.56 = 16.777216$. Then $S_6 = 50 \cdot \frac{16.777216-1}{0.6} = 50 \cdot \frac{15.777216}{0.6} = 50 \cdot 26.29536 = 1{,}314.768 \approx 1{,}315$ households.

(c) At month 6 the monthly households are $a_6 = 50\,(1.6)^5$. Compute $1.6^5 = 1.6^4 \cdot 1.6 = 6.5536 \cdot 1.6 = 10.48576$. So $a_6 \approx 50 \cdot 10.48576 = 524.288 \approx 524$. Month 7 value (start of tail) is $a_7 = a_6 \cdot r' = 524.288 \cdot 0.9 \approx 471.8592$. The tail is an infinite G.P. with first term $A = a_7$ and ratio $q = 0.9$. Since $|q| < 1$, sum $= \frac{A}{1-q} = \frac{471.8592}{1-0.9} = \frac{471.8592}{0.1} = 4{,}718.592 \approx 4{,}719$ additional households after month 6.

25. **Water tank filling schedule — A.P. pump schedule.**

A village pump office schedules inspections: first inspection takes 30 minutes, and each next inspection increases by 5 minutes (to allow for more checks) forming an arithmetic sequence. They plan 12 inspections.

(a) Give formula for inspection time in minutes for the n-th inspection and find the 12th inspection length.

(b) Compute total time spent on all 12 inspections.

(c) If two technicians share these inspections equally (each does alternate inspections: one does odd-numbered, other even-numbered), compute total minutes each technician spends.

Solution.

A.P. with $a_1 = 30$, $d = 5$.

(a) $a_n = 30+(n-1)5 = 25+5n$. For $n = 12$: $a_{12} = 30+11\cdot5 = 30+55 = 85$ minutes.

(b) $S_{12} = \frac{12}{2}(a_1 + a_{12}) = 6(30 + 85) = 6 \cdot 115 = 690$ minutes.

(c) Odd-indexed inspections: there are 6 odd (1,3,...,11). Their sum $S_{\text{odd}} = \frac{6}{2}(a_1 + a_{11})$, compute $a_{11} = 30+10\cdot5 = 80$. So $S_{\text{odd}} = 3(30 + 80) = 3 \cdot 110 = 330$ minutes. Even-indexed sum $= 690 - 330 = 360$ minutes. So technicians spend 330 and 360 minutes respectively.

26. **Construction brick stacking — G.P. stock reduction.**

A construction site uses bricks from a stockpile. On day 1 they use 4,000 bricks; each subsequent day consumption decreases by a factor of 0.85 as they finish detailed works, for a planned long tail of consumption.

(a) Model daily consumption as a G.P. and compute day-5 consumption.

(b) Compute total bricks used in the first 10 days.

(c) Estimate the total bricks that will be used in future days from day 11 onward as an infinite GP (if appropriate).

Solution.

G.P. with $a_1 = 4000$, ratio $r = 0.85$.

(a) $a_n = 4000 \cdot (0.85)^{n-1}$. Day-5: $a_5 = 4000 \cdot (0.85)^4$. Compute $(0.85)^2 = 0.7225$, $(0.85)^4 = 0.52200625$. So $a_5 \approx 4000 \cdot 0.52200625 = 2088.025 \approx 2{,}088$ bricks.

(b) Sum first 10 days: $S_{10} = 4000 \cdot \frac{1-0.85^{10}}{1-0.85}$. Compute $0.85^{10} \approx (0.85^5)^2$. First $0.85^5 = 0.85^4 \cdot 0.85 \approx 0.52200625 \cdot 0.85 \approx 0.4437053125$.

Then square: $(0.4437053)^2 \approx 0.196873$. So $S_{10} \approx 4000 \cdot \dfrac{1-0.196873}{0.15} = 4000 \cdot \dfrac{0.803127}{0.15} = 4000 \cdot 5.35418 \approx 21{,}416.7 \approx 21{,}417$ bricks.

(c) From day 11 onward the sequence continues as a G.P. with first term $a_{11} = 4000 \cdot 0.85^{10} \approx 4000 \cdot 0.196873 \approx 787.49$. Ratio $r = 0.85$ with $|r| < 1$ so infinite sum $= \dfrac{a_{11}}{1-r} \approx \dfrac{787.49}{0.15} \approx 5{,}249.93 \approx 5{,}250$ bricks expected after day 10.

27. **Mobile data bundle decay — G.P. usage over days.**

A student buys a 10 GB data bundle and uses 20% of the remaining data on day 1, and then each day uses 20% of that day's remaining balance (i.e., remaining fraction multiplies by 0.8 daily).

(a) Show that the remaining data after n days follows a geometric progression. Compute remaining data after 7 days (in GB).

(b) Find total data used in the first 7 days.

(c) Does the series of daily usage (amount used each day) form a geometric progression? If yes, find the infinite total data used if the pattern continued forever (should equal 10 GB). Show the structure.

Solution.

Remaining after n days: $R_n = 10 \cdot (0.8)^n$ GB.

(a) After 7 days: $R_7 = 10 \cdot 0.8^7$. Compute $0.8^2 = 0.64$, $0.8^4 = 0.4096$, $0.8^7 = 0.8^4 \cdot 0.8^3 = 0.4096 \cdot 0.512 = 0.2097152$. So $R_7 \approx 10 \cdot 0.2097152 = 2.097152 \approx 2.097$ GB.

(b) Total used in first 7 days $= 10 - R_7 \approx 10 - 2.097152 = 7.902848 \approx 7.903$ GB.

(c) Daily usage amounts: Day 1 used $10(1-0.8) = 10 \cdot 0.2 = 2$ GB. Day 2 used $10 \cdot 0.8(1-0.8) = 10 \cdot 0.8 \cdot 0.2 = 2 \cdot 0.8$ GB, etc. So daily usages form a G.P. with first term 2 and ratio 0.8. Infinite sum $= \dfrac{2}{1-0.8} = \dfrac{2}{0.2} = 10$ GB, which equals the bundle size, as expected.

28. **School textbook distribution — A.P. delivery schedule.**

The district education office receives textbooks and delivers them to schools in a schedule where the first delivery batch contains 120 textbooks, and each subsequent delivery batch increases by 15 textbooks to cover more schools. They plan 9 batch deliveries.

(a) Give the formula for the n-th batch size and compute the 9th batch.

(b) Total textbooks delivered in 9 batches.

(c) If each textbook costs UGX 28,000, compute the total procurement cost for the 9 batches.

Solution.

A.P. with $a_1 = 120$, $d = 15$.

(a) $a_n = 120 + (n-1)15$. For $n = 9$: $a_9 = 120 + 8 \cdot 15 = 120 + 120 = 240$ textbooks.

(b) $S_9 = \dfrac{9}{2}(a_1 + a_9) = \dfrac{9}{2}(120 + 240) = \dfrac{9}{2} \cdot 360 = 9 \cdot 180 = 1{,}620$ textbooks.

(c) Cost $= 1{,}620 \times 28{,}000 = 45{,}360{,}000$ UGX.

29. **Fish pond yield growth — G.P. stock.**

A fish farmer stocks 1,000 fingerlings. Each breeding cycle (every 6 months) the population multiplies by a factor of 1.25 due to controlled breeding, for four cycles planned in two years.

(a) Express fish population after n cycles and compute population after 4 cycles.

(b) Total fish added (net increase) after 4 cycles compared to initial stocking.

(c) If the farmer plans to harvest 60% of the final population at cycle 4, how many fish will be harvested? If each sells at UGX 4,200, calculate gross revenue.

Solution.

G.P. with $a_1 = 1000$, $r = 1.25$.

(a) $a_n = 1000 \cdot (1.25)^{n-1}$. After 4 cycles population $= 1000 \cdot (1.25)^3$. Compute $(1.25)^2 = 1.5625$, $(1.25)^3 = 1.953125$. So $a_4 = 1000 \cdot 1.953125 = 1{,}953.125 \approx 1{,}953$ fish.

(b) Net increase $\approx 1{,}953 - 1{,}000 = 953$ fish.

(c) Harvest $= 0.6 \times 1{,}953 \approx 1{,}171.8 \approx 1{,}172$ fish. Gross revenue $\approx 1{,}172 \times 4{,}200 = 4{,}922{,}400$ UGX.

30. **Market stall pricing for sukuma wiki bundles — mixed A.P./G.P.**

A vegetable vendor prices bundles of sukuma

wiki (collard greens) starting at UGX 2,000 for a small bundle. To encourage bulk purchases she decides: if a buyer purchases multiple bundles in a single transaction, the price of each successive bundle in that transaction reduces by 5% multiplicatively (i.e., geometric discount per additional bundle). However, the vendor also increases the base price by UGX 50 each week (A.P. week-on-week) due to rising costs. Consider a buyer who purchases 5 bundles in week 6.

(a) Compute the base price in week 6.

(b) Given the geometric discount of 5% per additional bundle within the transaction, compute the prices of the 5 bundles (bundle 1 at full base price, bundle 2 at base×0.95, etc.) and total transaction cost.

(c) If the buyer instead buys one bundle every week for 6 weeks (no multi-bundle discount but A.P. weekly increases apply), compare the total cost of the two strategies. Which is cheaper and by how much?

Solution.

Base price week 1 is UGX 2,000 and weekly A.P. increase is $d = 50$.

(a) Base price in week n: $P_n = 2000 + (n-1)\cdot 50$. For $n = 6$: $P_6 = 2000 + 5\cdot 50 = 2000 + 250 = 2{,}250$ UGX.

(b) Prices with 5% multiplicative discount: Bundle 1: 2,250. Bundle 2: $2{,}250 \cdot 0.95 = 2{,}137.5$. Bundle 3: $2{,}250 \cdot 0.95^2 = 2{,}030.625$. Bundle 4: $2{,}250 \cdot 0.95^3 \approx 1{,}929.09375$. Bundle 5: $2{,}250 \cdot 0.95^4 \approx 1{,}832.63906$. Total cost $\approx 2{,}250 + 2{,}137.5 + 2{,}030.625 + 1{,}929.09375 + 1{,}832.63906 \approx 10{,}179.85781 \approx 10{,}180$ UGX.

(c) If buying one bundle each week for 6 weeks: week prices are $P_1 = 2000$, $P_2 = 2050$, $P_3 = 2100$, $P_4 = 2150$, $P_5 = 2200$, $P_6 = 2250$. Total $= 2000 + 2050 + 2100 + 2150 + 2200 + 2250 = 12{,}750$ UGX. Comparison: multi-bundle in week 6 costs $\approx 10{,}180$ UGX vs spread purchases cost 12,750 UGX. Multi-bundle strategy is cheaper by $\approx 12{,}750 - 10{,}180 = 2{,}570$ UGX.

31. **Coffee Farmers' Cooperative Savings Scheme in Mbale District**

Context: The Mbale Coffee Farmers' Cooperative encourages members to save progressively. A farmer starts by saving UGX 50,000 in the first month and increases savings by UGX 25,000 each subsequent month.

(a) Identify this as an arithmetic progression and derive the formula for the farmer's savings in the nth month. Calculate savings in the 12th month.

(b) Derive and apply the formula for the sum of an arithmetic progression to find total savings after 2 years (24 months).

(c) The cooperative offers a bonus scheme where monthly bonuses form a geometric progression: UGX 5,000, UGX 10,000, UGX 20,000, etc. Find the total bonus received in the first 8 months.

Solution:

(a) **Identifying the Arithmetic Progression:**

Monthly savings: 50,000, 75,000, 100,000, 125,000, ...

This is an A.P. with: - First term: $a = 50{,}000$ - Common difference: $d = 25{,}000$

Formula for nth term of A.P.: $a_n = a + (n-1)d$

Derivation:

$$a_1 = a = 50{,}000 \tag{9.1}$$

$$a_2 = a + d = 50{,}000 + 25{,}000 = 75{,}000 \tag{9.2}$$

$$a_3 = a + 2d = 50{,}000 + 2(25{,}000) = 100{,}000 \tag{9.3}$$

$$\vdots \tag{9.4}$$

$$a_n = a + (n-1)d \tag{9.5}$$

12th month savings: $a_{12} = 50{,}000 + (12-1)\times 25{,}000 = 50{,}000 + 275{,}000 =$ UGX 325,000

(b) **Sum of Arithmetic Progression:**

Formula derivation: Let $S_n = a_1 + a_2 + a_3 + \ldots + a_n$

Writing in two ways:

$$S_n = a + (a+d) + (a+2d) + \ldots + (a+(n-1)d) \tag{9.6}$$

$$S_n = (a+(n-1)d) + (a+(n-2)d) + \ldots + (a+d) + a \tag{9.7}$$

Adding these equations:

$$2S_n = n[a + (a+(n-1)d)] = n[2a + (n-1)d] \tag{9.8}$$

$$S_n = \frac{n}{2}[2a + (n-1)d] \tag{9.9}$$

Alternative form: $S_n = \frac{n}{2}(a_1 + a_n)$

Total savings after 24 months:

$$S_{24} = \frac{24}{2}[2(50,000) + (24-1)(25,000)] \tag{9.10}$$

$$= 12[100,000 + 23 \times 25,000] \tag{9.11}$$

$$= 12[100,000 + 575,000] \tag{9.12}$$

$$= 12 \times 675,000 = \text{UGX } 8,100,000 \tag{9.13}$$

(c) **Geometric Progression for Bonuses:**
Bonus sequence: 5,000, 10,000, 20,000, 40,000, ...
This is a G.P. with: - First term: $a = 5,000$ - Common ratio: $r = 2$
Sum of G.P. formula: For $r \neq 1$: $S_n = \frac{a(r^n-1)}{r-1}$
Total bonus in 8 months:

$$S_8 = \frac{5,000(2^8-1)}{2-1} \tag{9.14}$$

$$= 5,000(256-1) \tag{9.15}$$

$$= 5,000 \times 255 = \text{UGX } 1,275,000 \tag{9.16}$$

32. **Boda Boda SACCO Loan Repayment Plan in Kampala**
Context: Kampala Boda Boda SACCO offers loans with a repayment plan where monthly payments decrease arithmetically. A member borrows UGX 2,000,000 and repays UGX 200,000 in the first month, then UGX 15,000 less each subsequent month.
(a) Set up the arithmetic progression for monthly payments and find the payment in the 10th month. Determine when payments become zero.
(b) Calculate the total amount repaid over the complete loan period.
(c) If interest compounds geometrically at 2

Solution:

(a) **Arithmetic Progression Setup:**
Monthly payments: 200,000, 185,000, 170,000, 155,000, ...
A.P. with $a = 200,000$, $d = -15,000$
10th month payment: $a_{10} = 200,000 + (10-1)(-15,000) = 200,000 - 135,000 = \text{UGX } 65,000$
When payments become zero: $a_n = 200,000 + (n-1)(-15,000) = 0$
$200,000 = 15,000(n-1)$ $n-1 = \frac{200,000}{15,000} = 13.33$ $n = 14.33$
Payments become zero after month 14 (in the 15th month).

(b) **Total amount repaid:**
Since payments go to zero in month 15, we calculate sum for 14 months:

$$S_{14} = \frac{14}{2}[2(200,000) + (14-1)(-15,000)] \tag{9.17}$$

$$= 7[400,000 + 13(-15,000)] \tag{9.18}$$

$$= 7[400,000 - 195,000] \tag{9.19}$$

$$= 7 \times 205,000 = \text{UGX } 1,435,000 \tag{9.20}$$

(c) **Geometric Interest Pattern:**
Let B_n be the balance after month n.
Interest in month $n+1 = 0.02 \times B_n$
If balance reduces by payment minus interest, the remaining balance follows a pattern.
Month 1: Balance = UGX 2,000,000
Interest $= 0.02 \times 2,000,000 =$ UGX 40,000
For the 5th month (assuming geometric growth of 2Interest $= 40,000 \times (1.02)^4 = 40,000 \times 1.0824 =$ UGX 43,297

33. **Makerere University Tuition Fee Structure**
Context: Makerere University implements a progressive fee structure where tuition increases arithmetically for private students. First-year fees are UGX 1,500,000, increasing by UGX 200,000 each year.
(a) Establish the arithmetic progression for tuition fees and calculate the fee for a

student's final (4th) year.

(b) Find the total tuition cost for a complete 4-year degree program.

(c) The university's endowment fund grows geometrically at 15

Solution:

(a) **Tuition Fee Progression:**
Annual fees: 1,500,000, 1,700,000, 1,900,000, 2,100,000
A.P. with $a = 1,500,000$, $d = 200,000$
4th year fee: $a_4 = 1,500,000 + (4-1)(200,000) = 1,500,000 + 600,000 =$ UGX $2,100,000$

(b) **Total tuition cost:**

$$S_4 = \frac{4}{2}[2(1,500,000) + (4-1)(200,000)] \quad (9.21)$$

$$= 2[3,000,000 + 600,000] \quad (9.22)$$

$$= 2 \times 3,600,000 = \text{UGX } 7,200,000 \quad (9.23)$$

(c) **Endowment Fund Growth:**
This is geometric progression with $a = 10,000,000,000$, $r = 1.15$
Value after 10 years: $a_{11} = 10,000,000,000 \times (1.15)^{10} = 10,000,000,000 \times 4.046 =$ UGX 40.46 billion

34. **Solar Panel Installation Business in Arua District**
Context: A solar panel business in Arua plans expansion with installations increasing arithmetically: 10 panels in month 1, 15 in month 2, 20 in month 3, etc. Revenue per panel follows geometric growth due to improved efficiency.

(a) Find the number of panels installed in the 12th month and total installations in the first year.

(b) If revenue per panel starts at UGX 500,000 and increases by 8

(c) Determine the total revenue in the first 6 months, combining both progressions.

Solution:

(a) **Panel Installation Pattern:**
Monthly installations: 10, 15, 20, 25, ...
A.P. with $a = 10$, $d = 5$
12th month installations: $a_{12} = 10 + (12-1)(5) = 10 + 55 = 65$ panels
Total installations in first year:

$$S_{12} = \frac{12}{2}[2(10) + (12-1)(5)] \quad (9.24)$$

$$= 6[20 + 55] \quad (9.25)$$

$$= 6 \times 75 = 450 \text{ panels} \quad (9.26)$$

(b) **Revenue per Panel (Geometric):**
Revenue progression: 500,000, 500,000(1.08), 500,000$(1.08)^2$, ...
G.P. with $a = 500,000$, $r = 1.08$
Month 6 revenue per panel: $a_6 = 500,000 \times (1.08)^5 = 500,000 \times 1.469 =$ UGX 734, 664

(c) **Total Revenue (First 6 Months):**
Month-by-month calculation:

Month 1: $10 \times 500,000 = 5,000,000$ (9.27)

Month 2: $15 \times 500,000(1.08) = 15 \times 540,000 = 8,100$ (9.28)

Month 3: $20 \times 500,000(1.08)^2 = 20 \times 583,200 = 11,6$ (9.29)

Month 4: $25 \times 500,000(1.08)^3 = 25 \times 629,856 = 15,7$ (9.30)

Month 5: $30 \times 500,000(1.08)^4 = 30 \times 680,245 = 20,4$ (9.31)

Month 6: $35 \times 500,000(1.08)^5 = 35 \times 734,665 = 25,7$ (9.32)

Total Revenue: UGX 86,630,025

35. **Fish Farming Cooperative in Jinja District**
Context: A fish farming cooperative plans to increase pond construction arithmetically and stock fish in geometric progression. They build 5 ponds in year 1, 8 in year 2, 11 in year 3, etc. Each pond is initially stocked with 1000 fish, doubling each year.

(a) Determine the number of ponds built in year 10 and total ponds after 10 years.

(b) Find the fish population per pond in year 8, assuming geometric growth.

(c) Calculate the total fish population across all ponds at the end of year 5.

Solution:

(a) **Pond Construction (Arithmetic):**
Annual ponds: 5, 8, 11, 14, ...
A.P. with $a = 5$, $d = 3$

Year 10 ponds: $a_{10} = 5+(10-1)(3) = 5+27 = 32$ ponds

Total ponds after 10 years:

$$S_{10} = \frac{10}{2}[2(5)+(10-1)(3)] \quad (9.33)$$

$$= 5[10+27] \quad (9.34)$$

$$= 5 \times 37 = 185 \text{ ponds} \quad (9.35)$$

(b) **Fish Population per Pond (Geometric):**
Fish per pond: 1000, 2000, 4000, 8000, ...
G.P. with $a = 1000$, $r = 2$
Year 8 fish per pond: $a_8 = 1000 \times 2^{8-1} = 1000 \times 2^7 = 1000 \times 128 = 128,000$ fish

(c) **Total Fish Population (End of Year 5):**
Total ponds after 5 years: $S_5 = \frac{5}{2}[2(5)+(5-1)(3)] = \frac{5}{2}[10+12] = \frac{5\times 22}{2} = 55$ ponds
Fish per pond in year 5: $1000 \times 2^4 = 1000 \times 16 = 16,000$ fish per pond
Total fish population: $55 \times 16,000 = 880,000$ fish

36. **Matoke Plantation Expansion in Mbarara District**
Context: A matoke plantation increases planting arithmetically: 100 stems in month 1, 125 in month 2, 150 in month 3, etc. Yield per stem grows geometrically due to improved farming techniques.
(a) Find the monthly planting pattern and calculate stems planted in month 15.
(b) If yield starts at 5 kg per stem and increases by 10
(c) Determine total expected harvest in month 12, combining both progressions.
Solution:
(a) **Planting Pattern (Arithmetic):**
Monthly planting: 100, 125, 150, 175, ...
A.P. with $a = 100$, $d = 25$
Month 15 planting: $a_{15} = 100+(15-1)(25) = 100+350 = 450$ stems

(b) **Yield per Stem (Geometric):**
Yield progression: $5, 5(1.1), 5(1.1)^2, \ldots$
G.P. with $a = 5$, $r = 1.1$
Month 8 yield per stem: $a_8 = 5 \times (1.1)^7 = 5 \times 1.949 = 9.745$ kg per stem

(c) **Total Harvest in Month 12:**
Stems planted in month 12: $a_{12} = 100 + (12-1)(25) = 100 + 275 = 375$ stems
Yield per stem in month 12: $5 \times (1.1)^{11} = 5 \times 2.839 = 14.195$ kg per stem
Total harvest: $375 \times 14.195 = 5,323$ kg

37. **Taxi Park Revenue Analysis in Kampala**
Context: Old Taxi Park generates revenue from parking fees that increase arithmetically during peak hours. Base fare is UGX 1,000, increasing by UGX 500 each hour. Daily passenger volume grows geometrically due to urbanization.
(a) Model the hourly parking fee structure as an arithmetic progression for 8 peak hours.
(b) If daily passengers start at 10,000 and grow by 5
(c) Calculate total revenue for one peak day, combining both progressions.
Solution:
(a) **Parking Fee Structure:**
Hourly fees: 1000, 1500, 2000, 2500, 3000, 3500, 4000, 4500
A.P. with $a = 1000$, $d = 500$
8th hour fee: $a_8 = 1000 + (8-1)(500) = 1000+3500 =$ UGX 4500
Total fees for 8 hours: $S_8 = \frac{8}{2}[2(1000)+(8-1)(500)] = 4[2000+3500] = 4 \times 5500 =$ UGX 22,000

(b) **Passenger Growth (Geometric):**
G.P. with $a = 10,000$, $r = 1.05$, monthly growth
After 24 months: $a_{25} = 10,000 \times (1.05)^{24} = 10,000 \times 3.225 = 32,251$ passengers

(c) **Total Peak Day Revenue:**
Assuming current passenger base and average fee: Average hourly fee = $\frac{22,000}{8} = 2,750$ UGX
If 10,000 passengers pay average fee: Total revenue $= 10,000 \times 2,750 =$ UGX 27,500,000

38. **Mobile Money Agent Network in Gulu District**
Context: A mobile money company plans agent recruitment with arithmetic growth: 20

agents in month 1, 35 in month 2, 50 in month 3, etc. Transaction volume per agent grows geometrically at 12

(a) Find the recruitment pattern and total agents after 18 months.

(b) Calculate transaction volume per agent in month 10, starting from UGX 2,000,000.

(c) Determine total network transaction volume in month 6.

Solution:

(a) **Agent Recruitment (Arithmetic):**
Monthly recruitment: 20, 35, 50, 65, ...
A.P. with $a = 20$, $d = 15$
Total agents after 18 months: $S_{18} = \frac{18}{2}[2(20) + (18-1)(15)] = 9[40 + 255] = 9 \times 295 = 2,655$ agents

(b) **Transaction Volume per Agent (Geometric):**
G.P. with $a = 2,000,000$, $r = 1.12$
Month 10 volume per agent: $a_{10} = 2,000,000 \times (1.12)^9 = 2,000,000 \times 2.476 =$ UGX $4,952,492$

(c) **Total Network Volume (Month 6):**
Total agents by month 6: $S_6 = \frac{6}{2}[2(20) + (6-1)(15)] = 3[40 + 75] = 3 \times 115 = 345$ agents
Volume per agent in month 6: $2,000,000 \times (1.12)^5 = 2,000,000 \times 1.762 = 3,524,578$ UGX
Total network volume: $345 \times 3,524,578 =$ UGX $1,215,979,410$

39. **Tea Processing Factory in Fort Portal**
Context: A tea factory increases daily processing capacity arithmetically while efficiency improvements follow geometric progression. Daily processing starts at 5,000 kg, increasing by 200 kg daily.

(a) Model the daily processing capacity for 30 days and find total tea processed.

(b) If processing efficiency starts at 85

(c) Calculate actual tea output on day 20, combining capacity and efficiency.

Solution:

(a) **Processing Capacity (Arithmetic):**
Daily capacity: 5000, 5200, 5400, 5600, ...
A.P. with $a = 5000$, $d = 200$
Total processed in 30 days: $S_{30} = \frac{30}{2}[2(5000) + (30-1)(200)] = 15[10000 + 5800] = 15 \times 15800 = 237,000$ kg

(b) **Efficiency Improvement (Geometric):**
Efficiency: $0.85, 0.85(1.02), 0.85(1.02)^2$, ...
G.P. with $a = 0.85$, $r = 1.02$
Day 15 efficiency: $a_{15} = 0.85 \times (1.02)^{14} = 0.85 \times 1.319 = 1.121 = 112.1\%$

(c) **Day 20 Actual Output:**
Day 20 capacity: $a_{20} = 5000 + (20-1)(200) = 5000 + 3800 = 8800$ kg
Day 20 efficiency: $0.85 \times (1.02)^{19} = 0.85 \times 1.456 = 1.238 = 123.8\%$
Actual output: $8800 \times 1.238 = 10,894$ kg

40. **Hydroelectric Power Generation Planning at Owen Falls**
Context: Owen Falls Dam plans to increase power generation arithmetically over 10 years while demand grows geometrically. Current generation is 180 MW, increasing by 15 MW annually.

(a) Calculate power generation capacity in year 10 and total power generated over 10 years.

(b) If power demand starts at 200 MW and grows by 8

(c) Determine when generation capacity will meet projected demand.

Solution:

(a) **Power Generation (Arithmetic):**
Annual capacity: 180, 195, 210, 225, ...
A.P. with $a = 180$, $d = 15$
Year 10 capacity: $a_{10} = 180 + (10-1)(15) = 180 + 135 = 315$ MW
Total generated over 10 years: $S_{10} = \frac{10}{2}[2(180) + (10-1)(15)] = 5[360 + 135] = 5 \times 495 = 2475$ MW-years

(b) **Power Demand (Geometric):**
G.P. with $a = 200$, $r = 1.08$
Year 12 demand: $a_{12} = 200 \times (1.08)^{11} = 200 \times 2.332 = 466.4$ MW

(c) **When Generation Meets Demand:**
Generation in year n: $180+(n-1)(15) = 165 + 15n$ Demand in year n: $200 \times (1.08)^{n-1}$

Setting equal: $165 + 15n = 200 \times (1.08)^{n-1}$

This requires numerical solution. Testing values: - Year 5: Generation = 240 MW, Demand = 272 MW - Year 10: Generation = 315 MW, Demand = 432 MW

Generation will never meet demand as geometric growth outpaces arithmetic growth.

41. **Mobile Money Adoption in Kampala**

Scenario: A mobile money company in Kampala observes that their user base increases by 20% each quarter. Initially, they have 10,000 users. The company wants to understand the long-term growth potential.

(a) Show that this forms a geometric progression and find the common ratio.

(b) Calculate the theoretical maximum number of users if this growth continues indefinitely (sum to infinity).

(c) Express the total users after n quarters using summation notation.

Solution 1

(a) This is a GP with:

$$a = 10,000, \quad r = 1.20$$

Sequence: $10,000, 12,000, 14,400, 17,280, \ldots$

(b) Since $|r| = 1.20 > 1$, the series diverges and sum to infinity does not exist.

$S_\infty = \dfrac{a}{1-r}$ only converges when $|r| < 1$

(c) Summation notation:

$$S_n = \sum_{k=0}^{n-1} ar^k = \sum_{k=0}^{n-1} 10,000 \times (1.20)^k$$

42. **Agricultural Investment Return**

Scenario: An agricultural investment in Mbale district returns 15% of the initial investment each year, with the returns reinvested annually. The initial investment is 5,000,000 UGX.

(a) Show that the total value forms a GP and find the common ratio.

(b) Calculate the total value after 10 years.

(c) Determine if the sum to infinity exists and interpret the result.

Solution 2

(a) This forms a GP with:

$a = 5,000,$

Value after n years: $V_n = 5,000,000 \times (1.15)^n$

(b) After 10 years:

$$V_{10} = 5,000,000 \times (1.15)^{10}$$
$$= 5,000,000 \times 4.0456 = 20,228,000 \text{ UGX}$$

(c) Since $|r| = 1.15 > 1$, the sum to infinity diverges, meaning the investment grows without bound theoretically.

43. **Malaria Spread Model in Gulu**

Scenario: In a malaria outbreak in Gulu, each infected person infects 0.8 new people every week. Initially, there are 100 infected persons.

(a) Show that the total infections form a GP and find the common ratio.

(b) Calculate the theoretical maximum total infections if this spread continues.

(c) Express the total infections after n weeks using summation notation.

Solution 3

(a) This forms a GP with:

$$a = 100, \quad r = 0.8$$

Sequence: $100, 80, 64, 51.2, \ldots$

(b) Since $|r| = 0.8 < 1$, the sum to infinity converges:

$$S_\infty = \frac{a}{1-r} = \frac{100}{1-0.8} = \frac{100}{0.2} = 500 \text{ infections}$$

(c) Summation notation:

$$S_n = \sum_{k=0}^{n-1} ar^k = \sum_{k=0}^{n-1} 100 \times (0.8)^k$$

44. **Radioactive Decay in Medical Equipment**

Scenario: A hospital in Kampala uses radioactive material that decays by 25% each year. Initially, there are 800 grams of material.

(a) Show that the remaining material forms a GP and find the common ratio.

(b) Calculate the amount that will remain after 6 years.

(c) Determine the theoretical minimum amount that will always remain.

Solution 4

(a) This forms a GP with:

$$a = 800, \quad r = 0.75$$

Sequence: $800, 600, 450, 337.5, \ldots$

(b) After 6 years:

$$a_6 = ar^5 = 800 \times (0.75)^5$$
$$= 800 \times 0.2373 = 189.84 \text{ grams}$$

(c) Since $|r| = 0.75 < 1$, the sum to infinity converges, but for decay, we consider the limit:

$$\lim_{n\to\infty} ar^n = 800 \times 0 = 0$$

The material will eventually completely decay.

45. **Microfinance Loan Growth**
Scenario: A microfinance institution in Jinja has a loan portfolio that grows at 18% annually. The initial portfolio is 20,000,000 UGX.
(a) Prove by induction that the portfolio value after n years is $20,000,000 \times (1.18)^n$.
(b) Calculate the portfolio value after 7 years.
(c) Express the growth using summation notation.

Solution 5

(a) Proof by induction:
Base case (n=1): $20,000,000 \times 1.18 = 23,600,000$
Inductive step: Assume true for n=k: $V_k = 20,000,000 \times (1.18)^k$
For n=k+1: $V_{k+1} = V_k \times 1.18 = 20,000,000 \times (1.18)^{k+1}$

(b) After 7 years:

$$V_7 = 20,000,000 \times (1.18)^7$$
$$= 20,000,000 \times 3.1855 = 63,710,000 \text{ UGX}$$

(c) Summation notation for total growth:

$$G_n = \sum_{k=1}^{n} 20,000,000 \times (0.18) \times (1.18)^{k-1}$$

46. **Compound Interest Savings Plan**
Scenario: A bank in Entebbe offers a savings plan with 12% annual compound interest. A client invests 2,000,000 UGX.
(a) Prove by induction that the amount after n years is $2,000,000 \times (1.12)^n$.
(b) Calculate the amount after 10 years.
(c) Determine if the sum to infinity exists and interpret the result.

Solution 6

(a) Proof by induction:
Base case (n=1): $2,000,000 \times 1.12 = 2,240,000$
Inductive step: Assume true for n=k: $A_k = 2,000,000 \times (1.12)^k$
For n=k+1: $A_{k+1} = A_k \times 1.12 = 2,000,000 \times (1.12)^{k+1}$

(b) After 10 years:

$$A_{10} = 2,000,000 \times (1.12)^{10}$$
$$= 2,000,000 \times 3.1058 = 6,211,600 \text{ UGX}$$

(c) Since $|r| = 1.12 > 1$, the sum to infinity diverges, meaning the investment grows without bound theoretically.

47. **Population Growth Model**
Scenario: The population of a village in Nakaseke district grows at 3% annually. The current population is 5,000 people.
(a) Prove by induction that the population after n years is $5,000 \times (1.03)^n$.
(b) Calculate the population after 15 years.
(c) Express the total population growth using summation notation.

Solution 7

(a) Proof by induction:
Base case (n=1): $5,000 \times 1.03 = 5,150$
Inductive step: Assume true for n=k: $P_k = 5,000 \times (1.03)^k$
For n=k+1: $P_{k+1} = P_k \times 1.03 = 5,000 \times (1.03)^{k+1}$

(b) After 15 years:

$$P_{15} = 5,000 \times (1.03)^{15}$$
$$= 5,000 \times 1.5580 = 7,790 \text{ people}$$

(c) Summation notation for growth:

$$G_n = \sum_{k=1}^{n} 5,000 \times (0.03) \times (1.03)^{k-1}$$

48. **Technology Adoption Curve**
Scenario: The adoption of solar technology in a rural community follows a pattern where each

adopter influences 0.6 new adopters monthly. Initially, 50 households have adopted.

(a) Show that the adoptions form a GP and find the common ratio.

(b) Calculate the theoretical maximum number of adopters.

(c) Express the total adoptions after n months using summation notation.

Solution 8

(a) This forms a GP with:

$$a = 50, \quad r = 0.6$$

Sequence: $50, 30, 18, 10.8, \ldots$

(b) Since $|r| = 0.6 < 1$, the sum to infinity converges:

$$S_\infty = \frac{a}{1-r} = \frac{50}{1-0.6} = \frac{50}{0.4} = 125 \text{ households}$$

(c) Summation notation:

$$S_n = \sum_{k=0}^{n-1} ar^k = \sum_{k=0}^{n-1} 50 \times (0.6)^k$$

49. **Water Conservation Impact**

Scenario: A water conservation program in Karamoja reduces water consumption by 15% each year through improved practices. Initial consumption is 1,000,000 liters monthly.

(a) Prove by induction that the consumption after n years is $1,000,000 \times (0.85)^n$.

(b) Calculate the consumption after 5 years.

(c) Determine the theoretical minimum consumption.

Solution 9

(a) Proof by induction:

Base case (n=1): $1,000,000 \times 0.85 = 850,000$

Inductive step: Assume true for n=k: $C_k = 1,000,000 \times (0.85)^k$

For n=k+1: $C_{k+1} = C_k \times 0.85 = 1,000,000 \times (0.85)^{k+1}$

(b) After 5 years:

$$C_5 = 1,000,000 \times (0.85)^5$$
$$= 1,000,000 \times 0.4437 = 443,700 \text{ liters/month}$$

(c) Since $|r| = 0.85 < 1$:

$$\lim_{n\to\infty} C_n = 1,000,000 \times 0 = 0$$

The consumption approaches zero theoretically.

50. **Agricultural Yield Improvement**

Scenario: A farming cooperative in Masaka improves yields by 8% annually through better techniques. Initial yield is 2,000 kg/hectare.

(a) Prove by induction that the yield after n years is $2,000 \times (1.08)^n$.

(b) Calculate the yield after 8 years.

(c) Express the total yield improvement using summation notation.

Solution 10

(a) Proof by induction:

Base case (n=1): $2,000 \times 1.08 = 2,160$

Inductive step: Assume true for n=k: $Y_k = 2,000 \times (1.08)^k$

For n=k+1: $Y_{k+1} = Y_k \times 1.08 = 2,000 \times (1.08)^{k+1}$

(b) After 8 years:

$$Y_8 = 2,000 \times (1.08)^8$$
$$= 2,000 \times 1.8509 = 3,701.8 \text{ kg/hectare}$$

(c) Summation notation for improvement:

$$I_n = \sum_{k=1}^{n} 2,000 \times (0.08) \times (1.08)^{k-1}$$

51. **Bank of Uganda Monetary Policy and Inflation Control**

Context: Bank of Uganda aims to control inflation by reducing money supply growth geometrically. Current money supply growth is 15% quarterly, reduced by multiplying by 0.8 each quarter.

(a) Analyze the convergence of this geometric progression and find the sum to infinity of growth rates using the formula $S_\infty = \frac{a}{1-r}$ where $|r| < 1$.

(b) Express the total cumulative effect using summation notation $\sum_{n=1}^{\infty} ar^{n-1}$ and calculate the long-term impact.

(c) Use mathematical induction to prove that for any geometric sequence with $|r| < 1$, the sum $S_n = \frac{a(1-r^n)}{1-r}$ approaches $\frac{a}{1-r}$ as $n \to \infty$.

Solution:

(a) **Convergence Analysis:**

Growth rates form G.P.: $15\%, 15\% \times 0.8, 15\% \times (0.8)^2, \ldots$

This is G.P. with $a = 0.15$, $r = 0.8$

Since $|r| = 0.8 < 1$, the series converges.

Sum to infinity:

$$S_\infty = \frac{a}{1-r} = \frac{0.15}{1-0.8} = \frac{0.15}{0.2} = 0.75 = 75\% \tag{9.36}$$

The total cumulative growth effect will never exceed 75%.

(b) **Summation Notation:**

Total effect: $\sum_{n=1}^{\infty} 0.15 \times (0.8)^{n-1}$

$$\sum_{n=1}^{\infty} 0.15 \times (0.8)^{n-1} = 0.15 \sum_{n=1}^{\infty} (0.8)^{n-1} \tag{9.37}$$

$$= 0.15 \times \frac{1}{1-0.8} = 0.15 \times 5 = 0.75 \tag{9.38}$$

Long-term impact: Money supply growth will stabilize at cumulative 75% increase.

(c) **Mathematical Induction Proof:**

Statement: For G.P. with $|r| < 1$, $\lim_{n\to\infty} S_n = \frac{a}{1-r}$

Given: $S_n = \frac{a(1-r^n)}{1-r}$

Base Case (n=1): $S_1 = \frac{a(1-r^1)}{1-r} = \frac{a(1-r)}{1-r} = a$

Inductive Step: Assume $S_k = \frac{a(1-r^k)}{1-r}$ for some $k \geq 1$.

Need to prove: $S_{k+1} = \frac{a(1-r^{k+1})}{1-r}$

$$S_{k+1} = S_k + ar^k \tag{9.39}$$

$$= \frac{a(1-r^k)}{1-r} + ar^k \tag{9.40}$$

$$= \frac{a(1-r^k) + ar^k(1-r)}{1-r} \tag{9.41}$$

$$= \frac{a(1-r^k+r^k-r^{k+1})}{1-r} \tag{9.42}$$

$$= \frac{a(1-r^{k+1})}{1-r} \quad \checkmark \tag{9.43}$$

Limit Analysis: Since $|r| < 1$, $\lim_{n\to\infty} r^n = 0$

Therefore: $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}$

52. **COVID-19 Vaccination Campaign in Uganda**

Context: Uganda's vaccination campaign follows a geometric progression where each vaccination center vaccinates 1000 people initially, then 90% of the previous day's number due to supply constraints.

(a) Determine if this vaccination pattern converges and find the total number of people that can be vaccinated at one center.

(b) Express the cumulative vaccination using summation notation and analyze the effectiveness.

(c) Prove by induction that the sum of first n terms of any geometric series is $S_n = \frac{a(r^n-1)}{r-1}$ for $r \neq 1$.

Solution:

(a) **Vaccination Pattern Analysis:**

Daily vaccinations: $1000, 1000(0.9), 1000(0.9)^2, \ldots$

G.P. with $a = 1000$, $r = 0.9$

Since $|r| = 0.9 < 1$, the series converges.

Total vaccinations at one center:

$$S_\infty = \frac{1000}{1-0.9} = \frac{1000}{0.1} = 10,000 \text{ people} \tag{9.44}$$

(b) **Summation Notation:**

Cumulative vaccinations: $\sum_{n=1}^{\infty} 1000 \times (0.9)^{n-1}$

$$\sum_{n=1}^{\infty} 1000 \times (0.9)^{n-1} = 1000 \sum_{n=1}^{\infty} (0.9)^{n-1} \tag{9.45}$$

$$= 1000 \times \frac{1}{1-0.9} = 10,000 \tag{9.46}$$

Effectiveness Analysis: Each center can vaccinate maximum 10,000 people, providing herd immunity coverage for a small community.

(c) **Mathematical Induction for G.P. Sum:**

Statement: $S_n = a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^n-1)}{r-1}$ for $r \neq 1$

Base Case (n=1): $S_1 = a = \frac{a(r^1-1)}{r-1} = \frac{a(r-1)}{r-1} = a$

Inductive Hypothesis: Assume $S_k = \frac{a(r^k-1)}{r-1}$ for some $k \geq 1$.

Inductive Step: Need to prove $S_{k+1} = \frac{a(r^{k+1}-1)}{r-1}$

$$S_{k+1} = S_k + ar^k \tag{9.47}$$
$$= \frac{a(r^k - 1)}{r - 1} + ar^k \tag{9.48}$$
$$= \frac{a(r^k - 1) + ar^k(r - 1)}{r - 1} \tag{9.49}$$
$$= \frac{a(r^k - 1 + r^{k+1} - r^k)}{r - 1} \tag{9.50}$$
$$= \frac{a(r^{k+1} - 1)}{r - 1} \quad \checkmark \tag{9.51}$$

By mathematical induction, the formula holds for all positive integers n.

53. **Microfinance Interest Compounding in Rural Uganda**

Context: A microfinance institution offers compound interest at 3% monthly. A farmer borrows UGX 1,000,000 and the interest compounds geometrically.

(a) Analyze the convergence of the interest payment series if only interest is paid monthly (principal remains constant).

(b) Design two different repayment plans: one with equal monthly payments and another with increasing payments, using G.P. principles.

(c) Use induction to prove that compound interest formula $A = P(1 + r)^n$ correctly represents exponential growth.

Solution:

(a) **Interest Payment Series:**

Monthly interest on constant principal: UGX 30,000 each month

This forms an arithmetic sequence, not geometric.

However, if principal compounds: $1,000,000, 1,000,000(1.03), 1,000,000(1.03)^2, \ldots$

This is G.P. with $a = 1,000,000$, $r = 1.03 > 1$

Since $|r| = 1.03 > 1$, this diverges (debt grows indefinitely without payment).

(b) **Repayment Plan Design:**

Plan A - Equal Monthly Payments: Using annuity formula: $PMT = \frac{P \cdot r(1+r)^n}{(1+r)^n - 1}$

For 12 months: $PMT = \frac{1,000,000 \times 0.03 \times (1.03)^{12}}{(1.03)^{12} - 1} = \frac{42,576}{0.4258} = 99,992$ UGX

Plan B - Increasing Payments (G.P.): Start with UGX 70,000, increase by 5% monthly

Payments: $70,000, 70,000(1.05), 70,000(1.05)^2, \ldots$

Total payment over 12 months: $S_{12} = \frac{70,000((1.05)^{12} - 1)}{1.05 - 1} = \frac{70,000(0.7958)}{0.05} = 1,114,120$ UGX

(c) **Induction Proof for Compound Interest:**

Statement: After n periods, amount is $A_n = P(1 + r)^n$

Base Case (n=1): After 1 period: $A_1 = P + Pr = P(1 + r) = P(1 + r)^1$

Inductive Hypothesis: Assume $A_k = P(1 + r)^k$ for some $k \geq 1$.

Inductive Step: Need to prove $A_{k+1} = P(1 + r)^{k+1}$

$$A_{k+1} = A_k + A_k \cdot r = A_k(1 + r) \tag{9.52}$$
$$= P(1 + r)^k(1 + r) = P(1 + r)^{k+1} \quad \checkmark \tag{9.53}$$

Therefore, compound interest follows geometric progression by mathematical induction.

54. **Coffee Export Price Volatility Analysis**

Context: Uganda's coffee export prices fluctuate geometrically due to market conditions. Price changes follow a G.P. with initial drop of 10%, then each subsequent change is 80% of the previous change.

(a) Determine the convergence of total price change and find the maximum possible price decline.

(b) Express the price volatility using summation notation and calculate cumulative market impact.

(c) Prove by induction that the sum of an infinite geometric series with first term a and ratio r (where $|r| < 1$) converges to $\frac{a}{1-r}$.

Solution:

(a) **Price Change Convergence:**

Price changes: $-10\%, -10\% \times 0.8, -10\% \times (0.8)^2, \ldots$

G.P. with $a = -0.10$, $r = 0.8$

Since $|r| = 0.8 < 1$, the series converges.

Maximum possible price decline:

$$S_\infty = \frac{-0.10}{1-0.8} = \frac{-0.10}{0.2} = -0.5 = -50\% \tag{9.54}$$

Maximum price decline is 50%.

(b) **Summation Notation:**
Cumulative price impact: $\sum_{n=1}^{\infty}(-0.10)\times(0.8)^{n-1}$

$$\sum_{n=1}^{\infty}(-0.10)\times(0.8)^{n-1} = -0.10\sum_{n=1}^{\infty}(0.8)^{n-1} \tag{9.55}$$

$$= -0.10\times\frac{1}{1-0.8} = -0.50 = -50\% \tag{9.56}$$

(c) **Convergence Proof by Induction:**
Statement: For $|r|<1$, $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$
First, prove finite sum formula: $S_n = \frac{a(1-r^n)}{1-r}$
Base Case (n=1): $S_1 = a = \frac{a(1-r)}{1-r}$
Inductive Step: Assume true for k, prove for $k+1$:

$$S_{k+1} = S_k + ar^k = \frac{a(1-r^k)}{1-r} + ar^k \tag{9.57}$$

$$= \frac{a(1-r^k)+ar^k(1-r)}{1-r} = \frac{a(1-r^{k+1})}{1-r} \quad \checkmark \tag{9.58}$$

Limit for Convergence: Since $|r|<1$, $\lim_{n\to\infty} r^n = 0$
Therefore: $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{a(1-r^n)}{1-r} = \frac{a}{1-r}$

55. **Solar Panel Efficiency Degradation in Kampala**
Context: Solar panels in Kampala lose efficiency geometrically due to dust and weather. Initial efficiency is 95%, degrading by 2% of current efficiency each month.
(a) Analyze the efficiency loss pattern and determine if total loss converges.
(b) Calculate the long-term minimum efficiency using infinite series convergence.
(c) Prove by induction that exponential decay follows the pattern $A_n = A_0\times r^n$ where $0<r<1$.
Solution:

(a) **Efficiency Pattern:**
Monthly efficiency: $95\%, 95\%\times 0.98, 95\%\times(0.98)^2, \ldots$
This is G.P. with $a = 0.95$, $r = 0.98$
Efficiency after n months: $0.95\times(0.98)^n$
As $n\to\infty$: $\lim_{n\to\infty} 0.95\times(0.98)^n = 0$

(b) **Long-term Analysis:**
Theoretical minimum efficiency approaches 0, but practically stabilizes around 70-80% due to physical limits.
Monthly efficiency loss: $5\%, 5\%\times 0.98, 5\%\times(0.98)^2, \ldots$
Total theoretical efficiency loss: This would approach 100%, but physical constraints prevent complete degradation.

(c) **Exponential Decay Proof:**
Statement: $A_n = A_0\times r^n$ for decay with ratio r
Base Case (n=0): $A_0 = A_0\times r^0 = A_0\times 1 = A_0$
Base Case (n=1): $A_1 = A_0\times r = A_0\times r^1$
Inductive Hypothesis: Assume $A_k = A_0\times r^k$ for some $k\geq 0$.
Inductive Step: Prove $A_{k+1} = A_0\times r^{k+1}$

$$A_{k+1} = A_k\times r \text{ (decay by factor } r) \tag{9.59}$$

$$= (A_0\times r^k)\times r = A_0\times r^{k+1} \quad \checkmark \tag{9.60}$$

By induction, exponential decay follows this pattern for all $n\geq 0$.

56. **Mobile Money Transaction Growth in Uganda**
Context: Mobile money transactions grow geometrically with initial volume of UGX 100 billion monthly, increasing by 15% monthly due to financial inclusion initiatives.
(a) Since growth ratio $r>1$, analyze why this series diverges and calculate transaction volume after 2 years.
(b) Express the cumulative transaction volume using summation notation for finite terms.
(c) Prove by induction that for geometric growth with $r>1$, the sum $S_n = \frac{a(r^n-1)}{r-1}$ grows without bound as n increases.
Solution:
(a) **Transaction Growth Analysis:**

Monthly transactions: $100, 100(1.15), 100(1.15)^2, \ldots$
G.P. with $a = 100$ billion UGX, $r = 1.15$
Since $r = 1.15 > 1$, the series diverges (grows indefinitely).
Volume after 24 months: $A_{24} = 100 \times (1.15)^{24} = 100 \times 27.23 = 2,723$ billion UGX

(b) **Cumulative Volume (Finite Sum):**
Sum for first n months: $\sum_{k=1}^{n} 100 \times (1.15)^{k-1}$

$$S_n = 100 \sum_{k=1}^{n} (1.15)^{k-1} = 100 \times \frac{(1.15)^n - 1}{1.15 - 1} \tag{9.61}$$

$$= 100 \times \frac{(1.15)^n - 1}{0.15} = \frac{100((1.15)^n - 1)}{0.15} \tag{9.62}$$

Cumulative volume for 24 months: $S_{24} = \frac{100((1.15)^{24}-1)}{0.15} = \frac{100(27.23-1)}{0.15} = 17,487$ billion UGX

(c) **Divergence Proof by Induction:**
Statement: For $r > 1$, $S_n = \frac{a(r^n-1)}{r-1}$ and $\lim_{n\to\infty} S_n = \infty$
First prove the sum formula:
Base Case (n=1): $S_1 = a = \frac{a(r-1)}{r-1} = a$
Inductive Step: Assume $S_k = \frac{a(r^k-1)}{r-1}$

$$S_{k+1} = S_k + ar^k = \frac{a(r^k - 1)}{r - 1} + ar^k \tag{9.63}$$

$$= \frac{a(r^k - 1) + ar^k(r - 1)}{r - 1} = \frac{a(r^{k+1} - 1)}{r - 1} \quad \checkmark \tag{9.64}$$

Divergence Analysis: Since $r > 1$, $\lim_{n\to\infty} r^n = \infty$
Therefore: $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(r^n-1)}{r-1} = \infty$

57. **Malaria Vector Control Program Effectiveness**
Context: A malaria control program reduces mosquito population by 25% monthly through systematic intervention. Initial population is 1 million mosquitoes in a district.
(a) Determine the convergence of mosquito population and find the theoretical minimum population.
(b) Express population reduction using summation notation and calculate total mosquitoes eliminated over infinite time.
(c) Prove by induction that population reduction follows exponential decay model $P_n = P_0(1 - r)^n$ where r is reduction rate.

Solution:

(a) **Population Convergence:**
Monthly population: $1,000,000, 1,000,000(0.75), 1,000$
G.P. with $a = 1,000,000$, $r = 0.75$
Since $|r| = 0.75 < 1$, population converges to 0.
Theoretical minimum: $\lim_{n\to\infty} 1,000,000 \times (0.75)^n = 0$

(b) **Total Mosquitoes Eliminated:**
Mosquitoes eliminated monthly: Month 1: $1,000,000 - 750,000 = 250,000$ Month 2: $750,000 - 562,500 = 187,500$ And so on...
Elimination forms G.P.: $250,000, 250,000(0.75), 250,000$
Total eliminated: $\sum_{n=1}^{\infty} 250,000 \times (0.75)^{n-1}$

$$S_\infty = \frac{250,000}{1 - 0.75} = \frac{250,000}{0.25} = 1,000,000 \text{ mosquitoes} \tag{9.65}$$

All mosquitoes are eventually eliminated.

(c) **Exponential Decay Model Proof:**
Statement: $P_n = P_0(1 - r)^n$ where r is monthly reduction rate
Base Case (n=0): $P_0 = P_0(1 - r)^0 = P_0 \times 1 = P_0$
Base Case (n=1): $P_1 = P_0 - rP_0 = P_0(1 - r) = P_0(1 - r)^1$
Inductive Hypothesis: Assume $P_k = P_0(1 - r)^k$ for some $k \geq 0$.
Inductive Step: Prove $P_{k+1} = P_0(1 - r)^{k+1}$

$$P_{k+1} = P_k - rP_k = P_k(1 - r) \tag{9.66}$$

$$= P_0(1 - r)^k(1 - r) = P_0(1 - r)^{k+1} \quad \checkmark \tag{9.67}$$

By induction, the exponential decay model holds for all $n \geq 0$.

58. **Hydroelectric Dam Water Flow Management**
Context: Bujagali Dam regulates water flow during dry season by releasing 80% of previ-

ous day's volume. Initial release is 1000 cubic meters per second.

(a) Analyze flow reduction convergence and determine minimum sustainable flow rate.

(b) Calculate total water released over infinite time using geometric series convergence.

(c) Use induction to prove that the general formula for geometric series $\sum_{k=0}^{n-1} ar^k = \frac{a(1-r^n)}{1-r}$ is correct.

Solution:

(a) **Flow Reduction Analysis:**
Daily flow rates: $1000, 1000(0.8), 1000(0.8)^2, \ldots$
G.P. with $a = 1000$, $r = 0.8$
Since $|r| = 0.8 < 1$, flow converges to 0.
Theoretical minimum: 0 m^3/s, but practical minimum ecological flow maintained.

(b) **Total Water Released:**
Infinite sum: $\sum_{n=1}^{\infty} 1000 \times (0.8)^{n-1}$

$$S_\infty = \frac{1000}{1-0.8} = \frac{1000}{0.2} = 5000 \text{ m}^3\text{/s total capacity} \tag{9.68}$$

(c) **Geometric Series Formula Proof:**
Statement: $\sum_{k=0}^{n-1} ar^k = \frac{a(1-r^n)}{1-r}$ for $r \neq 1$
Base Case (n=1): $\sum_{k=0}^{0} ar^k = ar^0 = a = \frac{a(1-r^1)}{1-r} = \frac{a(1-r)}{1-r} = a$
Inductive Hypothesis: Assume $\sum_{k=0}^{m-1} ar^k = \frac{a(1-r^m)}{1-r}$ for some $m \geq 1$.
Inductive Step: Prove $\sum_{k=0}^{m} ar^k = \frac{a(1-r^{m+1})}{1-r}$

$$\sum_{k=0}^{m} ar^k = \sum_{k=0}^{m-1} ar^k + ar^m \tag{9.69}$$

$$= \frac{a(1-r^m)}{1-r} + ar^m \tag{9.70}$$

$$= \frac{a(1-r^m) + ar^m(1-r)}{1-r} \tag{9.71}$$

$$= \frac{a(1-r^m+r^m-r^{m+1})}{1-r} \tag{9.72}$$

$$= \frac{a(1-r^{m+1})}{1-r} \quad \checkmark \tag{9.73}$$

By mathematical induction, the formula holds for all positive integers n.

59. **Tea Estate Investment Returns in Fort Portal**
Context: A tea estate investment offers returns that decrease geometrically over time. Initial return is 20% in year 1, then 85% of previous year's return due to market saturation.

(a) Determine if the investment returns converge and calculate total lifetime returns.

(b) Compare this investment with a fixed 8% annual return over 20 years using summation analysis.

(c) Prove by induction that the sum of first n terms of series $\sum_{k=1}^{n} k \cdot r^{k-1}$ equals $\frac{1-(n+1)r^n+nr^{n+1}}{(1-r)^2}$ for $r \neq 1$.

Solution:

(a) **Investment Returns Analysis:**
Annual returns: $20\%, 20\%\times0.85, 20\%\times(0.85)^2, \ldots$
G.P. with $a = 0.20$, $r = 0.85$
Since $|r| = 0.85 < 1$, returns converge.
Total lifetime returns:

$$S_\infty = \frac{0.20}{1-0.85} = \frac{0.20}{0.15} = 1.333 = 133.3\% \tag{9.74}$$

(b) **Investment Comparison:**
Geometric returns (20 years): $S_{20} = \frac{0.20(1-(0.85)^{20})}{1-0.85} = \frac{0.20(1-0.0388)}{0.15} = \frac{0.1922}{0.15} = 128.1\%$
Fixed 8% returns (20 years): $S_{20} = 20 \times 0.08 = 160\%$
Conclusion: Fixed 8% return is better than geometric declining return.

(c) **Series Induction Proof:**
Statement: $\sum_{k=1}^{n} k \cdot r^{k-1} = \frac{1-(n+1)r^n+nr^{n+1}}{(1-r)^2}$
Base Case (n=1): LHS: $\sum_{k=1}^{1} k \cdot r^{k-1} = 1 \cdot r^0 = 1$
RHS: $\frac{1-(1+1)r^1+1\cdot r^{1+1}}{(1-r)^2} = \frac{1-2r+r^2}{(1-r)^2} = \frac{(1-r)^2}{(1-r)^2} = 1$
Inductive Hypothesis: Assume formula holds for some $m \geq 1$.
Inductive Step: Through algebraic manipulation (shown conceptually):

$$\sum_{k=1}^{m+1} k \cdot r^{k-1} = \sum_{k=1}^{m} k \cdot r^{k-1} + (m+1)r^m \tag{9.75}$$

$$= \frac{1-(m+1)r^m + mr^{m+1}}{(1-r)^2} + (m+1)r^m \tag{9.76}$$

After extensive algebraic simplification, this equals $\frac{1-(m+2)r^{m+1}+(m+1)r^{m+2}}{(1-r)^2}$

60. **Population Growth Model for Kampala Metropolitan Area**

Context: Kampala's population growth rate starts at 8% annually but decreases by 5% of the rate each year due to urbanization limits and resource constraints.

(a) Model population growth rates as geometric progression and analyze long-term convergence.

(b) Calculate cumulative population increase over infinite time using current population of 3 million.

(c) Use induction to prove that population growth with decreasing rates follows the model $P_n = P_0 \prod_{k=1}^{n}(1+r_k)$ where r_k are the growth rates.

Solution:

(a) **Growth Rate Analysis:**

Annual growth rates: $8\%, 8\% \times 0.95, 8\% \times (0.95)^2, \ldots$

G.P. with $a = 0.08$, $r = 0.95$

Since $|r| = 0.95 < 1$, growth rates converge to 0.

Total cumulative growth rate: $S_\infty = \frac{0.08}{1-0.95} = \frac{0.08}{0.05} = 1.6 = 160\%$

(b) **Population Calculation:**

Current population: 3 million

Maximum population increase: $3,000,000 \times 1.6 = 4,800,000$

Maximum total population: $3,000,000 + 4,800,000 = 7,800,000$ people

(c) **Population Growth Model Proof:**

Statement: $P_n = P_0 \prod_{k=1}^{n}(1+r_k)$ where r_k are annual growth rates

Base Case (n=1): $P_1 = P_0(1+r_1) = P_0 \prod_{k=1}^{1}(1+r_k)$

Inductive Hypothesis: Assume $P_m = P_0 \prod_{k=1}^{m}(1+r_k)$ for some $m \geq 1$.

Inductive Step: Prove $P_{m+1} = P_0 \prod_{k=1}^{m+1}(1+r_k)$

$$P_{m+1} = P_m(1+r_{m+1}) \text{ (growth in year } m+1) \tag{9.77}$$

$$= P_0 \prod_{k=1}^{m}(1+r_k) \times (1+r_{m+1}) \tag{9.78}$$

$$= P_0 \prod_{k=1}^{m+1}(1+r_k) \quad \checkmark \tag{9.79}$$

By induction, the population growth model is correct for all years n.

Question 61: Community Water Tank Leakage (G.P. Conver

A community water tank in Pallisa has a small leak. The leak causes a water loss of 100 liters in the first hour. In the second hour, the loss is 50 liters, and in the third, it is 25 liters. This pattern continues.

1. Find the amount of water lost in the 8th hour.
2. Find the total amount of water lost after an infinitely long time.
3. What is the practical significance of the sum to infinity in this scenario?

Solution 61

1. The water loss forms a geometric progression (G.P.) with the first term $a = 100$ and common ratio $r = \frac{50}{100} = \frac{1}{2}$. The n^{th} term is given by $a_n = ar^{n-1}$. The water loss in the 8th hour is $a_8 = 100\left(\frac{1}{2}\right)^{8-1} = 100\left(\frac{1}{2}\right)^7 = 100\left(\frac{1}{128}\right) = \frac{100}{128} = \frac{25}{32} = \mathbf{0.78125}$ liters.
2. The total amount of water lost after an infinitely long time is the sum to infinity of the G.P. The series converges because $|r| < 1$. The formula for the sum to infinity is $S_\infty = \frac{a}{1-r}$. $S_\infty = \frac{100}{1-\frac{1}{2}} = \frac{100}{\frac{1}{2}} = \mathbf{200}$ liters.
3. The sum to infinity represents the maximum possible amount of water that can be lost from the leak if it is not repaired. It provides a crucial value for the community, as it tells them the finite amount of water they will lose over a long period. This information is important for managing resources and deciding on the urgency of the repair.

Question 62: Disease Spread in a Community (G.P.)

A highly contagious disease spreads in a rural community. Initially, 10 people are infected. In the next week, each infected person infects 3 other people. This pattern continues.

1. Use summation notation to represent the total number of new infections over the first 5 weeks.
2. Find the total number of people who have been infected over the first 5 weeks.
3. Explain the conditions under which the total number of infected people would not approach infinity.

Solution 62

1. The number of new infections per week forms a G.P. with $a = 10$ and $r = 3$. The number of new infections in week k is $10 \cdot 3^{k-1}$. The total number of new infections over the first 5 weeks can be represented by the summation notation: $\sum_{\mathbf{k=1}}^{\mathbf{5}} \mathbf{10 \cdot 3^{k-1}}$
2. To find the total number of people infected over 5 weeks, we calculate the sum of the G.P. for the first 5 terms. $S_5 = \frac{a(r^n-1)}{r-1} = \frac{10(3^5-1)}{3-1} = \frac{10(243-1)}{2} = \frac{10(242)}{2} = 5(242) = \mathbf{1210}$ people.
3. The total number of infected people would not approach infinity if the common ratio, r, were less than 1 ($|r| < 1$). This would mean that each infected person is infecting, on average, less than one other person. In such a scenario, the disease would naturally die out over time, and the total number of cases would converge to a finite number. This is the goal of public health measures like vaccination and social distancing.

Question 63: Investment Plan Comparison (G.P. & A.P.)

A young entrepreneur has two investment options:

- Plan A: Start with a deposit of 1,000,000 shillings and increase the deposit by 200,000 shillings each year.
- Plan B: Start with a deposit of 1,000,000 shillings and increase the deposit by 10% each year.

1. Write a summation notation to represent the total money deposited in Plan A after 5 years.
2. Find the total amount of money deposited in both plans after 5 years.
3. Which plan would you recommend to the entrepreneur and why?

Solution 63

1. Plan A is an A.P. with $a = 1,000,000$ and $d = 200,000$. The amount deposited in year k is $1,000,000 + (k-1)200,000$. The total amount can be represented as $\sum_{\mathbf{k=1}}^{\mathbf{5}} (\mathbf{1,000,000} + (\mathbf{k-1})\mathbf{200,000})$.
2. Plan A: Total deposited is the sum of an A.P. $S_5 = \frac{5}{2}(2(1,000,000) + (5-1)200,000) = 2.5(2,000,000 + 800,000) = 2.5(2,800,000) = \mathbf{7,000,000}$ shillings. Plan B: This is a G.P. with $a = 1,000,000$ and $r = 1.10$. Total deposited is the sum of the G.P. $S_5 = \frac{a(r^5-1)}{r-1} = \frac{1,000,000(1.1^5-1)}{1.1-1} = \frac{1,000,000(1.61051-1)}{0.1} = \frac{1,000,000(0.61051)}{0.1} = \mathbf{6,105,100}$ shillings.
3. Plan A would be recommended. Although Plan B's deposits grow exponentially, after 5 years, the total amount deposited in Plan A is higher. This is a good choice for someone with a consistent income who wants to make a significant total deposit over a short to medium-term horizon. However, it's important to note that if the timeline were longer, the exponential growth of Plan B would eventually surpass Plan A.

Question 64: Water Well Drilling (Inductive Reasoning)

A drilling company charges a fixed price of 1,000,000 shillings for the first meter of a well and 50,000 shillings for each subsequent meter. A community needs to prove that the cost to drill to any depth n meters is given by the formula $C(n) = 50,000n + 950,000$.

1. Show that the formula holds for the first three meters.
2. Assuming the formula holds for $n = k$, prove by induction that it also holds for $n = k + 1$.
3. Why is proof by induction a reliable method for proving such a statement?

Solution 64

1. Let's calculate the costs and check the formula.
   - For $n = 1$: Cost = 1,000,000. Formula: $C(1) = 50,000(1) + 950,000 = 1,000,000$. It holds.
   - For $n = 2$: Cost = 1,000,000 (first meter) + 50,000 (second meter) = 1,050,000.

Formula: $C(2) = 50,000(2) + 950,000 = 100,000 + 950,000 = 1,050,000$. It holds.

- For $n = 3$: Cost = 1,050,000 + 50,000 = 1,100,000. Formula: $C(3) = 50,000(3) + 950,000 = 150,000 + 950,000 = 1,100,000$. It holds.

2. Proof by Induction:
   - Base Case: We have shown above that the formula holds for $n = 1, 2, 3$.
   - Inductive Hypothesis: Assume the formula holds for some integer $k \geq 1$. That is, assume $C(k) = 50,000k + 950,000$.
   - Inductive Step: We want to show that the formula holds for $n = k + 1$. The cost to drill to a depth of $k + 1$ meters is the cost to drill to k meters plus the cost of the $(k+1)^{th}$ meter. The cost of each subsequent meter is 50,000. $C(k + 1) = C(k) + 50,000$ By our hypothesis, we substitute $C(k)$: $C(k+1) = (50,000k+950,000)+50,000 = 50,000k + 1,000,000$ We need to show this is equal to the formula for $n = k + 1$: $50,000(k + 1) + 950,000 = 50,000k + 50,000 + 950,000 = 50,000k + 1,000,000$. Since both expressions are equal, the formula holds for $n = k+1$. By the principle of mathematical induction, the formula holds for all integers $n \geq 1$.
3. Proof by induction is reliable because it establishes a logical chain of truth. It proves that if a statement is true for an initial case, and if its truth for any given case implies its truth for the next case, then the statement must be true for all cases in that sequence. This method is like a domino effect: if the first domino falls, and the fall of any domino makes the next one fall, then all the dominoes will fall.

Question 65: Agricultural Yield Decay (G.P. Convergence)

A farm in Gulu experiences a decrease in soil fertility, causing a drop in maize yield each year. The yield in the first year is 10 tons. In the second year, it is 9 tons, and in the third, it is 8.1 tons. This pattern continues.

1. Find the yield in the 6th year.
2. Find the total maize yield from the farm after an infinitely long period.
3. What does the sum to infinity imply for the farm's long-term sustainability?

Solution 65

1. The annual yield forms a G.P. with $a = 10$ and $r = \frac{9}{10} = 0.9$. The yield in the 6th year is $a_6 = ar^{6-1} = 10(0.9)^5 = 10(0.59049) = \mathbf{5.9049}$ tons.
2. The total yield after an infinitely long period is the sum to infinity. The series converges because $|r| < 1$. $S_\infty = \frac{a}{1-r} = \frac{10}{1-0.9} = \frac{10}{0.1} = \mathbf{100}$ tons.
3. The sum to infinity implies that the farm can only produce a finite amount of total maize if the soil fertility is not addressed. This model warns the farmer that continuous cultivation without intervention (like fertilization or crop rotation) will eventually lead to a minimal, unsustainable yield. The finite sum highlights the necessity of implementing sustainable farming practices to avoid reaching a point of zero productivity.

Question 66: Public Health Vaccination Campaign (G.P.)

A public health team in a rural district launches a vaccination campaign. In the first week, they vaccinate 500 people. In the second week, they aim to vaccinate 10

1. Write a summation notation to represent the total number of people vaccinated in the first 8 weeks.
2. Find the total number of people vaccinated over the first 8 weeks.
3. What is the advantage of using a proportional increase (G.P.) for a campaign's target over a fixed increase (A.P.)?

Solution 66

1. The number of vaccinations per week forms a G.P. with $a = 500$ and $r = 1 + \frac{10}{100} = 1.1$. The number of people vaccinated in week k is $500(1.1)^{k-1}$. The total vaccinated over the first 8 weeks can be represented as: $\sum_{\mathbf{k=1}}^{\mathbf{8}} \mathbf{500(1.1)^{k-1}}$.
2. The total number of people vaccinated is the sum of the G.P. for the first 8 terms. $S_8 = \frac{a(r^n-1)}{r-1} = \frac{500(1.1^8-1)}{1.1-1} = \frac{500(2.143588-1)}{0.1} = \frac{500(1.143588)}{0.1} = \mathbf{5717.94}$ people. Approximately 5,718 people would be vaccinated.
3. A proportional increase is more ambitious and dynamic than a fixed increase. It encourages

the campaign team to leverage their success, using the momentum from one week to set a higher target for the next. This exponential growth can lead to a much larger impact and is more realistic for campaigns that become more efficient over time.

Question 67: Savings for a Borehole (G.P.)

A village in Busoga is saving money to buy a borehole. In the first month, they save 200,000 shillings. In each subsequent month, they save 5

1. Find the amount they save in the 15th month.
2. Find the total amount of money they will have saved after 20 months.
3. Will they have enough money to buy the borehole after 20 months? Justify your answer.

Solution 67

1. The monthly savings form a G.P. with $a = 200,000$ and $r = 1.05$. Savings in the 15th month: $a_{15} = ar^{15-1} = 200,000(1.05)^{14} \approx 200,000(1.9799) = \mathbf{395,980}$ shillings.
2. The total savings after 20 months is the sum of the G.P. $S_{20} = \frac{a(r^{20}-1)}{r-1} = \frac{200,000(1.05^{20}-1)}{1.05-1} = \frac{200,000(2.6533-1)}{0.05} = \frac{200,000(1.6533)}{0.05} = \mathbf{6,613,200}$ shillings.
3. No, they will not have enough money. The total savings after 20 months is 6,613,200 shillings, which is less than the borehole's cost of 10,000,000 shillings. They will need to continue saving for a longer period.

Question 68: Land Value Appreciation (G.P. Convergence)

The value of a land plot in Kampala appreciates each year. It is valued at 50,000,000 shillings initially, and the value is expected to increase by 8% annually. However, a new policy is implemented, causing the rate of increase to drop to 4% from the second year onwards.

1. Find the value of the land after 3 years.
2. Does the value of the land converge to a specific amount over an infinitely long period? Justify your answer.
3. What is the practical implication of a common ratio greater than 1?

Solution 68

1. This is a G.P. with a changing common ratio. The initial value is $a = 50,000,000$.
   - Year 1 value: $50,000,000(1.08) = 54,000,000$.
   - Year 2 value: $54,000,000(1.04) = 56,160,000$.
   - Year 3 value: $56,160,000(1.04) = \mathbf{58,406,400}$ shillings.
2. No, the value of the land will not converge to a specific amount. After the first year, the common ratio is $r = 1.04$. Since the common ratio is greater than 1 ($|r| > 1$), the series is divergent, meaning the sum will approach infinity. The value of the land will continue to increase without bound.
3. The practical implication of a common ratio greater than 1 is that the value of the asset grows exponentially and will not stop at a finite value. This means that for an investor, the land is a highly valuable long-term asset whose value will continue to appreciate, providing a solid return on investment.

Question 69: Deforestation Rate (G.P. Convergence)

The rate of deforestation in a protected forest is measured in hectares per year. In the first year, 100 hectares are lost. In the second year, 80 hectares are lost, and in the third, 64 hectares. This trend continues.

1. Use summation notation to represent the total area deforested over the first 10 years.
2. Find the total area deforested after a very long time.
3. What assumption is made about the rate of deforestation, and is this realistic?

Solution 69

1. The deforestation rate forms a G.P. with $a = 100$ and $r = \frac{80}{100} = 0.8$. The area deforested in year k is $100(0.8)^{k-1}$. The total area deforested over 10 years is: $\sum_{k=1}^{10} \mathbf{100(0.8)^{k-1}}$.
2. To find the total area deforested after a very long time, we find the sum to infinity since $|r| < 1$. $S_\infty = \frac{a}{1-r} = \frac{100}{1-0.8} = \frac{100}{0.2} = \mathbf{500}$ hectares.
3. The assumption made is that the rate of deforestation is decreasing at a constant rate, which may not be entirely realistic. In reality, factors like government intervention, community awareness, and alternative livelihoods can affect the rate of deforestation, making it less predictable. However, this model serves as a

useful benchmark to show that a continued decrease in the rate will eventually lead to a finite, manageable amount of total deforestation.

Question 70: Proving a Business Revenue Formula (Inductive Reasoning)

A small business sells roasted maize cobs. They earn 500 shillings for the first cob sold, and the price of each subsequent cob is 20 shillings less than the previous one, down to a minimum price of 300 shillings. The business owner claims that the total revenue from selling the first n cobs is given by the formula $R(n) = -10n^2 + 510n$.

1. Show that the formula holds for the first three cobs sold.
2. Assuming the formula is true for $n = k$, use induction to prove it is also true for $n = k + 1$.
3. What is the benefit of using proof by induction for a business owner?

Solution 70

1. Let's check the formula for the first three cobs.
   - Price of 1st cob: 500 shillings. Revenue: 500. Formula: $R(1) = -10(1)^2 + 510(1) = -10 + 510 = 500$. It holds.
   - Price of 2nd cob: $500 - 20 = 480$. Revenue: $500 + 480 = 980$. Formula: $R(2) = -10(2)^2 + 510(2) = -40 + 1020 = 980$. It holds.
   - Price of 3rd cob: $480 - 20 = 460$. Revenue: $980 + 460 = 1440$. Formula: $R(3) = -10(3)^2 + 510(3) = -90 + 1530 = 1440$. It holds.
2. Proof by Induction:
   - Base Case: Shown above for $n = 1, 2, 3$.
   - Inductive Hypothesis: Assume the formula holds for $n = k$. That is, assume $R(k) = -10k^2 + 510k$.
   - Inductive Step: We want to prove it holds for $n = k + 1$. The revenue from selling $k + 1$ cobs is the revenue from selling k cobs plus the price of the $(k + 1)^{th}$ cob. The price of the $(k+1)^{th}$ cob is $500 + (k + 1 - 1)(-20) = 500 - 20k$. $R(k + 1) = R(k) + (500 - 20k)$ By our hypothesis, we substitute $R(k)$: $R(k+1) = (-10k^2+510k)+(500-20k) = -10k^2 + 490k + 500$. Now we show that the formula for $n = k + 1$ is the same: $R(k + 1) = -10(k + 1)^2 + 510(k + 1) = -10(k^2 + 2k + 1) + 510k + 510 = -10k^2 - 20k - 10+510k+510 = -10k^2+490k+500$. Since the expressions are the same, the formula holds for $n = k + 1$. The proof is complete.
3. Proof by induction provides a business owner with a high degree of certainty that a formula derived from a few examples holds for all cases. This is crucial for long-term planning, as it allows them to confidently predict revenue, manage inventory, and make strategic decisions based on a formula that has been rigorously proven to be true for any number of sales.

71. **Coffee harvest probability (Agriculture).**
In a coffee plantation in Masaka, each tree has a 0.7 probability of producing high-quality berries in a season, independent of others. A farmer randomly samples 5 trees.
   (a) State the probability model that describes the number of trees yielding high-quality berries.
   (b) Compute the probability that exactly 3 of the 5 trees yield high-quality berries.
   (c) Find the probability that at least 4 trees yield high-quality berries.

   Solution.
   Model: Binomial with $n = 5$, $p = 0.7$.
   (a) $X \sim Bin(5, 0.7)$.
   (b) $P(X = 3) = \binom{5}{3}(0.7)^3(0.3)^2 = 10 \cdot 0.343 \cdot 0.09 = 0.3087$.
   (c) $P(X \geq 4) = P(X = 4) + P(X = 5)$.
   $P(X = 4) = \binom{5}{4}(0.7^4)(0.3) = 5 \cdot 0.2401 \cdot 0.3 = 0.36015$.
   $P(X = 5) = \binom{5}{5}(0.7^5) = 0.16807$.
   So $P(X \geq 4) = 0.52822$.

72. **Microfinance loan repayment (Finance).**
In a village SACCO, each borrower has a 0.9 probability of fully repaying a loan installment on time. For 6 borrowers chosen at random:
   (a) Write down the binomial model.
   (b) Find the probability that all 6 repay on time.
   (c) Find the probability that at most 4 repay on time.

   Solution.
   $X \sim Bin(6, 0.9)$.
   (a) All repay: $P(X = 6) = 0.9^6 = 0.531441$.
   (b) At most 4: $P(X \leq 4) = \sum_{k=0}^{4} \binom{6}{k}(0.9)^k(0.1)^{6-k}$.

Compute: $P(4) = \binom{6}{4}0.9^4 \cdot 0.1^2 = 15 \cdot 0.6561 \cdot 0.01 = 0.098415$.
$P(3) = \binom{6}{3}0.9^3 \cdot 0.1^3 = 20 \cdot 0.729 \cdot 0.001 = 0.01458$.
$P(2) = 15 \cdot 0.81 \cdot 10^{-4} = 0.001215$.
$P(1) = 6 \cdot 0.9 \cdot 10^{-5} = 0.000054$.
$P(0) = 10^{-6}$.
Total ≈ 0.114265.

73. **Malaria test kits (Health).**
A malaria rapid diagnostic kit has 95% accuracy for detecting infection. Suppose 8 patients are tested, all infected.
   (a) Define the binomial model for the number of correct results.
   (b) Find probability that exactly 7 patients test positive.
   (c) Find probability that all 8 test positive.

   Solution.
   $X \sim Bin(8, 0.95)$.
   (a) $P(X = 7) = \binom{8}{7}0.95^7 \cdot 0.05 = 8 \cdot 0.6983 \cdot 0.05 \approx 0.2793$.
   (b) $P(X = 8) = 0.95^8 = 0.6634$.

74. **Mobile network drop rate (Telecoms).**
A telecom provider records that each call has a 0.02 probability of dropping. For 12 calls:
   (a) Model the number of dropped calls.
   (b) Compute the probability that none drops.
   (c) Compute the probability that at most 1 call drops.

   Solution.
   $X \sim Bin(12, 0.02)$.
   (a) $P(X = 0) = (0.98)^{12} = 0.784$.
   (b) $P(X \le 1) = P(0) + P(1)$. $P(1) = \binom{12}{1}(0.02)(0.98^{11}) = 12 \cdot 0.02 \cdot 0.800 = 0.192$.
   So $P(X \le 1) = 0.784 + 0.192 = 0.976$.

75. **Bus breakdowns (Transport).**
A bus company finds each bus has a 0.1 probability of breakdown on a long trip. For 10 buses sent:
   (a) Define the model.
   (b) Find probability exactly 2 buses break down.
   (c) Find expected number of buses that break down.

   Solution.
   $X \sim Bin(10, 0.1)$.
   (a) $P(X = 2) = \binom{10}{2}(0.1)^2(0.9)^8 = 45 \cdot 0.01 \cdot 0.4305 = 0.1937$.
   (b) $E[X] = np = 10 \cdot 0.1 = 1$.

76. **Building site safety (Construction).**
On a construction site, each day there is a 0.15 probability of a minor safety incident. Over 20 days:
   (a) Write the binomial model.
   (b) Probability of exactly 5 incidents in 20 days.
   (c) Probability of fewer than 3 incidents.

   Solution.
   $X \sim Bin(20, 0.15)$.
   (a) $P(X = 5) = \binom{20}{5}(0.15^5)(0.85^{15})$.
   Compute: $\binom{20}{5} = 15504$, $0.15^5 = 7.59 \cdot 10^{-5}$, $0.85^{15} \approx 0.087$.
   So $P(5) \approx 15504 \cdot 7.59 \cdot 10^{-5} \cdot 0.087 = 0.102$.
   (b) $P(X < 3) = P(0) + P(1) + P(2)$.
   $P(0) = 0.85^{20} \approx 0.0388$.
   $P(1) = 20 \cdot 0.15 \cdot 0.85^{19} \approx 20 \cdot 0.15 \cdot 0.0457 = 0.1371$.
   $P(2) = 190 \cdot 0.15^2 \cdot 0.85^{18} \approx 190 \cdot 0.0225 \cdot 0.0538 = 0.230$.
   Total ≈ 0.406.

77. **Fish farm mortality (Fisheries).**
A hatchery estimates a 0.08 probability of fingerling mortality within the first month. From 15 stocked fish:
   (a) State the binomial model.
   (b) Probability that exactly 2 die.
   (c) Probability that none dies.

   Solution.
   $X \sim Bin(15, 0.08)$.
   (a) $P(X = 2) = \binom{15}{2}(0.08^2)(0.92^{13}) = 105 \cdot 0.0064 \cdot 0.339 \approx 0.228$.
   (b) $P(X = 0) = 0.92^{15} \approx 0.286$.

78. **Election polling (Politics).**
In a district, 60% of voters support candidate A. A sample of 10 voters is taken.
   (a) Define binomial model.
   (b) Probability exactly 7 support A.
   (c) Probability at least 8 support A.

   Solution.
   $X \sim Bin(10, 0.6)$.
   (a) $P(X = 7) = \binom{10}{7}0.6^7 \cdot 0.4^3 = 120 \cdot 0.0280 \cdot 0.064 = 0.215$.
   (b) $P(X \ge 8) = P(8) + P(9) + P(10)$.
   $P(8) = 45 \cdot 0.0168 \cdot 0.16 = 0.121$.
   $P(9) = 10 \cdot 0.0101 \cdot 0.4 = 0.040$.
   $P(10) = 0.0060$.
   Total $= 0.167$.

79. **Rainfall prediction (Climate).**
Meteorological data show a 30% chance of rainfall on any given day in Kampala in October. For 12

days sampled:

(a) Model the number of rainy days.

(b) Probability of exactly 4 rainy days.

(c) Expected number of rainy days.

Solution.

$X \sim Bin(12, 0.3)$.

(a) $P(X = 4) = \binom{12}{4}(0.3)^4(0.7)^8 = 495 \cdot 0.0081 \cdot 0.05765 \approx 0.231$.

(b) $E[X] = np = 12 \cdot 0.3 = 3.6$ days.

80. **Business risk insurance (Entrepreneurship).**
A start-up estimates a 0.2 probability that a new project will fail. It invests in 5 independent projects.

(a) Model the number of failed projects.

(b) Probability that at most 2 projects fail.

(c) Probability that all projects succeed.

Solution.

$X \sim Bin(5, 0.2)$.

(a) $P(X \le 2) = \sum_{k=0}^{2} \binom{5}{k} 0.2^k 0.8^{5-k}$.

$P(0) = 0.8^5 = 0.3277$.

$P(1) = 5 \cdot 0.2 \cdot 0.8^4 = 5 \cdot 0.2 \cdot 0.4096 = 0.4096$.

$P(2) = 10 \cdot 0.04 \cdot 0.512 = 0.2048$.

Total = 0.9421.

(b) $P(\text{all succeed}) = P(X = 0) = 0.3277$.

81. **Coffee Yield Prediction in Mbale**
Scenario: A coffee farmer in Mbale estimates that her yield will increase by a factor of $(1+0.1x)^5$ over 5 years, where x represents the annual improvement factor.

(a) Expand $(1 + 0.1x)^5$ using the binomial theorem.

(b) Find the coefficient of the x^3 term in the expansion.

(c) If the initial yield is 500 kg and x = 2, estimate the yield after 5 years using the first three terms of the expansion.

Solution 1

(a) Using the binomial theorem:

$$(1 + 0.1x)^5 = \sum_{k=0}^{5} \binom{5}{k} (1)^{5-k}(0.1x)^k$$
$$= 1 + 5(0.1x) + 10(0.1x)^2 + 10(0.1x)^3 + 5(0.1x)^4 + (0.1x)^5$$
$$= 1 + 0.5x + 0.1x^2 + 0.01x^3 + 0.0005x^4 + 0.00001x^5$$

(b) The coefficient of x^3 is:

$$\binom{5}{3}(0.1)^3 = 10 \times 0.001 = 0.01$$

(c) Using the first three terms with x = 2:

$$\text{Yield} = 500 \times (1 + 0.5(2) + 0.1(2)^2)$$
$$= 500 \times (1 + 1 + 0.4) = 500 \times 2.4 = 1200 \text{ kg}$$

82. **Population Growth Model in Kampala**
Scenario: The population growth in a Kampala suburb is modeled by $(1 + 0.08)^7$, representing 7 years of 8% annual growth.

(a) Expand $(1 + 0.08)^7$ using the binomial theorem.

(b) Calculate the approximate population multiplier using the first four terms.

(c) If the initial population is 10,000, estimate the population after 7 years.

Solution 2

(a) Using the binomial theorem:

$$(1 + 0.08)^7 = \sum_{k=0}^{7} \binom{7}{k} (1)^{7-k}(0.08)^k$$
$$= 1 + 7(0.08) + 21(0.08)^2 + 35(0.08)^3 + 35(0.08$$

(b) Using the first four terms:

$$\text{Multiplier} \approx 1 + 0.56 + 21(0.0064) + 35(0.000512)$$
$$= 1 + 0.56 + 0.1344 + 0.01792 = 1.71232$$

(c) Estimated population:

$$\text{Population} = 10,000 \times 1.71232 = 17,123 \text{ people}$$

83. **Agricultural Investment Return**
Scenario: An agricultural investment in Jinja is expected to grow according to $(1 + 0.12x)^4$, where x is the risk factor (0 < x < 1).

(a) Expand $(1 + 0.12x)^4$ using the binomial theorem.

(b) Find the term containing x^2 in the expansion.

(c) If the initial investment is 2,000,000 UGX and x = 0.8, estimate the value after 4 years using the expansion.

Solution 3

(a) Using the binomial theorem:

$$(1 + 0.12x)^4 = \sum_{k=0}^{4} \binom{4}{k} (1)^{4-k}(0.12x)^k$$
$$= 1 + 4(0.12x) + 6(0.12x)^2 + 4(0.12x)^3 + (0.1$$
$$= 1 + 0.48x + 0.0864x^2 + 0.006912x^3 + 0.0002$$

(b) The term containing x^2 is:

$$\binom{4}{2}(0.12)^2x^2 = 6 \times 0.0144x^2 = 0.0864x^2$$

(c) With x = 0.8:

$$\text{Value} = 2,000,000 \times (1 + 0.48(0.8) + 0.0864(0.8)^2)$$
$$= 2,000,000 \times (1 + 0.384 + 0.0553) = 2,000,000 \times 1.4393 = 2,878,600 \text{ UGX}$$

84. **Disease Spread Model in Gulu**

Scenario: The spread of a disease in Gulu is modeled by $(1 + 0.2)^6$, where 0.2 represents the daily infection rate.

(a) Expand $(1 + 0.2)^6$ using the binomial theorem.

(b) Calculate the approximate spread factor using the first three terms.

(c) If initially 100 people are infected, estimate the number after 6 days.

Solution 4

(a) Using the binomial theorem:

$$(1 + 0.2)^6 = \sum_{k=0}^{6} \binom{6}{k} (1)^{6-k}(0.2)^k$$
$$= 1 + 6(0.2) + 15(0.2)^2 + 20(0.2)^3 + 15(0.2)^4 + 6(0.2)^5 + (0.2)^6$$

(b) Using the first three terms:

$$\text{Spread factor} \approx 1 + 1.2 + 15(0.04) = 1 + 1.2 + 0.6 = 2.8$$

(c) Estimated infections:

$$\text{Infections} = 100 \times 2.8 = 280 \text{ people}$$

85. **Water Reservoir Capacity Planning**

Scenario: The capacity of a water reservoir in Karamoja is expected to increase by a factor of $(1 + 0.15)^5$ after expansion works.

(a) Expand $(1 + 0.15)^5$ using the binomial theorem.

(b) Find the fourth term in the expansion.

(c) If the initial capacity is 500,000 liters, estimate the new capacity using the expansion.

Solution 5

(a) Using the binomial theorem:

$$(1 + 0.15)^5 = \sum_{k=0}^{5} \binom{5}{k} (1)^{5-k}(0.15)^k$$
$$= 1 + 5(0.15) + 10(0.15)^2 + 10(0.15)^3 + 5(0.15)^4 + (0.15)^5$$

(b) The fourth term is:

$$\binom{5}{3}(0.15)^3 = 10 \times 0.003375 = 0.03375$$

(c) Estimated capacity:

$$\text{Capacity} = 500,000 \times (1 + 0.75 + 0.225 + 0.03375)$$
$$= 2,500,000 \times 0.40875 = 1,004,375 \text{ liters}$$

86. **Microfinance Loan Growth**

Scenario: A microfinance institution in Mbale models its loan growth as $(1 + 0.25)^4$, representing 4 years of 25% annual growth.

(a) Expand $(1 + 0.25)^4$ using the binomial theorem.

(b) Calculate the exact value using the expansion.

(c) If the initial loan portfolio is 50,000,000 UGX, what will it be after 4 years?

Solution 6

(a) Using the binomial theorem:

$$(1 + 0.25)^4 = \sum_{k=0}^{4} \binom{4}{k} (1)^{4-k}(0.25)^k$$
$$= 1 + 4(0.25) + 6(0.25)^2 + 4(0.25)^3 + (0.25)^4$$
$$= 1 + 1 + 6(0.0625) + 4(0.015625) + 0.0039062$$
$$= 1 + 1 + 0.375 + 0.0625 + 0.00390625 = 2.441$$

(b) The exact value is:

$$(1.25)^4 = 2.44140625$$

(c) Final loan portfolio:

$$\text{Portfolio} = 50,000,000 \times 2.44140625 = 122,070,312.50 \text{ UG}$$

87. **Agricultural Productivity Improvement**

Scenario: A new farming technique in Lira is expected to improve productivity by a factor of $(1 + 0.1x)^3$, where x is the adoption rate ($0 \le x \le 1$).

(a) Expand $(1 + 0.1x)^3$ using the binomial theorem.

(b) Find the coefficient of the x^2 term.

(c) If the initial yield is 1,000 kg/hectare and x = 0.7, estimate the new yield.

Solution 7

(a) Using the binomial theorem:

$$(1 + 0.1x)^3 = \sum_{k=0}^{3} \binom{3}{k} (1)^{3-k}(0.1x)^k$$
$$= 1 + 3(0.1x) + 3(0.1x)^2 + (0.1x)^3$$
$$= 1 + 0.3x + 0.03x^2 + 0.001x^3$$

(b) The coefficient of x^2 is:

$$\binom{3}{2}(0.1)^2 = 3 \times 0.01 = 0.03$$

(c) With x = 0.7:

$$\text{Yield} = 1,000 \times (1 + 0.3(0.7) + 0.03(0.7)^2)$$
$$= 1,000 \times (1 + 0.21 + 0.0147) = 1,000 \times 1.2247 = 1,224.7 \text{ kg/ha}$$

88. **Solar Energy Adoption Model**
Scenario: Solar energy adoption in a Kampala neighborhood is modeled by $(1 + 0.3)^5$, representing 5 years of 30% annual adoption growth.
(a) Expand $(1 + 0.3)^5$ using the binomial theorem.
(b) Calculate the adoption factor using the first four terms.
(c) If initially 50 households use solar, estimate the number after 5 years.

Solution 8

(a) Using the binomial theorem:

$$(1 + 0.3)^5 = \sum_{k=0}^{5} \binom{5}{k}(1)^{5-k}(0.3)^k$$
$$= 1 + 5(0.3) + 10(0.3)^2 + 10(0.3)^3 + 5(0.3)^4 + (0.3)^5$$

(b) Using the first four terms:

$$\text{Adoption factor} \approx 1 + 1.5 + 10(0.09) + 10(0.027)$$
$$= 1 + 1.5 + 0.9 + 0.27 = 3.67$$

(c) Estimated households:

$$\text{Households} = 50 \times 3.67 = 183.5 \approx 184 \text{ household}$$

89. **Fishery Stock Growth Prediction**
Scenario: The fish stock in Lake Victoria is expected to grow by a factor of $(1 + 0.08)^6$ over 6 years with proper conservation measures.
(a) Expand $(1 + 0.08)^6$ using the binomial theorem.
(b) Find the fifth term in the expansion.
(c) If the initial stock is 500,000 fish, estimate the stock after 6 years.

Solution 9

(a) Using the binomial theorem:

$$(1 + 0.08)^6 = \sum_{k=0}^{6} \binom{6}{k}(1)^{6-k}(0.08)^k$$
$$= 1 + 6(0.08) + 15(0.08)^2 + 20(0.08)^3 + 15(0.08)^4 + 6(0.08)^5 + (0.08)^6$$

(b) The fifth term is:

$$\binom{6}{4}(0.08)^4 = 15 \times 0.00004096 = 0.0006144$$

(c) Estimated stock:

$$\text{Stock} = 500,000 \times (1 + 0.48 + 15(0.0064) + 20(0.000512))$$
$$= 500,000 \times (1 + 0.48 + 0.096 + 0.01024)$$
$$= 500,000 \times 1.58624 = 793,120 \text{ fish}$$

90. **Education Enrollment Projection**
Scenario: School enrollment in a rural district is projected to grow by a factor of $(1 + 0.12)^4$ over 4 years with new government initiatives.
(a) Expand $(1 + 0.12)^4$ using the binomial theorem.
(b) Calculate the growth factor using the expansion.
(c) If the initial enrollment is 2,000 students, estimate the enrollment after 4 years.

Solution 10

(a) Using the binomial theorem:

$$(1 + 0.12)^4 = \sum_{k=0}^{4} \binom{4}{k}(1)^{4-k}(0.12)^k$$
$$= 1 + 4(0.12) + 6(0.12)^2 + 4(0.12)^3 + (0.12)^4$$
$$= 1 + 0.48 + 6(0.0144) + 4(0.001728) + 0.0002$$
$$= 1 + 0.48 + 0.0864 + 0.006912 + 0.00020736 =$$

(b) The growth factor is approximately 1.5735
(c) Estimated enrollment:

$$\text{Enrollment} = 2,000 \times 1.5735 = 3,147 \text{ students}$$

Question 91: Probability of a Successful Harvest in Masindi

A farmer in Masindi plants 8 maize seeds. Based on historical data, the probability of a single seed germinating and growing into a healthy plant is 0.7.

1. Find the probability that exactly 5 out of the 8 seeds grow into healthy plants.
2. Find the probability that at least 6 seeds grow into healthy plants.
3. Explain how the binomial theorem is related to this type of probability calculation.

Solution 91

1. This is a binomial probability problem, where the number of trials $n = 8$, the probability of success $p = 0.7$, and the probability of failure $q = 0.3$. The probability of exactly k

successes is given by the binomial probability formula, which is a term from the binomial expansion: $P(X = k) = \binom{n}{k} p^k q^{n-k}$. $P(X = 5) = \binom{8}{5}(0.7)^5(0.3)^{8-5} = \binom{8}{5}(0.7)^5(0.3)^3$ $\binom{8}{5} = \frac{8!}{5!3!} = \frac{8\cdot7\cdot6}{3\cdot2\cdot1} = 56$. $P(X = 5) = 56(0.7)^5(0.3)^3 = 56(0.16807)(0.027) \approx$ **0.2541**.

2. The probability of at least 6 successes is the sum of the probabilities of exactly 6, 7, or 8 successes. $P(X \geq 6) = P(X = 6) + P(X = 7) + P(X = 8)$ $P(X = 6) = \binom{8}{6}(0.7)^6(0.3)^2 = 28(0.117649)(0.09) \approx 0.2964$ $P(X = 7) = \binom{8}{7}(0.7)^7(0.3)^1 = 8(0.0823543)(0.3) \approx 0.1977$ $P(X = 8) = \binom{8}{8}(0.7)^8(0.3)^0 = 1(0.057648) \approx 0.0576$ $P(X \geq 6) = 0.2964 + 0.1977 + 0.0576 =$ **0.5517**.
3. The binomial theorem, $(p + q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$, represents the sum of all possible probabilities in an experiment with n trials. Each term in the expansion, $\binom{n}{k} p^k q^{n-k}$, corresponds to the probability of obtaining a specific number of successes (k) and failures ($n - k$). The theorem states that the sum of these probabilities for all possible outcomes is equal to $(p + q)^n = (0.7 + 0.3)^8 = 1^8 = 1$, which is the total probability of all outcomes.

Question 92: Quality Control in a Factory in Jinja

A factory in Jinja produces electronic components. The probability that a single component is defective is 0.05. A sample of 10 components is taken.

1. Find the probability that exactly 2 components are defective.
2. Find the probability that at most 1 component is defective.
3. Discuss how a quality assurance manager can use this concept to make decisions about production batches.

Solution 92

1. The binomial parameters are $n = 10$, $p = 0.05$, and $q = 0.95$. $P(X = 2) = \binom{10}{2}(0.05)^2(0.95)^{10-2} = 45(0.05)^2(0.95)^8$ $P(X = 2) = 45(0.0025)(0.6634) \approx$ **0.0746**.
2. At most 1 defective component means either 0 or 1 defective component. $P(X \leq 1) = P(X = 0) + P(X = 1)$ $P(X = 0) = \binom{10}{0}(0.05)^0(0.95)^{10} = 1(1)(0.5987) \approx 0.5987$ $P(X = 1) = \binom{10}{1}(0.05)^1(0.95)^9 = 10(0.05)(0.6302) \approx 0.3151$ $P(X \leq 1) = 0.5987 + 0.3151 =$ **0.9138**.
3. A quality assurance manager can set a benchmark probability for a batch to be considered acceptable. For example, they might decide that if the probability of having more than a certain number of defects is too high, the entire batch must be re-inspected or scrapped. By using the binomial theorem, they can quantify the risk of sending a faulty batch to market and set a reasonable quality threshold.

Question 93: Investment Portfolio Success Rate

An investor in Uganda diversifies his portfolio with 15 different small ventures. Each venture has a 60% chance of being profitable in the first year.

1. Find the probability that exactly 10 of his ventures are profitable.
2. Find the probability that at least 12 of his ventures are profitable.
3. Explain the significance of this type of stochastic model for an investor.

Solution 93

1. This is a binomial problem with $n = 15$, $p = 0.6$, and $q = 0.4$. $P(X = 10) = \binom{15}{10}(0.6)^{10}(0.4)^{15-10} = \binom{15}{10}(0.6)^{10}(0.4)^5$ $\binom{15}{10} = \frac{15!}{10!5!} = 3003$. $P(X = 10) = 3003(0.0060466)(0.01024) \approx$ **0.1859**.
2. At least 12 ventures means exactly 12, 13, 14, or 15 ventures. $P(X \geq 12) = P(X = 12) + P(X = 13) + P(X = 14) + P(X = 15)$ $P(X = 12) = \binom{15}{12}(0.6)^{12}(0.4)^3 = 455(0.00217678)(0.064) \approx 0.0634$ $P(X = 13) = \binom{15}{13}(0.6)^{13}(0.4)^2 = 105(0.001306) \approx 0.035$ $P(X = 14) = \binom{15}{14}(0.6)^{14}(0.4)^1 = 15(0.0007836) \approx 0.01175$ $P(X = 15) = \binom{15}{15}(0.6)^{15}(0.4)^0 = 1(0.00047) \approx 0.00047$ $P(X \geq 12) \approx 0.0634 + 0.035 + 0.01175 + 0.00047 =$ **0.11062**.
3. This stochastic model helps an investor quantify risk and potential returns. It provides a probability distribution for the number of successful ventures, allowing the investor to see that while a very high number of successes (e.g., all 15) is unlikely, a reasonably high number (e.g., at

least 12) is possible. This supports the strategy of diversification by showing that even with individual risks, the overall portfolio has a good chance of yielding a positive outcome.

Question 94: Estimating an Irrational Number

A civil engineer in Uganda needs to find an accurate approximation for $\sqrt[3]{28}$. She uses the binomial theorem for this calculation.

1. Rewrite $\sqrt[3]{28}$ in a form suitable for binomial expansion, such that the expansion is valid.
2. Find the first four terms of the expansion.
3. Use the first four terms to find an approximate value of $\sqrt[3]{28}$.

Solution 94

1. To use the binomial theorem for non-integer powers, we need the form $(1+x)^n$. We can rewrite $\sqrt[3]{28}$ as $\sqrt[3]{27+1} = \sqrt[3]{27(1+\frac{1}{27})} = 3(1+\frac{1}{27})^{1/3}$. The expression is $\mathbf{3(1+\frac{1}{27})^{1/3}}$. The expansion is valid because $|\frac{1}{27}| < 1$.
2. The expansion of $(1+x)^n$ is $1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots$ Here, $n = 1/3$ and $x = 1/27$.
   - First term: 1
   - Second term: $nx = \frac{1}{3}\cdot\frac{1}{27} = \frac{1}{81}$
   - Third term: $\frac{n(n-1)}{2!}x^2 = \frac{\frac{1}{3}(\frac{1}{3}-1)}{2}(\frac{1}{27})^2 = \frac{\frac{1}{3}(-\frac{2}{3})}{2}\cdot\frac{1}{729} = \frac{-2/9}{2}\cdot\frac{1}{729} = -\frac{1}{9}\cdot\frac{1}{729} = -\frac{1}{6561}$
   - Fourth term: $\frac{n(n-1)(n-2)}{3!}x^3 = \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{6}(\frac{1}{27})^3 = \frac{10/27}{6}\cdot\frac{1}{19683} = \frac{5}{81}\cdot\frac{1}{19683} = \frac{5}{1594323}$

   The first four terms of $(1+\frac{1}{27})^{1/3}$ are $\mathbf{1+\frac{1}{81}-\frac{1}{6561}+\frac{5}{1594323}}$.
3. To get the approximation for $\sqrt[3]{28}$, we multiply the sum of the terms by 3. $\sqrt[3]{28} \approx 3\left(1+\frac{1}{81}-\frac{1}{6561}+\frac{5}{1594323}\right)$ $\sqrt[3]{28} \approx 3(1+0.0123457-0.0001524+0.0000031) = 3(1.0121964) \approx \mathbf{3.0365892}$.

Question 95: Public Health Campaign Success

A local public health organization launches a campaign to encourage handwashing in primary schools. The campaign targets 20 schools, and the probability of a school implementing a handwashing station is 0.8.

1. Find the probability that exactly 15 schools implement the station.
2. Find the probability that more than 18 schools implement the station.
3. Explain the practical implications of a high probability of success for such a campaign.

Solution 95

1. This is a binomial problem with $n = 20$, $p = 0.8$, and $q = 0.2$. $P(X = 15) = \binom{20}{15}(0.8)^{15}(0.2)^5 = 15504(0.035184)(0.00032) \approx \mathbf{0.1746}$.
2. More than 18 schools means exactly 19 or 20 schools. $P(X > 18) = P(X = 19) + P(X = 20)$ $P(X = 19) = \binom{20}{19}(0.8)^{19}(0.2)^1 = 20(0.018014)(0.2) \approx 0.072056$ $P(X = 20) = \binom{20}{20}(0.8)^{20}(0.2)^0 = 1(0.011529) \approx 0.011529$ $P(X > 18) = 0.072056 + 0.011529 = \mathbf{0.083585}$.
3. A high probability of success, such as $p = 0.8$, implies that the campaign strategy is very effective. The organization can use this information to secure more funding, justify scaling up the campaign to more schools, and prove to stakeholders that their efforts have a strong likelihood of leading to a positive outcome, which in this case is improved public health.

Question 96: Agricultural Irrigation System

An agricultural engineer is designing an irrigation system for a farm in Mukono. He uses the binomial theorem to model the expected flow rate, which is given by the expansion of $(1+0.01x)^8$.

1. Find the first three terms of the expansion.
2. Find the coefficient of the term containing x^2.
3. If $x = 10$, find the value of the expansion. What does this value represent?

Solution 96

1. The binomial expansion of $(1+0.01x)^8$ is $1+nx+\frac{n(n-1)}{2!}x^2+\ldots$ Here, $n = 8$ and the variable is $0.01x$.
   - First term: $1^8 = \mathbf{1}$
   - Second term: $\binom{8}{1}(1)^7(0.01x)^1 = 8(0.01x) = \mathbf{0.08x}$
   - Third term: $\binom{8}{2}(1)^6(0.01x)^2 = 28(0.0001x^2) = \mathbf{0.0028x^2}$
2. The coefficient of the term containing x^2 is $\binom{8}{2}(0.01)^2 = 28(0.0001) = \mathbf{0.0028}$.
3. If $x = 10$, we evaluate the expansion: $(1+0.01(10))^8 = (1+0.1)^8 = (1.1)^8 \approx \mathbf{2.1436}$. This value represents the expected flow rate

factor. If the initial flow rate is, say, 100 liters per hour, then the final flow rate under these conditions would be $100 \times 2.1436 = 214.36$ liters per hour. The engineer can use this to predict the system's performance and ensure it meets the farm's needs.

Question 97: Approximation of an Interest Rate

A microfinance institution needs to approximate the value of $(1.025)^{-5}$ to calculate a loan's present value.

1. Rewrite the expression in a form suitable for binomial expansion.
2. Find the first three terms of the expansion.
3. Use the first three terms to find an approximate value of the expression.

Solution 97

1. We can rewrite the expression as $(1+0.025)^{-5}$. This fits the form $(1+x)^n$ where $x = 0.025$ and $n = -5$. The expression is $(\mathbf{1 + 0.025})^{\mathbf{-5}}$.
2. The expansion is $1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$
   - First term: 1
   - Second term: $nx = (-5)(0.025) = \mathbf{-0.125}$
   - Third term: $\frac{n(n-1)}{2!}x^2 = \frac{(-5)(-5-1)}{2}(0.025)^2 = \frac{(-5)(-6)}{2}(0.000625) = 15(0.000625) = \mathbf{0.009375}$
3. To get the approximation, we sum the first three terms. $(1.025)^{-5} \approx 1 - 0.125 + 0.009375 = \mathbf{0.884375}$. The actual value is approximately 0.88385. The approximation is quite close.

Question 98: Vaccine Efficacy Trial

A new vaccine is being trialed in a community. The probability that a vaccinated person does not get a particular disease is 0.9. A sample of 12 vaccinated people is randomly selected.

1. Find the probability that exactly 10 of these 12 people do not get the disease.
2. Find the probability that fewer than 9 of these people do not get the disease.
3. How does this type of statistical modeling support a public health authority's decision-making?

Solution 98

1. This is a binomial problem with $n = 12$, $p = 0.9$, and $q = 0.1$. $P(X = 10) = \binom{12}{10}(0.9)^{10}(0.1)^2 = 66(0.348678)(0.01) \approx \mathbf{0.2299}$.
2. Fewer than 9 people means exactly 0 to 8 people. This is too long to compute manually. We can use the complement rule: $P(X < 9) = 1 - P(X \geq 9)$. $P(X \geq 9) = P(X = 9) + P(X = 10) + P(X = 11) + P(X = 12)$ $P(X = 9) = \binom{12}{9}(0.9)^9(0.1)^3 = 220(0.38742)(0.001) \approx 0.0852$ $P(X = 10) \approx 0.2299$ (from part a) $P(X = 11) = \binom{12}{11}(0.9)^{11}(0.1)^1 = 12(0.31381)(0.1) \approx 0.3766$ $P(X = 12) = \binom{12}{12}(0.9)^{12}(0.1)^0 = 1(0.2824) \approx 0.2824$ $P(X \geq 9) = 0.0852 + 0.2299 + 0.3766 + 0.2824 = 0.9741$. $P(X < 9) = 1 - 0.9741 = \mathbf{0.0259}$.
3. This type of statistical modeling allows a public health authority to make evidence-based decisions. The low probability of fewer than 9 people being protected (only 2.59%) confirms the vaccine's high efficacy. This provides the authority with the confidence to roll out the vaccine to the general public, knowing it is highly likely to have a positive impact on public health.

Question 99: Bridge Maintenance and Safety

Engineers are assessing the safety of a bridge. The probability that a random support pillar is structurally sound is 0.95. The bridge has 25 support pillars.

1. Find the probability that exactly 24 pillars are structurally sound.
2. What is the probability that all 25 pillars are structurally sound?
3. Why is this type of analysis important for bridge maintenance and public safety?

Solution 99

1. This is a binomial problem with $n = 25$, $p = 0.95$, and $q = 0.05$. $P(X = 24) = \binom{25}{24}(0.95)^{24}(0.05)^1 = 25(0.29245)(0.05) \approx \mathbf{0.3656}$.
2. The probability that all 25 pillars are sound: $P(X = 25) = \binom{25}{25}(0.95)^{25}(0.05)^0 = 1(0.95)^{25} \approx \mathbf{0.2774}$.
3. This analysis is crucial for public safety. It allows engineers to quantify the risk of structural failure. The probabilities show that it is more likely for a few pillars to be unsound than for all of them to be sound. By calculating

the probability of a specific number of failures, engineers can establish a maintenance schedule. They can use this to justify the need for regular inspections and repairs to mitigate risks and ensure the long-term safety of the bridge for commuters.

Question 100: Telecommunication Signal Strength

A telecommunications engineer is testing a new cell tower. The signal strength at a given location is modeled by the expansion of $(1.05)^5$.

1. Find the coefficient of the third term in the binomial expansion of $(1+x)^5$.
2. Use the binomial theorem to expand $(1.05)^5$.
3. What does the full expansion tell the engineer about the signal strength?

Solution 100

1. The third term in the expansion of $(1+x)^5$ is given by $\binom{5}{2}x^2$. The coefficient is $\binom{5}{2} = \frac{5!}{2!3!} = \frac{5\cdot4}{2\cdot1} = \mathbf{10}$.
2. We expand $(1.05)^5 = (1+0.05)^5$. $(1+0.05)^5 = \binom{5}{0}(1)^5(0.05)^0 + \binom{5}{1}(1)^4(0.05)^1 + \binom{5}{2}(1)^3(0.05)^2 + \binom{5}{3}(1)^2(0.05)^3 + \binom{5}{4}(1)^1(0.05)^4 + \binom{5}{5}(1)^0(0.05)^5 = 1(1) + 5(0.05) + 10(0.0025) + 10(0.000125) + 5(0.00000625) + 1(0.0000003125) = 1 + 0.25 + 0.025 + 0.00125 + 0.00003125 + 0.0000003125 = \mathbf{1.2762815625}$.
3. The full expansion shows the engineer that the signal strength is the sum of several factors. The first term (1) represents the baseline strength. The second term (0.25) is the most significant increase, while subsequent terms get progressively smaller. This shows that the initial increases are the most crucial to overall performance, and any additional factors beyond the first few have a diminishing return. This insight can help the engineer focus on the most impactful factors for improving signal quality.

Chapter 10

VECTORS IN TWO AND THREE DIMENSIONS

10.0.1 Introduction to Vectors

Vectors are fundamental mathematical objects that describe quantities having both magnitude and direction. They are essential in physics, engineering, navigation, and many real-world applications in Uganda, from surveying land boundaries to planning transportation routes.

Definition: A vector is a quantity that has both magnitude (size) and direction. It is represented geometrically by a directed line segment or algebraically by its components.

10.0.2 Vector Notation and Representation

A vector can be represented in several ways:

- Bold lowercase letters: $\mathbf{a}$, $\mathbf{v}$
- Arrows over letters: $\vec{a}$, $\vec{AB}$ (from point A to point B)
- Column vectors: $\begin{pmatrix} x \\ y \end{pmatrix}$ in 2D, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in 3D
- Component form: $\mathbf{a} = x\mathbf{i}+y\mathbf{j}$ in 2D, or $\mathbf{a} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ in 3D

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors along the x, y, and z axes respectively.

Example 1.1: Navigation in Kampala City

A boda-boda rider in Kampala travels 3 km east from the Old Taxi Park along Kampala Road, then 4 km north along Bombo Road. Represent this journey as vectors.

Solution:

Let east be the positive x-direction and north be the positive y-direction.

First displacement: $\mathbf{d_1} = 3\mathbf{i} + 0\mathbf{j} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ km

Second displacement: $\mathbf{d_2} = 0\mathbf{i} + 4\mathbf{j} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$ km

Total displacement: $\mathbf{d} = \mathbf{d_1} + \mathbf{d_2} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ km

The rider is 3 km east and 4 km north of the starting point.

10.0.3 Vector Operations in 2D and 3D

Vector Addition and Subtraction

For vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$:

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$

$$\mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix}$$

Example 1.2: Coffee Farm Survey in Mbale

A surveyor in Mbale district is mapping a coffee plantation. From a reference point P, one corner A of the farm is located at position vector $\mathbf{a} = \begin{pmatrix} 120 \\ 80 \\ 5 \end{pmatrix}$ meters, and another corner B is at $\mathbf{b} = \begin{pmatrix} 200 \\ 150 \\ 3 \end{pmatrix}$ meters (where the third component represents elevation above sea level reference).

Find the vector $\overrightarrow{AB}$ and interpret its meaning.

Solution:

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 200 - 120 \\ 150 - 80 \\ 3 - 5 \end{pmatrix} = \begin{pmatrix} 80 \\ 70 \\ -2 \end{pmatrix} \text{ meters}$$

Interpretation: To go from corner A to corner B, one must move 80 m in the x-direction (east), 70 m in the y-direction (north), and descend 2 m in elevation. This helps in planning irrigation systems and understanding water flow patterns on the farm.

Scalar Multiplication

For a vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and scalar k:

$$k\mathbf{a} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}$$

The magnitude is multiplied by $|k|$, and the direction is reversed if $k < 0$.

10.0.4 Magnitude and Direction of Vectors

Magnitude of a Vector

The magnitude (or modulus or length) of a vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

In 2D: $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$

Unit Vector

A unit vector is a vector with magnitude 1. The unit vector in the direction of $\mathbf{a}$ is:

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

Example 1.3: Distance Between Cities

The position of Kampala can be taken as the origin. Jinja is located at coordinates (80, 0, 0) km and Mbarara at (-240, -80, 50) km relative to Kampala (with elevation component). Calculate:

(a) The distance from Kampala to Jinja
(b) The distance from Kampala to Mbarara
(c) A unit vector in the direction from Kampala to Mbarara

Solution:

(a) For Jinja: $\mathbf{r}_J = \begin{pmatrix} 80 \\ 0 \\ 0 \end{pmatrix}$ km

Distance $= |\mathbf{r}_J| = \sqrt{80^2 + 0^2 + 0^2} = 80$ km

(b) For Mbarara: $\mathbf{r}_M = \begin{pmatrix} -240 \\ -80 \\ 50 \end{pmatrix}$ km

Distance $= |\mathbf{r}_M| = \sqrt{(-240)^2 + (-80)^2 + 50^2} = \sqrt{57600 + 6400 + 2500}$

$$= \sqrt{66500} = 257.88 \text{ km}$$

(c) Unit vector: $\hat{\mathbf{r}}_M = \frac{1}{257.88}\begin{pmatrix} -240 \\ -80 \\ 50 \end{pmatrix} = \begin{pmatrix} -0.931 \\ -0.310 \\ 0.194 \end{pmatrix}$

Direction Angles and Direction Cosines

The angles α, β, γ that a vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ makes with the positive x, y, z axes are given by:

$$\cos\alpha = \frac{a_1}{|\mathbf{a}|}, \quad \cos\beta = \frac{a_2}{|\mathbf{a}|}, \quad \cos\gamma = \frac{a_3}{|\mathbf{a}|}$$

These are called direction cosines, and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

10.0.5 Distance Between Two Points

Distance Formula: The distance between points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is:

$$d = |AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 1.4: Telecommunication Tower Planning

Uganda Telecom is planning to install a telecommunication relay. Tower A is at coordinates (5, 3, 120) km where the z-coordinate represents the tower height in meters above ground. Tower B is at (12, 8, 150) km. Calculate the straight-line distance between the tops of the two towers.

Solution:

Using the distance formula:

$$d = \sqrt{(12 - 5)^2 + (8 - 3)^2 + (150 - 120)^2}$$

$$d = \sqrt{7^2 + 5^2 + 30^2} = \sqrt{49 + 25 + 900} = \sqrt{974} = 31.21 \text{ km}$$

The signal needs to travel approximately 31.21 km through the air between the two towers.

10.0.6 The Scalar (Dot) Product

Definition: The scalar (or dot) product of two vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Alternatively: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$ where θ is the angle between the vectors.

Properties of Dot Product:

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (commutative)
2. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ (distributive)
3. $(k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b})$
4. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
5. If $\mathbf{a} \perp \mathbf{b}$, then $\mathbf{a} \cdot \mathbf{b} = 0$

Angle Between Two Vectors

The angle θ between vectors $\mathbf{a}$ and $\mathbf{b}$ is:

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Example 1.5: Solar Panel Orientation at Makerere University

At Makerere University's solar energy research facility, a solar panel's normal vector (perpendicular to its surface) is $\mathbf{n} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$. The direction of sunlight at noon is $\mathbf{s} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$.

Find the angle between the panel normal and the sunlight direction. What does this mean for energy efficiency?

Solution:

$$\mathbf{n} \cdot \mathbf{s} = (2)(1) + (1)(0) + (3)(-2) = 2 + 0 - 6 = -4$$

$$|\mathbf{n}| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$$

$$|\mathbf{s}| = \sqrt{1^2 + 0^2 + (-2)^2} = \sqrt{5}$$

$$\cos\theta = \frac{-4}{\sqrt{14} \times \sqrt{5}} = \frac{-4}{\sqrt{70}} = -0.478$$

$$\theta = \cos^{-1}(-0.478) = 118.6$$

Interpretation: The angle between the panel normal and sunlight is 118.6 degrees. For optimal energy collection, we want the angle between the panel surface and sunlight (90 degrees minus angle with normal). The angle of incidence is $180 - 118.6 = 61.4$ from the panel surface. The panel should be adjusted for better perpendicular alignment to maximize energy absorption.

Example 1.6: Perpendicular Vectors

Show that the vectors $\mathbf{a} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$ are perpendicular.

Solution:

$$\mathbf{a} \cdot \mathbf{b} = (3)(2) + (-2)(4) + (1)(2) = 6 - 8 + 2 = 0$$

Since the dot product is zero, the vectors are perpendicular.

10.0.7 The Vector (Cross) Product

Definition: The vector (or cross) product of vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

Properties of Cross Product:

1. $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$ (anti-commutative)
2. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (distributive)
3. $(k\mathbf{a}) \times \mathbf{b} = k(\mathbf{a} \times \mathbf{b})$
4. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
5. $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$
6. $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$
7. If $\mathbf{a} \parallel \mathbf{b}$, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$

Example 1.7: Land Area Calculation in Gulu

A farmer in Gulu district owns a plot of land. Two adjacent sides of the plot can be represented by vectors $\mathbf{p} = \begin{pmatrix} 50 \\ 30 \\ 0 \end{pmatrix}$ meters and $\mathbf{q} = \begin{pmatrix} 20 \\ 60 \\ 0 \end{pmatrix}$ meters from a corner point.

Calculate the area of the parallelogram formed by these sides.

Solution:

$$\mathbf{p} \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 50 & 30 & 0 \\ 20 & 60 & 0 \end{vmatrix}$$

$$= \mathbf{i}(30 \cdot 0 - 0 \cdot 60) - \mathbf{j}(50 \cdot 0 - 0 \cdot 20) + \mathbf{k}(50 \cdot 60 - 30 \cdot 20)$$

$$= 0\mathbf{i} - 0\mathbf{j} + 2400\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 2400 \end{pmatrix}$$

$$\text{Area} = |\mathbf{p} \times \mathbf{q}| = \sqrt{0^2 + 0^2 + 2400^2} = 2400 \text{ m}^2$$

The plot has an area of 2400 square meters, or 0.24 hectares.

Example 1.8: Force and Torque at Kiira Motors

At Kiira Motors Corporation's assembly plant in Jinja, a mechanic applies a force $\mathbf{F} = \begin{pmatrix} 10 \\ 5 \\ 0 \end{pmatrix}$ N using a wrench. The position vector from the bolt center to the point of application is $\mathbf{r} = \begin{pmatrix} 0.3 \\ 0 \\ 0 \end{pmatrix}$ meters.

Calculate the torque (moment) about the bolt.

Solution:

The torque is given by $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$

$$\boldsymbol{\tau} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.3 & 0 & 0 \\ 10 & 5 & 0 \end{vmatrix}$$

$$= \mathbf{i}(0 - 0) - \mathbf{j}(0 - 0) + \mathbf{k}(0.3 \cdot 5 - 0 \cdot 10)$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1.5 \end{pmatrix} \text{ N·m}$$

The torque is 1.5 N·m in the direction perpendicular to the plane containing $\mathbf{r}$ and $\mathbf{F}$ (following the right-hand rule, this would be counterclockwise when viewed from above).

10.0.8 The Ratio Theorem (Section Formula)

Ratio Theorem: If point P divides the line segment joining points A and B in the ratio $m : n$, then the position vector of P is:

$$\mathbf{r}_P = \frac{n\mathbf{r}_A + m\mathbf{r}_B}{m + n}$$

where $\mathbf{r}_A$ and $\mathbf{r}_B$ are the position vectors of A and B.

For internal division (P between A and B): use $m : n$

For external division (P outside segment): use $m : (-n)$

Midpoint Formula: When $m = n = 1$, we get the midpoint:

$$\mathbf{r}_M = \frac{\mathbf{r}_A + \mathbf{r}_B}{2}$$

Example 1.9: Water Distribution Pipeline in Kampala

The National Water and Sewerage Corporation is installing a water distribution point along a main pipeline. The pipeline connects reservoir A at position $\begin{pmatrix}100\\200\\350\end{pmatrix}$ meters to pumping station B at $\begin{pmatrix}700\\800\\380\end{pmatrix}$ meters.

The distribution point P must divide the pipeline in the ratio 2:3 from A to B. Find the position of P.

Solution:

Using the ratio theorem with $m = 2$ and $n = 3$:

$$\mathbf{r}_P = \frac{3\mathbf{r}_A + 2\mathbf{r}_B}{2+3} = \frac{3\begin{pmatrix}100\\200\\350\end{pmatrix} + 2\begin{pmatrix}700\\800\\380\end{pmatrix}}{5}$$

$$= \frac{\begin{pmatrix}300\\600\\1050\end{pmatrix} + \begin{pmatrix}1400\\1600\\760\end{pmatrix}}{5} = \frac{\begin{pmatrix}1700\\2200\\1810\end{pmatrix}}{5} = \begin{pmatrix}340\\440\\362\end{pmatrix}$$

The distribution point should be located at coordinates (340, 440, 362) meters.

10.1 LINES IN TWO AND THREE DIMENSIONS

10.1.1 Equations of Lines

A line in space can be described in three main forms: vector form, parametric form, and Cartesian form.

Vector Equation of a Line

A line passing through point A with position vector $\mathbf{a}$ and parallel to vector $\mathbf{d}$ (direction vector) has vector equation:

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$$

where λ is a scalar parameter and $\mathbf{r}$ is the position vector of any point on the line.

Alternatively, for a line through two points A and B:

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$$

Parametric Equations

If $\mathbf{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$, $\mathbf{a} = \begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}$, and $\mathbf{d} = \begin{pmatrix}l\\m\\n\end{pmatrix}$, then:

$$x = x_0 + \lambda l$$

$$y = y_0 + \lambda m$$

$$z = z_0 + \lambda n$$

These are the parametric equations of the line.

Cartesian Equation

Eliminating λ from the parametric equations gives the Cartesian (symmetric) form:

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$$

where (l, m, n) are the direction ratios of the line.

Example 2.1: Uganda Railway Line Survey

The Uganda Railways Corporation is surveying a new railway line. The line passes through Kampala at coordinates (0, 0, 1200) meters (elevation 1200m) and must run in the direction of vector $\mathbf{d} = \begin{pmatrix}5\\3\\-0.2\end{pmatrix}$ km.

Write the equation of the railway line in:

(a) Vector form
(b) Parametric form
(c) Cartesian form

Solution:

(a) Vector form: $\mathbf{r} = \begin{pmatrix}0\\0\\1.2\end{pmatrix} + \lambda\begin{pmatrix}5\\3\\-0.2\end{pmatrix}$ km

(b) Parametric form:

$$x = 5\lambda$$
$$y = 3\lambda$$
$$z = 1.2 - 0.2\lambda$$

(c) Cartesian form: $\frac{x}{5} = \frac{y}{3} = \frac{z-1.2}{-0.2}$

10.1.2 Angle Between Two Lines

If two lines have direction vectors $\mathbf{d_1}$ and $\mathbf{d_2}$, the acute angle θ between them is:

$$\cos\theta = \frac{|\mathbf{d_1} \cdot \mathbf{d_2}|}{|\mathbf{d_1}||\mathbf{d_2}|}$$

Note: We use absolute value to ensure we get the acute angle.

Parallel and Perpendicular Lines:

- Lines are parallel if $\mathbf{d_1} = k\mathbf{d_2}$ for some scalar k
- Lines are perpendicular if $\mathbf{d_1} \cdot \mathbf{d_2} = 0$

Example 2.2: Road Intersection near Entebbe Airport

Two roads near Entebbe International Airport have direction vectors:

- Road 1 (Kampala-Entebbe Highway): $\mathbf{d_1} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$
- Road 2 (Access road): $\mathbf{d_2} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$

Find the angle at which these roads intersect.

Solution:

$$\mathbf{d_1} \cdot \mathbf{d_2} = (4)(1) + (1)(-2) + (0)(0) = 4 - 2 = 2$$

$$|\mathbf{d_1}| = \sqrt{16 + 1 + 0} = \sqrt{17}$$

$$|\mathbf{d_2}| = \sqrt{1 + 4 + 0} = \sqrt{5}$$

$$\cos\theta = \frac{|2|}{\sqrt{17} \times \sqrt{5}} = \frac{2}{\sqrt{85}} = 0.217$$

$$\theta = \cos^{-1}(0.217) = 77.5$$

The roads intersect at an angle of 77.5 degrees, which is close to perpendicular (90 degrees) and provides good visibility for drivers.

10.1.3 Distance from a Point to a Line

The perpendicular distance from point P with position vector $\mathbf{p}$ to a line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ is:

$$\text{Distance} = \frac{|(\mathbf{p} - \mathbf{a}) \times \mathbf{d}|}{|\mathbf{d}|}$$

Example 2.3: Distance of Hospital from Highway

A new hospital in Mbarara is located at point H with position vector $\mathbf{h} = \begin{pmatrix} 8 \\ 12 \\ 0 \end{pmatrix}$ km. The Mbarara-Kabale highway has equation:

$$\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

Find the perpendicular distance from the hospital to the highway.

Solution:

Let $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

$$\mathbf{h} - \mathbf{a} = \begin{pmatrix} 8 - 2 \\ 12 - 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ 0 \end{pmatrix}$$

$$(\mathbf{h} - \mathbf{a}) \times \mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 9 & 0 \\ 3 & 1 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6 - 27 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -21 \end{pmatrix}$$

$$|(\mathbf{h} - \mathbf{a}) \times \mathbf{d}| = 21$$

$$|\mathbf{d}| = \sqrt{9 + 1} = \sqrt{10}$$

$$\text{Distance} = \frac{21}{\sqrt{10}} = \frac{21\sqrt{10}}{10} = 6.64 \text{ km}$$

The hospital is approximately 6.64 km from the highway. This is important for emergency response planning.

10.1.4 Intersection of Lines

Lines in 2D: Two lines in 2D will either:

- Intersect at a single point
- Be parallel (no intersection)
- Be coincident (infinite intersections)

Lines in 3D: Two lines in 3D may:

- Intersect at a point
- Be parallel
- Be skew (not parallel and not intersecting)

Finding Intersection Point:

To find if lines $\mathbf{r} = \mathbf{a_1} + \lambda\mathbf{d_1}$ and $\mathbf{r} = \mathbf{a_2} + \mu\mathbf{d_2}$ intersect:

1. Set $\mathbf{a_1} + \lambda\mathbf{d_1} = \mathbf{a_2} + \mu\mathbf{d_2}$
2. Solve the resulting system for λ and μ
3. If consistent solution exists, lines intersect; substitute to find point
4. If inconsistent, lines are skew or parallel

Example 2.4: Line Intersection

Find the point of intersection of lines:

$$L_1 : \mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

$$L_2 : \mathbf{r} = \begin{pmatrix}3\\1\\4\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

Solution:

At intersection: $\begin{pmatrix}1+2\lambda\\2+\lambda\\3-\lambda\end{pmatrix} = \begin{pmatrix}3+\mu\\1-\mu\\4+\mu\end{pmatrix}$

From x: $1 + 2\lambda = 3 + \mu$ which gives $2\lambda - \mu = 2$ ... (1)

From y: $2 + \lambda = 1 - \mu$ which gives $\lambda + \mu = -1$ ... (2)

From (1) + (2): $3\lambda = 1$ so $\lambda = \frac{1}{3}$

Substituting in (2): $\frac{1}{3} + \mu = -1$ so $\mu = -\frac{4}{3}$

Check with z: $3 - \frac{1}{3} = \frac{8}{3}$ and $4 - \frac{4}{3} = \frac{8}{3}$ (correct)

Point of intersection: $\begin{pmatrix}1+\frac{2}{3}\\2+\frac{1}{3}\\3-\frac{1}{3}\end{pmatrix} = \begin{pmatrix}\frac{5}{3}\\\frac{7}{3}\\\frac{8}{3}\end{pmatrix}$

Skew Lines

Two lines are skew if they are not parallel and do not intersect. Skew lines exist only in 3D space.

Example 2.5: Power Lines in Kampala

Two power transmission lines in Kampala have paths:

$$L_1 : \mathbf{r} = \begin{pmatrix}0\\0\\20\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$L_2 : \mathbf{r} = \begin{pmatrix}0\\5\\15\end{pmatrix} + \mu\begin{pmatrix}0\\1\\0\end{pmatrix}$$

(Heights in meters, horizontal distances in kilometers)

Show that these lines are skew.

Solution:

Direction vectors: $\mathbf{d_1} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{d_2} = \begin{pmatrix}0\\1\\0\end{pmatrix}$

Since $\mathbf{d_1} \neq k\mathbf{d_2}$ for any k, the lines are not parallel.

For intersection: $\begin{pmatrix}\lambda\\0\\20\end{pmatrix} = \begin{pmatrix}0\\5+\mu\\15\end{pmatrix}$

From x: $\lambda = 0$

From y: $0 = 5 + \mu$ so $\mu = -5$

From z: $20 = 15$ (Contradiction!)

Since the system is inconsistent, the lines do not intersect. Therefore, they are skew lines.

Interpretation: The power lines run perpendicular to each other but at different heights (20m and 15m) and positions, never crossing. This is a safe configuration that avoids electrical interference.

10.2 PLANES IN THREE DIMENSIONS

10.2.1 Equations of Planes

Vector Equation of a Plane

A plane can be defined by:

1. A point A on the plane with position vector $\mathbf{a}$
2. A normal vector $\mathbf{n}$ perpendicular to the plane

The vector equation is: $\mathbf{n} \cdot (\mathbf{r} - \mathbf{a}) = 0$

or equivalently: $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{a}$

Parametric Equation

A plane through point A with position vector $\mathbf{a}$ and parallel to vectors $\mathbf{b}$ and $\mathbf{c}$ has parametric equation:

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

where λ and μ are scalar parameters.

Cartesian Equation of a Plane

If $\mathbf{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$ is normal to the plane, the Cartesian equation is:

$$Ax + By + Cz = D$$

where $D = \mathbf{n} \cdot \mathbf{a}$ for any point $\mathbf{a}$ on the plane.

Example 3.1: Airport Runway at Entebbe

A runway at Entebbe International Airport passes through point $(100, 50, 0)$ meters and has a slight incline. The normal vector to the runway surface is $\mathbf{n} = \begin{pmatrix} 0 \\ 0.02 \\ 1 \end{pmatrix}$ (indicating a 2 percent grade in the y-direction).

Find the Cartesian equation of the plane containing the runway surface.

Solution:

Using $Ax + By + Cz = D$ with $\mathbf{n} = \begin{pmatrix} 0 \\ 0.02 \\ 1 \end{pmatrix}$:

$$0 \cdot x + 0.02y + 1 \cdot z = D$$

At point $(100, 50, 0)$:

$$D = 0(100) + 0.02(50) + 1(0) = 1$$

Equation of runway surface: $0.02y + z = 1$ or $y + 50z = 50$

Interpretation: For every 50 meters traveled in the y-direction, the elevation changes by 1 meter, confirming the 2 percent grade.

10.2.2 Angle Between Two Planes

The acute angle θ between two planes with normal vectors $\mathbf{n_1}$ and $\mathbf{n_2}$ is:

$$\cos\theta = \frac{|\mathbf{n_1} \cdot \mathbf{n_2}|}{|\mathbf{n_1}||\mathbf{n_2}|}$$

Properties:

- Planes are parallel if $\mathbf{n_1} = k\mathbf{n_2}$
- Planes are perpendicular if $\mathbf{n_1} \cdot \mathbf{n_2} = 0$

Example 3.2: Geological Survey in Rwenzori Mountains

During a geological survey in the Rwenzori Mountains, two rock strata are identified:

- Stratum 1: $2x + y - z = 5$
- Stratum 2: $x - 2y + 2z = 3$

Find the angle between these geological planes.

Solution:

Normal vectors: $\mathbf{n_1} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{n_2} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

$$\mathbf{n_1} \cdot \mathbf{n_2} = (2)(1) + (1)(-2) + (-1)(2) = 2 - 2 - 2 = -2$$

$$|\mathbf{n_1}| = \sqrt{4 + 1 + 1} = \sqrt{6}$$

$$|\mathbf{n_2}| = \sqrt{1 + 4 + 4} = 3$$

$$\cos\theta = \frac{|-2|}{\sqrt{6} \times 3} = \frac{2}{3\sqrt{6}} = 0.272$$

$$\theta = \cos^{-1}(0.272) = 74.2$$

The rock strata intersect at an angle of 74.2 degrees, which is important for understanding the geological history and potential mineral deposits in the region.

10.2.3 Angle Between a Line and a Plane

The angle α between a line with direction vector $\mathbf{d}$ and a plane with normal $\mathbf{n}$ is:

$$\sin\alpha = \frac{|\mathbf{d}\cdot\mathbf{n}|}{|\mathbf{d}||\mathbf{n}|}$$

Note: If θ is the angle between $\mathbf{d}$ and $\mathbf{n}$, then $\alpha = 90-\theta$.

Example 3.3: Aircraft Approach to Entebbe Airport

An aircraft is approaching Entebbe Airport along a path with direction vector $\mathbf{d} = \begin{pmatrix}10\\8\\-2\end{pmatrix}$ km. The runway lies in a plane with equation $0.01y + z = 5$.

Find the angle at which the aircraft approaches the runway.

Solution:

Normal to runway: $\mathbf{n} = \begin{pmatrix}0\\0.01\\1\end{pmatrix}$

$\mathbf{d}\cdot\mathbf{n} = (10)(0)+(8)(0.01)+(-2)(1) = 0+0.08-2 = -1.92$

$$|\mathbf{d}| = \sqrt{100+64+4} = \sqrt{168} = 12.96$$

$$|\mathbf{n}| = \sqrt{0+0.0001+1} \approx 1$$

$$\sin\alpha = \frac{|-1.92|}{12.96\times 1} = 0.148$$

$$\alpha = \sin^{-1}(0.148) = 8.5$$

The aircraft approaches at an angle of 8.5 degrees to the runway, which is within the safe approach angle range (typically 3-5 degrees for most airports, though larger aircraft may use steeper approaches).

10.2.4 Distance from a Point to a Plane

The perpendicular distance from point $P(x_1, y_1, z_1)$ to the plane $Ax + By + Cz = D$ is:

$$\text{Distance} = \frac{|Ax_1 + By_1 + Cz_1 - D|}{\sqrt{A^2+B^2+C^2}}$$

Example 3.4: Weather Station Height above Ground

A weather monitoring station is located at coordinates $(20, 30, 150)$ meters. The ground surface in that area can be approximated by the plane $x+2y+100z = 5000$.

Calculate the height of the station above the ground.

Solution:

Using the distance formula with $A = 1$, $B = 2$, $C = 100$, $D = 5000$:

Point: $(20, 30, 150)$

$$\text{Distance} = \frac{|1(20)+2(30)+100(150)-5000|}{\sqrt{1^2+2^2+100^2}}$$

$$= \frac{|20+60+15000-5000|}{\sqrt{1+4+10000}} = \frac{|10080|}{\sqrt{10005}}$$

$$= \frac{10080}{100.02} = 100.78 \text{ meters}$$

The weather station is approximately 100.78 meters above the ground level. This height is suitable for accurate atmospheric measurements while being safely accessible for maintenance.

10.2.5 Intersection of Lines and Planes

Line-Plane Intersection

To find where line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ intersects plane $\mathbf{n}\cdot\mathbf{r} = p$:

1. Substitute the line equation into the plane equation
2. Solve for λ
3. Substitute λ back into line equation to find point

Special cases:

- If $\mathbf{n}\cdot\mathbf{d} = 0$: line is parallel to plane (no intersection or line lies in plane)
- If line lies in plane: infinite solutions

Example 3.5: Underground Cable through Foundation

An underground fiber optic cable follows the line:

$$\mathbf{r} = \begin{pmatrix} 10 \\ 20 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

It must pass through a building foundation represented by plane $x + 2y - z = 35$.

Find the point where the cable passes through the foundation.

Solution:

Point on line: $\begin{pmatrix} 10 + 2\lambda \\ 20 + \lambda \\ -5 + 3\lambda \end{pmatrix}$

Substitute into plane equation:

$$(10 + 2\lambda) + 2(20 + \lambda) - (-5 + 3\lambda) = 35$$

$$10 + 2\lambda + 40 + 2\lambda + 5 - 3\lambda = 35$$

$$55 + \lambda = 35$$

$$\lambda = -20$$

Intersection point:

$$\begin{pmatrix} 10 + 2(-20) \\ 20 + (-20) \\ -5 + 3(-20) \end{pmatrix} = \begin{pmatrix} -30 \\ 0 \\ -65 \end{pmatrix}$$

The cable passes through the foundation at coordinates $(-30, 0, -65)$ meters. The negative z-coordinate indicates it's 65 meters below the reference level.

10.2.6 Intersection of Two Planes

When two planes intersect, they form a line. To find this line:

1. Find the direction vector: $\mathbf{d} = \mathbf{n_1} \times \mathbf{n_2}$
2. Find one point on the line by setting one variable (usually z) to zero and solving
3. Write line equation: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$

Special case: If $\mathbf{n_1} = k\mathbf{n_2}$, planes are parallel (no intersection or coincident)

Example 3.6: Intersection of Two Planes

Find the line of intersection of planes:

$$\pi_1 : x + 2y + z = 6$$

$$\pi_2 : 2x - y + 3z = 10$$

Solution:

Direction vector of line:

$$\mathbf{d} = \mathbf{n_1} \times \mathbf{n_2} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 2 & -1 & 3 \end{vmatrix} = \begin{pmatrix} 6 - (-1) \\ -(3 - 2) \\ -1 - 4 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ -5 \end{pmatrix}$$

To find a point, set $z = 0$:

$$x + 2y = 6$$

... (1)

$$2x - y = 10$$

... (2)

From (1): $x = 6 - 2y$

Substitute in (2): $2(6 - 2y) - y = 10$

$$12 - 4y - y = 10$$

$$5y = 2$$

so $y = \frac{2}{5}$

$$x = 6 - 2\left(\frac{2}{5}\right) = \frac{26}{5}$$

Point: $A\left(\frac{26}{5}, \frac{2}{5}, 0\right)$

Line of intersection: $\mathbf{r} = \begin{pmatrix} 26/5 \\ 2/5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -1 \\ -5 \end{pmatrix}$

10.2.7 Intersection of Three Planes

Three planes can intersect in various ways:

1. At a single point (unique solution)
2. Along a line (infinite solutions)
3. No common point (parallel planes or forming a triangular prism)

Finding Intersection Point:

Solve the system of three equations simultaneously:

$$A_1x + B_1y + C_1z = D_1$$

$$A_2x + B_2y + C_2z = D_2$$

$$A_3x + B_3y + C_3z = D_3$$

Use elimination, substitution, or matrix methods (Gaussian elimination, Cramer's rule).

Example 3.7: Water Pipeline Junction in Kampala

Three water distribution pipelines in Kampala lie in planes:

$$\pi_1 : x + y + z = 6$$

$$\pi_2 : 2x - y + z = 3$$

$$\pi_3 : x + 2y - z = 4$$

Find the coordinates of the junction point where all three pipes meet.

Solution:

From π_1: $x + y + z = 6$ ... (1)

From π_2: $2x - y + z = 3$ ... (2)

From π_3: $x + 2y - z = 4$ ... (3)

Add (1) and (3):

$$2x + 3y = 10$$

... (4)

Subtract (1) from (2):

$$x - 2y = -3$$

... (5)

From (5): $x = 2y - 3$

Substitute in (4):

$$2(2y - 3) + 3y = 10$$

$$4y - 6 + 3y = 10$$

$$7y = 16$$

so $y = \frac{16}{7}$

$$x = 2 \times \frac{16}{7} - 3 = \frac{32 - 21}{7} = \frac{11}{7}$$

From (1): $z = 6 - x - y = 6 - \frac{11}{7} - \frac{16}{7} = 6 - \frac{27}{7} = \frac{15}{7}$

Junction point: $\left(\frac{11}{7}, \frac{16}{7}, \frac{15}{7}\right) \approx (1.57, 2.29, 2.14)$ meters

This is where the maintenance valve should be installed.

10.3 PRACTICE PROBLEMS

10.3.1 Vector Operations

1. Given $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$:
   (a) Find $\mathbf{a} + \mathbf{b}$ and $2\mathbf{a} - 3\mathbf{b}$
   (b) Calculate $|\mathbf{a}|$ and $|\mathbf{b}|$
   (c) Find $\mathbf{a} \cdot \mathbf{b}$ and the angle between the vectors
   (d) Calculate $\mathbf{a} \times \mathbf{b}$
2. A surveyor in Fort Portal measures two boundary vectors of a plot:

$$\mathbf{p} = \begin{pmatrix} 40 \\ 30 \\ 0 \end{pmatrix} \text{ m}, \quad \mathbf{q} = \begin{pmatrix} 20 \\ 50 \\ 0 \end{pmatrix} \text{ m}$$

   Calculate the area of the plot.
3. Points A, B, and C have position vectors:

$$\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

   (a) Find $|\overrightarrow{AB}|$ and $|\overrightarrow{BC}|$
   (b) Calculate the angle ABC
   (c) Find the area of triangle ABC
4. A point P divides the line segment joining A(1, 2, 3) and B(7, 8, 9) in ratio 2:1. Find the coordinates of P.
5. In Jinja, two forces act on a machinery part:

$$\mathbf{F_1} = \begin{pmatrix} 50 \\ 30 \\ 0 \end{pmatrix} \text{ N}, \quad \mathbf{F_2} = \begin{pmatrix} -20 \\ 40 \\ 0 \end{pmatrix} \text{ N}$$

   Find the magnitude and direction of the resultant force.

10.3.2 Lines in 3D

6. Write the equation of the line passing through $(2, -1, 3)$ and parallel to $\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ in:
   (a) Vector form
   (b) Parametric form
   (c) Cartesian form
7. Find the equation of the line passing through points $A(1, 2, 3)$ and $B(4, 0, 5)$.
8. Two roads in Masaka have equations:
$$L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$
$$L_2 : \mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$
Find their intersection point.
9. Calculate the perpendicular distance from point $(3, 4, 5)$ to the line:
$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$
10. Determine whether the following lines are parallel, intersecting, or skew:
$$L_1 : \frac{x-1}{2} = \frac{y+1}{3} = \frac{z-2}{1}$$
$$L_2 : \frac{x-3}{4} = \frac{y-2}{6} = \frac{z+1}{2}$$

10.3.3 Planes in 3D

11. Find the Cartesian equation of the plane:
   (a) Passing through $(2, 1, -1)$ with normal $\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$
   (b) Containing points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$
12. Find the angle between planes $x + 2y - 2z = 5$ and $3x - 6y - 2z = 7$.
13. A radio signal tower at coordinates $(10, 20, 80)$ m needs to transmit perpendicular to the ground plane $2x + y - z = 50$. Find the perpendicular distance from the tower to the ground.
14. Find the line of intersection of planes:
$$2x + y - z = 4$$
$$x - y + 2z = 1$$
15. Determine where the line $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ intersects the plane $x + y + z = 6$.
16. Find the angle between the line $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and the plane $2x - y + z = 5$.
17. Three planes in a building design are:
$$x + 2y + z = 4$$
$$2x - y + 3z = 5$$
$$3x + y + 4z = 9$$
Find their common point of intersection.
18. A construction crane's boom can be modeled by the line:
$$\frac{x-5}{3} = \frac{y-2}{4} = \frac{z-10}{-5}$$
At what point does it intersect the ground plane $z = 0$?
19. Determine whether the planes $2x - y + 3z = 4$ and $4x - 2y + 6z = 7$ are parallel, perpendicular, or neither.
20. In a mining operation near Kasese, drill holes are planned along:
$$L : \mathbf{r} = \begin{pmatrix} 100 \\ 50 \\ 200 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$$
The ore deposit lies in plane $x + 2y + z = 350$. Find where the drill will strike the deposit.

10.3.4 Advanced Applications

21. **Agricultural Planning:** A farm in Mukono has corners at $A(0, 0, 100)$, $B(200, 0, 100)$, $C(200, 150, 105)$, and $D(0, 150, 105)$ meters. Calculate:
   (a) The area of the farm
   (b) The angle of slope
   (c) The length of fencing required
22. **Navigation Problem:** A boat travels from Entebbe $(0, 0)$ with velocity $\mathbf{v_b} = 10\mathbf{i} + 5\mathbf{j}$ km/h. The current has velocity $\mathbf{v_c} = -2\mathbf{i} + 3\mathbf{j}$ km/h. Find:
   (a) The resultant velocity
   (b) The boat's position after 2 hours
   (c) The distance traveled

23. **Structural Engineering:** Three support beams in a building frame meet at angles. Their direction vectors are:

$$\mathbf{d_1} = \begin{pmatrix}1\\0\\1\end{pmatrix}, \quad \mathbf{d_2} = \begin{pmatrix}0\\1\\1\end{pmatrix}, \quad \mathbf{d_3} = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

Calculate all three angles between the beams.

24. **Telecommunications:** Two cell towers are at positions:

$$T_1 : (0, 0, 50) \text{ m}, \quad T_2 : (1000, 800, 60) \text{ m}$$

A building is at $(400, 300, 0)$ with height 30 m. Will the building block the direct signal between towers?

25. **Coffee Processing:** In a coffee drying yard in Mbale, coffee beans are spread on a sloped platform. The platform's surface equation is $x + 2y + 20z = 1000$ (measurements in cm). A water drainage point is at $(100, 200, 30)$ cm. Find the shortest distance water must flow to reach the edge of the platform at $z = 0$.

10.4 SUMMARY AND KEY FORMULAS

10.4.1 Vector Operations Summary

Essential Vector Formulas:

Magnitude:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Unit Vector:

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

Dot Product:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\mathbf{a}||\mathbf{b}| \cos\theta$$

Cross Product:

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix}a_2b_3 - a_3b_2\\a_3b_1 - a_1b_3\\a_1b_2 - a_2b_1\end{pmatrix}$$

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

Angle Between Vectors:

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Ratio Theorem:

$$\mathbf{r}_P = \frac{n\mathbf{r}_A + m\mathbf{r}_B}{m + n}$$

10.4.2 Lines Summary

Line Equations:

Vector Form:

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$$

Parametric Form:

$$x = x_0 + \lambda l, \quad y = y_0 + \lambda m, \quad z = z_0 + \lambda n$$

Cartesian Form:

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$$

Distance from Point to Line:

$$d = \frac{|(\mathbf{p} - \mathbf{a}) \times \mathbf{d}|}{|\mathbf{d}|}$$

10.4.3 Planes Summary

Plane Equations:

Vector Form:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{a}) = 0$$

Cartesian Form:

$$Ax + By + Cz = D$$

Distance from Point to Plane:

$$d = \frac{|Ax_1 + By_1 + Cz_1 - D|}{\sqrt{A^2 + B^2 + C^2}}$$

Angle Between Planes:

$$\cos\theta = \frac{|\mathbf{n_1} \cdot \mathbf{n_2}|}{|\mathbf{n_1}||\mathbf{n_2}|}$$

Angle Between Line and Plane:

$$\sin\alpha = \frac{|\mathbf{d} \cdot \mathbf{n}|}{|\mathbf{d}||\mathbf{n}|}$$

10.4.4 Problem-Solving Strategies

1. Always draw a diagram when possible, even for 3D problems
2. Identify what is given and what needs to be found
3. Choose the appropriate form (vector, parametric, or Cartesian)
4. Check your answer by substituting back into original equations
5. Consider the context for real-world problems
6. Use unit vectors to simplify direction problems
7. Remember special cases: parallel, perpendicular, skew lines

10.5 ANSWERS TO SELECTED PROBLEMS

Section 1: Vector Operations

1. (a) $4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$; $3\mathbf{i} - 19\mathbf{j} + 14\mathbf{k}$ (b) $|\mathbf{a}| = \sqrt{29}$; $|\mathbf{b}| = \sqrt{30}$ (c) $\mathbf{a} \cdot \mathbf{b} = -15$; $\theta = 106.2$ (d) $\mathbf{a} \times \mathbf{b} = -16\mathbf{i} + 10\mathbf{j} + 17\mathbf{k}$

2. Area = 1400 m^2 = 0.14 hectares

3. (a) $|\overrightarrow{AB}| = \sqrt{18}$; $|\overrightarrow{BC}| = \sqrt{30}$ (b) $\angle ABC = 116.6$ (c) Area = $\frac{1}{2}\sqrt{234} \approx 7.65$ square units

4. P(5, 6, 7)

5. Resultant: $\begin{pmatrix} 30 \\ 70 \end{pmatrix}$ N; Magnitude = 76.16 N; Direction = 66.8 degrees from x-axis

Section 2: Lines in 3D

6. (a) $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ (b) $x = 2 + 4\lambda$; $y = -1 + \lambda$; $z = 3 - 2\lambda$ (c) $\frac{x-2}{4} = \frac{y+1}{1} = \frac{z-3}{-2}$

8. Intersection at (3, 2, 0)

9. Distance = $\sqrt{6}$ units

10. Lines are parallel

Section 3: Planes in 3D

11. (a) $3x - 2y + 4z = 0$ (b) $x + y + z = 1$

12. $\theta = 81.9$

13. Distance = 33.17 m

15. Intersection at (2, 2, 2)

16. $\alpha = 19.5$

17. Common point: (1, 1, 1)

18. Intersection at (11, 10, 0)

19. Parallel (normals are scalar multiples, but constant terms don't match)

20. Drill strikes at (200, 150, 50) m

End of Topic 17: Vectors

Remember: Vectors are powerful tools for solving real-world problems in Uganda and beyond. Master these concepts and you'll be well-prepared for engineering, physics, computer science, and many other fields!

10.6 More scenario based revision questions with solutions

Item 1. Market Navigation in Kampala

Owino Market in Kampala can be modeled on a coordinate system where each unit represents 10 meters. A vendor at point $A(3, 5)$ needs to deliver goods to point $B(11, 17)$.

(a) Find the displacement vector from A to B.

(b) Calculate the distance the vendor must travel.

(c) Determine the direction of travel (angle from the positive x-axis).

Solution:

(a) The displacement vector is:

$$\vec{AB} = B - A = (11-3)\mathbf{i} + (17-5)\mathbf{j} = 8\mathbf{i} + 12\mathbf{j}$$

(b) The distance (magnitude) is:

$$|\vec{AB}| = \sqrt{8^2 + 12^2} = \sqrt{64 + 144} = \sqrt{208} = 4\sqrt{13} \approx 14.4$$

Actual distance = 14.42×10 = 144.2 meters

(c) The direction angle θ is:

$$\tan\theta = \frac{12}{8} = \frac{3}{2} = 1.5$$

$$\theta = \arctan(1.5) \approx 56.31$$

The vendor travels approximately 56.31 north of east.

Item 2. High-Rise Construction in Kampala

Pearl Tower, a new building in Kampala, has a communication mast. The base of the building is at point $O(0, 0, 0)$, a structural support beam starts at point $A(12, 15, 0)$ on the ground floor and extends to point $B(12, 15, 45)$ at the top, where units are in meters.

(a) Express the position vectors $\vec{OA}$ and $\vec{OB}$.

(b) Find the vector $\vec{AB}$ and its magnitude.

(c) Another beam goes from $C(8, 20, 0)$ to $D(8, 20, 45)$. Verify that $\vec{AB}$ and $\vec{CD}$ are parallel.

Solution:

(a) Position vectors:

$$\vec{OA} = 12\mathbf{i} + 15\mathbf{j} + 0\mathbf{k} = 12\mathbf{i} + 15\mathbf{j}$$

$$\vec{OB} = 12\mathbf{i} + 15\mathbf{j} + 45\mathbf{k}$$

(b) The displacement vector:

$$\vec{AB} = \vec{OB} - \vec{OA} = (12-12)\mathbf{i} + (15-15)\mathbf{j} + (45-0)\mathbf{k} = 45\mathbf{k}$$

$$|\vec{AB}| = \sqrt{0^2 + 0^2 + 45^2} = 45 \text{ meters}$$

(c) For the second beam:

$$\vec{CD} = \vec{OD} - \vec{OC} = (8-8)\mathbf{i} + (20-20)\mathbf{j} + (45-0)\mathbf{k} = 45\mathbf{k}$$

Since $\vec{AB} = \vec{CD} = 45\mathbf{k}$, the vectors are parallel (in fact, equal).

Item 3. Coffee Plantation Work

In a coffee plantation in Mbale, a worker pulls a sack with a force vector $\vec{F} = 50\mathbf{i} + 30\mathbf{j}$ Newtons at an angle, causing a displacement $\vec{d} = 8\mathbf{i} + 3\mathbf{j}$ meters along the ground.

(a) Calculate the work done (Work $= \vec{F} \cdot \vec{d}$).

(b) Find the angle between the force and displacement vectors.

Solution:

(a) The work done is:

$$W = \vec{F} \cdot \vec{d} = (50)(8) + (30)(3) = 400 + 90 = 490 \text{ Joules}$$

(b) Using the dot product formula:

$$\vec{F} \cdot \vec{d} = |\vec{F}||\vec{d}| \cos\theta$$

$$|\vec{F}| = \sqrt{50^2 + 30^2} = \sqrt{2500 + 900} = \sqrt{3400} = 10\sqrt{34}$$

$$|\vec{d}| = \sqrt{8^2 + 3^2} = \sqrt{64 + 9} = \sqrt{73}$$

$$\cos\theta = \frac{490}{10\sqrt{34} \times \sqrt{73}} = \frac{490}{10\sqrt{2482}} = \frac{49}{\sqrt{2482}}$$

$$\theta = \arccos\left(\frac{49}{\sqrt{2482}}\right) \approx 10.3$$

Item 4. Hydroelectric Dam in Jinja

At the Isimba Dam, two force vectors act on a turbine blade: $\vec{F_1} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ kN and position vector $\vec{r} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ meters from the pivot.

(a) Calculate the torque $\vec{\tau} = \vec{r} \times \vec{F_1}$.

(b) Find the magnitude of the torque.

Solution:

(a) The torque (cross product):

$$\vec{\tau} = \vec{r} \times \vec{F_1} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= \mathbf{i}(1\cdot 1 - 2\cdot 3) - \mathbf{j}(4\cdot 1 - 2\cdot 2) + \mathbf{k}(4\cdot 3 - 1\cdot 2)$$

$$= \mathbf{i}(1-6) - \mathbf{j}(4-4) + \mathbf{k}(12-2)$$

$$= -5\mathbf{i} + 0\mathbf{j} + 10\mathbf{k} = -5\mathbf{i} + 10\mathbf{k} \text{ kN·m}$$

(b) The magnitude:

$$|\vec{\tau}| = \sqrt{(-5)^2 + 0^2 + 10^2} = \sqrt{25 + 100} = \sqrt{125} = 5\sqrt{5} \approx$$

Item 5. Land Division in Mukono

A farmer in Mukono has land with corners at $A(2, 3)$ and $B(14, 11)$ km on a map. He wants to divide the land such that point P divides AB in the ratio $3 : 5$ from A.

(a) Find the position of point P.

(b) If a water well is at $W(8, 6)$, find the distance from P to the well.

Solution:

(a) Using the ratio theorem, if P divides AB in ratio $m : n$:

$$\vec{OP} = \frac{n\vec{OA} + m\vec{OB}}{m+n} = \frac{5(2\mathbf{i} + 3\mathbf{j}) + 3(14\mathbf{i} + 11\mathbf{j})}{3+5}$$

$$= \frac{(10\mathbf{i} + 15\mathbf{j}) + (42\mathbf{i} + 33\mathbf{j})}{8} = \frac{52\mathbf{i} + 48\mathbf{j}}{8}$$

$$= 6.5\mathbf{i} + 6\mathbf{j}$$

Therefore, P is at $(6.5, 6)$ km.

(b) Distance from P to W:

$$\vec{PW} = (8 - 6.5)\mathbf{i} + (6 - 6)\mathbf{j} = 1.5\mathbf{i}$$

$$|\vec{PW}| = 1.5 \text{ km} = 1500 \text{ meters}$$

Item 6. Oil Pipeline in Hoima

An oil pipeline in Hoima runs from point $A(10, 20, 5)$ to point $B(30, 50, 15)$ where coordinates are in kilometers.

(a) Write the vector equation of the pipeline.

(b) Write the parametric equations.

(c) Find the position of a monitoring station located at parameter $t = 0.4$.

Solution:

(a) The direction vector is:

$$\vec{AB} = (30-10)\mathbf{i}+(50-20)\mathbf{j}+(15-5)\mathbf{k} = 20\mathbf{i}+30\mathbf{j}+10\mathbf{k}$$

Vector equation:

$$\vec{r} = (10\mathbf{i}+20\mathbf{j}+5\mathbf{k})+t(20\mathbf{i}+30\mathbf{j}+10\mathbf{k})$$

or $\vec{r} = \vec{a} + t\vec{b}$ where $\vec{a} = 10\mathbf{i} + 20\mathbf{j} + 5\mathbf{k}$ and $\vec{b} = 20\mathbf{i} + 30\mathbf{j} + 10\mathbf{k}$

(b) Parametric equations:

$$x = 10 + 20t$$

$$y = 20 + 30t$$

$$z = 5 + 10t$$

(c) At $t = 0.4$:

$$x = 10 + 20(0.4) = 10 + 8 = 18$$

$$y = 20 + 30(0.4) = 20 + 12 = 32$$

$$z = 5 + 10(0.4) = 5 + 4 = 9$$

The monitoring station is at $(18, 32, 9)$ km.

Item 7. Flight Path over Uganda

An aircraft flies from Entebbe at position $(0, 0, 2)$ with velocity vector $6\mathbf{i} + 8\mathbf{j} + \mathbf{k}$ where horizontal units are in km and vertical is in km above sea level.

(a) Write the Cartesian equation of the flight path.

(b) Determine if the plane passes through point $(12, 16, 4)$.

Solution:

(a) Vector equation: $\vec{r} = (0, 0, 2) + t(6, 8, 1)$

Parametric form:

$$x = 6t, \quad y = 8t, \quad z = 2 + t$$

From first equation: $t = \frac{x}{6}$

From second equation: $t = \frac{y}{8}$

From third equation: $t = z - 2$

Cartesian equation:

$$\frac{x}{6} = \frac{y}{8} = \frac{z-2}{1}$$

or equivalently: $\frac{x}{6} = \frac{y}{8} = z - 2$

(b) Check if $(12, 16, 4)$ satisfies the equation:

$$\frac{12}{6} = 2, \quad \frac{16}{8} = 2, \quad 4 - 2 = 2$$

Since all equal 2, yes, the plane passes through this point.

Item 8. Road Intersection in Gulu

Two roads in Gulu can be modeled as lines:

$$L_1 : \vec{r} = (2\mathbf{i} + 3\mathbf{j}) + s(3\mathbf{i} + 4\mathbf{j})$$

$$L_2 : \vec{r} = (5\mathbf{i} + \mathbf{j}) + t(2\mathbf{i} - \mathbf{j})$$

Find the acute angle at which these roads intersect.

Solution:

The direction vectors are:

$$\vec{d}_1 = 3\mathbf{i} + 4\mathbf{j}, \quad \vec{d}_2 = 2\mathbf{i} - \mathbf{j}$$

Using the dot product:

$$\vec{d}_1 \cdot \vec{d}_2 = (3)(2) + (4)(-1) = 6 - 4 = 2$$

$$|\vec{d}_1| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5$$

$$|\vec{d}_2| = \sqrt{2^2 + (-1)^2} = \sqrt{4 + 1} = \sqrt{5}$$

$$\cos\theta = \frac{\vec{d}_1 \cdot \vec{d}_2}{|\vec{d}_1||\vec{d}_2|} = \frac{2}{5\sqrt{5}} = \frac{2}{5\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{25}$$

$$\theta = \arccos\left(\frac{2\sqrt{5}}{25}\right) \approx 79.7$$

The roads intersect at an acute angle of approximately 79.7.

Item 9. Shortest Distance to Railway Line

The Uganda Railway line from Kampala to Malaba follows the path:

$$\vec{r} = (10\mathbf{i} + 5\mathbf{j}) + t(6\mathbf{i} + 8\mathbf{j})$$

where units are in kilometers. A village is located at point $V(25, 30)$.

(a) Find the shortest distance from the village to the railway.

(b) Determine the point on the railway closest to the village.

Solution:

(a) Let $A = (10, 5)$ be a point on the line and $\vec{b} = 6\mathbf{i} + 8\mathbf{j}$ be the direction vector.

$$\vec{AV} = (25-10)\mathbf{i}+(30-5)\mathbf{j} = 15\mathbf{i}+25\mathbf{j}$$

The distance formula is:

$$d = \frac{|\vec{AV} \times \vec{b}|}{|\vec{b}|}$$

For 2D vectors, the cross product magnitude is:

$$|\vec{AV}\times\vec{b}| = |15(8)-25(6)| = |120-150| = |-30| = 30$$

$$|\vec{b}| = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = 10$$

$$d = \frac{30}{10} = 3 \text{ km}$$

(b) To find the closest point, we need t where:

$$\vec{AV} \cdot \vec{b} = |\vec{b}|^2 t$$

$$t = \frac{\vec{AV} \cdot \vec{b}}{|\vec{b}|^2} = \frac{(15)(6) + (25)(8)}{100} = \frac{90 + 200}{100} = \frac{290}{100} = 2.9$$

Closest point:

$$P = (10, 5)+2.9(6, 8) = (10+17.4, 5+23.2) = (27.4, 28.2)$$

Item 10. Utility Lines Intersection

Two power lines in Kampala are represented by:

$$L_1 : \frac{x-2}{3} = \frac{y-1}{2} = \frac{z-4}{1}$$

$$L_2 : \frac{x-5}{2} = \frac{y-4}{-1} = \frac{z-2}{3}$$

Determine if these lines intersect and find the point of intersection if they do.

Solution:

Convert to parametric form:

L_1: $x = 2 + 3s$, $y = 1 + 2s$, $z = 4 + s$

L_2: $x = 5 + 2t$, $y = 4 - t$, $z = 2 + 3t$

At intersection:

$$2 + 3s = 5 + 2t \quad \Rightarrow \quad 3s - 2t = 3 \quad ...(i)$$

$$1 + 2s = 4 - t \quad \Rightarrow \quad 2s + t = 3 \quad ...(ii)$$

$$4 + s = 2 + 3t \quad \Rightarrow \quad s - 3t = -2 \quad ...(iii)$$

From (ii): $t = 3 - 2s$

Substitute into (i):

$$3s - 2(3 - 2s) = 3$$

$$3s - 6 + 4s = 3$$

$$7s = 9$$

$$s = \frac{9}{7}$$

Then: $t = 3 - 2(\frac{9}{7}) = 3 - \frac{18}{7} = \frac{21-18}{7} = \frac{3}{7}$

Check in (iii):

$$\frac{9}{7} - 3(\frac{3}{7}) = \frac{9}{7} - \frac{9}{7} = 0 \neq -2$$

The lines do not intersect (they are skew lines).

Item 11. Electrical Lines in Mbarara

Two electrical cables in Mbarara follow paths:

$$L_1 : \vec{r} = (1, 2, 3) + s(2, 1, 1)$$

$$L_2 : \vec{r} = (0, 3, 1) + t(1, 1, 2)$$

(a) Verify that these lines are skew.

(b) Find the shortest distance between the lines.

Solution:

(a) Check if parallel: $\vec{d_1} = (2, 1, 1)$ and $\vec{d_2} = (1, 1, 2)$

$\frac{2}{1} \neq \frac{1}{1} \neq \frac{1}{2}$, so not parallel.

Check if intersecting:

$$1 + 2s = 0 + t \quad \Rightarrow \quad 2s - t = -1$$

$$2 + s = 3 + t \quad \Rightarrow \quad s - t = 1$$

$$3 + s = 1 + 2t \quad \Rightarrow \quad s - 2t = -2$$

From second equation: $s = 1 + t$

Substitute into first: $2(1+t)-t = -1 \Rightarrow 2+t=-1 \Rightarrow t=-3$

Then $s = 1+(-3) = -2$

Check in third: $-2-2(-3) = -2+6 = 4 \neq -2$

Lines don't intersect and aren't parallel, so they are skew.

(b) Distance between skew lines:

$$d = \frac{|(\vec{a_2}-\vec{a_1})\cdot(\vec{d_1}\times\vec{d_2})|}{|\vec{d_1}\times\vec{d_2}|}$$

$$\vec{a_2}-\vec{a_1} = (0-1, 3-2, 1-3) = (-1, 1, -2)$$

$$\vec{d_1}\times\vec{d_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \mathbf{i}(2-1)-\mathbf{j}(4-1)+\mathbf{k}(2-1)$$

$$= \mathbf{i} - 3\mathbf{j} + \mathbf{k} = (1, -3, 1)$$

$$(\vec{a_2}-\vec{a_1})\cdot(\vec{d_1}\times\vec{d_2}) = (-1)(1)+(1)(-3)+(-2)(1) = -1-3-2 = -6$$

$$|\vec{d_1}\times\vec{d_2}| = \sqrt{1+9+1} = \sqrt{11}$$

$$d = \frac{|-6|}{\sqrt{11}} = \frac{6}{\sqrt{11}} = \frac{6\sqrt{11}}{11} \approx 1.81 \text{ units}$$

$$\vec{r} = (2,1,3) + s(2,2,2) + t(-1,4,1)$$

Or in component form:

$$x = 2 + 2s - t$$

$$y = 1 + 2s + 4t$$

$$z = 3 + 2s + t$$

(b) To find the Cartesian equation, find the normal vector:

$$\vec{n} = \vec{AB}\times\vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 2 \\ -1 & 4 & 1 \end{vmatrix}$$

$$= \mathbf{i}(2-8) - \mathbf{j}(2-(-2)) + \mathbf{k}(8-(-2))$$

$$= -6\mathbf{i} - 4\mathbf{j} + 10\mathbf{k}$$

Simplify by dividing by -2: $\vec{n} = (3, 2, -5)$

Plane equation: $3(x-2)+2(y-1)-5(z-3) = 0$

$$3x - 6 + 2y - 2 - 5z + 15 = 0$$

$$3x + 2y - 5z + 7 = 0$$

Check $D(3,3,4)$:

$$3(3)+2(3)-5(4)+7 = 9+6-20+7 = 2 \neq 0$$

Point D does not lie on the plane.

Item 12. Roof Design in Kampala

A building roof in Kampala passes through points $A(2,1,3)$, $B(4,3,5)$, and $C(1,5,4)$ meters.

(a) Find the parametric equation of the plane containing the roof.

(b) Determine if point $D(3,3,4)$ lies on the same plane.

Solution:

(a) Find two direction vectors in the plane:

$$\vec{AB} = (4-2, 3-1, 5-3) = (2,2,2)$$

$$\vec{AC} = (1-2, 5-1, 4-3) = (-1,4,1)$$

Parametric equation:

$$\vec{r} = \vec{OA} + s\vec{AB} + t\vec{AC}$$

Item 13. Mining Slope in Kasese

In Kasese's copper mine, a sloped surface is perpendicular to vector $\vec{n} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ and passes through point $P(1,2,-1)$.

(a) Find the Cartesian equation of the plane.

(b) Find where the plane intersects the z-axis.

Solution:

(a) Using $\vec{n} = (2,-3,6)$ and point $P(1,2,-1)$:

$$2(x-1)-3(y-2)+6(z-(-1)) = 0$$

$$2x - 2 - 3y + 6 + 6z + 6 = 0$$

$$2x - 3y + 6z + 10 = 0$$

(b) At z-axis intersection, $x = 0$ and $y = 0$:

$$2(0) - 3(0) + 6z + 10 = 0$$

$$6z = -10$$

$$z = -\frac{10}{6} = -\frac{5}{3}$$

The plane intersects the z-axis at $(0, 0, -\frac{5}{3})$.

Item 14. Borehole Drilling in Karamoja

A drilling company in Karamoja is drilling a borehole along the line:

$$\vec{r} = (5, 3, 10) + t(1, 2, -3)$$

The water table forms a plane with equation $2x + y - 2z + 6 = 0$, where all measurements are in meters.

(a) Find the point where the borehole meets the water table.

(b) Calculate the depth drilled from the starting point to the water table.

Solution:

(a) Parametric equations of the line:

$$x = 5 + t, \quad y = 3 + 2t, \quad z = 10 - 3t$$

Substitute into plane equation:

$$2(5+t) + (3+2t) - 2(10-3t) + 6 = 0$$

$$10 + 2t + 3 + 2t - 20 + 6t + 6 = 0$$

$$10t - 1 = 0$$

$$t = 0.1$$

Intersection point:

$$x = 5 + 0.1 = 5.1$$

$$y = 3 + 2(0.1) = 3.2$$

$$z = 10 - 3(0.1) = 9.7$$

The borehole meets the water table at $(5.1, 3.2, 9.7)$ meters.

(b) Distance drilled:

$$d = |t| \times |\vec{d}| = 0.1 \times \sqrt{1^2 + 2^2 + (-3)^2}$$

$$= 0.1 \times \sqrt{1 + 4 + 9} = 0.1 \times \sqrt{14} \approx 0.374 \text{ meters}$$

Item 15. Valley Formation in Rwenzori Mountains

Two mountain slopes in the Rwenzori range are modeled by planes:

$$P_1 : x + 2y - z = 5$$

$$P_2 : 2x - y + 3z = 1$$

(a) Find the line of intersection (the valley floor).

(b) Express your answer in vector form.

Solution:

(a) The direction vector of the line is perpendicular to both normals:

$$\vec{n_1} = (1, 2, -1), \quad \vec{n_2} = (2, -1, 3)$$

$$\vec{d} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= \mathbf{i}(6 - 1) - \mathbf{j}(3 - (-2)) + \mathbf{k}(-1 - 4)$$

$$= 5\mathbf{i} - 5\mathbf{j} - 5\mathbf{k} = 5(1, -1, -1)$$

Simplify: $\vec{d} = (1, -1, -1)$

Find a point on the line. Let $z = 0$:

$$x + 2y = 5 \quad \text{...(i)}$$

$$2x - y = 1 \quad \text{...(ii)}$$

From (i): $x = 5 - 2y$

Substitute into (ii): $2(5 - 2y) - y = 1$

$$10 - 4y - y = 1$$

$$-5y = -9$$

$$y = \frac{9}{5}$$

$$x = 5 - 2(\frac{9}{5}) = 5 - \frac{18}{5} = \frac{7}{5}$$

Point on line: $(\frac{7}{5}, \frac{9}{5}, 0)$

(b) Vector equation:

$$\vec{r} = (\frac{7}{5}, \frac{9}{5}, 0) + t(1, -1, -1)$$

Or: $\vec{r} = \frac{1}{5}(7, 9, 0) + t(1, -1, -1)$

Item 16. Utility Network Junction in Kampala

Three utility networks (water, electricity, sewage) in Kampala intersect at a common point. Their paths are modeled by planes:

$$P_1 : x + y + z = 6$$

$$P_2 : 2x - y + 3z = 14$$

$$P_3 : x + 2y - z = 2$$

Find the coordinates of the junction point.

Solution:

Solve the system of three equations:

$$x + y + z = 6 \quad ...(1)$$

$$2x - y + 3z = 14 \quad ...(2)$$

$$x + 2y - z = 2 \quad ...(3)$$

From (1): $z = 6 - x - y$ ...(4)

Substitute (4) into (2):

$$2x - y + 3(6 - x - y) = 14$$

$$2x - y + 18 - 3x - 3y = 14$$

$$-x - 4y = -4$$

$$x + 4y = 4 \quad ...(5)$$

Substitute (4) into (3):

$$x + 2y - (6 - x - y) = 2$$

$$x + 2y - 6 + x + y = 2$$

$$2x + 3y = 8 \quad ...(6)$$

From (5): $x = 4 - 4y$

Substitute into (6):

$$2(4 - 4y) + 3y = 8$$

$$8 - 8y + 3y = 8$$

$$-5y = 0$$

$$y = 0$$

Then: $x = 4 - 4(0) = 4$

And: $z = 6 - 4 - 0 = 2$

The junction point is at $(4, 0, 2)$.

Verification:

$$P_1 : 4 + 0 + 2 = 6 \checkmark$$

$$P_2 : 2(4) - 0 + 3(2) = 8 + 6 = 14 \checkmark$$

$$P_3 : 4 + 2(0) - 2 = 2 \checkmark$$

Item 17. Solar Panel Installation in Soroti

A solar panel support beam in Soroti follows the line:

$$\vec{r} = (2, 1, 0) + t(1, 2, 2)$$

The ground is modeled by the plane $x + y + z = 0$.

(a) Find the angle between the beam and the ground.

(b) What is the angle of elevation of the solar panel?

Solution:

(a) Direction vector of line: $\vec{d} = (1, 2, 2)$

Normal to plane: $\vec{n} = (1, 1, 1)$

The angle α between line and plane is related to angle β between line and normal by:

$$\alpha = 90 - \beta$$

$$\sin\alpha = \cos\beta = \frac{|\vec{d} \cdot \vec{n}|}{|\vec{d}||\vec{n}|}$$

$$\vec{d}\cdot\vec{n} = (1)(1)+(2)(1)+(2)(1) = 1+2+2 = 5$$

$$|\vec{d}| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{1 + 4 + 4} = 3$$

$$|\vec{n}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\sin\alpha = \frac{5}{3\sqrt{3}} = \frac{5\sqrt{3}}{9}$$

$$\alpha = \arcsin\left(\frac{5\sqrt{3}}{9}\right) \approx 73.2$$

(b) The angle of elevation is the same as the angle between the line and plane: approximately 73.2.

This is a good angle for solar panels in Uganda (near the equator).

Item 18. Roof Valley Design in Fort Portal

A house in Fort Portal has two roof sections modeled by planes:

$$P_1 : 2x + 3y - z = 10$$

$$P_2 : x - y + 2z = 5$$

Find the angle between the two roof sections (dihedral angle).

Solution:

The normal vectors are:

$$\vec{n_1} = (2, 3, -1), \quad \vec{n_2} = (1, -1, 2)$$

The angle θ between planes equals the angle between their normals:

$$\cos\theta = \frac{|\vec{n_1} \cdot \vec{n_2}|}{|\vec{n_1}||\vec{n_2}|}$$

$$\vec{n_1} \cdot \vec{n_2} = (2)(1)+(3)(-1)+(-1)(2) = 2-3-2 = -3$$

$$|\vec{n_1}| = \sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$$

$$|\vec{n_2}| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{1 + 1 + 4} = \sqrt{6}$$

$$\cos\theta = \frac{|-3|}{\sqrt{14} \times \sqrt{6}} = \frac{3}{\sqrt{84}} = \frac{3}{2\sqrt{21}} = \frac{3\sqrt{21}}{42} = \frac{\sqrt{21}}{14}$$

$$\theta = \arccos\left(\frac{\sqrt{21}}{14}\right) \approx 71.1$$

The dihedral angle between the roof sections is approximately 71.1.

Item 19. Telecommunications Tower in Masaka

A telecommunications tower in Masaka is to be built at point $T(4, 5, 0)$. A restricted flight zone is defined by the plane $3x+4y-12z = 26$, where heights are in meters.

(a) Find the perpendicular distance from the tower base to the plane.

(b) If the tower is 50m tall, will it penetrate the restricted zone?

Solution:

(a) Distance from point (x_0, y_0, z_0) to plane $ax + by + cz = d$:

$$D = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

For point $T(4, 5, 0)$ and plane $3x+4y-12z = 26$:

$$D = \frac{|3(4) + 4(5) - 12(0) - 26|}{\sqrt{3^2 + 4^2 + (-12)^2}}$$

$$= \frac{|12 + 20 - 0 - 26|}{\sqrt{9 + 16 + 144}}$$

$$= \frac{|6|}{\sqrt{169}} = \frac{6}{13} \approx 0.46 \text{ meters}$$

(b) The tower extends from $(4, 5, 0)$ to $(4, 5, 50)$.

Check if top point satisfies the plane inequality:

$$3(4)+4(5)-12(50) = 12+20-600 = -568$$

Since $-568 < 26$, and the base point gives $32 > 26$, the tower crosses the plane and will penetrate the restricted zone.

The tower penetrates at height z where:

$$3(4) + 4(5) - 12z = 26$$

$$32 - 12z = 26$$

$$12z = 6$$

$$z = 0.5 \text{ meters}$$

The tower penetrates the restricted zone at a height of 0.5 meters and above.

Item 20. Quarry Operations in Tororo

A limestone quarry in Tororo has:

- Drilling site at $A(10, 15, 30)$ meters
- Processing plant at $B(50, 35, 10)$ meters
- Conveyor belt following: $\vec{r} = (20, 20, 25) + t(6, 4, -3)$
- Ground level plane: $z = 0$

(a) Find the distance from the drilling site to the processing plant.

(b) Determine the shortest distance from point A to the conveyor belt.

(c) Find where the conveyor belt reaches ground level.

(d) Calculate the angle the conveyor belt makes with the ground.

Solution:

(a) Distance from A to B:

$$\vec{AB} = (50-10, 35-15, 10-30) = (40, 20, -20)$$

$$|\vec{AB}| = \sqrt{40^2 + 20^2 + (-20)^2} = \sqrt{1600 + 400 + 400} = \sqrt{2400}$$

$$= \sqrt{400 \times 6} = 20\sqrt{6} \approx 48.99 \text{ meters}$$

(b) For distance from A to line $\vec{r} = \vec{a} + t\vec{d}$:

$$\vec{a} = (20, 20, 25), \quad \vec{d} = (6, 4, -3)$$

$$\vec{PA} = A - \vec{a} = (10-20, 15-20, 30-25) = (-10, -5, 5)$$

$$\vec{PA} \times \vec{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -10 & -5 & 5 \\ 6 & 4 & -3 \end{vmatrix}$$

$$= \mathbf{i}(15-20) - \mathbf{j}(30-30) + \mathbf{k}(-40+30)$$

$$= -5\mathbf{i} + 0\mathbf{j} - 10\mathbf{k} = (-5, 0, -10)$$

$$|\vec{PA} \times \vec{d}| = \sqrt{25 + 0 + 100} = \sqrt{125} = 5\sqrt{5}$$

$$|\vec{d}| = \sqrt{36 + 16 + 9} = \sqrt{61}$$

$$\text{Distance} = \frac{5\sqrt{5}}{\sqrt{61}} = \frac{5\sqrt{5} \times \sqrt{61}}{61} = \frac{5\sqrt{305}}{61} \approx 1.43 \text{ meters}$$

(c) At ground level, $z = 0$:

$$25 - 3t = 0$$

$$t = \frac{25}{3}$$

Position:

$$x = 20 + 6(\frac{25}{3}) = 20 + 50 = 70$$

$$y = 20+4(\frac{25}{3}) = 20+\frac{100}{3} = \frac{160}{3} \approx 53.33$$

Conveyor reaches ground at approximately $(70, 53.33, 0)$ meters.

(d) Angle with ground (xy-plane with normal $\vec{n} = (0, 0, 1)$):

$$\sin\alpha = \frac{|\vec{d} \cdot \vec{n}|}{|\vec{d}||\vec{n}|} = \frac{|(6)(0) + (4)(0) + (-3)(1)|}{\sqrt{61} \times 1} = \frac{3}{\sqrt{61}}$$

$$\alpha = \arcsin\left(\frac{3}{\sqrt{61}}\right) \approx 22.6$$

The conveyor belt descends at an angle of approximately 22.6 below horizontal.

Key Concepts Demonstrated

These problems illustrate the application of vectors to real situations in Uganda, covering:

- Vector operations in 2D and 3D spaces (magnitude, direction, components)
- Dot and cross products with physical applications
- Ratio theorem for position division
- Vector, parametric, and Cartesian equations of lines
- Properties of lines (angles, distances, intersections, skew lines)
- Parametric and Cartesian equations of planes
- Line-plane and plane-plane intersections
- Angles between geometric objects
- Perpendicular distances

Students are encouraged to visualize these scenarios and understand how mathematical vectors model real physical situations in Uganda's infrastructure, agriculture, mining, construction, and telecommunications industries.

Chapter 11

Differentiation 2: Advanced Techniques

Chapter Overview

In this chapter, we extend our understanding of differentiation to more complex functions including trigonometric, exponential, logarithmic, and their inverses. We will explore Maclaurin's series for approximating functions and develop advanced curve sketching techniques. These powerful tools have applications in engineering, physics, economics, and various fields relevant to Uganda's development needs.

Competency

By the end of this chapter, you will be able to apply differentiation techniques to non-algebraic functions, Maclaurin's series, and irrational curves to analyse real-world phenomena in various fields.

11.1 Trigonometric Derivatives

11.1.1 Introduction

Trigonometric functions are essential in modelling periodic phenomena such as alternating current in electrical systems, seasonal variations in agricultural output, and wave motion in physics. Understanding their derivatives is crucial for optimization and analysis.

11.1.2 Derivatives from First Principles

Recall that the derivative of a function $f(x)$ from first principles is:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

11.1.3 Derivative of $\sin x$

Theorem: $\frac{d}{dx}(\sin x) = \cos x$

Proof:

Let $f(x) = \sin x$. Then:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\ &= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} \\ &= \sin x(0) + \cos x(1) \\ &= \cos x \end{aligned}$$

Note: We use the standard limits $\lim_{h\to 0} \frac{\sin h}{h} = 1$ and $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$.

11.1.4 Derivative of $\cos x$

Theorem: $\frac{d}{dx}(\cos x) = -\sin x$

Proof:

Let $f(x) = \cos x$. Then:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\ &= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\ &= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\ &= \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h} \\ &= \cos x(0) - \sin x(1) \\ &= -\sin x \end{aligned}$$

11.1.5 Derivative of $\tan x$

Theorem: $\frac{d}{dx}(\tan x) = \sec^2 x$

Proof:

Using the quotient rule with $\tan x = \frac{\sin x}{\cos x}$:

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)\\ &= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x}\\ &= \sec^2 x\end{aligned}$$

11.1.6 Summary of Trigonometric Derivatives

The following are the standard derivatives of trigonometric functions:

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

11.1.7 Derivatives of Inverse Trigonometric Functions

The derivatives of the main inverse trigonometric functions are:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \quad |x| < 1$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, \quad |x| < 1$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

Proof for $\sin^{-1} x$:

Let $y = \sin^{-1} x$, then $x = \sin y$ where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Differentiating both sides with respect to x:

$$1 = \cos y \frac{dy}{dx}$$

Therefore:

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

Since $\cos^2 y + \sin^2 y = 1$, we have $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$ (positive in the given range).

Thus:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

11.1.8 Example 1: Kampala-Entebbe Expressway Traffic Flow

The traffic volume $V(t)$ on the Kampala-Entebbe Expressway varies throughout the day and can be modelled by:

$$V(t) = 5000 + 3000 \sin\left(\frac{\pi(t-6)}{12}\right)$$

where t is the time in hours (0 to 24) and V is the number of vehicles per hour.

Find:

(a) The rate of change of traffic volume at 8:00 AM
(b) The time when traffic volume is increasing most rapidly

Solution:

(a) First, find $V'(t)$:

$$V'(t) = 3000 \cdot \cos\left(\frac{\pi(t-6)}{12}\right) \cdot \frac{\pi}{12} = \frac{250\pi \cos\left(\frac{\pi(t-6)}{12}\right)}{1}$$

At $t = 8$ (8:00 AM):

$$V'(8) = 250\pi \cos\left(\frac{\pi(2)}{12}\right) = 250\pi \cos\left(\frac{\pi}{6}\right) = 250\pi \cdot \frac{\sqrt{3}}{2}$$

$$V'(8) = 125\sqrt{3}\pi \approx 680 \text{ vehicles per hour per hour}$$

The traffic volume is increasing at approximately 680 vehicles per hour squared at 8:00 AM.

(b) Traffic volume increases most rapidly when $V'(t)$ is maximum. Since $V'(t)$ is maximum when the cosine term equals 1:

$$\cos\left(\frac{\pi(t-6)}{12}\right) = 1$$

This occurs when:

$$\frac{\pi(t-6)}{12} = 0$$

$$t - 6 = 0$$

$$t = 6$$

Therefore, traffic volume increases most rapidly at 6:00 AM (sunrise time when people start commuting).

11.1.9 Example 2: Solar Panel Optimization in Kampala

A solar panel installation company in Kampala models the angle of elevation θ of the sun as:

$$\theta(t) = 30 + 50\sin\left(\frac{\pi t}{12}\right)$$

where t is hours after sunrise (0 to 12) and θ is in degrees.

Find the rate of change of the sun's elevation at:

(a) 3 hours after sunrise
(b) 6 hours after sunrise

Solution:

First, find $\theta'(t)$:

$$\theta'(t) = 50 \cdot \cos\left(\frac{\pi t}{12}\right) \cdot \frac{\pi}{12} = \frac{25\pi}{6}\cos\left(\frac{\pi t}{12}\right)$$

(a) At $t = 3$:

$$\theta'(3) = \frac{25\pi}{6}\cos\left(\frac{\pi \cdot 3}{12}\right) = \frac{25\pi}{6}\cos\left(\frac{\pi}{4}\right)$$

$$= \frac{25\pi}{6} \cdot \frac{\sqrt{2}}{2} = \frac{25\pi\sqrt{2}}{12} \approx 9.25 \text{ degrees per hour}$$

(b) At $t = 6$:

$$\theta'(6) = \frac{25\pi}{6}\cos\left(\frac{\pi \cdot 6}{12}\right) = \frac{25\pi}{6}\cos\left(\frac{\pi}{2}\right)$$

$$= \frac{25\pi}{6} \cdot 0 = 0 \text{ degrees per hour}$$

This makes sense as the sun reaches its highest point at midday (6 hours after sunrise), so the rate of change of elevation is zero at this moment.

11.1.10 Example 3: Lake Victoria Water Level

The height h (in meters) of water in Lake Victoria at a particular monitoring station varies with time according to:

$$h(t) = 1134 + 0.5\sin\left(\frac{2\pi t}{365}\right)$$

where t is the number of days from January 1st.

Find:

(a) The rate of change of water level on day 90 (approximately March 31st)
(b) When the water level is rising most rapidly

Solution:

(a) First, find $h'(t)$:

$$h'(t) = 0.5 \cdot \cos\left(\frac{2\pi t}{365}\right) \cdot \frac{2\pi}{365} = \frac{\pi}{365}\cos\left(\frac{2\pi t}{365}\right)$$

At $t = 90$:

$$h'(90) = \frac{\pi}{365}\cos\left(\frac{180\pi}{365}\right) = \frac{\pi}{365}\cos(0.494\pi)$$

$$\approx \frac{\pi}{365} \times (-0.156) \approx -0.0013 \text{ meters per day}$$

The water level is decreasing at about 1.3 mm per day on March 31st.

(b) The water level rises most rapidly when $h'(t)$ is maximum:

$$\cos\left(\frac{2\pi t}{365}\right) = 1$$

$$\frac{2\pi t}{365} = 0 \text{ or } 2\pi$$

$$t = 0 \text{ or } 365$$

This corresponds to January 1st or December 31st, the beginning of the year.

11.1.11 Exercises

Exercise 1: Differentiate the following functions:

(a) $f(x) = 3\sin x + 4\cos x$
(b) $g(x) = \sin^2 x$
(c) $h(x) = \tan(3x^2)$
(d) $k(x) = x^2 \sin x$
(e) $m(x) = \frac{\sin x}{1+\cos x}$
(f) $n(x) = \sin^{-1}(2x)$

Exercise 2: A telecommunications tower in Kampala needs to maintain its signal angle $\alpha(t) = 45+10\sin(2t)$ degrees, where t is time in hours. Find $\alpha'(t)$ and determine when the rate of change is zero.

Exercise 3: The temperature T in degrees Celsius in Kampala during a day can be modelled by:

$$T(t) = 25 + 5\cos\left(\frac{\pi(t-14)}{12}\right)$$

where t is time in hours (0 to 24). Find the rate of temperature change at 10:00 AM and 2:00 PM.

11.2 Exponential and Logarithmic Functions

11.2.1 Introduction

Exponential and logarithmic functions are fundamental in modelling growth and decay processes such as population growth, compound interest in banking, radioactive decay, and the spread of diseases. Their unique properties make them invaluable in applied mathematics.

11.2.2 The Natural Exponential Function

The natural exponential function is defined as:

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

where $e \approx 2.71828$ is Euler's number, also known as the base of natural logarithms.

Theorem: The derivative of the exponential function e^x is itself:

$$\frac{d}{dx}(e^x) = e^x$$

Proof:

From first principles:

$$\begin{aligned}\frac{d}{dx}(e^x) &= \lim_{h\to0}\frac{e^{x+h}-e^x}{h}\\ &= \lim_{h\to0}\frac{e^x\cdot e^h - e^x}{h}\\ &= e^x\lim_{h\to0}\frac{e^h-1}{h}\\ &= e^x\cdot 1\\ &= e^x\end{aligned}$$

The limit $\lim_{h\to0}\frac{e^h-1}{h} = 1$ is a standard result.

This remarkable property means that the rate of change of e^x is equal to e^x itself at every point.

11.2.3 General Exponential Functions

For the exponential function a^x where $a > 0$ and $a \neq 1$:

$$\frac{d}{dx}(a^x) = a^x\ln a$$

Proof:

Since $a^x = e^{x\ln a}$, using the chain rule:

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x\ln a}) = e^{x\ln a}\cdot\ln a = a^x\ln a$$

11.2.4 The Natural Logarithm

The natural logarithm $\ln x$ is the inverse function of e^x. That is:

$$y = \ln x \iff e^y = x, \quad x > 0$$

Theorem: The derivative of the natural logarithm is:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad x > 0$$

Proof:

Let $y = \ln x$, then $x = e^y$.

Differentiating both sides with respect to x:

$$1 = e^y\frac{dy}{dx}$$

Therefore:

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

11.2.5 General Logarithmic Functions

For logarithms to any base $a > 0$, $a \neq 1$:

$$\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$$

This follows from the change of base formula: $\log_a x = \frac{\ln x}{\ln a}$

11.2.6 Logarithmic Differentiation

For complex products, quotients, or powers, logarithmic differentiation is a useful technique.

Steps for Logarithmic Differentiation:

1. Take the natural logarithm of both sides of the equation
2. Use logarithm properties to simplify (product rule: $\ln(AB) = \ln A + \ln B$, quotient rule: $\ln(A/B) = \ln A - \ln B$, power rule: $\ln(A^n) = n\ln A$)

3. Differentiate both sides implicitly with respect to x
4. Solve for $\frac{dy}{dx}$
5. Express the answer in terms of x (substitute back y if needed)

Example: Find $\frac{dy}{dx}$ if $y = x^x$, $x > 0$.

Solution:

Step 1: Take logarithms of both sides:

$$\ln y = \ln(x^x)$$

Step 2: Simplify using logarithm properties:

$$\ln y = x \ln x$$

Step 3: Differentiate both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

Step 4: Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y(\ln x + 1)$$

Step 5: Substitute back $y = x^x$:

$$\frac{dy}{dx} = x^x(\ln x + 1)$$

11.2.7 Example 4: Banking and Compound Interest in Uganda

Centenary Bank offers a savings account with continuous compounding at 8% per annum. If you invest UGX 10,000,000, the amount $A(t)$ after t years is given by the formula:

$$A(t) = 10,000,000e^{0.08t}$$

Find:

(a) The rate at which your investment is growing after 5 years
(b) When the investment will be growing at UGX 1,000,000 per year

Solution:

(a) Find $A'(t)$:

$$A'(t) = 10,000,000 \cdot 0.08 \cdot e^{0.08t} = 800,000e^{0.08t}$$

At $t = 5$:

$$A'(5) = 800,000e^{0.08\times5} = 800,000e^{0.4}$$

$$= 800,000 \times 1.492 = \text{UGX } 1,193,600 \text{ per year}$$

After 5 years, the investment is growing at approximately UGX 1,193,600 per year.

(b) We need to find when $A'(t) = 1,000,000$:

$$800,000e^{0.08t} = 1,000,000$$

$$e^{0.08t} = \frac{1,000,000}{800,000} = 1.25$$

Taking natural logarithms:

$$0.08t = \ln(1.25)$$

$$t = \frac{\ln(1.25)}{0.08} = \frac{0.2231}{0.08} \approx 2.79 \text{ years}$$

The investment will be growing at UGX 1,000,000 per year after approximately 2 years and 9 months.

11.2.8 Example 5: Coffee Production Growth in Uganda

Uganda's coffee production (in thousands of 60kg bags) can be modelled by:

$$P(t) = 4500e^{0.03t}$$

where t is years since 2020.

Find:

(a) The rate of production growth in 2025
(b) The percentage rate of growth

Solution:

(a) First, find $P'(t)$:

$$P'(t) = 4500 \cdot 0.03 \cdot e^{0.03t} = 135e^{0.03t}$$

At $t = 5$ (year 2025):

$$P'(5) = 135e^{0.03\times5} = 135e^{0.15}$$

$$= 135 \times 1.162 = 156.9 \text{ thousand bags per year}$$

In 2025, Uganda's coffee production is growing at approximately 157,000 bags per year.

(b) The percentage rate of growth is:

$$\frac{P'(t)}{P(t)} \times 100\% = \frac{135e^{0.03t}}{4500e^{0.03t}} \times 100\%$$

$$= \frac{135}{4500} \times 100\% = 3\%$$

The percentage rate of growth is 3% per year. Note that this is constant for exponential growth, which is a characteristic property of such models.

11.2.9 Example 6: Population Growth in Kampala

The population of Kampala metropolitan area (in millions) is modelled by:

$$P(t) = 2.5e^{0.04t}$$

where t is years since 2020. Find when the population will be growing at 120,000 people per year.

Solution:

First, find $P'(t)$:

$$P'(t) = 2.5 \cdot 0.04 \cdot e^{0.04t} = 0.1e^{0.04t}$$

We need $P'(t) = 0.12$ (since population is in millions, 120,000 people = 0.12 million):

$$0.1e^{0.04t} = 0.12$$

$$e^{0.04t} = 1.2$$

Taking natural logarithms:

$$0.04t = \ln(1.2)$$

$$t = \frac{\ln(1.2)}{0.04} = \frac{0.1823}{0.04} \approx 4.56 \text{ years}$$

The population will be growing at 120,000 people per year approximately 4.6 years after 2020, which is around mid-2024.

11.2.10 Example 7: Drug Concentration at Mulago Hospital

A patient at Mulago Hospital is given a drug. The concentration $C(t)$ of the drug in the bloodstream (in mg/L) is given by:

$$C(t) = 10e^{-0.2t}$$

where t is time in hours after administration.

Find:

(a) The rate at which the concentration is changing after 2 hours

(b) When the concentration is decreasing at 1 mg/L per hour

Solution:

(a) Find $C'(t)$:

$$C'(t) = 10 \cdot (-0.2) \cdot e^{-0.2t} = -2e^{-0.2t}$$

At $t = 2$:

$$C'(2) = -2e^{-0.2\times 2} = -2e^{-0.4}$$

$$= -2 \times 0.6703 = -1.34 \text{ mg/L per hour}$$

After 2 hours, the concentration is decreasing at approximately 1.34 mg/L per hour. The negative sign indicates a decrease.

(b) We need $C'(t) = -1$ (negative because decreasing):

$$-2e^{-0.2t} = -1$$

$$e^{-0.2t} = 0.5$$

Taking natural logarithms:

$$-0.2t = \ln(0.5)$$

$$t = \frac{\ln(0.5)}{-0.2} = \frac{-0.6931}{-0.2} \approx 3.47 \text{ hours}$$

The concentration is decreasing at exactly 1 mg/L per hour after approximately 3.5 hours.

11.2.11 Exercises

Exercise 4: Differentiate the following functions:

(a) $f(x) = e^{3x}$
(b) $g(x) = e^{x^2+2x}$
(c) $h(x) = \ln(5x + 3)$
(d) $k(x) = x^2e^x$
(e) $m(x) = \frac{e^x}{x^2}$
(f) $n(x) = \ln\left(\frac{x^2+1}{x-1}\right)$
(g) $p(x) = 3^x$
(h) $q(x) = \log_{10}(x)$

Exercise 5: Use logarithmic differentiation to find $\frac{dy}{dx}$ for:

(a) $y = (x + 1)^x$
(b) $y = \frac{x^2(x-1)^3}{(2x+1)^4}$
(c) $y = x^{\sin x}$
(d) $y = (\sin x)^x$

Exercise 6: The value of a vehicle in Uganda depreciates according to:

$$V(t) = 50,000,000e^{-0.15t}$$

where V is in UGX and t is in years.

(a) Find the rate of depreciation after 3 years
(b) What is the percentage rate of depreciation?
(c) When will the vehicle be depreciating at UGX 5,000,000 per year?

Exercise 7: The number of mobile phone subscribers N (in millions) in Uganda grows according to:

$$N(t) = 15e^{0.06t}$$

where t is years since 2020. Find the growth rate in 2028 and the percentage growth rate.

11.3 Maclaurin's Series

11.3.1 Introduction

Maclaurin's series provides a powerful method for approximating functions using polynomials. This technique is essential in numerical analysis, computer algorithms, and situations where we need to calculate function values efficiently without using calculators or complex computations.

11.3.2 Maclaurin's Theorem

Theorem (Maclaurin's Series):

If a function $f(x)$ has derivatives of all orders at $x = 0$, then the function can be expressed as an infinite series:

$$f(x) = f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\cdots+\frac{x^n}{n!}f^{(n)}(0)+\cdots$$

or more compactly:

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

where $f^{(n)}(0)$ denotes the n-th derivative of f evaluated at $x = 0$, and $n! = n\times(n-1)\times(n-2)\times\cdots\times2\times1$ (with $0! = 1$).

11.3.3 Standard Maclaurin Series

The following are some important standard Maclaurin series expansions:

1. Exponential function:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

2. Sine function:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

3. Cosine function:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

4. Natural logarithm:

$$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}, \quad |x| < 1$$

5. Binomial series:

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$$

For any real number n and $|x| < 1$.

11.3.4 Deriving Maclaurin Series

Example 8: Derive the Maclaurin series for $f(x) = e^x$.

Solution:

We need to find $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$, etc.

For $f(x) = e^x$:

$$\begin{aligned} f(x) &= e^x & f(0) &= e^0 = 1 \\ f'(x) &= e^x & f'(0) &= e^0 = 1 \\ f''(x) &= e^x & f''(0) &= e^0 = 1 \\ f'''(x) &= e^x & f'''(0) &= e^0 = 1 \\ f^{(4)}(x) &= e^x & f^{(4)}(0) &= e^0 = 1 \end{aligned}$$

We observe that for all values of n, $f^{(n)}(0) = 1$.

Therefore, using Maclaurin's formula:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Example 9: Derive the Maclaurin series for $f(x) = \sin x$.

Solution:

For $f(x) = \sin x$:

$$\begin{aligned} f(x) &= \sin x & f(0) &= \sin 0 = 0 \\ f'(x) &= \cos x & f'(0) &= \cos 0 = 1 \\ f''(x) &= -\sin x & f''(0) &= -\sin 0 = 0 \\ f'''(x) &= -\cos x & f'''(0) &= -\cos 0 = -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= \sin 0 = 0 \\ f^{(5)}(x) &= \cos x & f^{(5)}(0) &= \cos 0 = 1 \end{aligned}$$

The pattern repeats every four derivatives: 0, 1, 0, -1, 0, 1, 0, -1, ...

Therefore:

$$\sin x = 0 + x + \frac{0 \cdot x^2}{2!} + \frac{(-1) \cdot x^3}{3!} + \frac{0 \cdot x^4}{4!} + \frac{1 \cdot x^5}{5!} + \cdots$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Example 10: Derive the Maclaurin series for $f(x) = \cos x$ up to the x^6 term.

Solution:

For $f(x) = \cos x$:

$$\begin{aligned} f(x) &= \cos x & f(0) &= \cos 0 = 1 \\ f'(x) &= -\sin x & f'(0) &= -\sin 0 = 0 \\ f''(x) &= -\cos x & f''(0) &= -\cos 0 = -1 \\ f'''(x) &= \sin x & f'''(0) &= \sin 0 = 0 \\ f^{(4)}(x) &= \cos x & f^{(4)}(0) &= \cos 0 = 1 \\ f^{(5)}(x) &= -\sin x & f^{(5)}(0) &= -\sin 0 = 0 \\ f^{(6)}(x) &= -\cos x & f^{(6)}(0) &= -\cos 0 = -1 \end{aligned}$$

Therefore:

$$\cos x = 1 + 0 \cdot x + \frac{(-1) \cdot x^2}{2!} + \frac{0 \cdot x^3}{3!} + \frac{1 \cdot x^4}{4!} + \frac{0 \cdot x^5}{5!} + \frac{(-1) \cdot x^6}{6!} + \cdots$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

11.3.5 Series Approximation and Truncation

In practice, we often truncate the series after a few terms to get a polynomial approximation of the function. The more terms we include, the more accurate our approximation becomes, especially for values of x close to 0.

Example 11: Approximate $e^{0.5}$ using the first 5 terms of the Maclaurin series.

Solution:

Using $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

For $x = 0.5$, taking the first 5 terms:

$$\begin{aligned} e^{0.5} &\approx 1 + 0.5 + \frac{(0.5)^2}{2} + \frac{(0.5)^3}{6} + \frac{(0.5)^4}{24} \\ &= 1 + 0.5 + \frac{0.25}{2} + \frac{0.125}{6} + \frac{0.0625}{24} \\ &= 1 + 0.5 + 0.125 + 0.0208333 + 0.0026042 \\ &\approx 1.6484 \end{aligned}$$

The actual value is $e^{0.5} \approx 1.6487$, so our approximation is accurate to 3 decimal places.

Example 12: Approximate $\sin(0.3)$ using the first 4 non-zero terms of the Maclaurin series.

Solution:

Using $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

For $x = 0.3$:

$$\begin{aligned} \sin(0.3) &\approx 0.3 - \frac{(0.3)^3}{6} + \frac{(0.3)^5}{120} - \frac{(0.3)^7}{5040} \\ &= 0.3 - \frac{0.027}{6} + \frac{0.00243}{120} - \frac{0.0002187}{5040} \\ &= 0.3 - 0.0045 + 0.00002025 - 0.000000043 \\ &\approx 0.2955 \end{aligned}$$

The actual value is $\sin(0.3) \approx 0.2955$, which matches our approximation to 4 decimal places.

11.3.6 Example 13: Mobile Money Transaction Processing

A mobile money platform (like MTN MoMo or Airtel Money) models transaction processing efficiency $T(x)$ where x is a small delay parameter (in seconds):

$$T(x) = e^{-x}$$

For small delays, expand $T(x)$ as a Maclaurin series up to the x^3 term and use it to estimate the processing efficiency when $x = 0.1$ seconds.

Solution:

For $f(x) = e^{-x}$, we need to find the derivatives at $x = 0$:

$$\begin{aligned} f(x) &= e^{-x} & f(0) &= e^0 = 1 \\ f'(x) &= -e^{-x} & f'(0) &= -e^0 = -1 \\ f''(x) &= e^{-x} & f''(0) &= e^0 = 1 \\ f'''(x) &= -e^{-x} & f'''(0) &= -e^0 = -1 \end{aligned}$$

Therefore:

$$T(x) = 1 + (-1)x + \frac{1 \cdot x^2}{2!} + \frac{(-1) \cdot x^3}{3!} + \cdots$$
$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$$

For $x = 0.1$:

$$T(0.1) \approx 1 - 0.1 + \frac{(0.1)^2}{2} - \frac{(0.1)^3}{6}$$
$$= 1 - 0.1 + 0.005 - 0.000167$$
$$\approx 0.9048$$

This means the processing efficiency is about 90.48% of the base rate when there is a 0.1 second delay.

11.3.7 Example 14: Solar Energy Calculation

The angle of incidence effect on solar panel efficiency in Kampala can be modelled using:

$$E(\theta) = \cos\theta$$

where θ is in radians. For small angles, approximate this using a Maclaurin series up to θ^4.

Solution:

Using the Maclaurin series for cosine:

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

Up to θ^4:

$$E(\theta) \approx 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$$

For $\theta = 0.2$ radians (about 11.5 degrees):

$$E(0.2) \approx 1 - \frac{(0.2)^2}{2} + \frac{(0.2)^4}{24}$$
$$= 1 - \frac{0.04}{2} + \frac{0.0016}{24}$$
$$= 1 - 0.02 + 0.000067$$
$$\approx 0.9801$$

This suggests the panel operates at approximately 98.01% efficiency at this angle.

11.3.8 Example 15: Approximating $\ln(1.1)$

Use the Maclaurin series for $\ln(1+x)$ to approximate $\ln(1.1)$.

Solution:

The Maclaurin series for $\ln(1+x)$ is:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

For $\ln(1.1)$, we have $x = 0.1$:

$$\ln(1.1) \approx 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4}$$
$$= 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4}$$
$$= 0.1 - 0.005 + 0.000333 - 0.000025$$
$$\approx 0.0953$$

The actual value is $\ln(1.1) \approx 0.0953$, which matches our approximation.

11.3.9 Exercises

Exercise 8: Derive the Maclaurin series for:

(a) $f(x) = e^{-x}$ (up to x^4 term)
(b) $f(x) = \ln(1-x)$ (up to x^5 term)
(c) $f(x) = (1+x)^{-1}$ (up to x^4 term)
(d) $f(x) = (1+x)^{1/2}$ (up to x^3 term)

Exercise 9: Use the first 4 terms of the appropriate Maclaurin series to approximate:

(a) $e^{-0.2}$
(b) $\cos(0.4)$
(c) $\ln(1.2)$
(d) $\sqrt{1.1}$

Compare your answers with calculator values.

Exercise 10: The population growth rate in a Ugandan district is modelled by:

$$r(t) = 0.03e^{0.01t}$$

where t is in years. Use a Maclaurin series (up to t^3) to approximate $r(t)$ for small t. What is the approximate growth rate at $t = 0.5$ years?

Exercise 11: A telecommunications signal strength S varies with distance d (in km) according to:

$$S(d) = \frac{1}{1+d} = (1+d)^{-1}$$

(a) Find the Maclaurin series for $S(d)$ up to the d^4 term
(b) Use this to approximate the signal strength at $d = 0.2$ km
(c) Compare with the exact value

11.4 Graphing and Curve Sketching

11.4.1 Introduction

Accurate curve sketching is essential for visualizing mathematical relationships and understanding function behavior. This skill combines knowledge of derivatives, limits, asymptotes, and function properties to create a complete picture of how a function behaves.

11.4.2 Key Features of Curves

When sketching curves, we systematically analyze the following features:

1. Domain and Range

- Domain: The set of all possible x values for which the function is defined
- Range: The set of all possible y values that the function can take

2. Intercepts

- x-intercepts: Points where the curve crosses the x-axis (set $y = 0$)
- y-intercept: Point where the curve crosses the y-axis (set $x = 0$)

3. Asymptotes

- Vertical asymptotes: Occur where the denominator equals zero (for rational functions)
- Horizontal asymptotes: Found by examining $\lim_{x\to\pm\infty} f(x)$
- Oblique (slant) asymptotes: Occur when degree of numerator is one more than denominator

4. Critical Points

- Points where $f'(x) = 0$ or $f'(x)$ is undefined
- These are potential locations for local maxima, local minima, or points of inflection

5. Increasing and Decreasing Intervals

- If $f'(x) > 0$ on an interval, the function is increasing
- If $f'(x) < 0$ on an interval, the function is decreasing

6. Concavity

- If $f''(x) > 0$ on an interval, the function is concave up (curves upward)
- If $f''(x) < 0$ on an interval, the function is concave down (curves downward)

7. Points of Inflection

- Points where concavity changes
- Occur where $f''(x) = 0$ or $f''(x)$ is undefined (but must check for sign change)

8. Symmetry

- Even function: $f(-x) = f(x)$ (symmetric about y-axis)
- Odd function: $f(-x) = -f(x)$ (symmetric about origin)
- Periodic function: $f(x+T) = f(x)$ for some period T

11.4.3 Procedure for Curve Sketching

Step-by-Step Approach:

Step 1: Find the domain and identify any restrictions on x.

Step 2: Find intercepts.

- Set $y = 0$ to find x-intercepts
- Set $x = 0$ to find the y-intercept

Step 3: Identify asymptotes.

- Vertical: where denominator = 0
- Horizontal: find $\lim_{x\to\pm\infty} f(x)$
- Oblique: use polynomial division if applicable

Step 4: Find $f'(x)$ and locate critical points (where $f'(x) = 0$ or undefined).

Step 5: Determine intervals of increase/decrease using sign analysis of $f'(x)$.

Step 6: Find $f''(x)$ and determine concavity using sign analysis.

Step 7: Identify inflection points (where $f''(x) = 0$ and concavity changes).

Step 8: Check for symmetry or periodicity if applicable.

Step 9: Plot key points and sketch the curve, ensuring all features are incorporated.

11.4.4 Example 16: Sketching a Rational Function

Sketch the curve $y = \frac{x^2}{x-1}$.

Solution:

Step 1: Domain

The function is undefined when $x - 1 = 0$, i.e., when $x = 1$.

Domain: $x \in \mathbb{R}, x \neq 1$ or $(-\infty, 1) \cup (1, \infty)$

Step 2: Intercepts

y-intercept: Set $x = 0$

$$y = \frac{0^2}{0-1} = \frac{0}{-1} = 0$$

So $(0, 0)$ is the y-intercept.

x-intercepts: Set $y = 0$

$$\frac{x^2}{x-1} = 0 \implies x^2 = 0 \implies x = 0$$

So $(0, 0)$ is the only intercept.

Step 3: Asymptotes

Vertical asymptote: $x = 1$ (where denominator = 0)

For horizontal/oblique asymptote, we divide:

$$y = \frac{x^2}{x-1}$$

Using polynomial long division:

$$x^2 = (x-1)(x+1) + 1$$

$$\frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}$$

As $x \to \pm\infty$, $\frac{1}{x-1} \to 0$, so $y \to x + 1$.

Therefore, $y = x + 1$ is an oblique asymptote.

Step 4: First Derivative

Using the quotient rule:

$$y' = \frac{2x(x-1) - x^2 \cdot 1}{(x-1)^2} = \frac{2x^2 - 2x - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}$$

$$= \frac{x(x-2)}{(x-1)^2}$$

Critical points: $y' = 0$ when $x = 0$ or $x = 2$

Step 5: Increasing/Decreasing

Test intervals:

- $x < 0$: Choose $x = -1$: $y' = \frac{(-1)(-3)}{4} = \frac{3}{4} > 0$ (increasing)
- $0 < x < 1$: Choose $x = 0.5$: $y' = \frac{(0.5)(-1.5)}{0.25} = -3 < 0$ (decreasing)
- $1 < x < 2$: Choose $x = 1.5$: $y' = \frac{(1.5)(-0.5)}{0.25} = -3 < 0$ (decreasing)
- $x > 2$: Choose $x = 3$: $y' = \frac{(3)(1)}{4} = 0.75 > 0$ (increasing)

Local maximum at $x = 0$: $y(0) = 0$, so $(0, 0)$ is a local maximum.

Local minimum at $x = 2$: $y(2) = \frac{4}{1} = 4$, so $(2, 4)$ is a local minimum.

Step 6-7: Concavity (Optional for basic sketch)

$$y'' = \frac{d}{dx}\left[\frac{x(x-2)}{(x-1)^2}\right]$$

This calculation is complex but shows the curve's curvature changes.

Step 8: Sketch

Plot the key features:

- Intercept at origin (0, 0) which is also a local maximum
- Vertical asymptote at $x = 1$
- Oblique asymptote $y = x + 1$
- Local minimum at (2, 4)
- Function increases on $(-\infty, 0)$ and $(2, \infty)$
- Function decreases on $(0, 1)$ and $(1, 2)$

11.4.5 Example 17: Sketching an Exponential Function

Sketch the curve $y = xe^{-x}$.

Solution:

Step 1: Domain

All real numbers: $x \in \mathbb{R}$ or $(-\infty, \infty)$

Step 2: Intercepts

y-intercept: Set $x = 0$

$$y = 0 \cdot e^0 = 0$$

x-intercept: Set $y = 0$

$$xe^{-x} = 0 \implies x = 0$$

(since $e^{-x} \neq 0$ for any x)

So $(0, 0)$ is the only intercept.

Step 3: Asymptotes

As $x \to -\infty$: $y = xe^{-x} \to -\infty \cdot \infty = -\infty$ (no horizontal asymptote on left)

As $x \to \infty$: $y = xe^{-x} = \frac{x}{e^x} \to 0$ (exponential dominates)

Therefore, $y = 0$ (the x-axis) is a horizontal asymptote as $x \to \infty$.

No vertical asymptotes.

Step 4: First Derivative

Using the product rule:

$$y' = e^{-x} + x \cdot (-e^{-x}) = e^{-x}(1 - x)$$

Critical point: $y' = 0$ when $1 - x = 0$, so $x = 1$

At $x = 1$: $y = 1 \cdot e^{-1} = \frac{1}{e} \approx 0.368$

Step 5: Increasing/Decreasing

Since $e^{-x} > 0$ for all x, the sign of y' depends on $(1 - x)$:

- $x < 1$: $y' > 0$ (increasing)
- $x > 1$: $y' < 0$ (decreasing)

Maximum at $(1, e^{-1}) \approx (1, 0.368)$

Step 6: Second Derivative

$$\begin{aligned} y'' &= \frac{d}{dx}[e^{-x}(1 - x)] \\ &= -e^{-x}(1 - x) + e^{-x}(-1) \\ &= e^{-x}[-(1 - x) - 1] \\ &= e^{-x}(-1 + x - 1) \\ &= e^{-x}(x - 2) \end{aligned}$$

Step 7: Inflection Point

$y'' = 0$ when $x = 2$

At $x = 2$: $y = 2e^{-2} \approx 0.271$

Concavity:

- $x < 2$: $y'' < 0$ (concave down)
- $x > 2$: $y'' > 0$ (concave up)

Inflection point at $(2, 2e^{-2}) \approx (2, 0.271)$

Step 8: Sketch

Key features:

- Passes through origin
- Maximum at $(1, 0.368)$
- Inflection point at $(2, 0.271)$
- Horizontal asymptote $y = 0$ as $x \to \infty$
- Goes to $-\infty$ as $x \to -\infty$

11.4.6 Example 18: Bwindi Tourism Revenue

The monthly tourism revenue R (in millions of UGX) at Bwindi Impenetrable Forest depends on the number of gorilla permits sold x according to:

$$R(x) = \frac{50x}{x + 10}, \quad 0 \le x \le 100$$

Sketch this curve and identify:

(a) The maximum possible revenue
(b) The rate of increase at different permit levels

Solution:

Domain: $0 \le x \le 100$ (restricted by context)

Intercepts: At $x = 0$: $R(0) = 0$, so $(0, 0)$ is the intercept.

Asymptote: As $x \to \infty$:

$$R(x) = \frac{50x}{x + 10} = \frac{50}{1 + \frac{10}{x}} \to 50$$

Horizontal asymptote: $R = 50$ (theoretical maximum revenue is 50 million UGX)

First Derivative:

$$R'(x) = \frac{50(x + 10) - 50x \cdot 1}{(x + 10)^2} = \frac{50x + 500 - 50x}{(x + 10)^2} = \frac{500}{(x + 10)^2}$$

Since $R'(x) > 0$ for all $x \ge 0$, revenue is always increasing (no critical points in domain).

However, $R'(x)$ decreases as x increases, meaning the rate of increase slows down.

Second Derivative:

$$R''(x) = \frac{d}{dx}\left[\frac{500}{(x + 10)^2}\right] = 500 \cdot \frac{-2}{(x + 10)^3} = \frac{-1000}{(x + 10)^3}$$

Since $R''(x) < 0$ for all $x \ge 0$, the curve is concave down everywhere.

Analysis:

(a) At $x = 100$ permits: $R(100) = \frac{5000}{110} \approx 45.45$ million UGX

This is the maximum revenue within the practical range, though theoretically it approaches 50 million.

(b) Rate of increase at different levels:

- At $x = 10$: $R'(10) = \frac{500}{400} = 1.25$ million UGX per permit
- At $x = 50$: $R'(50) = \frac{500}{3600} \approx 0.139$ million UGX per permit
- At $x = 90$: $R'(90) = \frac{500}{10000} = 0.05$ million UGX per permit

The rate of revenue increase diminishes significantly as more permits are sold, exhibiting diminishing returns.

11.4.7 Example 19: Matoke Production Efficiency

A farmer's matoke (plantain) production efficiency E varies with fertilizer amount f (in kg per acre):

$$E(f) = \frac{f^2}{f^2 + 100}, \quad f \geq 0$$

Sketch this curve and determine optimal fertilizer strategy.

Solution:

Domain: $f \geq 0$ (cannot have negative fertilizer)

Intercept: At $f = 0$: $E(0) = \frac{0}{100} = 0$

Asymptote: As $f \to \infty$:

$$E(f) = \frac{f^2}{f^2 + 100} = \frac{1}{1 + \frac{100}{f^2}} \to 1$$

Horizontal asymptote: $E = 1$ (100% efficiency is the theoretical maximum)

First Derivative:

$$E'(f) = \frac{2f(f^2 + 100) - f^2 \cdot 2f}{(f^2 + 100)^2} = \frac{2f^3 + 200f - 2f^3}{(f^2 + 100)^2} = \frac{200f}{(f^2 + 100)^2}$$

Since $E'(f) > 0$ for all $f > 0$, efficiency always increases with fertilizer.

Analysis of Practical Optimal Point:

While efficiency always increases, the rate of increase diminishes. Let's evaluate at key points:

- At $f = 10$: $E(10) = \frac{100}{200} = 0.5$ (50% efficiency)
- At $f = 20$: $E(20) = \frac{400}{500} = 0.8$ (80% efficiency)
- At $f = 30$: $E(30) = \frac{900}{1000} = 0.9$ (90% efficiency)
- At $f = 50$: $E(50) = \frac{2500}{2600} \approx 0.962$ (96.2% efficiency)

Optimal Strategy:

Beyond $f = 30$ kg/acre, additional fertilizer provides minimal benefit (only 6.2% increase from 30 to 50 kg). For cost-effectiveness, farmers should use approximately 25-30 kg/acre to achieve 90% efficiency.

11.4.8 Example 20: Logarithmic Function

Sketch the curve $y = \frac{\ln x}{x}$ for $x > 0$.

Solution:

Domain: $x > 0$ (logarithm restriction)

Intercepts: $y = 0$ when $\ln x = 0$, so $x = 1$. Intercept at $(1, 0)$.

Asymptotes: As $x \to 0^+$: $\ln x \to -\infty$ and $x \to 0^+$, so $y \to -\infty$ (vertical asymptote at $x = 0$)

As $x \to \infty$: Using L'Hôpital's rule or the fact that $\ln x$ grows slower than x:

$$\lim_{x \to \infty} \frac{\ln x}{x} = 0$$

Horizontal asymptote: $y = 0$ as $x \to \infty$

First Derivative:

$$y' = \frac{\frac{1}{x} \cdot x - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$$

Critical point: $y' = 0$ when $1 - \ln x = 0$, so $\ln x = 1$, giving $x = e \approx 2.718$

At $x = e$: $y = \frac{\ln e}{e} = \frac{1}{e} \approx 0.368$

Increasing/Decreasing:

- $x < e$: $\ln x < 1$, so $y' > 0$ (increasing)
- $x > e$: $\ln x > 1$, so $y' < 0$ (decreasing)

Maximum at $(e, 1/e) \approx (2.718, 0.368)$

11.4.9 Exercises

Exercise 12: Sketch the following curves, showing all key features:

(a) $y = \frac{x^2+1}{x^2-4}$
(b) $y = x^3 - 3x^2 + 2$
(c) $y = x\sqrt{4 - x^2}$

(d) $y = e^{-x^2}$
(e) $y = x - \ln x$

Exercise 13: The cost C (in thousands of UGX) of producing x tonnes of coffee is:

$$C(x) = \frac{x^2 + 10x + 50}{x + 1}$$

for $x > 0$.

(a) Sketch the curve
(b) Find the average cost per tonne: $\bar{C}(x) = \frac{C(x)}{x}$
(c) Determine when average cost is minimized

Exercise 14: The efficiency η of a hydroelectric power station on the Nile varies with water flow rate r (in thousands of cubic meters per second):

$$\eta(r) = r(10 - r)e^{-0.1r}, \quad 0 \leq r \leq 20$$

(a) Sketch this curve
(b) Find the flow rate that maximizes efficiency
(c) What is the maximum efficiency?

Exercise 15: Sketch the curve $y = \frac{x^3}{x^2-9}$ showing:

(a) All asymptotes
(b) Critical points and their nature
(c) Inflection points (if any)
(d) Intervals of increase/decrease
(e) Concavity

Chapter Summary

In this chapter, we have explored advanced differentiation techniques and their applications:

1. Trigonometric Derivatives

- Derivatives of $\sin x$, $\cos x$, $\tan x$ and other trigonometric functions from first principles
- Derivatives of inverse trigonometric functions: $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$
- Applications in periodic phenomena such as traffic flow, solar panel optimization, and water level variations

2. Exponential and Logarithmic Functions

- The remarkable property that $\frac{d}{dx}(e^x) = e^x$
- Derivatives of general exponential functions: $\frac{d}{dx}(a^x) = a^x \ln a$
- Derivative of natural logarithm: $\frac{d}{dx}(\ln x) = \frac{1}{x}$
- Logarithmic differentiation technique for complex expressions
- Applications in compound interest, population growth, drug concentration, and depreciation

3. Maclaurin's Series

- Representation of functions as infinite polynomial series
- Standard series for e^x, $\sin x$, $\cos x$, $\ln(1 + x)$, and $(1 + x)^n$
- Series truncation for practical approximations
- Applications in mobile money processing, solar energy calculations, and numerical computations

4. Curve Sketching

- Systematic approach to analyzing functions: domain, intercepts, asymptotes, critical points
- Using first derivative to determine increasing/decreasing intervals and extrema
- Using second derivative to determine concavity and inflection points
- Applications in tourism revenue modeling, agricultural efficiency, and cost optimization

These techniques are fundamental to advanced mathematics and have direct applications in engineering, economics, natural sciences, and technology sectors relevant to Uganda's development. They enable us to model, analyze, and optimize real-world systems ranging from banking to agriculture, from telecommunications to environmental monitoring.

Review Exercises

Exercise 16: The altitude h (in meters) of a Ugandan Airways flight is modelled by:

$$h(t) = 10000 + 2000 \sin\left(\frac{\pi t}{30}\right) + 500 \cos\left(\frac{\pi t}{15}\right)$$

where t is time in minutes.

(a) Find the rate of altitude change at $t = 10$ minutes
(b) When is the aircraft climbing most rapidly in the first 30 minutes?

Exercise 17: The value of land (in millions of UGX per acre) in Kampala suburbs grows according to:

$$V(t) = 50e^{0.12t}$$

where t is years from 2020.

(a) What is the growth rate in 2025?

(b) When will land be appreciating at 10 million UGX per acre per year?
(c) What is the percentage rate of appreciation?

Exercise 18: Use Maclaurin series to:

(a) Approximate $\sin(0.2)$ to 4 decimal places using the first 3 non-zero terms
(b) Show that for small x: $(1+x)^3 \approx 1+3x+3x^2$ and verify for $x = 0.1$
(c) Approximate $\sqrt{1.2}$ using the $(1+x)^{1/2}$ expansion with $x = 0.2$

Exercise 19: A tea processing plant's efficiency E depends on temperature T (in °C):

$$E(T) = \frac{T(100-T)}{50}, \quad 20 \le T \le 100$$

(a) Sketch this curve showing all key features
(b) Find the optimal operating temperature
(c) What is the efficiency at this temperature?
(d) What is the efficiency at the boundary temperatures?

Exercise 20: The concentration of fertilizer in an irrigation system decreases according to:

$$C(t) = 100e^{-0.15t}$$

where C is in mg/L and t is in hours.

(a) Find the rate of concentration decrease after 3 hours
(b) When will the concentration be decreasing at 5 mg/L per hour?
(c) Use a Maclaurin series (up to t^3) to approximate $C(t)$ for small t
(d) Use the approximation to estimate $C(0.5)$

Exercise 21: Differentiate using appropriate techniques:

(a) $y = \sin(e^x)$
(b) $y = e^{\sin x}$
(c) $y = \ln(\cos x)$
(d) $y = x^{\cos x}$
(e) $y = \tan^{-1}(e^x)$
(f) $y = e^x \sin x$

Exercise 22: For the function $f(x) = x^2e^{-x}$:

(a) Find all critical points
(b) Determine intervals of increase and decrease
(c) Find inflection points
(d) Sketch the curve showing all key features

Exercise 23: The monthly profit P (in millions of UGX) from a mobile phone business depends on the number of phones sold n (in thousands):

$$P(n) = 5ne^{-0.1n}, \quad n \ge 0$$

(a) Find the number of phones that maximizes profit
(b) What is the maximum profit?
(c) Sketch the profit function

Exercise 24: The spread of a disease in a community of 10,000 people is modelled by:

$$N(t) = \frac{10000}{1+9999e^{-0.5t}}$$

where N is the number infected after t days.

(a) How many are initially infected?
(b) Find the rate of infection after 5 days
(c) When is the disease spreading most rapidly?
(d) What is the long-term prediction for total infections?

Exercise 25: Use logarithmic differentiation to find $\frac{dy}{dx}$ for:

(a) $y = \frac{(x+1)^3(x-2)^4}{(x+3)^5}$
(b) $y = x^{e^x}$
(c) $y = (\sin x)^{\cos x}$

11.5 More scenario based revision question with solutions

Question 1: Kampala-Entebbe Expressway Bridge Design

Engineers designing a bridge support arch for the Kampala-Entebbe Expressway model the arch using the function $y = 20\sin\left(\frac{\pi x}{40}\right)$ metres, where x is the horizontal distance from one end in metres.

(a) Find the rate of change of the arch height at $x = 10$ m.
(b) Determine the maximum slope of the arch.
(c) At what horizontal distance does the arch reach its maximum height?

Solution:

(a) Given: $y = 20\sin\left(\frac{\pi x}{40}\right)$
$\frac{dy}{dx} = 20 \cdot \cos\left(\frac{\pi x}{40}\right) \cdot \frac{\pi}{40} = \frac{\pi}{2}\cos\left(\frac{\pi x}{40}\right)$
At $x = 10$: $\frac{dy}{dx} = \frac{\pi}{2}\cos\left(\frac{\pi}{4}\right) = \frac{\pi}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\pi\sqrt{2}}{4} \approx 1.11$ m per metre

(b) Maximum slope occurs when $\left|\frac{dy}{dx}\right|$ is maximum:
$\frac{dy}{dx} = \frac{\pi}{2}\cos\left(\frac{\pi x}{40}\right)$
Maximum value is $\frac{\pi}{2}$ when $\cos\left(\frac{\pi x}{40}\right) = 1$, i.e., when $x = 0$ or $x = 40$ m.
Maximum slope $= \frac{\pi}{2} \approx 1.57$ m per metre

(c) Maximum height occurs when $\frac{dy}{dx} = 0$:
$\frac{\pi}{2}\cos\left(\frac{\pi x}{40}\right) = 0$
$\cos\left(\frac{\pi x}{40}\right) = 0 \Rightarrow \frac{\pi x}{40} = \frac{\pi}{2}$
$x = 20$ m
Maximum height: $y = 20\sin\left(\frac{\pi}{2}\right) = 20$ m

Question 2: Karuma Hydroelectric Dam Water Flow

The water level in a section of the Karuma Dam fluctuates according to the function $h(t) = 5 + 3\cos\left(\frac{\pi t}{6}\right)$ metres, where t is time in hours after midnight.

(a) Find the rate of change of water level at 3:00 AM.
(b) At what time is the water level decreasing most rapidly?
(c) Calculate the maximum rate of water level decrease.

Solution:

(a) $h(t) = 5 + 3\cos\left(\frac{\pi t}{6}\right)$
$h'(t) = -3\sin\left(\frac{\pi t}{6}\right)\cdot\frac{\pi}{6} = -\frac{\pi}{2}\sin\left(\frac{\pi t}{6}\right)$
At $t = 3$ hours: $h'(3) = -\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) = -\frac{\pi}{2} \approx -1.57$ m/hour
The water level is decreasing at 1.57 m/hour.

(b) Water level decreases most rapidly when $h'(t)$ is minimum (most negative):
$h'(t) = -\frac{\pi}{2}\sin\left(\frac{\pi t}{6}\right)$ is minimum when $\sin\left(\frac{\pi t}{6}\right) = 1$
$\frac{\pi t}{6} = \frac{\pi}{2} \Rightarrow t = 3$ hours (3:00 AM)
Also at $t = 15$ hours (3:00 PM), etc.

(c) Maximum rate of decrease: $|h'(3)| = \frac{\pi}{2} \approx 1.57$ m/hour

Question 3: Telecommunications Tower Signal Strength

MTN Uganda's signal strength from a tower at angle θ from horizontal is modeled by $S(\theta) = 100\sin(2\theta)\cos(\theta)$ units, where $0 \le \theta \le \frac{\pi}{2}$.

(a) Find the rate of change of signal strength with respect to angle.
(b) Determine the angle that maximizes signal strength.

Solution:

(a) $S(\theta) = 100\sin(2\theta)\cos(\theta)$
Using product rule: $S'(\theta) = 100[\sin(2\theta)\cdot(-\sin\theta) + \cos(\theta)\cdot 2\cos(2\theta)]$
$= 100[-\sin(2\theta)\sin\theta + 2\cos\theta\cos(2\theta)]$
$= 100[2\cos\theta\cos(2\theta) - \sin(2\theta)\sin\theta]$

(b) For maximum, set $S'(\theta) = 0$:
$2\cos\theta\cos(2\theta) - \sin(2\theta)\sin\theta = 0$
Using $\sin(2\theta) = 2\sin\theta\cos\theta$:
$2\cos\theta\cos(2\theta) - 2\sin^2\theta\cos\theta = 0$
$2\cos\theta[\cos(2\theta) - \sin^2\theta] = 0$
Either $\cos\theta = 0$ (giving $\theta = \frac{\pi}{2}$) or $\cos(2\theta) = \sin^2\theta$
Using $\cos(2\theta) = 1 - 2\sin^2\theta$:
$1 - 2\sin^2\theta = \sin^2\theta \Rightarrow 3\sin^2\theta = 1 \Rightarrow \sin\theta = \frac{1}{\sqrt{3}}$
$\theta = \arcsin\left(\frac{1}{\sqrt{3}}\right) \approx 0.615$ radians ≈ 35.3

Question 4: Solar Panel Angle Optimization

A solar farm in Soroti has panels with energy output $E(\theta) = 250\sin\theta + 100\sin(2\theta)$ watts, where θ is the angle from horizontal.

Find the optimal angle for maximum energy output.

Solution:

$E(\theta) = 250\sin\theta + 100\sin(2\theta)$

$E'(\theta) = 250\cos\theta + 200\cos(2\theta)$

For maximum, set $E'(\theta) = 0$:

$250\cos\theta + 200\cos(2\theta) = 0$

Using $\cos(2\theta) = 2\cos^2\theta - 1$:

$250\cos\theta + 200(2\cos^2\theta - 1) = 0$

$250\cos\theta + 400\cos^2\theta - 200 = 0$

$400\cos^2\theta + 250\cos\theta - 200 = 0$

$8\cos^2\theta + 5\cos\theta - 4 = 0$

Using quadratic formula with $u = \cos\theta$:

$u = \frac{-5\pm\sqrt{25+128}}{16} = \frac{-5\pm\sqrt{153}}{16}$

$u = \frac{-5+12.37}{16} \approx 0.461$ (taking positive value)

$\theta = \arccos(0.461) \approx 1.089$ radians ≈ 62.4

11.6 Exponential and Logarithmic Derivatives

Question 5: Coffee Production Growth

Uganda's coffee export volume is modeled by $V(t) = 500e^{0.08t}$ thousand bags, where t is years since 2020.

(a) Find the rate of growth in 2025.
(b) When will the rate of growth reach 50 thousand bags per year?

Solution:

(a) $V(t) = 500e^{0.08t}$
$V'(t) = 500 \cdot 0.08e^{0.08t} = 40e^{0.08t}$
In 2025, $t = 5$:
$V'(5) = 40e^{0.4} = 40 \times 1.4918 \approx 59.67$ thousand bags per year
(b) Set $V'(t) = 50$:
$40e^{0.08t} = 50$
$e^{0.08t} = 1.25$
$0.08t = \ln(1.25)$
$t = \frac{\ln(1.25)}{0.08} = \frac{0.2231}{0.08} \approx 2.79$ years
This occurs approximately in late 2022 (about 2.79 years after 2020).

Question 6: Mobile Money Transaction Volume

Airtel Uganda's mobile money transactions follow $T(t) = 2000e^{0.15t}$ million shillings per day, where t is years since 2020.

(a) Find the rate of change in transaction volume in 2024.
(b) Calculate the percentage rate of growth.

Solution:

(a) $T(t) = 2000e^{0.15t}$
$T'(t) = 2000 \cdot 0.15e^{0.15t} = 300e^{0.15t}$
In 2024, $t = 4$:
$T'(4) = 300e^{0.6} = 300 \times 1.8221 \approx 546.64$ million shillings per day per year
(b) Percentage rate of growth:
Rate $= \frac{T'(t)}{T(t)} \times 100\% = \frac{300e^{0.15t}}{2000e^{0.15t}} \times 100\% = \frac{300}{2000} \times 100\% = 15\%$ per year
This is constant regardless of the year (characteristic of exponential growth).

Question 7: Population Growth in Kampala

The population of Kampala is modeled by $P(t) = 2.5e^{0.042t}$ million, where t is years since 2020.

(a) Find the growth rate in 2025.
(b) When will the growth rate reach 150,000 people per year?

Solution:

(a) $P(t) = 2.5e^{0.042t}$
$P'(t) = 2.5 \cdot 0.042e^{0.042t} = 0.105e^{0.042t}$ million per year
In 2025, $t = 5$:
$P'(5) = 0.105e^{0.21} = 0.105 \times 1.2337 \approx 0.1295$ million per year $= 129,500$ people per year
(b) Set $P'(t) = 0.15$ million:
$0.105e^{0.042t} = 0.15$
$e^{0.042t} = \frac{0.15}{0.105} = 1.4286$
$0.042t = \ln(1.4286) = 0.3567$
$t = \frac{0.3567}{0.042} \approx 8.49$ years
This occurs in mid-2028 (about 8.49 years after 2020).

Question 8: Maize Production Decay

Post-harvest losses in stored maize follow $M(t) = 1000e^{-0.05t}$ bags, where t is months after harvest.

(a) Find the rate of loss after 6 months.
(b) When is the rate of loss equal to 30 bags per month?

Solution:

(a) $M(t) = 1000e^{-0.05t}$
$M'(t) = 1000 \cdot (-0.05)e^{-0.05t} = -50e^{-0.05t}$ bags per month
At $t = 6$ months:
$M'(6) = -50e^{-0.3} = -50 \times 0.7408 \approx -37.04$ bags per month
The maize is being lost at a rate of 37.04 bags per month.
(b) Set $|M'(t)| = 30$:
$50e^{-0.05t} = 30$
$e^{-0.05t} = 0.6$
$-0.05t = \ln(0.6) = -0.5108$
$t = \frac{0.5108}{0.05} \approx 10.22$ months

Question 9: Logarithmic Differentiation in Manufacturing

A Jinja-based steel company's production cost function is $C(x) = 50x^2 \ln(x)$ thousand shillings for producing x tonnes.

Find the marginal cost when $x = 10$ tonnes.

Solution:

$C(x) = 50x^2 \ln(x)$

Using product rule:

$C'(x) = 50[x^2 \cdot \frac{1}{x} + \ln(x) \cdot 2x]$

$= 50[x + 2x\ln(x)]$

$= 50x[1 + 2\ln(x)]$

At $x = 10$:

$C'(10) = 50 \times 10[1 + 2\ln(10)]$

$= 500[1 + 2 \times 2.3026]$

$= 500[1 + 4.6052]$

$= 500 \times 5.6052 = 2802.6$ thousand shillings per tonne

The marginal cost is approximately 2,802,600 shillings per tonne.

Question 10: Logarithmic Growth in Tea Production

A tea estate in Mubende has production modeled by $P(t) = 500\ln(t + 1) + 1000$ kilograms per day, where t is days since planting season began.

(a) Find the rate of production growth after 30 days.
(b) When does the rate of growth drop to 10 kg per day per day?

Solution:

(a) $P(t) = 500\ln(t + 1) + 1000$
$P'(t) = 500 \cdot \frac{1}{t+1} = \frac{500}{t+1}$
At $t = 30$:
$P'(30) = \frac{500}{31} \approx 16.13$ kg per day per day
(b) Set $P'(t) = 10$:
$\frac{500}{t+1} = 10$
$t + 1 = 50$
$t = 49$ days

11.7 Maclaurin's Series

Question 11: Temperature Variation Approximation

The temperature in Mbarara during a day is approximately $T(t) = 20 + 5\sin\left(\frac{\pi t}{12}\right)$ degrees Celsius, where t is hours after midnight.

Use Maclaurin's series to approximate $\sin(x)$ up to the x^3 term and hence estimate the temperature at $t = 2$ hours.

Solution:

Maclaurin series for $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

Up to x^3 term: $\sin(x) \approx x - \frac{x^3}{6}$

For $t = 2$ hours:

$x = \frac{\pi \times 2}{12} = \frac{\pi}{6}$

$\sin\left(\frac{\pi}{6}\right) \approx \frac{\pi}{6} - \frac{1}{6}\left(\frac{\pi}{6}\right)^3$

$= \frac{\pi}{6} - \frac{\pi^3}{1296}$

$\approx 0.5236 - 0.0239 = 0.4997 \approx 0.5$

Therefore:

$T(2) \approx 20 + 5(0.5) = 22.5C$

(The exact value: $\sin(\pi/6) = 0.5$, so $T(2) = 22.5C$ exactly)

Question 12: Exponential Growth Approximation

The number of tourists visiting Bwindi Impenetrable National Park grows according to $N(t) = 5000e^{0.1t}$ visitors per year, where t is years.

Use Maclaurin's series for e^x up to the x^2 term to approximate the number of visitors in year $t = 1$.

Solution:

Maclaurin series for $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Up to x^2 term: $e^x \approx 1 + x + \frac{x^2}{2}$

For $t = 1$:

$e^{0.1} \approx 1 + 0.1 + \frac{(0.1)^2}{2}$

$= 1 + 0.1 + 0.005 = 1.105$

Therefore:

$N(1) \approx 5000 \times 1.105 = 5525$ visitors

(Exact value: $e^{0.1} \approx 1.1052$, so $N(1) \approx 5526$ visitors)

Question 13: Pendulum Motion

A pendulum at Makerere University's physics lab has displacement $\theta(t) = 0.2\cos(2t)$ radians at time t seconds.

Use Maclaurin's series for $\cos(x)$ up to the x^4 term to approximate the displacement at $t = 0.5$ seconds.

Solution:

Maclaurin series for $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

Up to x^4 term: $\cos(x) \approx 1 - \frac{x^2}{2} + \frac{x^4}{24}$

For $t = 0.5$:

$x = 2(0.5) = 1$ radian

$\cos(1) \approx 1 - \frac{1}{2} + \frac{1}{24}$

$= 1 - 0.5 + 0.0417 = 0.5417$

Therefore:

$\theta(0.5) \approx 0.2 \times 0.5417 = 0.1083$ radians

(Exact value: $\cos(1) \approx 0.5403$, so $\theta(0.5) \approx 0.1081$ radians)

Question 14: Small Angle Approximation in Engineering

A suspension bridge cable in Kampala sags by an amount $s = L\left[1 - \cos\left(\frac{x}{L}\right)\right]$ metres, where $L = 50$ m is the cable length and x is the horizontal distance.

Use Maclaurin's series to show that for small angles, $s \approx \frac{x^2}{2L}$.

Solution:

Given: $s = L\left[1 - \cos\left(\frac{x}{L}\right)\right]$

Maclaurin series for $\cos(u) = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \cdots$

For small angles, keeping only up to u^2 term:

$\cos(u) \approx 1 - \frac{u^2}{2}$

Let $u = \frac{x}{L}$:

$\cos\left(\frac{x}{L}\right) \approx 1 - \frac{1}{2}\left(\frac{x}{L}\right)^2 = 1 - \frac{x^2}{2L^2}$

Therefore:

$s = L\left[1 - \left(1 - \frac{x^2}{2L^2}\right)\right]$

$= L \cdot \frac{x^2}{2L^2}$

$= \frac{x^2}{2L}$

This approximation is valid for small horizontal distances compared to cable length.

Question 15: Power Series in Electrical Engineering

The voltage across a capacitor in a circuit at Kiira Motors is $V(t) = 12\left(1 - e^{-t/RC}\right)$ volts, where $RC = 2$ seconds.

Use Maclaurin's series to approximate the voltage for small values of t.

Solution:

Given: $V(t) = 12\left(1 - e^{-t/2}\right)$

Maclaurin series for $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

For $x = -\frac{t}{2}$:

$e^{-t/2} = 1 - \frac{t}{2} + \frac{1}{2!}\left(\frac{t}{2}\right)^2 - \frac{1}{3!}\left(\frac{t}{2}\right)^3 + \cdots$

$= 1 - \frac{t}{2} + \frac{t^2}{8} - \frac{t^3}{48} + \cdots$

For small t, keeping up to t^2 term:

$e^{-t/2} \approx 1 - \frac{t}{2} + \frac{t^2}{8}$

Therefore:

$V(t) \approx 12\left[1 - \left(1 - \frac{t}{2} + \frac{t^2}{8}\right)\right]$

$= 12\left[\frac{t}{2} - \frac{t^2}{8}\right]$

$= 6t - \frac{3t^2}{2}$ volts

For very small t, $V(t) \approx 6t$ volts (linear approximation).

11.8 Curve Sketching and Optimization

Question 16: Profit Maximization for a Coffee Farmer

A coffee farmer in Mbale has daily profit function $P(x) = -2x^3 + 30x^2 - 96x + 50$ thousand shillings, where x is the number of workers employed (1 to 8 workers).

(a) Find the critical points.
(b) Determine the number of workers that maximizes profit.
(c) Calculate the maximum profit.

Solution:

(a) $P(x) = -2x^3 + 30x^2 - 96x + 50$
$P'(x) = -6x^2 + 60x - 96 = -6(x^2 - 10x + 16)$
$= -6(x-2)(x-8)$
Critical points: $P'(x) = 0 \Rightarrow x = 2$ or $x = 8$

(b) $P''(x) = -12x + 60$
At $x = 2$: $P''(2) = -24 + 60 = 36 > 0$ (local minimum)
At $x = 8$: $P''(8) = -96 + 60 = -36 < 0$ (local maximum)
However, we need to check endpoints and critical points within the domain $[1, 8]$:
$P(1) = -2 + 30 - 96 + 50 = -18$ thousand shillings
$P(2) = -16 + 120 - 192 + 50 = -38$ thousand shillings
$P(8) = -1024 + 1920 - 768 + 50 = 178$ thousand shillings
Maximum profit occurs at $x = 8$ workers.

(c) Maximum profit = 178,000 shillings per day

Question 17: Container Design for Milk Packaging

A dairy company in Mbarara wants to design cylindrical milk containers with volume 1000 cm^3. Find the dimensions that minimize the surface area (and thus material cost).

Solution:

Let r = radius and h = height of cylinder.

Volume constraint: $\pi r^2 h = 1000 \Rightarrow h = \frac{1000}{\pi r^2}$

Surface area: $S = 2\pi r^2 + 2\pi r h$

Substituting h:

$S(r) = 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}$

To minimize:

$S'(r) = 4\pi r - \frac{2000}{r^2}$

Set $S'(r) = 0$:

$4\pi r = \frac{2000}{r^2}$

$4\pi r^3 = 2000$

$r^3 = \frac{500}{\pi}$

$r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42$ cm

$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi \times 29.38} \approx 10.84$ cm

Check second derivative:

$S''(r) = 4\pi + \frac{4000}{r^3} > 0$ for all $r > 0$ (confirming minimum)

Optimal dimensions: radius ≈ 5.42 cm, height ≈ 10.84 cm

Question 18: Road Design with Irrational Curves

A section of the Kampala Northern Bypass is modeled by the curve $y^2 = x^3 + 3x^2$ (in hundreds of metres).

(a) Find $\frac{dy}{dx}$ using implicit differentiation.
(b) Find the slope at the point $(1, 2)$.
(c) Identify any special features of the curve.

Solution:

(a) Given: $y^2 = x^3 + 3x^2$
Differentiating implicitly:
$2y\frac{dy}{dx} = 3x^2 + 6x$
$\frac{dy}{dx} = \frac{3x^2+6x}{2y} = \frac{3x(x+2)}{2y}$

(b) At point $(1, 2)$:
$\frac{dy}{dx} = \frac{3(1)(3)}{2(2)} = \frac{9}{4}$
The slope is $\frac{9}{4}$ or 2.25.

(c) Special features:
At $x = 0$: $y^2 = 0 \Rightarrow y = 0$
At $x = -2$: $y^2 = -8 + 12 = 4 \Rightarrow y = \pm 2$
When $y = 0$: $\frac{dy}{dx}$ is undefined (vertical tangent)
The curve has a cusp at the origin $(0, 0)$ and exists for $x \geq -3$ (where $x^3 + 3x^2 \geq 0$).

Question 19: Revenue Optimization for Mobile Network

Uganda Telecom's revenue from x thousand subscribers is $R(x) = 120x - 0.5x^2$ million shillings, and the cost is $C(x) = 20x + 30$ million shillings.

(a) Find the profit function.
(b) Determine the number of subscribers that maximizes profit.
(c) Calculate the maximum profit.

Solution:

(a) Profit function:
$P(x) = R(x) - C(x) = 120x - 0.5x^2 - 20x - 30$
$P(x) = -0.5x^2 + 100x - 30$ million shillings

(b) To maximize profit:
$P'(x) = -x + 100$
Set $P'(x) = 0$:

$-x + 100 = 0 \Rightarrow x = 100$

Check second derivative:

$P''(x) = -1 < 0$ (confirming maximum)

Optimal number: 100 thousand subscribers = 100,000 subscribers

(c) Maximum profit:

$P(100) = -0.5(100)^2 + 100(100) - 30$

$= -5000 + 10000 - 30 = 4970$ million shillings

Maximum profit is 4,970 million shillings (approximately 4.97 billion shillings).

Question 20: Irrigation Channel Design

A farmer in Arua wants to construct an open irrigation channel with trapezoidal cross-section. The bottom width is x metres, the slant sides make 45° with the horizontal, and the depth is h metres. If the cross-sectional area must be 10 m², find the dimensions that minimize the wetted perimeter (to reduce friction losses).

Solution:

For a trapezoid with bottom width x, depth h, and 45° sides:

Top width = $x + 2h$ (since at 45°, horizontal extension = vertical rise = h)

Area: $A = \frac{1}{2}(x + x + 2h) \times h = h(x + h) = 10$

Therefore: $x = \frac{10}{h} - h$

Wetted perimeter (bottom + two slant sides):

Slant side length = $\sqrt{h^2 + h^2} = h\sqrt{2}$

$P = x + 2h\sqrt{2}$

Substituting x:

$P(h) = \frac{10}{h} - h + 2h\sqrt{2} = \frac{10}{h} + h(2\sqrt{2} - 1)$

To minimize:

$P'(h) = -\frac{10}{h^2} + 2\sqrt{2} - 1$

Set $P'(h) = 0$:

$\frac{10}{h^2} = 2\sqrt{2} - 1 \approx 1.828$

$h^2 = \frac{10}{1.828} \approx 5.470$

$h \approx 2.34$ m

$x = \frac{10}{2.34} - 2.34 \approx 4.27 - 2.34 = 1.93$ m

Check second derivative:

$P''(h) = \frac{20}{h^3} > 0$ for all $h > 0$ (confirming minimum)

Optimal dimensions: bottom width ≈ 1.93 m, depth ≈ 2.34 m

Chapter 12

Integration 2

Introduction

Integration is a fundamental tool in mathematics that allows us to solve problems involving accumulation, areas, volumes, and rates of change. In this chapter, we extend our knowledge of integration techniques to handle more complex functions and apply these methods to real-world problems in engineering, economics, agriculture, and other fields relevant to East Africa.

This chapter builds upon basic integration concepts and introduces advanced techniques including substitution, integration by parts, partial fractions, and special trigonometric substitutions. We will see how these powerful methods enable us to solve practical problems such as calculating water flow rates in irrigation systems, determining crop yields, analyzing economic growth, and designing efficient structures.

12.1 Determining Definite and Indefinite Integrals Using Various Techniques

12.1.1 Integration by Recognition

Many integrals can be evaluated by recognizing a function and its derivative. This method relies on familiarity with standard derivatives and their corresponding integrals.

Key Principle: If we can identify that an integrand has the form $f'(x) \cdot g(f(x))$ where we know the antiderivative of g, we can integrate directly.

Standard Results to Remember:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

$$\int e^x \, dx = e^x + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

Example 1: Coffee Production Optimization

A coffee processing plant in Mbale district measures the rate of coffee bean flow through a sorting machine as $r(t) = 3t^2 + 4t$ kilograms per minute, where t is time in minutes. Find the total mass of coffee beans sorted in the first 5 minutes.

Solution:

The total mass is given by:

$$\begin{aligned} M &= \int_0^5 (3t^2 + 4t) \, dt \\ &= \left[t^3 + 2t^2\right]_0^5 \\ &= (5^3 + 2(5)^2) - (0) \\ &= 125 + 50 \\ &= 175 \text{ kg} \end{aligned}$$

Therefore, 175 kilograms of coffee beans are sorted in the first 5 minutes.

12.1.2 Integration of Odd and Even Powers of Trigonometric Functions

When integrating powers of sine and cosine, we use the following strategies:

Strategy 1: If the power of sine or cosine is odd, save one factor and convert the rest using $\sin^2 x + \cos^2 x = 1$.

Strategy 2: If both powers are even, use the half-angle formulas:

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$
$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

Example 2: Electrical Power Calculation

The instantaneous power in an AC circuit supplying electricity to a maize mill in Jinja is given by $P(t) = 200 \sin^2(50\pi t)$ watts, where t is in seconds. Find the average power over one complete cycle (period $T = 0.04$ seconds).

Solution:

Average power $= \frac{1}{T} \int_0^T P(t)\, dt$

Using $\sin^2 x = \frac{1-\cos 2x}{2}$:

$$P(t) = 200 \cdot \frac{1 - \cos(100\pi t)}{2}$$
$$= 100(1 - \cos(100\pi t))$$

$$\text{Average power} = \frac{1}{0.04} \int_0^{0.04} 100(1 - \cos(100\pi t))\, dt$$
$$= 25 \cdot 100 \left[t - \frac{\sin(100\pi t)}{100\pi} \right]_0^{0.04}$$
$$= 2500 \left[0.04 - \frac{\sin(4\pi)}{100\pi} - 0 + 0 \right]$$
$$= 2500(0.04)$$
$$= 100 \text{ watts}$$

The average power consumed by the maize mill is 100 watts.

12.1.3 Integration by Substitution (Change of Variables)

When dealing with composite functions, substitution simplifies the integral by transforming it into a more recognizable form.

Method:

1. Choose $u = g(x)$ where $g(x)$ is the inner function
2. Find $du = g'(x)dx$
3. Rewrite the integral in terms of u
4. Integrate with respect to u
5. Substitute back to get the answer in terms of x

Example 3: Water Flow in Lake Victoria

The rate of water inflow into a section of Lake Victoria from a tributary is modeled by $Q(t) = 50(2t + 1)^3$ cubic meters per hour, where t is time in hours after 6:00 AM. Find the total volume of water that flows in between 6:00 AM and 10:00 AM.

Solution:

We need to evaluate $\int_0^4 50(2t + 1)^3\, dt$.

Let $u = 2t + 1$, then $du = 2dt$, so $dt = \frac{1}{2}du$.

When $t = 0$: $u = 1$

When $t = 4$: $u = 9$

$$\int_0^4 50(2t + 1)^3\, dt = \int_1^9 50u^3 \cdot \frac{1}{2}\, du$$
$$= 25 \int_1^9 u^3\, du$$
$$= 25 \left[\frac{u^4}{4} \right]_1^9$$
$$= \frac{25}{4}(9^4 - 1^4)$$
$$= \frac{25}{4}(6561 - 1)$$
$$= \frac{25}{4}(6560)$$
$$= 41000 \text{ m}^3$$

The total volume of water that flows into the lake section is 41,000 cubic meters.

12.1.4 Integration of Exponential and Logarithmic Functions

Type 1: Exponential Functions

If the integrand is of the form $f'(x)e^{f(x)}$, then:

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + C$$

Example 4: Population Growth in Kampala

The rate of population growth in a suburb of Kampala is modeled by $P'(t) = 300te^{t^2}$ people per year, where t is years since 2020. Find the increase in population from 2020 to 2022.

Solution:

We need $\int_0^2 300te^{t^2}\,dt$.

Notice that if $f(t) = t^2$, then $f'(t) = 2t$.

We can write: $300te^{t^2} = 150 \cdot 2te^{t^2}$

$$\begin{aligned}\int_0^2 300te^{t^2}\,dt &= 150\int_0^2 2te^{t^2}\,dt\\ &= 150[e^{t^2}]_0^2\\ &= 150(e^4 - e^0)\\ &= 150(e^4 - 1)\\ &\approx 150(54.598 - 1)\\ &\approx 8040 \text{ people}\end{aligned}$$

The population increased by approximately 8,040 people.

Type 2: Logarithmic Functions

If the integrand is of the form $\frac{f'(x)}{f(x)}$, then:

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

Example 5: Tea Production Analysis

A tea estate in Fort Portal has a cost function where the marginal cost (rate of change of cost with respect to quantity) is given by $MC(x) = \frac{200}{2x+5}$ shillings per kilogram, where x is the number of kilograms produced (in hundreds). Find the increase in total cost when production increases from 100 kg to 300 kg.

Solution:

The increase in cost is:

$$\Delta C = \int_1^3 \frac{200}{2x+5}\,dx$$

Let $f(x) = 2x + 5$, then $f'(x) = 2$.

We can write: $\frac{200}{2x+5} = 100 \cdot \frac{2}{2x+5}$

$$\begin{aligned}\int_1^3 \frac{200}{2x+5}\,dx &= 100\int_1^3 \frac{2}{2x+5}\,dx\\ &= 100[\ln|2x+5|]_1^3\\ &= 100(\ln 11 - \ln 7)\\ &= 100\ln\left(\frac{11}{7}\right)\\ &\approx 100(0.4518)\\ &\approx 45.18 \text{ thousand shillings}\end{aligned}$$

The increase in total cost is approximately 45,180 shillings.

12.1.5 Integration of Standard Forms

Form 1: $\int \frac{1}{a^2-b^2x^2}\,dx$

This integral relates to inverse hyperbolic or logarithmic functions. We can use partial fractions:

$$\frac{1}{a^2 - b^2x^2} = \frac{1}{(a-bx)(a+bx)} = \frac{1}{2a}\left(\frac{1}{a-bx} + \frac{1}{a+bx}\right)$$

Therefore:

$$\int \frac{1}{a^2 - b^2x^2}\,dx = \frac{1}{2ab}\ln\left|\frac{a+bx}{a-bx}\right| + C$$

Form 2: $\int \frac{1}{a^2+b^2x^2}\,dx$

This integral yields an inverse tangent function:

$$\int \frac{1}{a^2 + b^2x^2}\,dx = \frac{1}{ab}\arctan\left(\frac{bx}{a}\right) + C$$

Example 6: Radio Wave Propagation

An engineer designing a telecommunications tower near Mbarara needs to calculate the field intensity which involves evaluating $\int_0^{0.5} \frac{4}{9+16x^2}\,dx$. Find this integral.

Solution:

Here $a^2 = 9$, so $a = 3$, and $b^2 = 16$, so $b = 4$.

Using the formula:

$$\begin{aligned}\int_0^{0.5} \frac{4}{9+16x^2}\,dx &= 4\int_0^{0.5} \frac{1}{3^2+4^2x^2}\,dx \\ &= 4\cdot\frac{1}{3\cdot 4}\left[\arctan\left(\frac{4x}{3}\right)\right]_0^{0.5} \\ &= \frac{1}{3}\left[\arctan\left(\frac{2}{3}\right)-\arctan(0)\right] \\ &= \frac{1}{3}\arctan\left(\frac{2}{3}\right) \\ &\approx \frac{1}{3}(0.5880) \\ &\approx 0.196\end{aligned}$$

Example 7: Vehicle Speed Analysis

Traffic engineers studying vehicle speeds on the Kampala-Entebbe Expressway need to evaluate $\int \frac{3}{25-9x^2}\,dx$ for speeds in a certain range. Evaluate this integral.

Solution:

Here $a^2 = 25$, so $a = 5$, and $b^2 = 9$, so $b = 3$.

$$\begin{aligned}\int \frac{3}{25-9x^2}\,dx &= 3\int \frac{1}{5^2-3^2x^2}\,dx \\ &= 3\cdot\frac{1}{2\cdot 5\cdot 3}\ln\left|\frac{5+3x}{5-3x}\right|+C \\ &= \frac{1}{10}\ln\left|\frac{5+3x}{5-3x}\right|+C\end{aligned}$$

12.1.6 The t-Substitution

The t-substitution, also known as the Weierstrass substitution, is particularly useful for integrals involving trigonometric functions. We use:

$$t = \tan\left(\frac{x}{2}\right)$$

This gives us:

$$\begin{aligned}\sin x &= \frac{2t}{1+t^2} \\ \cos x &= \frac{1-t^2}{1+t^2} \\ dx &= \frac{2}{1+t^2}\,dt\end{aligned}$$

Example 8: Solar Panel Efficiency

An engineer installing solar panels in Soroti needs to evaluate the integral $\int \frac{1}{2+\cos x}\,dx$ which appears in calculating optimal panel angles. Solve this integral.

Solution:

Let $t = \tan\left(\frac{x}{2}\right)$, then $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2}{1+t^2}\,dt$.

$$\begin{aligned}\int \frac{1}{2+\cos x}\,dx &= \int \frac{1}{2+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt \\ &= \int \frac{2}{(2+1+t^2)+(1-t^2)}\,dt \\ &= \int \frac{2}{2(1+t^2)+1-t^2}\,dt \\ &= \int \frac{2}{3+t^2}\,dt \\ &= 2\cdot\frac{1}{\sqrt{3}}\arctan\left(\frac{t}{\sqrt{3}}\right)+C \\ &= \frac{2}{\sqrt{3}}\arctan\left(\frac{\tan(x/2)}{\sqrt{3}}\right)+C\end{aligned}$$

12.1.7 Integration by Partial Fractions

Partial fractions allow us to decompose complex rational functions into simpler fractions that can be integrated individually.

Types of Partial Fraction Decomposition:

Type 1: Linear factors in denominator

$$\frac{P(x)}{(ax+b)(cx+d)} = \frac{A}{ax+b}+\frac{B}{cx+d}$$

Type 2: Repeated linear factors

$$\frac{P(x)}{(ax+b)^2} = \frac{A}{ax+b}+\frac{B}{(ax+b)^2}$$

Type 3: Quadratic factors

$$\frac{P(x)}{(ax+b)(cx^2+dx+e)} = \frac{A}{ax+b}+\frac{Bx+C}{cx^2+dx+e}$$

Example 9: Agricultural Economics

A cooperative in Masaka analyzing profit margins encounters the integral $\int \frac{5x+7}{x^2+3x+2}\,dx$. Evaluate this integral.

Solution:

First, factor the denominator: $x^2+3x+2 = (x+1)(x+2)$

Set up partial fractions:

$$\frac{5x+7}{(x+1)(x+2)} = \frac{A}{x+1}+\frac{B}{x+2}$$

Multiply both sides by $(x+1)(x+2)$:

$$5x+7 = A(x+2)+B(x+1)$$

Let $x=-1$: $5(-1)+7 = A(1)+0 \Rightarrow A=2$

Let $x=-2$: $5(-2)+7 = 0+B(-1) \Rightarrow B=3$

Therefore:

$$\begin{aligned}\int \frac{5x+7}{x^2+3x+2}\,dx &= \int \left(\frac{2}{x+1}+\frac{3}{x+2}\right)dx \\ &= 2\ln|x+1|+3\ln|x+2|+C \\ &= \ln|(x+1)^2(x+2)^3|+C\end{aligned}$$

Example 10: Manufacturing Process

A factory in Namanve Industrial Park has a production rate function that requires evaluating $\int_1^2 \frac{2x-1}{x^2-x}\,dx$. Find this integral.

Solution:

Factor: $x^2-x = x(x-1)$

Partial fractions:

$$\frac{2x-1}{x(x-1)} = \frac{A}{x}+\frac{B}{x-1}$$

$$2x-1 = A(x-1)+Bx$$

Let $x=0$: $-1=-A \Rightarrow A=1$

Let $x=1$: $1=B \Rightarrow B=1$

$$\begin{aligned}\int_1^2 \frac{2x-1}{x^2-x}\,dx &= \int_1^2 \left(\frac{1}{x}+\frac{1}{x-1}\right)dx \\ &= [\ln|x|+\ln|x-1|]_1^2 \\ &= (\ln 2+\ln 1)-(\ln 1+\ln 0^+)\end{aligned}$$

Note: The integral diverges because of $\ln 0$ at $x=1$. This indicates that the production rate model has a singularity at $x=1$ and needs to be revised.

12.1.8 Integration by Parts

Integration by parts is based on the product rule for differentiation and is given by:

$$\int u\,dv = uv - \int v\,du$$

Choosing u and dv (LIATE Rule):

Priority for choosing u (in order):

1. Logarithmic functions
2. Inverse trigonometric functions
3. Algebraic functions (polynomials)
4. Trigonometric functions
5. Exponential functions

Example 11: Hydroelectric Power Generation

The Bujagali Dam engineers need to calculate work done by water pressure, which involves evaluating $\int xe^{2x}\,dx$. Find this integral.

Solution:

Let $u=x$ and $dv=e^{2x}dx$

Then $du=dx$ and $v=\frac{1}{2}e^{2x}$

$$\begin{aligned}\int xe^{2x}\,dx &= x\cdot\frac{1}{2}e^{2x}-\int \frac{1}{2}e^{2x}\,dx \\ &= \frac{xe^{2x}}{2}-\frac{1}{2}\cdot\frac{e^{2x}}{2}+C \\ &= \frac{xe^{2x}}{2}-\frac{e^{2x}}{4}+C \\ &= \frac{e^{2x}}{4}(2x-1)+C\end{aligned}$$

Example 12: Construction Engineering

A construction project for a bridge over the Nile in Jinja requires calculating bending moments, leading to the integral $\int_0^1 x^2\ln x\,dx$. Evaluate this integral.

Solution:

Let $u=\ln x$ and $dv=x^2dx$

Then $du=\frac{1}{x}dx$ and $v=\frac{x^3}{3}$

$$\begin{aligned}\int x^2\ln x\,dx &= \ln x\cdot\frac{x^3}{3}-\int \frac{x^3}{3}\cdot\frac{1}{x}\,dx \\ &= \frac{x^3\ln x}{3}-\frac{1}{3}\int x^2\,dx \\ &= \frac{x^3\ln x}{3}-\frac{x^3}{9}+C\end{aligned}$$

For the definite integral:

$$\int_0^1 x^2\ln x\,dx = \left[\frac{x^3\ln x}{3}-\frac{x^3}{9}\right]_0^1$$

At $x = 1$: $\frac{1 \cdot 0}{3} - \frac{1}{9} = -\frac{1}{9}$

At $x = 0$: $\lim_{x \to 0^+} \left(\frac{x^3 \ln x}{3} - \frac{x^3}{9} \right) = 0$ (since $\lim_{x \to 0^+} x^3 \ln x = 0$)

Therefore: $\int_0^1 x^2 \ln x \, dx = -\frac{1}{9}$

Example 13: Repeated Integration by Parts

A telecommunications company analyzing signal propagation needs to evaluate $\int x^2 e^x \, dx$. Find this integral.

Solution:

First application: Let $u = x^2$, $dv = e^x dx$

Then $du = 2x dx$, $v = e^x$

$$\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx$$

Second application: Let $u = 2x$, $dv = e^x dx$

Then $du = 2dx$, $v = e^x$

$$\int 2x e^x \, dx = 2x e^x - \int 2e^x \, dx = 2x e^x - 2e^x$$

Combining:

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - (2x e^x - 2e^x) + C \\ &= x^2 e^x - 2x e^x + 2e^x + C \\ &= e^x (x^2 - 2x + 2) + C \end{aligned}$$

12.2 Applications of Integration to Real-World Problems

12.2.1 Applications in Engineering: Forces and Moments

Work Done by Variable Force

When a force $F(x)$ varies with position x, the work done in moving an object from position a to position b is:

$$W = \int_a^b F(x) \, dx$$

Example 14: Water Pumping System

A water pumping station near Lake Albert needs to pump water to a height of 20 meters. The force required to lift water at height h meters is $F(h) = 1000(20 - h)$ newtons (accounting for the decreasing amount of water above). Calculate the total work done.

Solution:

The work done is:

$$\begin{aligned} W &= \int_0^{20} 1000(20 - h) \, dh \\ &= 1000 \int_0^{20} (20 - h) \, dh \\ &= 1000 \left[20h - \frac{h^2}{2} \right]_0^{20} \\ &= 1000 \, (400 - 200) \\ &= 200000 \text{ joules} \end{aligned}$$

The total work done is 200 kilojoules.

Moments and Centers of Mass

The moment about a point and center of mass for a continuous distribution are found using integration.

Example 15: Bridge Design

A pedestrian bridge in Gulu has a variable cross-section. The mass per unit length at distance x meters from one end is $\rho(x) = 100 + 20x$ kg/m for a 10-meter span. Find the center of mass.

Solution:

Total mass: $M = \int_0^{10} (100 + 20x) \, dx = \left[100x + 10x^2 \right]_0^{10} = 1000 + 1000 = 2000$ kg

Moment about origin:

$$\begin{aligned} M_0 &= \int_0^{10} x(100 + 20x) \, dx \\ &= \int_0^{10} (100x + 20x^2) \, dx \\ &= \left[50x^2 + \frac{20x^3}{3} \right]_0^{10} \\ &= 5000 + \frac{20000}{3} \\ &= \frac{35000}{3} \end{aligned}$$

Center of mass: $\bar{x} = \frac{M_0}{M} = \frac{35000/3}{2000} = \frac{35}{6} \approx 5.83$ meters

The center of mass is approximately 5.83 meters from the starting end.

12.2.2 Applications in Economics

Consumer and Producer Surplus

Given a demand function $D(q)$ and supply function $S(q)$, at equilibrium quantity q_0:

Consumer Surplus = $\int_0^{q_0} D(q)\,dq - p_0 q_0$

Producer Surplus = $p_0 q_0 - \int_0^{q_0} S(q)\,dq$

where p_0 is the equilibrium price.

Example 16: Matoke Market Analysis

In a matoke (banana) market in Kampala, the demand function is $D(q) = 50 - 2q$ thousand shillings per bunch, and the supply function is $S(q) = 10 + q$ thousand shillings per bunch, where q is in hundreds of bunches. Find the consumer and producer surplus at equilibrium.

Solution:

Equilibrium: $50 - 2q = 10 + q$

$40 = 3q$, so $q_0 = \frac{40}{3}$ hundred bunches

$p_0 = 10 + \frac{40}{3} = \frac{70}{3}$ thousand shillings

Consumer Surplus:

$$\begin{aligned} CS &= \int_0^{40/3} (50 - 2q)\,dq - \frac{70}{3} \cdot \frac{40}{3} \\ &= \left[50q - q^2\right]_0^{40/3} - \frac{2800}{9} \\ &= 50 \cdot \frac{40}{3} - \left(\frac{40}{3}\right)^2 - \frac{2800}{9} \\ &= \frac{2000}{3} - \frac{1600}{9} - \frac{2800}{9} \\ &= \frac{6000 - 1600 - 2800}{9} \\ &= \frac{1600}{9} \approx 177.78 \text{ thousand shillings} \end{aligned}$$

Producer Surplus:

$$\begin{aligned} PS &= \frac{70}{3} \cdot \frac{40}{3} - \int_0^{40/3} (10 + q)\,dq \\ &= \frac{2800}{9} - \left[10q + \frac{q^2}{2}\right]_0^{40/3} \\ &= \frac{2800}{9} - \left(\frac{400}{3} + \frac{800}{9}\right) \\ &= \frac{2800}{9} - \frac{1200 + 800}{9} \\ &= \frac{800}{9} \approx 88.89 \text{ thousand shillings} \end{aligned}$$

Total Cost and Revenue Functions

If $MC(q)$ is the marginal cost and $MR(q)$ is the marginal revenue:

Total Cost = $\int MC(q)\,dq$+ Fixed Cost

Total Revenue = $\int MR(q)\,dq$

Example 17: Textile Manufacturing

A textile factory in Nytil, Jinja has marginal cost $MC(q) = 3q^2 - 8q + 10$ thousand shillings per unit, where q is in hundreds of units. Fixed costs are 50,000 shillings. Find the total cost of producing 500 units.

Solution:

Total variable cost for $q = 5$ (500 units):

$$\begin{aligned} VC &= \int_0^5 (3q^2 - 8q + 10)\,dq \\ &= \left[q^3 - 4q^2 + 10q\right]_0^5 \\ &= 125 - 100 + 50 \\ &= 75 \text{ thousand shillings} \end{aligned}$$

Total Cost = VC + FC = 75,000 + 50,000 = 125,000 shillings

12.2.3 Applications in Population Dynamics

Population Growth Models

If $P'(t)$ represents the rate of population change, then:

$$P(t) = P(0) + \int_0^t P'(s)\,ds$$

Example 18: Wildlife Conservation

The Uganda Wildlife Authority monitors the population of elephants in Queen Elizabeth National Park. The rate of population change is modeled by $P'(t) = 50e^{-0.1t}$ elephants per year, where t is years since 2020. If there were 800 elephants in 2020, predict the population in 2025.

Solution:

$$\begin{aligned} P(5) &= 800 + \int_0^5 50e^{-0.1t}\,dt \\ &= 800 + 50\left[\frac{e^{-0.1t}}{-0.1}\right]_0^5 \\ &= 800 - 500[e^{-0.5} - 1] \\ &= 800 - 500(0.6065 - 1) \\ &= 800 + 500(0.3935) \\ &= 800 + 196.75 \\ &\approx 997 \text{ elephants} \end{aligned}$$

The predicted elephant population in 2025 is approximately 997.

Example 19: Urban Population Growth

The population growth rate of Mbarara city is given by $P'(t) = \frac{1000t}{1+t^2}$ people per year. If the population in 2020 (t = 0) was 200,000, find the population after 5 years.

Solution:

$$\Delta P = \int_0^5 \frac{1000t}{1+t^2}\,dt$$

Let $u = 1 + t^2$, then $du = 2t\,dt$

$$\begin{aligned} \Delta P &= 500\int_{t=0}^{t=5} \frac{2t}{1+t^2}\,dt \\ &= 500[\ln(1+t^2)]_0^5 \\ &= 500(\ln 26 - \ln 1) \\ &= 500\ln 26 \\ &\approx 500(3.258) \\ &\approx 1629 \text{ people} \end{aligned}$$

Population in 2025 = 200,000 + 1,629 = 201,629 people

12.2.4 Applications in Statistical Modeling

Probability Distributions

For a continuous probability density function $f(x)$, the probability that X lies between a and b is:

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

The Normal Distribution

The standard normal distribution has probability density function:

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

Example 20: Crop Yield Analysis

Agricultural researchers studying maize yields in Mbale find that the yield per hectare follows approximately a normal distribution. Using statistical tables and integration, they need to find the probability that yield falls within certain ranges. If yields are standardized, find $P(0 \le Z \le 1)$ where Z is the standard normal variable.

Solution:

$$P(0 \le Z \le 1) = \int_0^1 \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx$$

This integral cannot be evaluated using elementary functions. Using numerical methods or standard normal tables:

$$P(0 \le Z \le 1) \approx 0.3413$$

This means approximately 34.13 percent of farms will have yields between the mean and one standard deviation above the mean.

Exponential Distribution in Service Times

Example 21: Banking Services

A bank in Kampala finds that customer service times follow an exponential distribution with rate parameter $\lambda = 0.2$ per minute. The probability density function is $f(t) = 0.2e^{-0.2t}$ for $t \ge 0$. Find the probability that a customer's service time is between 2 and 5 minutes.

Solution:

$$\begin{aligned} P(2 \le T \le 5) &= \int_2^5 0.2e^{-0.2t}\,dt \\ &= \left[-e^{-0.2t}\right]_2^5 \\ &= -e^{-1} + e^{-0.4} \\ &= e^{-0.4} - e^{-1} \\ &\approx 0.6703 - 0.3679 \\ &\approx 0.3024 \end{aligned}$$

There is approximately a 30.24 percent probability that service time is between 2 and 5 minutes.

12.2.5 Applications in Medicine and Health

Example 22: Drug Concentration

A hospital in Mulago administers a drug whose concentration in the bloodstream decreases according to $C(t) = 5e^{-0.3t}$ mg/L, where t is in hours. Calculate the total amount of drug absorbed by the body tissues in the first 4 hours if the absorption rate is proportional to concentration.

Solution:

If absorption rate is $kC(t)$ where $k = 0.5$ L/hour:

$$\begin{aligned} \text{Total absorbed} &= \int_0^4 0.5 \cdot 5e^{-0.3t}\,dt \\ &= 2.5\int_0^4 e^{-0.3t}\,dt \\ &= 2.5\left[\frac{e^{-0.3t}}{-0.3}\right]_0^4 \\ &= -\frac{2.5}{0.3}[e^{-1.2} - 1] \\ &= -8.333(0.3012 - 1) \\ &= -8.333(-0.6988) \\ &\approx 5.82 \text{ mg} \end{aligned}$$

Approximately 5.82 mg of the drug is absorbed in the first 4 hours.

12.2.6 Applications in Environmental Science

Example 23: Pollution Control

Lake Victoria receives pollutants from surrounding industries. The rate of pollutant flow into a section of the lake is $R(t) = 10 + 3\sin(0.5t)$ kg/hour, where t is hours after midnight. Find the total mass of pollutants entering between midnight and 8:00 AM.

Solution:

$$\begin{aligned} M &= \int_0^8 (10 + 3\sin(0.5t))\,dt \\ &= [10t - 6\cos(0.5t)]_0^8 \\ &= (80 - 6\cos 4) - (0 - 6\cos 0) \\ &= 80 - 6\cos 4 + 6 \\ &= 86 - 6\cos 4 \\ &\approx 86 - 6(-0.6536) \\ &\approx 86 + 3.92 \\ &\approx 89.92 \text{ kg} \end{aligned}$$

Approximately 89.92 kg of pollutants enter the lake section during this period.

12.3 Practice Exercises

12.3.1 Integration Techniques

Exercise 1: Evaluate the following integrals:

a) $\int (3x^2 + 2x - 5)\,dx$

b) $\int x\sqrt{x^2+1}\,dx$

c) $\int \frac{6x}{3x^2-5}\,dx$

d) $\int x^3 e^{x^2}\,dx$

e) $\int \sin^3 x \cos x\,dx$

f) $\int \frac{4}{16+9x^2}\,dx$

g) $\int \frac{2x+3}{x^2+5x+6}\,dx$

h) $\int x\cos x\,dx$

i) $\int \frac{1}{4-x^2}\,dx$ for $|x| < 2$

j) $\int xe^{-x}\,dx$

Exercise 2: Evaluate the definite integrals:

a) $\int_0^1 x^2 e^x\,dx$

b) $\int_1^4 \frac{1}{x(x+1)}\,dx$

c) $\int_0^{\pi/2} \sin^2 x\,dx$

d) $\int_0^2 \frac{x}{(x^2+1)^2}\,dx$

e) $\int_1^e x\ln x\,dx$

12.3.2 Application Problems

Exercise 3: A coffee washing station in Kasese processes coffee at a rate of $r(t) = 50 + 20t - t^2$ kg/hour, where t is hours after 8:00 AM. Find:

a) The total amount of coffee processed between 8:00 AM and 12:00 noon.

b) The average processing rate during this period.

Exercise 4: The velocity of a matatu (minibus) on Kampala roads is given by $v(t) = 20 - 5e^{-0.1t}$ m/s, where t is in seconds. Find:

a) The distance traveled in the first 10 seconds.

b) The time when the matatu reaches 95 percent of its maximum velocity.

Exercise 5: A reservoir supplying water to Arua town has an inflow rate of $I(t) = 100 + 30\cos(\pi t/12)$ cubic meters per hour and an outflow rate of $O(t) = 80 + 20t$ cubic meters per hour, where t is hours after 6:00 AM. Find:

a) The net change in water volume from 6:00 AM to 6:00 PM.

b) The time when inflow equals outflow.

Exercise 6: The marginal cost of producing iron sheets at a factory in Tororo is $MC(q) = 2q + \frac{50}{q+1}$ thousand shillings per unit, where q is production in hundreds of sheets. If fixed costs are 200,000 shillings, find:

a) The total cost of producing 1000 sheets.

b) The average cost per sheet at this production level.

Exercise 7: The population density of people living at distance r km from the center of Kampala is $\rho(r) = \frac{5000}{1+r^2}$ people per square kilometer. Find:

a) The total population within a 3 km radius (using circular rings).

b) The average population density within this radius.

Exercise 8: A dam on the Nile generates power according to $P(t) = 50 + 20\sin(0.2t)$ megawatts, where t is hours after midnight. Calculate:

a) The total energy generated in 24 hours.

b) The average power output.

Exercise 9: The concentration of a fertilizer in soil decreases according to $C(t) = 10e^{-0.15t}$ grams per cubic meter, where t is in days. A certain crop absorbs fertilizer at a rate proportional to concentration with constant $k = 0.4$ cubic meters per day. Find:

a) The total amount of fertilizer absorbed in the first week.

b) The time when half of the initial fertilizer has been absorbed.

Exercise 10: In a market for groundnuts (peanuts) in Lira, the demand function is $D(q) = 40 - 3q$ and the supply function is $S(q) = 5 + 2q$, where prices are in thousands of shillings and q is in hundreds of kilograms. Find:

a) The equilibrium price and quantity.

b) The consumer surplus and producer surplus at equilibrium.

c) The total economic surplus.

12.4 Summary

In this chapter, we have explored advanced integration techniques and their applications to real-world problems in Uganda and East Africa. The key concepts covered include:

Integration Techniques:

- Integration by recognition of standard forms
- Integration of trigonometric functions using odd and even powers
- Substitution method for composite functions
- Integration of exponential and logarithmic functions
- Standard forms involving $a^2 \pm b^2x^2$
- The t-substitution for trigonometric integrals
- Partial fractions for rational functions
- Integration by parts using the LIATE rule

Applications:

- Engineering: Work done by variable forces, moments, and centers of mass
- Economics: Consumer and producer surplus, cost and revenue analysis
- Population dynamics: Growth models and predictions

- Statistics: Probability distributions and normal distribution
- Environmental science: Pollution modeling
- Medicine: Drug concentration and absorption

Problem-Solving Strategy:

1. Identify the type of integral or application
2. Choose the most appropriate technique
3. Apply the technique systematically
4. Check your answer by differentiation
5. Interpret results in the context of the problem

Key Formulas for Quick Reference:

Substitution: If $u = g(x)$, then $\int f(g(x))g'(x)\,dx = \int f(u)\,du$

Integration by parts: $\int u\,dv = uv - \int v\,du$

Standard forms:

$$\int \frac{1}{a^2+b^2x^2}\,dx = \frac{1}{ab}\arctan\left(\frac{bx}{a}\right)+C$$

$$\int \frac{1}{a^2-b^2x^2}\,dx = \frac{1}{2ab}\ln\left|\frac{a+bx}{a-bx}\right|+C$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)}+C$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C$$

Mastery of these integration techniques and their applications will enable you to solve complex problems in various fields, from designing efficient irrigation systems to analyzing economic trends in Uganda's growing economy. Practice is essential to develop proficiency in selecting and applying the appropriate method for each problem.

12.5 More scenario based revision questions with solution

Question 1: Coffee Production in Bugisu Region

Scenario: A coffee cooperative in Mbale District finds that the rate of coffee harvesting (in kilograms per hour) follows the function $r(t) = 50 + 30\sin\left(\frac{\pi t}{6}\right)$, where t is time in hours from 8:00 AM.

Question: Calculate the total amount of coffee harvested between 8:00 AM and 2:00 PM (6 hours).

Solution:

We need to find $\int_0^6 \left(50 + 30\sin\left(\frac{\pi t}{6}\right)\right)dt$

$$\begin{aligned}\int_0^6 \left(50 + 30\sin\left(\frac{\pi t}{6}\right)\right)dt &= \left[50t - 30\cdot\frac{6}{\pi}\cos\left(\frac{\pi t}{6}\right)\right]_0^6\\ &= \left[50t - \frac{180}{\pi}\cos\left(\frac{\pi t}{6}\right)\right]_0^6\\ &= \left(300 - \frac{180}{\pi}\cos(\pi)\right) - \left(0 - \frac{180}{\pi}\cos(0)\right)\\ &= 300 + \frac{180}{\pi} - \left(-\frac{180}{\pi}\right)\\ &= 300 + \frac{360}{\pi}\\ &\approx 300 + 114.59 = 414.59 \text{ kg}\end{aligned}$$

Answer: Approximately 414.59 kg of coffee is harvested.

Question 2: Matatu Fuel Consumption in Kampala

Scenario: A matatu (taxi) operating on the Kampala-Entebbe route has a fuel consumption rate (in litres per kilometre) given by $f(x) = \frac{3x}{100+x^2}$, where x is the speed in km/h.

Question: Find the total fuel consumed when the taxi travels at speeds between 40 km/h and 60 km/h over a distance where speed varies uniformly.

Solution:

We need to integrate $\int_{40}^{60} \frac{3x}{100+x^2}dx$

This is of the form $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|+C$

Let $u = 100 + x^2$, then $du = 2x\,dx$, so $x\,dx = \frac{1}{2}du$

$$\begin{aligned}\int \frac{3x}{100+x^2}dx &= \int \frac{3}{2u}du\\ &= \frac{3}{2}\ln|u| + C\\ &= \frac{3}{2}\ln|100+x^2| + C\end{aligned}$$

Evaluating from 40 to 60:

$$\begin{aligned}\left[\frac{3}{2}\ln(100+x^2)\right]_{40}^{60} &= \frac{3}{2}\ln(3700) - \frac{3}{2}\ln(1700)\\ &= \frac{3}{2}\ln\left(\frac{3700}{1700}\right)\\ &= \frac{3}{2}\ln(2.176)\\ &\approx 1.17 \text{ litres}\end{aligned}$$

Answer: Approximately 1.17 litres of fuel.

Question 3: Hydroelectric Power at Bujagali Dam

Scenario: The rate of power generation (in megawatts) at Bujagali Dam varies with water flow rate according to $P(t) = 120e^{0.05t}$, where t is time in hours.

Question: Calculate the total energy generated in 8 hours.

Solution:

$$\text{Total energy} = \int_0^8 120e^{0.05t}\,dt$$

This is of the form $\int f'(x)e^{f(x)}dx = e^{f(x)} + C$

$$\begin{aligned}\int_0^8 120e^{0.05t}\,dt &= \left[\frac{120}{0.05}e^{0.05t}\right]_0^8 \\ &= 2400\left[e^{0.05t}\right]_0^8 \\ &= 2400(e^{0.4} - e^0) \\ &= 2400(1.4918 - 1) \\ &= 2400(0.4918) \\ &= 1180.4 \text{ MWh}\end{aligned}$$

Answer: Approximately 1180.4 megawatt-hours.

Question 4: Tea Production in Mbarara

Scenario: A tea estate in Mbarara experiences a production rate (kg/hour) of $r(t) = 200\cos^2\left(\frac{\pi t}{12}\right)$ during daylight hours.

Question: Find the total tea produced in 12 hours.

Solution:

We need $\int_0^{12} 200\cos^2\left(\frac{\pi t}{12}\right)dt$

Using the identity $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$:

$$\begin{aligned}\int_0^{12} 200\cos^2\left(\frac{\pi t}{12}\right)dt &= \int_0^{12} 200\cdot\frac{1+\cos\left(\frac{\pi t}{6}\right)}{2}dt \\ &= 100\int_0^{12}\left(1+\cos\left(\frac{\pi t}{6}\right)\right)dt \\ &= 100\left[t + \frac{6}{\pi}\sin\left(\frac{\pi t}{6}\right)\right]_0^{12} \\ &= 100\left[\left(12 + \frac{6}{\pi}\sin(2\pi)\right) - \left(0 + \frac{6}{\pi}\sin(0)\right)\right] \\ &= 100(12 + 0 - 0) \\ &= 1200 \text{ kg}\end{aligned}$$

Answer: 1200 kg of tea.

Question 5: Boda Boda Population Growth in Kampala

Scenario: The rate of increase of registered boda bodas in Kampala is modeled by $\frac{dN}{dt} = \frac{500t}{1+t^2}$, where N is the number of boda bodas and t is time in years since 2020.

Question: If there were 50,000 boda bodas in 2020, how many will there be in 2023?

Solution:

We need to find $N(3) - N(0) = \int_0^3 \frac{500t}{1+t^2}dt$

Let $u = 1 + t^2$, then $du = 2t\,dt$, so $t\,dt = \frac{1}{2}du$

$$\begin{aligned}\int \frac{500t}{1+t^2}dt &= \int \frac{250}{u}du \\ &= 250\ln|u| + C \\ &= 250\ln(1+t^2) + C\end{aligned}$$

Evaluating:

$$\begin{aligned}\left[250\ln(1+t^2)\right]_0^3 &= 250\ln(10) - 250\ln(1) \\ &= 250\ln(10) \\ &\approx 250(2.303) = 575.75\end{aligned}$$

$N(3) = 50000 + 575.75 \approx 50,576$ boda bodas

Answer: Approximately 50,576 boda bodas in 2023.

Question 6: Banana (Matooke) Sales Revenue

Scenario: A matooke vendor at Nakasero Market finds that marginal revenue (in UGX thousands per bunch)

is $MR(x) = \frac{50}{1+4x^2}$, where x is the number of bunches sold (in hundreds).

Question: Find the total revenue from selling 50 bunches (x = 0.5).

Solution:

$$\text{Total revenue} = \int_0^{0.5} \frac{50}{1+4x^2}\,dx$$

This is of the form $\int \frac{1}{a^2+b^2x^2}dx = \frac{1}{ab}\arctan\left(\frac{bx}{a}\right)+C$

Here $a^2 = 1$, so $a = 1$, and $b^2 = 4$, so $b = 2$

$$\begin{aligned}\int \frac{50}{1+4x^2}dx &= 50\cdot\frac{1}{1\cdot 2}\arctan(2x)+C\\ &= 25\arctan(2x)+C\end{aligned}$$

Evaluating:

$$\begin{aligned}[25\arctan(2x)]_0^{0.5} &= 25\arctan(1)-25\arctan(0)\\ &= 25\cdot\frac{\pi}{4}-0\\ &= \frac{25\pi}{4}\\ &\approx 19.63 \text{ thousand UGX}\end{aligned}$$

Answer: Approximately UGX 19,630.

Question 7: Water Flow at Karuma Dam

Scenario: The velocity of water flow (m/s) at Karuma Dam varies with depth d (meters) according to $v(d) = 12e^{-0.1d}$.

Question: Find the average velocity over a depth of 20 meters.

Solution:

$$\text{Average velocity} = \tfrac{1}{20}\int_0^{20} 12e^{-0.1d}dd$$

$$\begin{aligned}\int_0^{20} 12e^{-0.1d}dd &= \left[\frac{12}{-0.1}e^{-0.1d}\right]_0^{20}\\ &= -120\left[e^{-0.1d}\right]_0^{20}\\ &= -120(e^{-2}-e^{0})\\ &= -120(0.1353-1)\\ &= -120(-0.8647)\\ &= 103.76 \text{ m}\end{aligned}$$

$$\text{Average velocity} = \tfrac{103.76}{20} = 5.19 \text{ m/s}$$

Answer: 5.19 m/s average velocity.

Question 8: Milk Production in Ankole Region

Scenario: A dairy farm in Mbarara has milk production rate $P(t) = 100 + 40\sin^2\left(\frac{\pi t}{24}\right)$ litres per hour.

Question: Calculate milk produced in 24 hours.

Solution:

Using $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$:

$$\begin{aligned}\int_0^{24}\left(100+40\sin^2\left(\frac{\pi t}{24}\right)\right)dt &= \int_0^{24}\left(100+40\cdot\frac{1-\cos\left(\frac{\pi t}{12}\right)}{2}\right)dt\\ &= \int_0^{24}\left(100+20-20\cos\left(\frac{\pi t}{12}\right)\right)dt\\ &= \int_0^{24}\left(120-20\cos\left(\frac{\pi t}{12}\right)\right)dt\\ &= \left[120t-20\cdot\frac{12}{\pi}\sin\left(\frac{\pi t}{12}\right)\right]_0^{24}\\ &= \left[120t-\frac{240}{\pi}\sin\left(\frac{\pi t}{12}\right)\right]_0^{24}\\ &= \left(2880-\frac{240}{\pi}\sin(2\pi)\right)-\left(0-\frac{240}{\pi}\sin(\right.\\ &= 2880-0-0 = 2880 \text{ litres}\end{aligned}$$

Answer: 2880 litres of milk.

Question 9: Sugarcane Processing in Jinja

Scenario: A sugar factory processes sugarcane at a rate $r(t) = \frac{800t}{4+t^2}$ tonnes per hour.

Question: Find the total sugarcane processed between hours 2 and 6.

Solution:

$$\int_2^6 \frac{800t}{4+t^2}dt$$

Let $u = 4+t^2$, then $du = 2t\,dt$

$$\begin{aligned}\int \frac{800t}{4+t^2}dt &= \int \frac{400}{u}du\\ &= 400\ln|u|+C\\ &= 400\ln(4+t^2)+C\end{aligned}$$

Evaluating:

$$\begin{aligned}\left[400\ln(4+t^2)\right]_2^6 &= 400\ln(40) - 400\ln(8)\\ &= 400\ln\left(\frac{40}{8}\right)\\ &= 400\ln(5)\\ &\approx 400(1.609) = 643.6 \text{ tonnes}\end{aligned}$$

Answer: Approximately 643.6 tonnes.

Question 10: Hospital Admissions in Mulago

Scenario: The rate of patient admissions at Mulago Hospital follows $A(t) = \frac{30}{9+t^2}$ patients per hour, where t is hours since midnight.

Question: Calculate total admissions between 6 AM ($t = 6$) and 6 PM ($t = 18$).

Solution:

$$\int_6^{18} \frac{30}{9+t^2}dt$$

This is of the form $\int \frac{1}{a^2+t^2}dt = \frac{1}{a}\arctan\left(\frac{t}{a}\right) + C$

Here $a^2 = 9$, so $a = 3$

$$\begin{aligned}\int \frac{30}{9+t^2}dt &= 30\cdot\frac{1}{3}\arctan\left(\frac{t}{3}\right) + C\\ &= 10\arctan\left(\frac{t}{3}\right) + C\end{aligned}$$

Evaluating:

$$\begin{aligned}\left[10\arctan\left(\frac{t}{3}\right)\right]_6^{18} &= 10\arctan(6) - 10\arctan(2)\\ &\approx 10(1.4056) - 10(1.1071)\\ &= 14.056 - 11.071\\ &= 2.985 \text{ patients}\end{aligned}$$

Actually, let me recalculate more carefully:

$$\begin{aligned}10\arctan(6) &\approx 10(1.4056) = 14.056\\ 10\arctan(2) &\approx 10(1.1071) = 11.071\\ \text{Difference} &= 2.985\end{aligned}$$

Wait, this seems too small. Let me check the calculation properly for total patients per hour over 12 hours. The integral gives cumulative patients: $10[\arctan(6) - \arctan(2)] \approx 10[1.4056 - 1.1071] = 2.985$ is the result from the antiderivative formula. But we need to interpret this correctly - this value represents the integral outcome. Given the formula structure, let me recalculate:

Actually $\arctan(6) = 1.4056$ rad and $\arctan(2) = 1.1071$ rad, so the answer is correct but seems counterintuitive. Let me verify the magnitude is sensible - the average rate would be about $\frac{30}{9+144} = 0.196$ to $\frac{30}{9+36} = 0.667$ patients per hour, so over 12 hours we'd expect around 3-8 patients. The answer of 2.985 patients seems low but mathematically correct given the decreasing function.

Answer: Approximately 3 patients.

Question 11: Mobile Money Transaction Volume

Scenario: A mobile money agent in Kampala has transaction rate $T(x) = \frac{200x}{25-x^2}$ thousand UGX per minute, where $0 \le x < 5$.

Question: Find total transactions between $x = 1$ and $x = 3$ minutes.

Solution:

$$\int_1^3 \frac{200x}{25-x^2}dx$$

Let $u = 25 - x^2$, then $du = -2x\,dx$, so $x\,dx = -\frac{1}{2}du$

$$\begin{aligned}\int \frac{200x}{25-x^2}dx &= \int \frac{-100}{u}du\\ &= -100\ln|u| + C\\ &= -100\ln|25-x^2| + C\end{aligned}$$

Evaluating:

$$\begin{aligned}\left[-100\ln(25-x^2)\right]_1^3 &= -100\ln(16) - (-100\ln(24))\\ &= -100\ln(16) + 100\ln(24)\\ &= 100\ln\left(\frac{24}{16}\right)\\ &= 100\ln(1.5)\\ &\approx 100(0.4055) = 40.55 \text{ thousand UGX}\end{aligned}$$

Answer: UGX 40,550.

Question 12: Fish Farming in Lake Victoria

Scenario: A fish farm's harvest rate is $H(t) = 50te^{-0.5t}$ kg per day.

Question: Find total harvest over 10 days using integration by parts.

Solution:

$$\int_0^{10} 50te^{-0.5t}\,dt$$

Using integration by parts: $\int u\,dv = uv - \int v\,du$

Let $u = 50t$, so $du = 50\,dt$ Let $dv = e^{-0.5t}dt$, so $v = -2e^{-0.5t}$

$$\begin{aligned}\int 50te^{-0.5t}\,dt &= 50t\cdot(-2e^{-0.5t}) - \int(-2e^{-0.5t})\cdot 50\,dt\\ &= -100te^{-0.5t} + 100\int e^{-0.5t}\,dt\\ &= -100te^{-0.5t} + 100\cdot(-2e^{-0.5t}) + C\\ &= -100te^{-0.5t} - 200e^{-0.5t} + C\\ &= -100e^{-0.5t}(t+2) + C\end{aligned}$$

Evaluating:

$$\begin{aligned}\left[-100e^{-0.5t}(t+2)\right]_0^{10} &= -100e^{-5}(12) - (-100e^{0}(2))\\ &= -1200e^{-5} + 200\\ &= -1200(0.00674) + 200\\ &= -8.09 + 200\\ &= 191.91 \text{ kg}\end{aligned}$$

Answer: Approximately 191.91 kg.

Question 13: Tourism Revenue at Bwindi National Park

Scenario: Daily revenue (in million UGX) from gorilla trekking permits follows $R(x) = \frac{10x}{(x+2)^2}$, where x is the day number of the month.

Question: Find total revenue for the first 10 days using t-substitution.

Solution:

$$\int_0^{10} \frac{10x}{(x+2)^2}dx$$

Let $t = x + 2$, then $x = t - 2$ and $dx = dt$

When $x = 0$, $t = 2$; when $x = 10$, $t = 12$

$$\begin{aligned}\int_0^{10} \frac{10x}{(x+2)^2}dx &= \int_2^{12}\frac{10(t-2)}{t^2}dt\\ &= \int_2^{12}\left(\frac{10t}{t^2} - \frac{20}{t^2}\right)dt\\ &= \int_2^{12}\left(\frac{10}{t} - \frac{20}{t^2}\right)dt\\ &= \left[10\ln|t| + \frac{20}{t}\right]_2^{12}\\ &= \left(10\ln(12) + \frac{20}{12}\right) - \left(10\ln(2) + \frac{20}{2}\right)\\ &= 10\ln(12) + 1.667 - 10\ln(2) - 10\\ &= 10\ln(6) - 8.333\\ &\approx 10(1.792) - 8.333\\ &= 17.92 - 8.333 = 9.587 \text{ million UGX}\end{aligned}$$

Answer: Approximately UGX 9.587 million.

Question 14: Maize Yield in Eastern Uganda

Scenario: Maize yield (tonnes per hectare) as a function of fertilizer amount f kg is $Y(f) = \frac{8f}{16+f^2}$.

Question: Find total yield increase when fertilizer increases from 2 kg to 6 kg per hectare.

Solution:

$$\int_2^6 \frac{8f}{16+f^2}df$$

Let $u = 16 + f^2$, then $du = 2f\,df$

$$\begin{aligned}\int \frac{8f}{16+f^2}df &= \int\frac{4}{u}du\\ &= 4\ln|u| + C\\ &= 4\ln(16+f^2) + C\end{aligned}$$

Evaluating:

$$\begin{aligned}\left[4\ln(16+f^2)\right]_2^6 &= 4\ln(52) - 4\ln(20)\\ &= 4\ln\left(\frac{52}{20}\right)\\ &= 4\ln(2.6)\\ &\approx 4(0.9555) = 3.822 \text{ tonnes/hectare}\end{aligned}$$

Answer: Approximately 3.822 tonnes per hectare increase.

Question 15: Building Construction Rate in Kampala

Scenario: A construction company builds floors at rate $r(t) = \frac{15}{(t+1)\ln(t+1)}$ floors per month, starting from month $t = 0$.

Question: How many floors are completed in months 2 through 8?

Solution:

$$\int_2^8 \frac{15}{(t+1)\ln(t+1)}\,dt$$

Let $u = \ln(t+1)$, then $du = \frac{1}{t+1}dt$

$$\begin{aligned}\int \frac{15}{(t+1)\ln(t+1)}\,dt &= \int \frac{15}{u}\,du \\ &= 15\ln|u| + C \\ &= 15\ln|\ln(t+1)| + C\end{aligned}$$

Evaluating:

$$\begin{aligned}[15\ln|\ln(t+1)|]_2^8 &= 15\ln(\ln(9)) - 15\ln(\ln(3)) \\ &= 15\ln(2.197) - 15\ln(1.099) \\ &= 15(0.7875) - 15(0.0941) \\ &= 11.813 - 1.411 \\ &= 10.402 \text{ floors}\end{aligned}$$

Answer: Approximately 10 floors.

Question 16: Electricity Demand in Kampala

Scenario: Hourly electricity demand (MW) is $D(t) = 150 + 50\cos\left(\frac{\pi t}{12}\right) + 30\sin\left(\frac{\pi t}{6}\right)$.

Question: Calculate total energy consumption in 24 hours.

Solution:

$$\begin{aligned}&\int_0^{24}\left(150 + 50\cos\left(\frac{\pi t}{12}\right) + 30\sin\left(\frac{\pi t}{6}\right)\right)dt \\ &= \left[150t + 50\cdot\frac{12}{\pi}\sin\left(\frac{\pi t}{12}\right) - 30\cdot\frac{6}{\pi}\cos\left(\frac{\pi t}{6}\right)\right]_0^{24} \\ &= \left[150t + \frac{600}{\pi}\sin\left(\frac{\pi t}{12}\right) - \frac{180}{\pi}\cos\left(\frac{\pi t}{6}\right)\right]_0^{24} \\ &= \left(3600 + \frac{600}{\pi}\sin(2\pi) - \frac{180}{\pi}\cos(4\pi)\right) - \left(0 + 0 - \frac{180}{\pi}\right) \\ &= 3600 + 0 - \frac{180}{\pi} + \frac{180}{\pi} \\ &= 3600 \text{ MWh}\end{aligned}$$

Answer: 3600 MWh.

Question 17: Partial Fractions - Poultry Farm Revenue

Scenario: A poultry farm's marginal profit (million UGX per hundred chickens) is $P(x) = \frac{5x+7}{x^2+3x+2}$.

Question: Find total profit when production increases from 100 to 400 chickens (x from 1 to 4).

Solution:

First, factor: $x^2 + 3x + 2 = (x+1)(x+2)$

Using partial fractions:

$$\frac{5x+7}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$5x + 7 = A(x+2) + B(x+1)$

Setting $x = -1$: $2 = A(1)$, so $A = 2$ Setting $x = -2$: $-3 = B(-1)$, so $B = 3$

$$\begin{aligned}\int_1^4 \frac{5x+7}{x^2+3x+2}\,dx &= \int_1^4\left(\frac{2}{x+1} + \frac{3}{x+2}\right)dx \\ &= [2\ln|x+1| + 3\ln|x+2|]_1^4 \\ &= (2\ln(5) + 3\ln(6)) - (2\ln(2) + 3\ln(3)) \\ &= 2\ln\left(\frac{5}{2}\right) + 3\ln\left(\frac{6}{3}\right) \\ &= 2\ln(2.5) + 3\ln(2) \\ &\approx 2(0.9163) + 3(0.6931) \\ &= 1.833 + 2.079 = 3.912 \text{ million UGX}\end{aligned}$$

Answer: UGX 3.912 million.

Question 18: Vaccination Rate in Kampala District

Scenario: COVID-19 vaccination rate (people per day) is $V(t) = \frac{1200t}{4+t^2}$ where t is weeks since campaign start.

Question: Find total vaccinations in weeks 1 through 5.

Solution:

$$\int_1^5 \frac{1200t}{4+t^2}dt$$

Let $u = 4 + t^2$, then $du = 2t\,dt$

$$\begin{aligned}\int \frac{1200t}{4+t^2}dt &= \int \frac{600}{u}du \\ &= 600\ln|u| + C \\ &= 600\ln(4+t^2) + C\end{aligned}$$

Evaluating:

$$\begin{aligned}\left[600\ln(4+t^2)\right]_1^5 &= 600\ln(29) - 600\ln(5) \\ &= 600\ln\left(\frac{29}{5}\right) \\ &= 600\ln(5.8) \\ &\approx 600(1.758) = 1054.8 \text{ people}\end{aligned}$$

Answer: Approximately 1,055 people.

Question 19: Soil Erosion Control in Kabale

Scenario: Soil erosion rate (tonnes per hectare per month) is $E(t) = 8e^{-0.2t}$ after terrace construction at month $t = 0$.

Question: Calculate total soil loss prevented in first 12 months compared to constant rate of 8 tonnes/hectare/month.

Solution:

Actual soil loss: $\int_0^{12} 8e^{-0.2t}dt$

$$\begin{aligned}\int_0^{12} 8e^{-0.2t}dt &= \left[\frac{8}{-0.2}e^{-0.2t}\right]_0^{12} \\ &= -40\left[e^{-0.2t}\right]_0^{12} \\ &= -40(e^{-2.4} - 1) \\ &= -40(0.0907 - 1) \\ &= -40(-0.9093) \\ &= 36.37 \text{ tonnes/hectare}\end{aligned}$$

Without terraces: $8 \times 12 = 96$ tonnes/hectare

Soil saved: $96 - 36.37 = 59.63$ tonnes/hectare

Answer: 59.63 tonnes per hectare saved.

Question 20: Public Transport Optimization

Scenario: A new bus route in Kampala has passenger flow rate $F(t) = 80t\ln(t+1)$ passengers per hour.

Question: Find total passengers in first 4 hours using integration by parts.

Solution:

$$\int_0^4 80t\ln(t+1)dt$$

Let $u = \ln(t+1)$, so $du = \frac{1}{t+1}dt$ Let $dv = 80t\,dt$, so $v = 40t^2$

$$\begin{aligned}\int 80t\ln(t+1)dt &= 40t^2\ln(t+1) - \int 40t^2 \cdot \frac{1}{t+1}dt \\ &= 40t^2\ln(t+1) - 40\int \frac{t^2}{t+1}dt\end{aligned}$$

Dividing t^2 by $(t+1)$: $t^2 = (t+1)(t-1) + 1$

So $\frac{t^2}{t+1} = t - 1 + \frac{1}{t+1}$

$$\begin{aligned}\int 80t\ln(t+1)dt &= 40t^2\ln(t+1) - 40\int\left(t - 1 + \frac{1}{t+1}\right)dt \\ &= 40t^2\ln(t+1) - 40\left(\frac{t^2}{2} - t + \ln(t+1)\right) + C \\ &= 40t^2\ln(t+1) - 20t^2 + 40t - 40\ln(t+1) + C\end{aligned}$$

Evaluating from 0 to 4:

$$\left[40t^2 \ln(t+1) - 20t^2 + 40t - 40\ln(t+1)\right]_0^4$$

$$= (40(16)\ln(5) - 20(16) + 160 - 40\ln(5)) - (0 - 0 + 0 - 0)$$

$$= 640\ln(5) - 320 + 160 - 40\ln(5)$$

$$= 600\ln(5) - 160$$

$$\approx 600(1.609) - 160$$

$$= 965.4 - 160 = 805.4 \text{ passengers}$$

Answer: Approximately 805 passengers.

Chapter 13

COORDINATE GEOMETRY 2

Introduction

Coordinate geometry provides a powerful framework for analyzing spatial relationships by combining algebraic and geometric techniques. In this chapter, we explore the concept of locus and examine specific curves known as conic sections: circles, parabolas, and ellipses. These mathematical concepts have numerous applications in real-world contexts, from designing satellite communication systems to planning road networks and understanding planetary motion.

Understanding loci and conic sections is essential for solving problems in navigation, surveying, engineering, architecture, and various technological applications relevant to Uganda's development goals, including telecommunications infrastructure, agricultural planning, and urban development.

13.1 Analysis of Loci in Real-World Contexts

13.1.1 Definition and Concept of Locus

A locus is the path traced by a point moving under specified conditions. More formally, a locus is the set of all points satisfying a given geometric condition or set of conditions.

Key Principle: To find the equation of a locus, we typically:

1. Let $P(x, y)$ be a general point on the locus
2. Express the given condition(s) using distance formulas or other geometric relationships
3. Simplify the resulting equation to obtain the locus equation

13.1.2 Distance Formula Review

The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

13.1.3 Common Loci

Locus Equidistant from a Fixed Point

The locus of points equidistant from a fixed point is a circle with center at that fixed point.

Example 1: Find the equation of the locus of a point P that moves such that it is always 5 units from the point $A(2, 3)$.

Solution:

Let $P(x, y)$ be any point on the locus. The distance from P to A is 5 units.

Using the distance formula:

$$PA = 5$$

$$\sqrt{(x - 2)^2 + (y - 3)^2} = 5$$

Squaring both sides:

$$(x - 2)^2 + (y - 3)^2 = 25$$

This is the equation of a circle with center $(2, 3)$ and radius 5.

Locus Equidistant from Two Fixed Points

The locus of points equidistant from two fixed points is the perpendicular bisector of the line segment joining the two points.

Example 2: Find the equation of the locus of a point P that moves such that it is equidistant from the points $A(1,2)$ and $B(5,6)$.

Solution:

Let $P(x,y)$ be any point on the locus. Then $PA = PB$.

$$\sqrt{(x-1)^2+(y-2)^2} = \sqrt{(x-5)^2+(y-6)^2}$$

Squaring both sides:

$$(x-1)^2+(y-2)^2 = (x-5)^2+(y-6)^2$$

Expanding:

$$x^2-2x+1+y^2-4y+4 = x^2-10x+25+y^2-12y+36$$

Simplifying:

$$-2x-4y+5 = -10x-12y+61$$

$$8x+8y = 56$$

$$x+y = 7$$

This is the equation of a straight line, which is the perpendicular bisector of AB.

13.1.4 Ugandan Context: Navigation and Surveying Application

Example 3: The Uganda National Roads Authority (UNRA) is planning a new access road to serve communities around Lake Victoria. A telecommunication tower is located at point $T(4,2)$ where coordinates are in kilometers relative to a reference point in Kampala. The road must be constructed such that every point on it is equidistant from the tower T and a monitoring station at point $M(10,8)$. Find the equation of the path the road should follow.

Solution:

Let $P(x,y)$ be any point on the road. The condition is $PT = PM$.

$$\sqrt{(x-4)^2+(y-2)^2} = \sqrt{(x-10)^2+(y-8)^2}$$

Squaring both sides:

$$(x-4)^2+(y-2)^2 = (x-10)^2+(y-8)^2$$

Expanding:

$$x^2-8x+16+y^2-4y+4 = x^2-20x+100+y^2-16y+64$$

Simplifying:

$$-8x-4y+20 = -20x-16y+164$$

$$12x+12y = 144$$

$$x+y = 12$$

The road should follow the path defined by the equation $x+y=12$ (in kilometers).

13.1.5 Locus Maintaining a Constant Ratio of Distances

Example 4: Find the locus of a point P that moves such that its distance from $A(2,0)$ is twice its distance from $B(5,0)$.

Solution:

Let $P(x,y)$ be any point on the locus. The condition is $PA = 2PB$.

$$\sqrt{(x-2)^2+y^2} = 2\sqrt{(x-5)^2+y^2}$$

Squaring both sides:

$$(x-2)^2+y^2 = 4[(x-5)^2+y^2]$$

Expanding:

$$x^2-4x+4+y^2 = 4(x^2-10x+25+y^2)$$

$$x^2-4x+4+y^2 = 4x^2-40x+100+4y^2$$

Rearranging:

$$0 = 3x^2-36x+96+3y^2$$

$$3x^2+3y^2-36x+96 = 0$$

$$x^2+y^2-12x+32 = 0$$

This can be rewritten in standard form by completing the square:

$$(x-6)^2 + y^2 = 4$$

This is a circle with center $(6, 0)$ and radius 2.

13.1.6 Application in Robotics and Microphone Placement

Example 5: In a recording studio in Kampala, a sound engineer needs to position a microphone such that it is always 3 meters from a fixed speaker at position $S(0, 4)$ measured on a coordinate grid where units are in meters. Determine the locus of possible microphone positions.

Solution:

Let $M(x, y)$ represent the position of the microphone. The condition is that the distance from M to S is 3 meters.

$$SM = 3$$

$$\sqrt{x^2 + (y-4)^2} = 3$$

Squaring:

$$x^2 + (y-4)^2 = 9$$

The locus is a circle with center $(0, 4)$ and radius 3 meters.

13.2 Analysis of Circle Locus

13.2.1 Equation of a Circle

A circle is the locus of a point that moves such that its distance from a fixed point (the center) is constant (the radius).

The standard form of the equation of a circle with center (h, k) and radius r is:

$$(x-h)^2 + (y-k)^2 = r^2$$

For a circle with center at the origin $(0, 0)$:

$$x^2 + y^2 = r^2$$

13.2.2 General Equation of a Circle

Expanding the standard form gives the general equation:

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

where the center is $(-g, -f)$ and the radius is $r = \sqrt{g^2 + f^2 - c}$.

Note: For a valid circle, $g^2 + f^2 - c > 0$.

13.2.3 Finding the Equation of a Circle

Example 6: Find the equation of the circle with center $(3, -2)$ and radius 5.

Solution:

Using the standard form $(x-h)^2 + (y-k)^2 = r^2$:

$$(x-3)^2 + (y+2)^2 = 25$$

Expanding:

$$x^2 - 6x + 9 + y^2 + 4y + 4 = 25$$

$$x^2 + y^2 - 6x + 4y - 12 = 0$$

Example 7: Find the center and radius of the circle $x^2 + y^2 - 8x + 6y - 11 = 0$.

Solution:

Comparing with $x^2 + y^2 + 2gx + 2fy + c = 0$:

$$2g = -8 \implies g = -4$$

$$2f = 6 \implies f = 3$$

$$c = -11$$

Center: $(-g, -f) = (4, -3)$

Radius: $r = \sqrt{g^2 + f^2 - c} = \sqrt{16 + 9 + 11} = \sqrt{36} = 6$

13.2.4 Circle Through Three Points

Example 8: Find the equation of the circle passing through the points $A(1, 1)$, $B(2, 4)$, and $C(5, 3)$.

Solution:

Let the equation of the circle be $x^2+y^2+2gx+2fy+c = 0$.

Since the three points lie on the circle, they satisfy the equation:

For point $A(1, 1)$:

$$1 + 1 + 2g + 2f + c = 0$$

$$2g + 2f + c = -2 \quad ...(1)$$

For point $B(2,4)$:

$$4 + 16 + 4g + 8f + c = 0$$

$$4g + 8f + c = -20 \quad ...(2)$$

For point $C(5,3)$:

$$25 + 9 + 10g + 6f + c = 0$$

$$10g + 6f + c = -34 \quad ...(3)$$

Solving equations (1), (2), and (3):

From (2) - (1):

$$2g + 6f = -18$$

$$g + 3f = -9 \quad ...(4)$$

From (3) - (2):

$$6g - 2f = -14$$

$$3g - f = -7 \quad ...(5)$$

From (4): $g = -9 - 3f$

Substituting into (5):

$$3(-9 - 3f) - f = -7$$

$$-27 - 9f - f = -7$$

$$-10f = 20$$

$$f = -2$$

Therefore: $g = -9 - 3(-2) = -9 + 6 = -3$

From (1): $2(-3) + 2(-2) + c = -2$

$$-6 - 4 + c = -2$$

$$c = 8$$

The equation of the circle is:

$$x^2 + y^2 - 6x - 4y + 8 = 0$$

13.2.5 Tangent to a Circle

A tangent to a circle is a straight line that touches the circle at exactly one point. At the point of contact, the tangent is perpendicular to the radius.

Properties of Tangents

1. The tangent is perpendicular to the radius at the point of contact
2. The length of tangent from an external point to the circle is given by $\sqrt{g^2 + f^2 - c}$ for point (x_1, y_1): $L = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

Equation of Tangent at a Given Point

For a circle $x^2 + y^2 = r^2$, the equation of the tangent at point (x_1, y_1) is:

$$xx_1 + yy_1 = r^2$$

For a circle $(x - h)^2 + (y - k)^2 = r^2$, the tangent at (x_1, y_1) is:

$$(x - h)(x_1 - h) + (y - k)(y_1 - k) = r^2$$

Example 9: Find the equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3,4)$.

Solution:

Using the formula $xx_1 + yy_1 = r^2$:

$$3x + 4y = 25$$

This is the equation of the tangent.

Tangent with Given Gradient

To find the equation of a tangent to circle $x^2 + y^2 = r^2$ with gradient m:

$$y = mx \pm r\sqrt{1 + m^2}$$

Example 10: Find the equations of the tangents to the circle $x^2 + y^2 = 20$ with gradient 2.

Solution:

Here $r^2 = 20$, so $r = \sqrt{20} = 2\sqrt{5}$ and $m = 2$.

Using $y = mx \pm r\sqrt{1 + m^2}$:

$$y = 2x \pm 2\sqrt{5}\sqrt{1 + 4}$$

$$y = 2x \pm 2\sqrt{5} \cdot \sqrt{5}$$

$$y = 2x \pm 10$$

The equations of the tangents are $y = 2x + 10$ and $y = 2x - 10$.

13.2.6 Length of Tangent from an External Point

If a tangent is drawn from an external point $P(x_1, y_1)$ to a circle $x^2 + y^2 + 2gx + 2fy + c = 0$, the length of the tangent is:

$$L = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

Example 11: Find the length of the tangent from the point $(7, 5)$ to the circle $x^2 + y^2 - 4x - 6y - 12 = 0$.

Solution:

Using the formula with $(x_1, y_1) = (7, 5)$:

$$L = \sqrt{49 + 25 - 28 - 30 - 12}$$

$$L = \sqrt{4} = 2$$

The length of the tangent is 2 units.

13.2.7 Application: Telecommunications in Uganda

Example 12: MTN Uganda is installing a circular cellular network coverage area centered at coordinates $(20, 15)$ where units are in kilometers. The coverage radius is 8 km. A technician standing at location $(24, 22)$ needs to determine if they can establish a direct line-of-sight connection (tangent) to the edge of the coverage area and calculate the distance to the nearest point on the coverage boundary.

Solution:

The circle equation is $(x - 20)^2 + (y - 15)^2 = 64$.

Converting to general form:

$$x^2 - 40x + 400 + y^2 - 30y + 225 = 64$$

$$x^2 + y^2 - 40x - 30y + 561 = 0$$

So $g = -20$, $f = -15$, $c = 561$.

Length of tangent from $(24, 22)$:

$$L = \sqrt{24^2 + 22^2 - 40(24) - 30(22) + 561}$$

$$L = \sqrt{576 + 484 - 960 - 660 + 561}$$

$$L = \sqrt{1} = 1$$

The technician is 1 km from the nearest point on the coverage boundary along a tangent line.

13.2.8 Intersection of Two Circles

Two circles can intersect in different ways:

1. **External:** The circles do not touch or intersect (distance between centers $> r_1 + r_2$)
2. **External tangency:** The circles touch at one point externally (distance $= r_1 + r_2$)
3. **Intersection at two points:** The circles intersect at two distinct points ($|r_1 - r_2| <$ distance $< r_1 + r_2$)
4. **Internal tangency:** One circle touches the other internally (distance $= |r_1 - r_2|$)
5. **One inside the other:** No intersection (distance $< |r_1 - r_2|$)

Orthogonal Circles

Two circles are orthogonal if they intersect at right angles. For circles:

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

$$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

The condition for orthogonality is:

$$2g_1g_2 + 2f_1f_2 = c_1 + c_2$$

Example 13: Determine the nature of intersection of the circles:

$$C_1 : x^2 + y^2 - 6x + 4y - 12 = 0$$

$$C_2 : x^2 + y^2 + 2x - 8y + 8 = 0$$

Solution:

For C_1: Center $(3, -2)$, radius $r_1 = \sqrt{9 + 4 + 12} = 5$

For C_2: Center $(-1, 4)$, radius $r_2 = \sqrt{1 + 16 - 8} = 3$

Distance between centers:

$$d = \sqrt{(3 - (-1))^2 + (-2 - 4)^2} = \sqrt{16 + 36} = \sqrt{52} = 2\sqrt{13} \approx 7.21$$

Since $|r_1 - r_2| = 2$ and $r_1 + r_2 = 8$:

$$2 < 7.21 < 8$$

The circles intersect at two distinct points.

13.3 Analysis of Parabola Locus

13.3.1 Definition of a Parabola

A parabola is the locus of a point that moves such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed line (directrix).

Key elements of a parabola:

- **Focus:** The fixed point
- **Directrix:** The fixed line
- **Vertex:** The point on the parabola closest to the directrix
- **Axis of symmetry:** The line through the focus perpendicular to the directrix
- **Latus rectum:** A chord through the focus perpendicular to the axis

13.3.2 Standard Forms of Parabola

Parabola with Vertex at Origin

Type 1: Opening rightward

$$y^2 = 4ax$$

Focus: $(a, 0)$, Directrix: $x = -a$

Type 2: Opening leftward

$$y^2 = -4ax$$

Focus: $(-a, 0)$, Directrix: $x = a$

Type 3: Opening upward

$$x^2 = 4ay$$

Focus: $(0, a)$, Directrix: $y = -a$

Type 4: Opening downward

$$x^2 = -4ay$$

Focus: $(0, -a)$, Directrix: $y = a$

13.3.3 Deriving the Equation of a Parabola

Example 14: Derive the equation of the parabola with focus at $(3, 0)$ and directrix $x = -3$.

Solution:

Let $P(x, y)$ be any point on the parabola. The distance from P to the focus $F(3, 0)$ equals the perpendicular distance from P to the directrix $x = -3$.

$$PF = PM$$

where M is the foot of the perpendicular from P to the directrix, so $M(-3, y)$.

$$\sqrt{(x-3)^2 + y^2} = |x + 3|$$

Squaring both sides:

$$(x-3)^2 + y^2 = (x+3)^2$$

$$x^2 - 6x + 9 + y^2 = x^2 + 6x + 9$$

$$y^2 = 12x$$

This is the standard form with $4a = 12$, so $a = 3$.

13.3.4 Properties of a Parabola

For the parabola $y^2 = 4ax$:

- Vertex: $(0, 0)$
- Focus: $(a, 0)$
- Directrix: $x = -a$
- Axis of symmetry: $y = 0$ (x-axis)
- Length of latus rectum: $4a$
- Eccentricity: $e = 1$ (by definition of a parabola)

Example 15: For the parabola $y^2 = 16x$, find the focus, directrix, and length of latus rectum.

Solution:

Comparing with $y^2 = 4ax$:

$$4a = 16$$

$$a = 4$$

Focus: $(4, 0)$

Directrix: $x = -4$

Length of latus rectum: $4a = 16$

13.3.5 Parametric Equations of a Parabola

For the parabola $y^2 = 4ax$, any point on the parabola can be represented as:

$$P(at^2, 2at)$$

where t is a parameter.

This is useful because it automatically satisfies the parabola equation:

$$(2at)^2 = 4a^2t^2 = 4a(at^2)$$

Example 16: Find the coordinates of the point on the parabola $y^2 = 12x$ corresponding to parameter $t = 3$.

Solution:

Here $4a = 12$, so $a = 3$.

Using $(at^2, 2at)$:

$$x = 3(3)^2 = 27$$

$$y = 2(3)(3) = 18$$

The point is $(27, 18)$.

13.3.6 Tangent and Normal to a Parabola

Tangent at a Given Point

For the parabola $y^2 = 4ax$, the equation of the tangent at point (x_1, y_1) is:

$$yy_1 = 2a(x + x_1)$$

In parametric form, the tangent at $(at^2, 2at)$ is:

$$ty = x + at^2$$

Example 17: Find the equation of the tangent to the parabola $y^2 = 8x$ at the point $(2, 4)$.

Solution:

Here $4a = 8$, so $a = 2$, and $(x_1, y_1) = (2, 4)$.

Using $yy_1 = 2a(x + x_1)$:

$$4y = 4(x + 2)$$

$$4y = 4x + 8$$

$$y = x + 2$$

Normal to a Parabola

The normal is perpendicular to the tangent. For the parabola $y^2 = 4ax$, the equation of the normal at (x_1, y_1) is:

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

In parametric form, the normal at $(at^2, 2at)$ is:

$$y + tx = 2at + at^3$$

Example 18: Find the equation of the normal to the parabola $y^2 = 12x$ at the point where $t = 2$.

Solution:

Here $a = 3$ and $t = 2$.

The point is $(3 \cdot 4, 2 \cdot 3 \cdot 2) = (12, 12)$.

Using $y + tx = 2at + at^3$:

$$y + 2x = 2(3)(2) + 3(8)$$

$$y + 2x = 12 + 24$$

$$y + 2x = 36$$

13.3.7 Application: Parabolic Reflectors in Uganda

Example 19: A solar cooker designed for rural communities in Uganda uses a parabolic reflector. The cross-section of the reflector follows the equation $y^2 = 20x$ where measurements are in centimeters. Food is placed at the focus for maximum heat concentration. Determine:

1. The location of the focus
2. The depth of the reflector if its width is 40 cm

Solution:

(a) Comparing $y^2 = 20x$ with $y^2 = 4ax$:

$$4a = 20$$

$$a = 5$$

Focus: $(5, 0)$

The food should be placed 5 cm from the vertex along the axis.

(b) If the width is 40 cm, then $y = \pm 20$ cm from the axis.

At $y = 20$:

$$400 = 20x$$

$$x = 20$$

The depth of the reflector is 20 cm.

13.3.8 Application: Satellite Dishes

Example 20: A satellite dish used by a broadcasting company in Kampala has a parabolic cross-section. The dish is 2.4 meters in diameter and 0.4 meters deep. Find the equation of the parabola and determine where the receiver should be placed (at the focus).

Solution:

Place the vertex at the origin with the parabola opening upward: $x^2 = 4ay$.

The dish extends from $x = -1.2$ to $x = 1.2$ meters (radius = 1.2 m).

At the edge, $x = 1.2$ and $y = 0.4$:

$$(1.2)^2 = 4a(0.4)$$

$$1.44 = 1.6a$$

$$a = 0.9$$

The equation is $x^2 = 3.6y$.

The receiver should be placed at the focus $(0, 0.9)$, which is 0.9 meters above the vertex.

13.3.9 Chord of a Parabola

A chord is a line segment joining two points on the parabola. The equation of the chord joining points with parameters t_1 and t_2 on the parabola $y^2 = 4ax$ is:

$$y(t_1 + t_2) = 2x + 2at_1t_2$$

Example 21: Find the equation of the chord joining the points with parameters $t = 1$ and $t = 3$ on the parabola $y^2 = 8x$.

Solution:

Here $a = 2$, $t_1 = 1$, $t_2 = 3$.

Using $y(t_1 + t_2) = 2x + 2at_1t_2$:

$$y(1 + 3) = 2x + 2(2)(1)(3)$$

$$4y = 2x + 12$$

$$2y = x + 6$$

13.3.10 Application: Bridge Design

Example 22: The Nile Bridge under construction near Jinja has support cables that follow a parabolic shape. The span of the bridge is 80 meters, and the cables sag 20 meters at the center. Taking the vertex at the lowest point of the cable, find the equation of the parabola and the height of the cable at a point 20 meters from the center.

Solution:

Place the vertex at the origin with parabola opening upward: $x^2 = 4ay$.

At the end of the span, $x = 40$ m and $y = 20$ m:

$$1600 = 4a(20)$$

$$1600 = 80a$$

$$a = 20$$

The equation is $x^2 = 80y$ or $y = \frac{x^2}{80}$.

At $x = 20$ m from center:

$$y = \frac{400}{80} = 5$$

The cable is 5 meters above the lowest point.

13.4 Analysis of Ellipse Locus

13.4.1 Definition of an Ellipse

An ellipse is the locus of a point that moves such that the sum of its distances from two fixed points (foci) is constant.

If F_1 and F_2 are the foci and P is any point on the ellipse:

$$PF_1 + PF_2 = 2a$$

where $2a$ is a constant greater than the distance between the foci.

13.4.2 Standard Form of an Ellipse

Ellipse with Center at Origin

The standard form with major axis along the x-axis is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a > b > 0$.

Key properties:

- Major axis length: $2a$ (along x-axis)
- Minor axis length: $2b$ (along y-axis)
- Semi-major axis: a
- Semi-minor axis: b
- Foci: $F_1(-c, 0)$ and $F_2(c, 0)$ where $c^2 = a^2 - b^2$
- Eccentricity: $e = \frac{c}{a}$ where $0 < e < 1$
- Directrices: $x = \pm\frac{a}{e} = \pm\frac{a^2}{c}$
- Vertices: $(\pm a, 0)$
- Co-vertices: $(0, \pm b)$
- Length of latus rectum: $\frac{2b^2}{a}$

If the major axis is along the y-axis:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

where $a > b$, and the foci are at $(0, \pm c)$.

13.4.3 Deriving the Equation of an Ellipse

Example 23: Derive the equation of the ellipse with foci at $(\pm 3, 0)$ and the sum of distances from any point to the foci is 10.

Solution:

Let $P(x, y)$ be any point on the ellipse with foci $F_1(-3, 0)$ and $F_2(3, 0)$.

The condition is:

$$PF_1 + PF_2 = 10 = 2a$$

So $a = 5$.

Since the foci are at $(\pm 3, 0)$, we have $c = 3$.

Using $c^2 = a^2 - b^2$:

$$9 = 25 - b^2$$

$$b^2 = 16$$

$$b = 4$$

The equation is:

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

13.4.4 Properties and Characteristics

Example 24: For the ellipse $\frac{x^2}{36} + \frac{y^2}{20} = 1$, find:

1. The lengths of the major and minor axes
2. The coordinates of the foci
3. The eccentricity
4. The equations of the directrices

Solution:

Comparing with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$:

$$a^2 = 36 \implies a = 6$$

$$b^2 = 20 \implies b = 2\sqrt{5}$$

(a) Major axis length: $2a = 12$ Minor axis length: $2b = 4\sqrt{5} \approx 8.94$

(b) $c^2 = a^2 - b^2 = 36 - 20 = 16$ $c = 4$

Foci: $(\pm 4, 0)$

(c) Eccentricity: $e = \frac{c}{a} = \frac{4}{6} = \frac{2}{3}$

(d) Directrices: $x = \pm\frac{a}{e} = \pm\frac{6}{2/3} = \pm 9$

Equations: $x = 9$ and $x = -9$

13.4.5 Parametric Equations of an Ellipse

For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, any point can be represented as:

$$P(a\cos\theta, b\sin\theta)$$

where θ is the parameter (called the eccentric angle).

Example 25: Find the coordinates of the point on the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ when $\theta = 60$.

Solution:

Here $a = 5$ and $b = 3$.

$$x = 5\cos 60 = 5 \cdot \frac{1}{2} = 2.5$$

$$y = 3\sin 60 = 3 \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} \approx 2.598$$

The point is $(2.5, 2.598)$.

13.4.6 Tangent to an Ellipse

For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the equation of the tangent at point (x_1, y_1) is:

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

In parametric form, the tangent at $(a\cos\theta, b\sin\theta)$ is:

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

Example 26: Find the equation of the tangent to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at the point $(2, \frac{3\sqrt{3}}{2})$.

Solution:

Here $a^2 = 16$, $b^2 = 9$, and $(x_1, y_1) = (2, \frac{3\sqrt{3}}{2})$.

Using $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$:

$$\frac{2x}{16} + \frac{y \cdot \frac{3\sqrt{3}}{2}}{9} = 1$$

$$\frac{x}{8} + \frac{3\sqrt{3}y}{18} = 1$$

$$\frac{x}{8} + \frac{\sqrt{3}y}{6} = 1$$

Multiplying by 24:

$$3x + 4\sqrt{3}y = 24$$

13.4.7 Normal to an Ellipse

The normal at point (x_1, y_1) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has equation:

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

In parametric form, the normal at $(a\cos\theta, b\sin\theta)$ is:

$$ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$$

Example 27: Find the equation of the normal to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ at the point $(3, \frac{16}{5})$.

Solution:

Here $a^2 = 25$, $b^2 = 16$, and $(x_1, y_1) = (3, \frac{16}{5})$.

Using $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$:

$$\frac{25x}{3} - \frac{16y}{\frac{16}{5}} = 25 - 16$$

$$\frac{25x}{3} - 5y = 9$$

Multiplying by 3:

$$25x - 15y = 27$$

13.4.8 Chord of an Ellipse

The equation of the chord joining points $(a\cos\theta_1, b\sin\theta_1)$ and $(a\cos\theta_2, b\sin\theta_2)$ is:

$$\frac{x}{a}\cos\left(\frac{\theta_1 + \theta_2}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta_1 + \theta_2}{2}\right) = \cos\left(\frac{\theta_1 - \theta_2}{2}\right)$$

13.4.9 Application: Planetary Orbits

Example 28: A satellite tracking station in Uganda monitors a satellite in an elliptical orbit around Earth. The orbit can be modeled by the equation $\frac{x^2}{64000000} + \frac{y^2}{60000000} = 1$ where distances are in kilometers. Determine:

1. The closest and farthest distances of the satellite from the center of Earth
2. The eccentricity of the orbit

Solution:

(a) From the equation: $a^2 = 64000000$, so $a = 8000$ km

$b^2 = 60000000$, so $b \approx 7746$ km

Closest distance (perigee): $a - c$ Farthest distance (apogee): $a + c$

First find c:

$$c^2 = a^2 - b^2 = 64000000 - 60000000 = 4000000$$

$$c = 2000 \text{ km}$$

Closest: $8000 - 2000 = 6000$ km Farthest: $8000 + 2000 = 10000$ km

(b) Eccentricity: $e = \frac{c}{a} = \frac{2000}{8000} = 0.25$

13.4.10 Application: Architecture and Design

Example 29: The Uganda National Cultural Centre is designing an amphitheater with an elliptical dome. The dome has a major axis of 40 meters and a minor axis of 30 meters. If a speaker stands at one focus, where should a listener stand to hear the sound most clearly (at the other focus)?

Solution:

For the ellipse with major axis $2a = 40$ and minor axis $2b = 30$:

$$a = 20 \text{ m}, \quad b = 15 \text{ m}$$

Find the position of the foci:

$$c^2 = a^2 - b^2 = 400 - 225 = 175$$

$$c = \sqrt{175} = 5\sqrt{7} \approx 13.23 \text{ m}$$

The two foci are located at a distance of approximately 13.23 meters on either side of the center along the major axis. If the speaker is at one focus, the listener should stand at the other focus, approximately 26.46 meters away from the speaker, to hear most clearly due to the reflective property of the ellipse.

13.4.11 Application: Agricultural Planning

Example 30: An irrigation system in Kasese district uses elliptical spray patterns. The sprinkler produces an elliptical coverage area described by $\frac{x^2}{100} + \frac{y^2}{64} = 1$ where units are in meters. Calculate:

1. The maximum length and width of the coverage area
2. The area covered by the sprinkler (Area of ellipse = πab)

Solution:

(a) From the equation: $a^2 = 100$, so $a = 10$ m $b^2 = 64$, so $b = 8$ m

Maximum length (along major axis): $2a = 20$ meters
Maximum width (along minor axis): $2b = 16$ meters

(b) Area = $\pi ab = \pi(10)(8) = 80\pi \approx 251.3$ square meters

13.5 Summary of Key Concepts

13.5.1 Locus

A locus is the path traced by a point moving under specified geometric conditions. Common loci include circles (equidistant from a point) and straight lines (equidistant from two points).

13.5.2 Circle

The circle is the locus of points equidistant from a fixed point. Standard equation: $(x-h)^2 + (y-k)^2 = r^2$. Key concepts include tangents (perpendicular to radius at point of contact) and conditions for intersection of two circles.

13.5.3 Parabola

The parabola is the locus where distance to focus equals perpendicular distance to directrix. Standard form: $y^2 = 4ax$. Properties include vertex, focus, directrix, and axis of symmetry. Applications include reflectors and satellite dishes.

13.5.4 Ellipse

The ellipse is the locus where the sum of distances to two foci is constant. Standard form: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b$. Key properties include major and minor axes, foci, eccentricity, and directrices. Applications include planetary orbits and architectural designs.

13.6 Practice Exercises

13.6.1 Loci Problems

1. Find the equation of the locus of a point that moves so that it is equidistant from the points $A(2,3)$ and $B(4,7)$.

2. A point P moves such that its distance from $(1,0)$ is twice its distance from $(4,0)$. Find the equation of its locus.

3. In a telecommunications network, relay stations at coordinates $(0,0)$ and $(10,0)$ (in km) have equal signal strength. Find the locus of points where signals from both stations have equal intensity.

13.6.2 Circle Problems

4. Find the equation of the circle with center $(2,-3)$ and radius 5.

5. Find the center and radius of the circle $x^2 + y^2 - 6x + 8y + 9 = 0$.

6. Find the equation of the circle passing through points $(0,0)$, $(4,0)$, and $(0,3)$.

7. Find the equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3,4)$.

8. Determine whether the circles $x^2 + y^2 = 9$ and $x^2 + y^2 - 10x - 10y + 25 = 0$ intersect.

13.6.3 Parabola Problems

9. Find the focus and directrix of the parabola $y^2 = 12x$.

10. A parabolic solar collector is 3 meters wide and 1 meter deep. Find its equation and the location of the focus.

11. Find the equation of the tangent to $y^2 = 16x$ at the point $(4,8)$.

12. Find the equation of the parabola with focus $(0,3)$ and directrix $y = -3$.

13.6.4 Ellipse Problems

13. Find the equation of the ellipse with foci at $(\pm 4, 0)$ and major axis length 10.

14. For the ellipse $\frac{x^2}{49} + \frac{y^2}{24} = 1$, find the eccentricity and the coordinates of the foci.

15. Find the equation of the tangent to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ at the point $(3, \frac{16}{5})$.

16. A dam spillway has an elliptical opening with major axis 12 meters (horizontal) and minor axis 8 meters (vertical). Find its equation and eccentricity.

13.6.5 Mixed Application Problems

17. A water sprinkler covers a circular area with equation $x^2 + y^2 = 16$ (in meters). A second sprinkler covers the area $(x-6)^2 + y^2 = 9$. Determine if there are areas covered by both sprinklers.

18. A radio station tower at coordinates $(0, 0)$ has a broadcasting range following the equation $x^2 + y^2 = 100$ (km). A listener at $(8, 6)$ wants to know if they are within range.

19. A bridge arch follows the equation $\frac{x^2}{225} + \frac{y^2}{100} = 1$ where $y \geq 0$ and measurements are in meters. Find the maximum height and width of the arch.

20. A satellite dish for a TV station in Kampala has the cross-sectional equation $y^2 = 24x$ (cm). If the dish is 60 cm in diameter, how deep is it?

13.7 Conclusion

This chapter has explored the fundamental concepts of loci and conic sections with emphasis on practical applications relevant to Uganda's development in telecommunications, infrastructure, agriculture, and engineering. Understanding these geometric relationships provides essential tools for solving real-world spatial problems and designing efficient systems. The mathematical principles of circles, parabolas, and ellipses continue to play crucial roles in modern technology and construction, from satellite communications to architectural designs and agricultural planning.

13.8 More scenario based revision questions with solutions

1. **Irrigation System Design in Wakiso District**
   A farmer in Wakiso district wants to install a circular irrigation system. The system's sprinkler is located at point $A(3, 4)$ and needs to water crops within a radius of 50 meters. A water source is at point $B(20, 16)$.
   (a) Determine the equation of the circular irrigation boundary.
   (b) Find the distance from the water source to the sprinkler.
   (c) Will the water source be inside, outside, or on the irrigation boundary?
2. **Kampala Northern Bypass Construction**
   During the construction of the Kampala Northern Bypass, engineers need to establish the locus of points equidistant from two monitoring stations at $P(-2, 3)$ and $Q(4, -1)$.
   (a) Find the equation of the locus of points equidistant from P and Q.
   (b) Determine if the point $R(1, 1)$ lies on this locus.
3. **Entebbe Airport Radar System**
   The radar system at Entebbe International Airport tracks aircraft within a circular range. The radar station is at the origin, and an aircraft is detected at point $(12, 5)$ km. The radar has a maximum range of 15 km.
   (a) Write the equation of the circle representing the radar's maximum range.
   (b) Determine if the aircraft is within the radar range.
   (c) Find the equation of the tangent to the radar circle at the point where $x = 9$ in the first quadrant.
4. **Nile Breweries Water Pipeline**
   Nile Breweries in Jinja needs to lay a water pipeline. A point on the pipeline must be equidistant from the factory at $F(5, 8)$ and a water treatment plant at distance 10 units from point F.
   (a) If the point moves such that its distance from F is always 10 units, find the equation of its locus.
   (b) Determine the length of the tangent drawn from point $T(5, 20)$ to this circle.
5. **Parabolic Solar Reflector in Tororo**
   A solar energy project in Tororo uses parabolic reflectors. A parabola has its vertex at the origin and focus at $(0, 4)$ meters.
   (a) Find the equation of the parabola.
   (b) Determine the coordinates of a point on the parabola where $x = 8$.

(c) Find the equation of the directrix.

6. **Karuma Dam Arch Design**
The Karuma hydroelectric dam includes a parabolic arch. The arch can be modeled by the parabola $x^2 = -16y$ with vertex at the origin and opening downward.
   (a) Find the focus and directrix of this parabola.
   (b) Determine the equation of the tangent to the parabola at point $(8, -4)$.
   (c) Find the equation of the normal at the same point.

7. **Mbarara Referral Hospital Satellite Dish**
A satellite dish at Mbarara Regional Referral Hospital has a parabolic cross-section with equation $y^2 = 12x$. The receiver is placed at the focus.
   (a) Find the coordinates of the focus where the receiver should be placed.
   (b) Determine the parametric equations of this parabola.
   (c) If a signal hits the dish at point $(3, 6)$, find the equation of the tangent at this point.

8. **Source of the Nile Monument**
Engineers designing the plaza around the Source of the Nile monument in Jinja want to create an elliptical fountain. The fountain has its center at the origin, with the major axis along the x-axis of length 20 meters and minor axis of length 12 meters.
   (a) Find the equation of the ellipse.
   (b) Calculate the eccentricity of the ellipse.
   (c) Determine the coordinates of the foci.

9. **Namugongo Martyrs Shrine Dome**
The dome structure at Namugongo Martyrs Shrine can be approximated by an elliptical cross-section. The equation is $\frac{x^2}{400} + \frac{y^2}{225} = 1$ where measurements are in meters.
   (a) Find the lengths of the semi-major and semi-minor axes.
   (b) Calculate the eccentricity.
   (c) Determine the equations of the directrices.

10. **Kasese Cobalt Mining Survey**
A mining company in Kasese needs to establish a boundary. The locus of points is such that the distance from point $A(2, 3)$ is twice the distance from point $B(6, 7)$.
   (a) Derive the equation of this locus.
   (b) Identify the type of conic section this represents.

11. **Uganda Broadcasting Corporation Tower**
UBC's transmission tower in Kampala has a parabolic microphone array. The parabola has equation $y^2 = 16x$ and a microphone is to be placed at the focus.
   (a) Find the coordinates of the focus.
   (b) Determine the parametric equations using parameter t.
   (c) Find the chord of contact from external point $(4, 8)$ to the parabola.

12. **Kiira Motors Assembly Plant**
The Kiira Motors assembly plant in Jinja has an elliptical test track with equation $\frac{x^2}{900} + \frac{y^2}{400} = 1$ where units are in meters.
   (a) Find the parametric equations of the ellipse.
   (b) Determine the equation of the tangent at the point where $t = \frac{\pi}{4}$.
   (c) Calculate the total length of the major axis.

13. **Two Circular Water Tanks in Gulu**
Two circular water tanks in Gulu municipality have equations $x^2 + y^2 - 6x - 8y = 0$ and $x^2 + y^2 - 14x - 8y + 40 = 0$.
   (a) Find the centers and radii of both circles.
   (b) Determine if the circles intersect externally, internally, or orthogonally.
   (c) Find the equation of the radical axis.

14. **Bujagali Falls Suspension Bridge**
The main cable of the suspension bridge at Bujagali Falls forms a parabola. Taking the lowest point as the origin, the cable passes through points $(50, 20)$ and $(-50, 20)$ where measurements are in meters.
   (a) Find the equation of the parabola.
   (b) Determine the height of the cable at a horizontal distance of 30 meters from the center.
   (c) Find the equation of the normal at point $(50, 20)$.

15. **Makerere University Main Hall Dome**
The architectural design of Makerere University's main hall includes an elliptical dome. The ellipse has vertices at $(\pm 8, 0)$ and co-vertices

at $(0, \pm 6)$ meters.

(a) Write the equation of the ellipse.

(b) Find the eccentricity and foci.

(c) Determine the equation of the tangent at point $(4, 3\sqrt{3})$.

16. **Fort Portal Tea Estate Sprinkler System**
A tea estate in Fort Portal uses circular sprinklers. Two sprinklers have coverage areas with equations $x^2+y^2 = 100$ and $(x-15)^2+y^2 = 64$ where units are in meters.

(a) Determine if the coverage areas overlap.

(b) If they overlap, find the length of the common chord.

(c) Calculate the area watered by both sprinklers (intersection area).

17. **Mbale Coffee Processing Plant**
A coffee processing plant in Mbale uses parabolic reflectors for solar drying. The parabola has vertex at $(2, 3)$ and focus at $(2, 5)$.

(a) Find the equation of the parabola.

(b) Determine the equation of the directrix.

(c) Find the parametric equations of the parabola.

18. **Queen Elizabeth National Park Safari Route**
A safari route in Queen Elizabeth National Park is being designed. The route must maintain equal distance from a straight road (the x-axis) and a ranger station at point $(0, 4)$ km.

(a) Find the equation of the locus of the route.

(b) Identify the conic section formed.

(c) Find the coordinates of the point on the route where $x = 6$.

19. **Nakivubo Stadium Floodlight Coverage**
Two floodlights at Nakivubo Stadium illuminate circular areas. The first has center $(0, 0)$ and radius 30 meters, the second has center $(25, 0)$ and radius 20 meters.

(a) Write the equations of both circles.

(b) Determine the type of intersection.

(c) Find the equation of the common tangent to both circles.

20. **Elliptical Amphitheater at Ndere Centre**
The Ndere Cultural Centre designs an elliptical amphitheater with the stage at one focus. The ellipse has equation $\frac{x^2}{225} + \frac{y^2}{144} = 1$ where units are in meters.

(a) Find the coordinates of both foci.

(b) If a performer stands at one focus, at what point on the other side will sound focus?

(c) Determine the equation of the tangent at point $(9, 9.6)$.

(d) Find the parametric coordinates of a point on the ellipse where the parameter $t = \frac{\pi}{3}$.

Solutions

1. **Irrigation System Design in Wakiso District**
Solution:

(a) The equation of a circle with center $(h, k) = (3, 4)$ and radius $r = 50$ is:

$$(x-3)^2 + (y-4)^2 = 50^2$$

$$(x-3)^2 + (y-4)^2 = 2500$$

(b) Distance from $B(20, 16)$ to $A(3, 4)$:

$$d = \sqrt{(20-3)^2 + (16-4)^2} = \sqrt{17^2 + 12^2} = \sqrt{289 + 144}$$

(c) Since $\sqrt{433} \approx 20.81 < 50$, the water source is **inside** the irrigation boundary.

2. **Kampala Northern Bypass Construction**
Solution:

(a) Let (x, y) be equidistant from $P(-2, 3)$ and $Q(4, -1)$:

$$\sqrt{(x+2)^2 + (y-3)^2} = \sqrt{(x-4)^2 + (y+1)^2}$$

Squaring both sides:

$$(x+2)^2 + (y-3)^2 = (x-4)^2 + (y+1)^2$$

$$x^2+4x+4+y^2-6y+9 = x^2-8x+16+y^2+2y+1$$

$$4x - 6y + 13 = -8x + 2y + 17$$

$$12x - 8y = 4$$

$$3x - 2y = 1$$

(b) For $R(1, 1)$: $3(1) - 2(1) = 3 - 2 = 1$
Yes, point R lies on the locus.

3. **Entebbe Airport Radar System**
Solution:

(a) Equation of radar circle with center at origin and radius 15 km:

$$x^2 + y^2 = 225$$

(b) Distance of aircraft from origin:

$$d = \sqrt{12^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = 13 \text{ km}$$

Since $13 < 15$, the aircraft is **within** radar range.

(c) For $x = 9$ in first quadrant: $81 + y^2 = 225 \Rightarrow y = 12$
Point is $(9, 12)$.
Slope of radius: $m_r = \frac{12}{9} = \frac{4}{3}$
Slope of tangent: $m_t = -\frac{3}{4}$ (perpendicular)
Equation: $y - 12 = -\frac{3}{4}(x - 9)$

$$4y - 48 = -3x + 27$$

$$3x + 4y = 75$$

4. **Nile Breweries Water Pipeline**
Solution:
(a) Circle with center $F(5, 8)$ and radius 10:

$$(x - 5)^2 + (y - 8)^2 = 100$$

(b) Length of tangent from $T(5, 20)$:

$$L = \sqrt{d^2 - r^2}$$

where d is distance from T to center F:

$$d = \sqrt{(5 - 5)^2 + (20 - 8)^2} = 12$$

$$L = \sqrt{144 - 100} = \sqrt{44} = 2\sqrt{11} \approx 6.63 \text{ units}$$

5. **Parabolic Solar Reflector in Tororo**
Solution:
(a) Vertex at origin, focus at $(0, 4)$ means $a = 4$.
Standard form: $x^2 = 4ay$

$$x^2 = 16y$$

(b) When $x = 8$: $64 = 16y \Rightarrow y = 4$
Point: $(8, 4)$
(c) Directrix: $y = -a = -4$

6. **Karuma Dam Arch Design**
Solution:
Given: $x^2 = -16y$
(a) Comparing with $x^2 = -4ay$: $4a = 16 \Rightarrow a = 4$
Focus: $(0, -4)$
Directrix: $y = 4$
(b) For tangent at $(8, -4)$:
Using $xx_1 = -2a(y + y_1)$:

$$x(8) = -8(y + (-4))$$

$$8x = -8y + 32$$

$$x + y = 4$$

(c) Slope of tangent: $m_t = -1$
Slope of normal: $m_n = 1$
Normal: $y - (-4) = 1(x - 8)$

$$y = x - 12$$

7. **Mbarara Referral Hospital Satellite Dish**
Solution:
Given: $y^2 = 12x$
(a) Comparing with $y^2 = 4ax$: $4a = 12 \Rightarrow a = 3$
Focus: $(3, 0)$
(b) Parametric equations:

$$x = 3t^2, \quad y = 6t$$

(c) At $(3, 6)$: Using $yy_1 = 2a(x + x_1)$:

$$y(6) = 6(x + 3)$$

$$6y = 6x + 18$$

$$y = x + 3$$

8. **Source of the Nile Monument**
Solution:
(a) Semi-major axis: $a = 10$, Semi-minor axis: $b = 6$
Equation:

$$\frac{x^2}{100} + \frac{y^2}{36} = 1$$

(b) Eccentricity:

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{36}{100}} = \sqrt{0.64} = 0.8$$

(c) Foci: $(\pm ae, 0) = (\pm 8, 0)$

9. **Namugongo Martyrs Shrine Dome**
Solution:
Given: $\frac{x^2}{400} + \frac{y^2}{225} = 1$
(a) $a^2 = 400 \Rightarrow a = 20$ (semi-major axis)
$b^2 = 225 \Rightarrow b = 15$ (semi-minor axis)
(b) $e = \sqrt{1 - \frac{225}{400}} = \sqrt{\frac{175}{400}} = \frac{5\sqrt{7}}{20} = \frac{\sqrt{7}}{4} \approx 0.66$
(c) $c = ae = 20 \times \frac{\sqrt{7}}{4} = 5\sqrt{7}$
Directrices: $x = \pm\frac{a}{e} = \pm\frac{20}{\sqrt{7}/4} = \pm\frac{80}{\sqrt{7}} = \pm\frac{80\sqrt{7}}{7}$

10. **Kasese Cobalt Mining Survey**
Solution:
(a) Let (x, y) be any point. Distance from A is twice distance from B:

$$\sqrt{(x - 2)^2 + (y - 3)^2} = 2\sqrt{(x - 6)^2 + (y - 7)^2}$$

Squaring:

$$(x-2)^2+(y-3)^2 = 4[(x-6)^2+(y-7)^2]$$

$$x^2-4x+4+y^2-6y+9 = 4[x^2-12x+36+y^2-14y+49]$$

$$x^2-4x+y^2-6y+13 = 4x^2-48x+144+4y^2-56y+196$$

$$0 = 3x^2 - 44x + 3y^2 - 50y + 327$$

$$3x^2 + 3y^2 - 44x - 50y + 327 = 0$$

(b) Dividing by 3:

$$x^2 + y^2 - \frac{44}{3}x - \frac{50}{3}y + 109 = 0$$

This is a **circle** (Apollonius circle).

11. **Uganda Broadcasting Corporation Tower**

Solution:

Given: $y^2 = 16x$

(a) $4a = 16 \Rightarrow a = 4$

Focus: $(4, 0)$

(b) Parametric equations:

$$x = 4t^2, \quad y = 8t$$

(c) Chord of contact from $(4, 8)$ using $yy_1 = 2a(x + x_1)$:

$$y(8) = 8(x + 4)$$

$$8y = 8x + 32$$

$$y = x + 4$$

12. **Kiira Motors Assembly Plant**

Solution:

Given: $\frac{x^2}{900} + \frac{y^2}{400} = 1$

(a) $a = 30, b = 20$

Parametric equations:

$$x = 30\cos t, \quad y = 20\sin t$$

(b) At $t = \frac{\pi}{4}$:

$$x = 30\cdot\frac{1}{\sqrt{2}} = 15\sqrt{2}, \quad y = 20\cdot\frac{1}{\sqrt{2}} = 10\sqrt{2}$$

Tangent equation: $\frac{x\cdot 15\sqrt{2}}{900} + \frac{y\cdot 10\sqrt{2}}{400} = 1$

$$\frac{x\sqrt{2}}{60} + \frac{y\sqrt{2}}{40} = 1$$

Multiply by $\frac{1}{\sqrt{2}}$:

$$\frac{x}{60} + \frac{y}{40} = \frac{1}{\sqrt{2}}$$

$$2x + 3y = 60\sqrt{2}$$

(c) Major axis length: $2a = 60$ meters

13. **Two Circular Water Tanks in Gulu**

Solution:

Circle 1: $x^2 + y^2 - 6x - 8y = 0$

Circle 2: $x^2 + y^2 - 14x - 8y + 40 = 0$

(a) Circle 1: $(x - 3)^2 + (y - 4)^2 = 25$

Center: $C_1(3, 4)$, Radius: $r_1 = 5$

Circle 2: $(x - 7)^2 + (y - 4)^2 = 25$

Center: $C_2(7, 4)$, Radius: $r_2 = 5$

(b) Distance between centers: $d = \sqrt{(7 - 3)^2 + (4 - 4)^2} = 4$

Since $|r_1 - r_2| < d < r_1 + r_2$ (i.e., $0 < 4 < 10$), circles **intersect externally**.

To check orthogonality: $d^2 = r_1^2 + r_2^2$?

$16 \neq 50$, so not orthogonal.

(c) Radical axis (subtract equations):

$$-6x - 8y - (-14x - 8y + 40) = 0$$

$$8x - 40 = 0$$

$$x = 5$$

14. **Bujagali Falls Suspension Bridge**

Solution:

(a) Parabola with vertex at origin, passing through $(50, 20)$:

Form: $y = kx^2$

$20 = k(50)^2 \Rightarrow k = \frac{20}{2500} = \frac{1}{125}$

Equation: $y = \frac{x^2}{125}$ or $x^2 = 125y$

(b) At $x = 30$:

$$y = \frac{900}{125} = 7.2 \text{ meters}$$

(c) At $(50, 20)$:

$\frac{dy}{dx} = \frac{2x}{125} = \frac{100}{125} = \frac{4}{5}$

Slope of normal: $m_n = -\frac{5}{4}$

Normal: $y - 20 = -\frac{5}{4}(x - 50)$

$$4y - 80 = -5x + 250$$

$$5x + 4y = 330$$

15. **Makerere University Main Hall Dome**

Solution:

(a) Vertices at $(\pm 8, 0)$: $a = 8$

Co-vertices at $(0, \pm 6)$: $b = 6$

Equation: $\frac{x^2}{64} + \frac{y^2}{36} = 1$

(b) $e = \sqrt{1 - \frac{36}{64}} = \sqrt{\frac{28}{64}} = \frac{\sqrt{7}}{4} \approx 0.66$

Foci: $(\pm ae, 0) = (\pm 2\sqrt{7}, 0)$

(c) Tangent at $(4, 3\sqrt{3})$:

$$\frac{4x}{64} + \frac{3\sqrt{3} \cdot y}{36} = 1$$

$$\frac{x}{16} + \frac{\sqrt{3}y}{12} = 1$$

$$3x + 4\sqrt{3}y = 48$$

16. **Fort Portal Tea Estate Sprinkler System**

Solution:

Circle 1: $x^2 + y^2 = 100$ (center $(0, 0)$, radius 10)

Circle 2: $(x - 15)^2 + y^2 = 64$ (center $(15, 0)$, radius 8)

(a) Distance between centers: $d = 15$

Since $|r_1 - r_2| < d < r_1 + r_2$ (i.e., $2 < 15 < 18$), circles **overlap**.

(b) Common chord from radical axis:

$x^2 + y^2 - 100 = (x - 15)^2 + y^2 - 64$

$x^2 + y^2 - 100 = x^2 - 30x + 225 + y^2 - 64$

$-100 = -30x + 161$

$30x = 261$

$x = 8.7$

At $x = 8.7$: $y^2 = 100 - 75.69 = 24.31 \Rightarrow y = \pm 4.93$

Length of common chord: $2(4.93) \approx 9.86$ meters

(c) This requires integration; approximate answer: ≈ 38 square meters

17. **Mbale Coffee Processing Plant**

Solution:

(a) Vertex: $(2, 3)$, Focus: $(2, 5)$

Distance from vertex to focus: $a = 2$

Since focus is above vertex, opens upward:

$$(x - 2)^2 = 8(y - 3)$$

(b) Directrix is below vertex by distance a:

$$y = 3 - 2 = 1$$

(c) Using parameter t:

$$x = 2 + 4t, \quad y = 3 + 2t^2$$

18. **Queen Elizabeth National Park Safari Route**

Solution:

(a) Distance from (x, y) to x-axis equals distance to $(0, 4)$:

$$|y| = \sqrt{x^2 + (y - 4)^2}$$

$$y^2 = x^2 + y^2 - 8y + 16$$

$$0 = x^2 - 8y + 16$$

$$x^2 = 8y - 16 = 8(y - 2)$$

(b) This is a **parabola**.

(c) When $x = 6$:

$$36 = 8(y - 2)$$

$$y = 6.5$$

Point: $(6, 6.5)$

19. **Nakivubo Stadium Floodlight Coverage**

Solution:

(a) Circle 1: $x^2 + y^2 = 900$

Circle 2: $(x - 25)^2 + y^2 = 400$

(b) Distance between centers: $d = 25$

$r_1 + r_2 = 30 + 20 = 50$

$|r_1 - r_2| = 10$

Since $10 < 25 < 50$, circles **intersect at two points**.

(c) For external common tangent, using formula:

Slope perpendicular to line joining centers is 0 (centers on x-axis).

External tangent (upper): $y = 30$

External tangent (lower): $y = -30$

20. **Elliptical Amphitheater at Ndere Centre**

Solution:

Given: $\frac{x^2}{225} + \frac{y^2}{144} = 1$

(a) $a^2 = 225 \Rightarrow a = 15$, $b^2 = 144 \Rightarrow b = 12$

$c = \sqrt{a^2 - b^2} = \sqrt{225 - 144} = 9$

Foci: $(\pm 9, 0)$

(b) Sound from one focus reflects to the other focus: $(9, 0)$ and $(-9, 0)$

(c) Tangent at $(9, 9.6)$:

$$\frac{9x}{225} + \frac{9.6y}{144} = 1$$

$$\frac{x}{25} + \frac{y}{15} = 1$$

$$3x + 5y = 75$$

(d) At $t = \frac{\pi}{3}$:

$$x = 15\cos\frac{\pi}{3} = 15 \cdot \frac{1}{2} = 7.5$$

$$y = 12\sin\frac{\pi}{3} = 12 \cdot \frac{\sqrt{3}}{2} = 6\sqrt{3} \approx 10.39$$

Point: $(7.5, 6\sqrt{3})$ or approximately $(7.5, 10.39)$

Chapter 14

COMPLEX NUMBERS

14.1 Introduction

The evolution of the number system has been driven by the need to solve increasingly complex problems. Early civilizations developed natural numbers (1, 2, 3, ...) for counting cattle, people, and goods. The concept of zero emerged from the need to represent nothing or an empty position. Negative numbers arose from problems involving debt and temperatures below zero. Fractions were developed to handle division and sharing of resources.

However, when mathematicians attempted to solve equations like $x^2 = -1$, they encountered a fundamental problem: no real number, when squared, gives a negative result. This limitation led to the development of imaginary numbers and subsequently complex numbers, which have become indispensable tools in engineering, physics, and modern technology.

In Uganda, complex numbers play a crucial role in the electrical power systems managed by companies like Umeme and UETCL (Uganda Electricity Transmission Company Limited). They are essential for analyzing alternating current (AC) circuits in the hydroelectric power stations at Bujagali, Nalubaale, and Karuma. Telecommunications companies such as MTN Uganda and Airtel use complex number theory in signal processing and antenna design.

14.2 The Algebra of Complex Numbers

14.2.1 The Imaginary Unit

The imaginary unit, denoted by i, is defined as the solution to the equation $x^2 = -1$. Thus:

$$i = \sqrt{-1}$$

From this definition, we obtain:

$$i^2 = -1$$

14.2.2 Powers of i

The powers of i follow a cyclic pattern. Let us examine the first few powers:

$$\begin{aligned} i^1 &= i \\ i^2 &= -1 \\ i^3 &= i^2 \cdot i = -1 \cdot i = -i \\ i^4 &= i^2 \cdot i^2 = (-1)(-1) = 1 \\ i^5 &= i^4 \cdot i = 1 \cdot i = i \\ i^6 &= i^4 \cdot i^2 = 1 \cdot (-1) = -1 \end{aligned}$$

The pattern repeats every four powers. To find i^n for any positive integer n, we divide n by 4 and use the remainder:

- If $n \equiv 0 \pmod 4$, then $i^n = 1$
- If $n \equiv 1 \pmod 4$, then $i^n = i$
- If $n \equiv 2 \pmod 4$, then $i^n = -1$
- If $n \equiv 3 \pmod 4$, then $i^n = -i$

Example 1: Find the values of (a) i^{47} (b) i^{100} (c) i^{-23}

Solution:

(a) $47 = 4 \times 11 + 3$, so $47 \equiv 3 \pmod 4$

Therefore, $i^{47} = -i$

(b) $100 = 4 \times 25$, so $100 \equiv 0 \pmod 4$

Therefore, $i^{100} = 1$

(c) $i^{-23} = \frac{1}{i^{23}}$

$23 = 4 \times 5 + 3$, so $i^{23} = -i$

$i^{-23} = \frac{1}{-i} = \frac{1}{-i} \cdot \frac{i}{i} = \frac{i}{-i^2} = \frac{i}{-(-1)} = \frac{i}{1} = i$

14.2.3 Definition of Complex Numbers

A complex number is a number of the form:

$$z = a + bi$$

where a and b are real numbers, and i is the imaginary unit. The number a is called the **real part** of z, denoted $\text{Re}(z) = a$, and b is called the **imaginary part** of z, denoted $\text{Im}(z) = b$.

Note that:

- If $b = 0$, then $z = a$ is a real number
- If $a = 0$, then $z = bi$ is a purely imaginary number
- The set of complex numbers is denoted by $\mathbb{C}$

14.2.4 Equality of Complex Numbers

Two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ are equal if and only if their real parts are equal and their imaginary parts are equal:

$$a + bi = c + di \iff a = c \text{ and } b = d$$

Example 2: Find the values of x and y if $(2x - 3) + (y + 4)i = 7 + 9i$

Solution:

By equality of complex numbers:

$$2x - 3 = 7 \quad \text{and} \quad y + 4 = 9$$

From the first equation: $2x = 10$, so $x = 5$

From the second equation: $y = 5$

Therefore, $x = 5$ and $y = 5$.

14.2.5 Operations on Complex Numbers

Addition and Subtraction

For complex numbers $z_1 = a + bi$ and $z_2 = c + di$:

$$z_1 + z_2 = (a + c) + (b + d)i$$

$$z_1 - z_2 = (a - c) + (b - d)i$$

Example 3: If $z_1 = 3 + 5i$ and $z_2 = -2 + 7i$, find (a) $z_1 + z_2$ (b) $z_1 - z_2$

Solution:

(a) $z_1 + z_2 = (3 + (-2)) + (5 + 7)i = 1 + 12i$

(b) $z_1 - z_2 = (3 - (-2)) + (5 - 7)i = 5 - 2i$

Multiplication

To multiply complex numbers, we use the distributive property and the fact that $i^2 = -1$:

$(a+bi)(c+di) = ac+adi+bci+bdi^2 = ac+adi+bci-bd = (ac-bd)+(a$

Example 4: Find the product of $(3 + 2i)$ and $(4 - 5i)$

Solution:

$$\begin{aligned}(3 + 2i)(4 - 5i) &= 3(4) + 3(-5i) + 2i(4) + 2i(-5i) \\ &= 12 - 15i + 8i - 10i^2 \\ &= 12 - 7i - 10(-1) \\ &= 12 - 7i + 10 \\ &= 22 - 7i\end{aligned}$$

14.2.6 Complex Conjugate

The complex conjugate of $z = a + bi$, denoted by $\bar{z}$ or z^*, is defined as:

$$\bar{z} = a - bi$$

Properties of conjugates:

1. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
2. $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$
3. $\overline{(\bar{z})} = z$
4. $z + \bar{z} = 2a$ (a real number)
5. $z - \bar{z} = 2bi$ (a purely imaginary number)
6. $z \cdot \bar{z} = a^2 + b^2$ (a non-negative real number)

Example 5: If $z = 4 - 3i$, find (a) $\bar{z}$ (b) $z + \bar{z}$ (c) $z \cdot \bar{z}$

Solution:

(a) $\bar{z} = 4 + 3i$

(b) $z + \bar{z} = (4 - 3i) + (4 + 3i) = 8$

(c) $z \cdot \bar{z} = (4 - 3i)(4 + 3i) = 16 - 12i + 12i - 9i^2 = 16 + 9 = 25$

Division

To divide complex numbers, we multiply both numerator and denominator by the conjugate of the denominator:

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

Example 6: Simplify $\frac{3+2i}{4-5i}$

Solution:

$$\begin{aligned}\frac{3 + 2i}{4 - 5i} &= \frac{(3 + 2i)(4 + 5i)}{(4 - 5i)(4 + 5i)} \\ &= \frac{12 + 15i + 8i + 10i^2}{16 + 25} \\ &= \frac{12 + 23i - 10}{41} \\ &= \frac{2 + 23i}{41} \\ &= \frac{2}{41} + \frac{23}{41}i\end{aligned}$$

14.2.7 Equations with Complex Coefficients

Example 7: Solve the equation $(2 + i)z = 5 - 3i$

Solution:

$$z = \frac{5 - 3i}{2 + i} = \frac{(5 - 3i)(2 - i)}{(2 + i)(2 - i)} = \frac{10 - 5i - 6i + 3i^2}{4 + 1} = \frac{10 - 11i - 3}{5} = \frac{7}{5} - \frac{11}{5}i$$

14.2.8 Quadratic Equations with Complex Roots

For a quadratic equation $ax^2 + bx + c = 0$, the discriminant $\Delta = b^2 - 4ac$ determines the nature of the roots:

- If $\Delta > 0$, the equation has two distinct real roots
- If $\Delta = 0$, the equation has one repeated real root
- If $\Delta < 0$, the equation has two complex conjugate roots

When $\Delta < 0$, the roots are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}$$

Example 8: Solve the equation $x^2 - 4x + 13 = 0$

Solution:

Here, $a = 1$, $b = -4$, $c = 13$

$\Delta = (-4)^2 - 4(1)(13) = 16 - 52 = -36 < 0$

Since $\Delta < 0$, the equation has complex roots:

$$x = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

Therefore, $x = 2 + 3i$ or $x = 2 - 3i$

Scenario-Based Example 9: Electrical Engineering at Umeme

Engineers at Umeme's Lugogo substation are analyzing an AC circuit. The impedance Z (in ohms) of a circuit component satisfies the equation:

$$Z^2 - 8Z + 25 = 0$$

Find the impedance values.

Solution:

Using the quadratic formula with $a = 1$, $b = -8$, $c = 25$:

$$\Delta = (-8)^2 - 4(1)(25) = 64 - 100 = -36$$

$$Z = \frac{8 \pm \sqrt{-36}}{2} = \frac{8 \pm 6i}{2} = 4 \pm 3i \text{ ohms}$$

The impedances are $Z_1 = 4 + 3i$ ohms and $Z_2 = 4 - 3i$ ohms. These represent the resistance (real part = 4 ohms) and reactance (imaginary part = ±3 ohms) of the circuit components.

14.3 Different Forms of Complex Numbers

14.3.1 The Argand Diagram

A complex number $z = a + bi$ can be represented geometrically on the **Argand diagram** (or complex plane), where the horizontal axis represents the real part and the vertical axis represents the imaginary part. The point (a, b) corresponds to the complex number $a + bi$.

The Argand diagram provides a visual representation that connects complex numbers to the Cartesian coordinate system.

14.3.2 Modulus and Argument

Modulus

The **modulus** (or absolute value) of a complex number $z = a + bi$, denoted by $|z|$ or r, is the distance from the origin to the point (a, b) on the Argand diagram:

$$|z| = r = \sqrt{a^2 + b^2}$$

Properties of modulus:

1. $|z| \geq 0$ and $|z| = 0$ if and only if $z = 0$
2. $|z_1 z_2| = |z_1||z_2|$
3. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ for $z_2 \neq 0$
4. $|z|^2 = z\bar{z}$
5. $|z_1 + z_2| \leq |z_1| + |z_2|$ (triangle inequality)

Argument

The **argument** of a complex number $z = a + bi$, denoted by $\arg(z)$ or θ, is the angle (measured in radians) that the line from the origin to the point (a, b) makes with the positive real axis:

$$\tan\theta = \frac{b}{a}$$

The principal argument, denoted $\text{Arg}(z)$, is the unique value of θ in the range $-\pi < \theta \leq \pi$.

Important notes:

- The quadrant must be considered when finding the argument
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Example 10: Find the modulus and principal argument of $z = -3 + 4i$

Solution:

Modulus: $|z| = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

Argument: $\tan\theta = \frac{4}{-3}$

Since the real part is negative and the imaginary part is positive, z lies in the second quadrant.

$\alpha = \arctan\left(\frac{4}{3}\right) \approx 0.927$ radians

$\theta = \pi - \alpha = \pi - 0.927 \approx 2.214$ radians

Therefore, $|z| = 5$ and $\arg(z) \approx 2.214$ radians (or 126.87)

14.3.3 Polar Form

Any complex number $z = a + bi$ can be expressed in polar form as:

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z| = \sqrt{a^2 + b^2}$ and $\theta = \arg(z)$.

This can also be written compactly as:

$$z = r\,\text{cis}\,\theta$$

where $\text{cis}\,\theta = \cos\theta + i\sin\theta$.

To convert from Cartesian to polar form:

1. Calculate $r = \sqrt{a^2 + b^2}$
2. Calculate $\theta = \arg(z)$ considering the quadrant
3. Write $z = r(\cos\theta + i\sin\theta)$

To convert from polar to Cartesian form:

1. Calculate $a = r\cos\theta$
2. Calculate $b = r\sin\theta$
3. Write $z = a + bi$

Example 11: Express $z = 1 + \sqrt{3}i$ in polar form.

Solution:

$r = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1 + 3} = 2$

$\tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3}$

Since both real and imaginary parts are positive, z is in the first quadrant.

$\theta = \arctan(\sqrt{3}) = \frac{\pi}{3}$

Therefore, $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ or $z = 2\,\text{cis}\,\frac{\pi}{3}$

14.3.4 Exponential Form (Euler's Formula)

Using Euler's formula, a complex number can also be expressed as:

$$z = re^{i\theta}$$

where $e^{i\theta} = \cos\theta + i\sin\theta$.

This form is particularly useful in calculations involving products and quotients.

Example 12: Express $z = -2 - 2i$ in exponential form.

Solution:

$r = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$

$\tan\theta = \frac{-2}{-2} = 1$

Since both parts are negative, z is in the third quadrant.

$\theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$ or $\theta = -\frac{3\pi}{4}$ (principal argument)

Therefore, $z = 2\sqrt{2}e^{-i3\pi/4}$

14.3.5 Operations in Polar Form

For $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$:

Multiplication:

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Division:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

Example 13: If $z_1 = 2(\cos 30 + i\sin 30)$ and $z_2 = 3(\cos 45 + i\sin 45)$, find $z_1 z_2$ and $\frac{z_1}{z_2}$.

Solution:

$z_1 z_2 = 2 \times 3[\cos(30 + 45) + i\sin(30 + 45)] = 6(\cos 75 + i\sin 75)$

$\frac{z_1}{z_2} = \frac{2}{3}[\cos(30-45) + i\sin(30-45)] = \frac{2}{3}(\cos(-15) + i\sin(-15))$

14.3.6 Loci in the Complex Plane

A locus is the set of all points satisfying a given condition. In the complex plane, loci can represent geometric shapes.

Common Loci

1. Circle: $|z - z_0| = r$ represents a circle with center z_0 and radius r.

2. Perpendicular bisector: $|z - z_1| = |z - z_2|$ represents the perpendicular bisector of the line segment joining z_1 and z_2.

3. Half-plane: $\text{Re}(z) > a$ or $\text{Im}(z) > b$ represents a half-plane.

4. Ray: $\arg(z - z_0) = \alpha$ represents a ray from z_0 at angle α.

Example 14: Describe the locus of points satisfying $|z - 2 - 3i| = 4$.

Solution:

This represents a circle with center at $2 + 3i$ (point $(2, 3)$ on the Argand diagram) and radius 4.

Example 15: Find the locus of z if $|z - 1| = |z + 1|$.

Solution:

Let $z = x + yi$. Then:

$$|x + yi - 1| = |x + yi + 1|$$

$$|(x - 1) + yi| = |(x + 1) + yi|$$

$$\sqrt{(x-1)^2 + y^2} = \sqrt{(x+1)^2 + y^2}$$

Squaring both sides:

$$(x - 1)^2 + y^2 = (x + 1)^2 + y^2$$

$$x^2 - 2x + 1 = x^2 + 2x + 1$$

$$-2x = 2x$$

$$x = 0$$

The locus is the imaginary axis (the line $x = 0$), which is the perpendicular bisector of the segment joining 1 and -1.

Scenario-Based Example 16: MTN Cell Tower Coverage

MTN Uganda is planning cell tower placement in Kampala. A tower is located at position $z_0 = 3 + 2i$ on a coordinate map (in kilometers). The tower has a coverage radius of 5 km. Find the equation of the coverage boundary and determine if a location at $z = 6 + 5i$ receives signal.

Solution:

The coverage boundary is given by:

$$|z - (3 + 2i)| = 5$$

To check if $z = 6 + 5i$ receives signal:

$|6+5i-3-2i| = |3+3i| = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2} \approx 4.24$ km

Since $4.24 < 5$, the location at $6+5i$ is within coverage and receives signal.

14.3.7 De Moivre's Theorem

De Moivre's theorem states that for any real number θ and any integer n:

$$[\cos\theta + i\sin\theta]^n = \cos(n\theta) + i\sin(n\theta)$$

or equivalently:

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

This theorem is powerful for finding powers and roots of complex numbers.

Powers of Complex Numbers

Example 17: Find $(1+i)^{10}$.

Solution:

First, convert $1+i$ to polar form:

$$r = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\theta = \arctan(1) = \frac{\pi}{4}$$

So $1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

Using De Moivre's theorem:

$$(1+i)^{10} = (\sqrt{2})^{10}\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right)$$

$$= 2^5\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right)$$

$$= 32\left(\cos\left(2\pi+\frac{\pi}{2}\right) + i\sin\left(2\pi+\frac{\pi}{2}\right)\right)$$

$$= 32\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= 32(0 + i\cdot 1)$$

$$= 32i$$

Roots of Complex Numbers

To find the n-th roots of a complex number $z = r(\cos\theta + i\sin\theta)$, we use:

$$z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots, n-1$. This gives n distinct roots.

Example 18: Find all cube roots of $z = 8$.

Solution:

First, express 8 in polar form: $8 = 8(\cos 0 + i\sin 0)$

The cube roots are given by:

$$z^{1/3} = 8^{1/3}\left[\cos\left(\frac{0+2k\pi}{3}\right) + i\sin\left(\frac{0+2k\pi}{3}\right)\right] = 2\left[\cos\left(\frac{2k\pi}{3}\right) + i\sin\right.$$

For $k=0$: $z_0 = 2(\cos 0 + i\sin 0) = 2$

For $k = 1$: $z_1 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -1 + i\sqrt{3}$

For $k = 2$: $z_2 = 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = 2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -1 - i\sqrt{3}$

Therefore, the cube roots of 8 are: 2, $-1+i\sqrt{3}$, and $-1-i\sqrt{3}$.

Example 19: Solve the equation $z^4 = -16$.

Solution:

First, express -16 in polar form: $-16 = 16(\cos\pi + i\sin\pi)$

The fourth roots are:

$$z = 16^{1/4}\left[\cos\left(\frac{\pi+2k\pi}{4}\right) + i\sin\left(\frac{\pi+2k\pi}{4}\right)\right] = 2\left[\cos\left(\frac{(2k+1)\pi}{4}\right) + i\right.$$

For $k=0$: $z_0 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \sqrt{2} + i\sqrt{2}$

For $k = 1$: $z_1 = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = 2\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = -\sqrt{2} + i\sqrt{2}$

For $k = 2$: $z_2 = 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 2\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = -\sqrt{2} - i\sqrt{2}$

For $k = 3$: $z_3 = 2\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = \sqrt{2} - i\sqrt{2}$

Simplification of Trigonometric Expressions

De Moivre's theorem can be used to derive multiple angle formulas and simplify trigonometric expressions.

Example 20: Express $\cos 3\theta$ in terms of $\cos\theta$.

Solution:

By De Moivre's theorem:

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

Expanding the left side using the binomial theorem:

$$\begin{aligned}(\cos\theta + i\sin\theta)^3 &= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \\ &= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)\end{aligned}$$

Equating real parts:

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) \\ &= \cos^3\theta - 3\cos\theta + 3\cos^3\theta \\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

14.4 Applications of Complex Numbers

14.4.1 Alternating Current (AC) Circuits

Complex numbers are essential in electrical engineering for analyzing AC circuits. In Uganda's power grid, managed by UETCL and distributed by Umeme, AC electricity is transmitted at 50 Hz.

In AC circuit analysis:

- Resistance R is represented as a real number
- Reactance X (from inductors and capacitors) is represented as an imaginary number
- Impedance $Z = R + iX$ combines resistance and reactance
- Voltage and current are represented as phasors (complex numbers)

The impedance of circuit components:

- Resistor: $Z_R = R$
- Inductor: $Z_L = i\omega L$ where ω is angular frequency and L is inductance
- Capacitor: $Z_C = \frac{1}{i\omega C} = \frac{-i}{\omega C}$ where C is capacitance

For components in series: $Z_{total} = Z_1 + Z_2 + Z_3 + \ldots$

For components in parallel: $\frac{1}{Z_{total}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \ldots$

Ohm's law for AC circuits: $V = IZ$ where V is voltage, I is current, and Z is impedance.

Scenario-Based Example 21: Bujagali Hydroelectric Power Station

At the Bujagali power station, an AC generator produces voltage $V = 240(\cos\omega t + i\sin\omega t)$ volts. This voltage is applied to a circuit with impedance $Z = 30 + 40i$ ohms. Find:

(a) The magnitude of the impedance

(b) The current flowing through the circuit (as a phasor)

(c) The magnitude of the current

Solution:

(a) $|Z| = \sqrt{30^2 + 40^2} = \sqrt{900 + 1600} = \sqrt{2500} = 50$ ohms

(b) Using $I = \frac{V}{Z}$ (taking V at $t = 0$, so $V = 240$):

$$I = \frac{240}{30+40i} = \frac{240(30-40i)}{(30+40i)(30-40i)} = \frac{240(30-40i)}{900+1600} = \frac{240(30-40}{2500}$$

$$= \frac{7200 - 9600i}{2500} = 2.88 - 3.84i \text{ amperes}$$

(c) $|I| = \sqrt{(2.88)^2 + (-3.84)^2} = \sqrt{8.2944 + 14.7456} = \sqrt{23.04} = 4.8$ amperes

Scenario-Based Example 22: Industrial Motor at Kakira Sugar Works

A three-phase induction motor at Kakira Sugar Works in Jinja has the following impedances in each phase: resistance $R = 5$ ohms and inductive reactance $X_L = 12$ ohms. The supply voltage is 415 V per phase.

(a) Find the total impedance per phase

(b) Calculate the current per phase

(c) Find the power factor

Solution:

(a) $Z = R + iX_L = 5 + 12i$ ohms

$|Z| = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = 13$ ohms

(b) $I = \frac{V}{Z} = \frac{415}{5+12i}$

$I = \frac{415(5-12i)}{(5+12i)(5-12i)} = \frac{415(5-12i)}{169} = \frac{2075-4980i}{169}$

$I \approx 12.28 - 29.47i$ amperes

$|I| = \sqrt{(12.28)^2 + (29.47)^2} \approx 31.92$ amperes

(c) Power factor $= \cos\phi = \frac{R}{|Z|} = \frac{5}{13} \approx 0.385$ (lagging)

14.4.2 Signal Processing and Telecommunications

Telecommunications companies in Uganda (MTN, Airtel, Africell) use complex numbers extensively in signal processing, particularly in:

- Fourier transforms for frequency analysis
- Digital signal processing
- Antenna design and radiation patterns
- Modulation and demodulation of signals

A signal can be represented as: $s(t) = Ae^{i(\omega t+\phi)}$ where A is amplitude, ω is angular frequency, and ϕ is phase.

Scenario-Based Example 23: Mobile Network Signal Analysis

An Airtel base station in Mbarara transmits a signal represented by $s_1(t) = 5e^{i(2\pi\times10^9 t+\pi/4)}$ watts. A second signal from interference is $s_2(t) = 3e^{i(2\pi\times10^9 t-\pi/3)}$ watts. Find the resultant signal at $t = 0$.

Solution:

At $t = 0$:

$$s_1(0) = 5e^{i\pi/4} = 5\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 5\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \frac{5\sqrt{2}}{2}(1+i)$$

$$s_2(0) = 3e^{-i\pi/3} = 3\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) = 3\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = \frac{3}{2}(1-i\sqrt{3})$$

Resultant signal:

$$s(0) = s_1(0) + s_2(0) = \frac{5\sqrt{2}}{2}(1+i) + \frac{3}{2}(1-i\sqrt{3})$$

$$= \frac{5\sqrt{2}}{2} + i\frac{5\sqrt{2}}{2} + \frac{3}{2} - i\frac{3\sqrt{3}}{2}$$

$$= \left(\frac{5\sqrt{2}+3}{2}\right) + i\left(\frac{5\sqrt{2}-3\sqrt{3}}{2}\right)$$

The magnitude of the resultant signal:

$$|s(0)| = \sqrt{\left(\frac{5\sqrt{2}+3}{2}\right)^2 + \left(\frac{5\sqrt{2}-3\sqrt{3}}{2}\right)^2}$$

14.4.3 Mechanical Vibrations and Oscillations

Complex numbers are used to model periodic motion, such as:

- Vibrations in bridges and buildings
- Oscillations in vehicle suspension systems
- Pendulum motion
- Wave motion

The displacement of an oscillating system can be written as:

$$x(t) = Ae^{i\omega t}$$

where A is the amplitude and ω is the angular frequency.

Scenario-Based Example 24: Source of the Nile Bridge

The Source of the Nile Bridge in Jinja experiences periodic vibrations due to traffic. Engineers model the vertical displacement of a point on the bridge as:

$$y(t) = \text{Re}[10e^{i(0.5t+\pi/6)}]$$

meters

Find the displacement at $t = \pi$ seconds.

Solution:

$$y(\pi) = \text{Re}[10e^{i(0.5\pi+\pi/6)}] = \text{Re}[10e^{i(3\pi/6+\pi/6)}] = \text{Re}[10e^{i(4\pi/6)}]$$

$$= \text{Re}[10e^{i(2\pi/3)}] = \text{Re}\left[10\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]$$

$$= 10\cos\frac{2\pi}{3} = 10\times\left(-\frac{1}{2}\right) = -5 \text{ meters}$$

The negative sign indicates downward displacement of 5 meters.

14.4.4 Fluid Dynamics

Complex numbers are used in fluid flow analysis through complex potential functions. This is relevant for:

- Water flow in irrigation systems
- Air flow around structures
- Flow in pipes and channels

Scenario-Based Example 25: Lake Victoria Water Intake

The National Water and Sewerage Corporation (NWSC) uses complex potential theory to model water flow at the Gaba intake works on Lake Victoria. The complex potential is given by:

$$w(z) = A\ln(z)$$

where $z = x + iy$ represents position in meters and A is a constant. If streamlines are given by $\text{Im}[w(z)] =$ constant, find the equation of the streamline passing through $z = 1 + i$.

Solution:

$$w(z) = A\ln(z) = A[\ln|z| + i\arg(z)]$$

$$\text{Im}[w(z)] = A\arg(z)$$

At $z = 1 + i$:

$$\arg(1 + i) = \arctan(1) = \frac{\pi}{4}$$

Therefore, the streamline equation is:

$$\arg(z) = \frac{\pi}{4}$$

This represents a ray from the origin at 45 to the positive real axis.

14.4.5 Quantum Mechanics and Wave Functions

Although beyond the A-level syllabus scope, it's worth noting that complex numbers are fundamental in quantum mechanics, particularly in wave functions. This has applications in:

- Development of semiconductor devices
- Solar panel technology
- Modern electronics

14.5 Practice Exercises

14.5.1 Exercise Set 1: Algebra of Complex Numbers

1. Simplify: (a) i^{57} (b) i^{-44} (c) $(2i)^6$

2. If $z_1 = 3 - 2i$ and $z_2 = -1 + 4i$, find: (a) $z_1 + z_2$ (b) $z_1 - z_2$ (c) $z_1 z_2$ (d) $\frac{z_1}{z_2}$

3. Find the values of x and y if $(x+2)+(3y-1)i = 5-8i$

4. Solve for z: $(3 - 2i)z + (1 + i) = 7 - 5i$

5. If $z = 2 + 3i$, find: (a) $\bar{z}$ (b) $z + \bar{z}$ (c) $z\bar{z}$ (d) z^2

6. Solve the quadratic equation: $2x^2 - 6x + 5 = 0$

7. Prove that if z_1 and z_2 are complex numbers, then $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

8. Show that for any complex number z, $z + \bar{z}$ is real and $z - \bar{z}$ is purely imaginary.

9. A circuit has impedance $(4+3i)$ ohms in series with impedance $(2 - 5i)$ ohms. Find the total impedance.

10. Two impedances $Z_1 = 6+8i$ ohms and $Z_2 = 3-4i$ ohms are connected in parallel. Find the equivalent impedance using $\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$.

14.5.2 Exercise Set 2: Argand Diagram and Polar Form

11. Plot the following complex numbers on an Argand diagram: (a) $3 + 4i$ (b) $-2 + 5i$ (c) $-3 - 2i$ (d) $4 - 3i$

12. Find the modulus and principal argument of: (a) $z = 1 - i$ (b) $z = -\sqrt{3} + i$ (c) $z = -5$ (d) $z = 4i$

13. Express in polar form: (a) $z = 2 + 2i$ (b) $z = -1 + \sqrt{3}i$ (c) $z = 3 - 3\sqrt{3}i$

14. Convert to Cartesian form: (a) $z = 4(\cos 60 + i\sin 60)$ (b) $z = 2(\cos 135 + i\sin 135)$

15. If $z_1 = 2(\cos 30 + i\sin 30)$ and $z_2 = 3(\cos 120 + i\sin 120)$, find: (a) $z_1 z_2$ (b) $\frac{z_1}{z_2}$ (c) z_1^2

16. Describe and sketch the locus of points in the complex plane satisfying: (a) $|z| = 3$ (b) $|z - 1 - 2i| = 2$ (c) $\arg(z) = \frac{\pi}{3}$

17. Find the locus of z if $|z - 3i| = |z + 3i|$

18. A radio transmitter at position $2 + 3i$ km has a range of 10 km. Write the equation for the boundary

of its coverage area. Determine if a receiver at $8 + 10i$ km is within range.

14.5.3 Exercise Set 3: De Moivre's Theorem

19. Use De Moivre's theorem to find: (a) $(1 + i\sqrt{3})^6$ (b) $(\sqrt{3} - i)^{10}$ (c) $\left(\frac{1-i}{1+i}\right)^{20}$

20. Find all the fourth roots of $z = 16$.

21. Solve the equation $z^5 = 1$.

22. Find all cube roots of $-8i$.

23. Use De Moivre's theorem to show that:

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

24. Express $\cos 4\theta$ in terms of $\cos\theta$ using De Moivre's theorem.

25. Prove that $(\cos\theta + i\sin\theta)^n(\cos\theta - i\sin\theta)^n = 1$ for any integer n.

14.5.4 Exercise Set 4: Applications

26. An AC voltage $V = 220(\cos\omega t + i\sin\omega t)$ volts is applied to a circuit with impedance $Z = 40 + 30i$ ohms. Find: (a) The magnitude of impedance (b) The current (as a complex number) (c) The magnitude of current (d) The phase angle between voltage and current

27. At the Nalubaale power station, a transmission line has resistance $R = 8$ ohms and inductive reactance $X_L = 15$ ohms. If the current is $I = 20 - 10i$ amperes, find: (a) The impedance (b) The voltage across the line (c) The magnitude of the voltage

28. Two signals $s_1 = 4e^{i\pi/6}$ and $s_2 = 3e^{i\pi/3}$ are combined. Find: (a) The resultant signal in Cartesian form (b) The magnitude of the resultant signal

29. A suspension bridge cable oscillates with displacement:

$$y(t) = \text{Re}[15e^{i(0.2t+\pi/4)}]$$

meters Find the displacement at $t = 5\pi$ seconds.

30. The complex potential for flow around a cylinder is given by:

$$w(z) = z + \frac{1}{z}$$

Find $w(2i)$ and interpret the result.

14.5.5 Challenge Problems

31. Prove that for any complex number z, $|z|^2 = z\bar{z}$.

32. Show that the sum of the n-th roots of unity is zero.

33. If z_1, z_2, and z_3 are the vertices of an equilateral triangle in the complex plane, prove that:

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

34. Solve the equation $z^4 + 4 = 0$ and represent the solutions on an Argand diagram.

35. An electrical engineer at the Karuma hydroelectric project needs to analyze a complex circuit with three impedances: $Z_1 = 12 + 5i$ ohms, $Z_2 = 8 - 6i$ ohms, and $Z_3 = 15 + 20i$ ohms connected in series. A voltage of $V = 500(\cos\omega t + i\sin\omega t)$ volts is applied. Find the total impedance, current, and power dissipated (real power = $\text{Re}[VI^*]$ where I^* is the complex conjugate of current).

14.6 Summary

Complex numbers extend the real number system to include solutions to equations like $x^2 = -1$. Key concepts include:

Algebraic Form: $z = a + bi$ where $i^2 = -1$

Operations: Complex numbers can be added, subtracted, multiplied, and divided using algebraic rules and the conjugate.

Geometric Representation: Complex numbers can be represented on the Argand diagram, connecting algebra to geometry.

Polar Form: $z = r(\cos\theta + i\sin\theta)$ where $r = |z|$ and $\theta = \arg(z)$

Exponential Form: $z = re^{i\theta}$ using Euler's formula

De Moivre's Theorem: $[\cos\theta + i\sin\theta]^n = \cos(n\theta) + i\sin(n\theta)$

Applications: Complex numbers are essential in:

- Electrical engineering (AC circuit analysis)
- Signal processing and telecommunications
- Mechanical vibrations
- Fluid dynamics
- Control systems

In Uganda, these applications are evident in the electrical power sector (Umeme, UETCL, hydroelectric projects), telecommunications (MTN, Airtel), industrial operations (sugar factories, manufacturing), and infrastructure development (bridges, buildings).

Understanding complex numbers provides powerful tools for solving real-world engineering and scientific problems, making them indispensable in modern technology and development.

14.7 Further Reading and Resources

For deeper exploration of complex numbers and their applications:

- Complex Variables and Applications by Brown and Churchill
- Visual Complex Analysis by Tristan Needham
- Engineering Mathematics by K.A. Stroud
- Uganda's National Curriculum Development Centre (NCDC) resources
- IEEE publications on power systems analysis
- Research papers from Makerere University's College of Engineering, Design, Art and Technology (CEDAT)

Online resources from UETCL, Umeme, and telecommunications companies provide practical examples of complex number applications in Uganda's infrastructure and technology sectors.

14.8 More scenario based revision questions with solutions

1. **Karuma Hydropower Dam - AC Circuit Analysis**
   The Karuma hydropower station uses three-phase AC generators. The voltage in one phase is represented by the complex number $V = 240(\cos 60 + i \sin 60)$ volts, and the current is $I = 15(\cos 30 + i \sin 30)$ amperes.
   (a) Express V and I in rectangular form.
   (b) Calculate the complex power $S = VI^*$ where I^* is the complex conjugate of I.
   (c) Determine the real power (active power) delivered by this phase.

   Solution:

   (a) Converting to rectangular form:

$$\begin{aligned} V &= 240(\cos 60 + i \sin 60) \\ &= 240\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ &= 120 + 120\sqrt{3}i \text{ volts} \end{aligned}$$

$$\begin{aligned} I &= 15(\cos 30 + i \sin 30) \\ &= 15\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) \\ &= \frac{15\sqrt{3}}{2} + \frac{15}{2}i \text{ amperes} \end{aligned}$$

   (b) Finding I^*:

$$I^* = \frac{15\sqrt{3}}{2} - \frac{15}{2}i$$

   Complex power:

$$\begin{aligned} S &= VI^* \\ &= (120 + 120\sqrt{3}i)\left(\frac{15\sqrt{3}}{2} - \frac{15}{2}i\right) \\ &= 120 \cdot \frac{15\sqrt{3}}{2} - 120 \cdot \frac{15}{2}i + 120\sqrt{3}i \cdot \frac{15\sqrt{3}}{2} - 120\sqrt{3}i \cdot \\ &= 900\sqrt{3} - 900i + 2700i + 900\sqrt{3} \\ &= 1800\sqrt{3} + 1800i \\ &= 3117.69 + 1800i \text{ VA} \end{aligned}$$

   (c) The real power (active power) is the real part of S:

$$P = 1800\sqrt{3} \approx 3117.69 \text{ watts}$$

2. **MTN Uganda - Signal Processing**
   MTN Uganda's 4G network tower in Kampala transmits signals that can be modeled by the complex equation $z^4 = -16$. Find all possible signal frequencies (values of z) in polar form.
   Solution:
   First, express -16 in polar form:

$$-16 = 16(\cos 180 + i \sin 180)$$

   Using De Moivre's theorem, if $z^4 = 16(\cos 180 + i \sin 180)$, then:

$$z = \sqrt[4]{16}\left[\cos\left(\frac{180 + 360k}{4}\right) + i \sin\left(\frac{180 + 360k}{4}\right)\right]$$

   where $k = 0, 1, 2, 3$.

$$z = 2\left[\cos\left(\frac{180+360k}{4}\right) + i\sin\left(\frac{180+360k}{4}\right)\right]$$

For $k=0$: $z_1 = 2(\cos 45 + i\sin 45)$
For $k=1$: $z_2 = 2(\cos 135 + i\sin 135)$
For $k=2$: $z_3 = 2(\cos 225 + i\sin 225)$
For $k=3$: $z_4 = 2(\cos 315 + i\sin 315)$

3. **Coffee Processing - Temperature Oscillations**
Kaweri Coffee Plantation in Mubende uses drying chambers where temperature oscillates. The temperature T (in °C above ambient) at time t hours is modeled by $T(t) = 5e^{i\pi t/6}$.
   (a) Express this in the form $T(t) = A\cos(\omega t) + iB\sin(\omega t)$.
   (b) Find the temperature oscillation at $t = 3$ hours.
   (c) Determine the period of oscillation.

   Solution:
   (a) Using Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

$$T(t) = 5\left(\cos\frac{\pi t}{6} + i\sin\frac{\pi t}{6}\right)$$

$$T(t) = 5\cos\frac{\pi t}{6} + 5i\sin\frac{\pi t}{6}$$

   where $A = 5$, $B = 5$, and $\omega = \frac{\pi}{6}$ rad/hour.
   (b) At $t = 3$ hours:

$$\begin{aligned} T(3) &= 5\cos\frac{\pi\cdot 3}{6} + 5i\sin\frac{\pi\cdot 3}{6} \\ &= 5\cos\frac{\pi}{2} + 5i\sin\frac{\pi}{2} \\ &= 5(0) + 5i(1) \\ &= 5i \end{aligned}$$

   The real temperature oscillation is $\mathrm{Re}(T(3)) = 0$°C above ambient.
   (c) The period is:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi/6} = 12 \text{ hours}$$

4. **Entebbe Airport - Aircraft Navigation**
An aircraft leaving Entebbe International Airport follows a flight path where its position relative to the control tower is given by $z(t) = (3+4i) + (2-i)t$ km, where t is time in hours.
   (a) Find the initial position and velocity of the aircraft.
   (b) Calculate the distance of the aircraft from the control tower after 2 hours.
   (c) Determine the direction (argument) of the aircraft's velocity vector.

   Solution:
   (a) Initial position at $t = 0$:

$$z(0) = 3 + 4i$$

   This means 3 km East and 4 km North of the control tower.
   Velocity vector: $v = 2 - i$ km/h This means 2 km/h East and 1 km/h South.
   (b) At $t = 2$ hours:

$$\begin{aligned} z(2) &= (3+4i) + (2-i)(2) \\ &= 3 + 4i + 4 - 2i \\ &= 7 + 2i \end{aligned}$$

   Distance from control tower:

$$|z(2)| = \sqrt{7^2 + 2^2} = \sqrt{49+4} = \sqrt{53} \approx 7.28 \text{ km}$$

   (c) Direction of velocity:

$$\arg(v) = \arg(2-i) = \tan^{-1}\left(\frac{-1}{2}\right) = -26.57$$

   The aircraft is traveling at 26.57° South of East.

5. **Kampala Traffic Flow - Wave Propagation**
Traffic flow along Kampala Northern Bypass can be modeled as a wave. If the density of vehicles is represented by $(3+4i)$ and the flow rate by $(1-2i)$, find:
   (a) The impedance $Z = \frac{\text{density}}{\text{flow rate}}$ in rectangular and polar forms.
   (b) The modulus and argument of Z.

   Solution:
   (a) Impedance in rectangular form:

$$\begin{aligned} Z &= \frac{3+4i}{1-2i} \\ &= \frac{(3+4i)(1+2i)}{(1-2i)(1+2i)} \\ &= \frac{3+6i+4i+8i^2}{1-4i^2} \\ &= \frac{3+10i-8}{1+4} \\ &= \frac{-5+10i}{5} \\ &= -1+2i \end{aligned}$$

   For polar form, we need modulus and argument:

$$|Z| = \sqrt{(-1)^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$$

$$\arg(Z) = \tan^{-1}\left(\frac{2}{-1}\right) = 180-\tan^{-1}(2) = 180-63.43 = 116.57$$

Therefore:

$$Z = \sqrt{5}(\cos 116.57 + i \sin 116.57)$$

(b) Modulus: $|Z| = \sqrt{5} \approx 2.236$
Argument: $\arg(Z) = 116.57$

6. **Kakira Sugar Works - Centrifuge Motion**
A sugar centrifuge at Kakira Sugar Works rotates such that a sugar crystal's position is given by $z = 5e^{i3t}$ meters, where t is time in seconds. Use De Moivre's theorem to find:
(a) The position after $t = \frac{\pi}{3}$ seconds.
(b) The angular velocity.
(c) Express z^3 in terms of t.
Solution:
(a) At $t = \frac{\pi}{3}$:

$$\begin{aligned} z &= 5e^{i3\cdot\frac{\pi}{3}} \\ &= 5e^{i\pi} \\ &= 5(\cos\pi + i\sin\pi) \\ &= 5(-1+0i) \\ &= -5 \text{ meters} \end{aligned}$$

(b) The angular velocity is the coefficient of t in the exponent:

$$\omega = 3 \text{ rad/s}$$

(c) Using De Moivre's theorem:

$$\begin{aligned} z^3 &= (5e^{i3t})^3 \\ &= 5^3 e^{i9t} \\ &= 125e^{i9t} \end{aligned}$$

7. **Bujagali Hydropower - Impedance Matching**
At Bujagali hydropower station, the impedance of a transmission line is $Z_1 = 40 + 30i$ ohms and the load impedance is $Z_2 = 20 - 10i$ ohms.
(a) Find the total impedance $Z_T = Z_1 + Z_2$.
(b) Calculate the reflection coefficient $\Gamma = \frac{Z_2 - Z_1^*}{Z_2 + Z_1^*}$ where Z_1^* is the conjugate of Z_1.
(c) Express Γ in polar form.
Solution:
(a) Total impedance:

$$Z_T = (40+30i)+(20-10i) = 60+20i \text{ ohms}$$

(b) First, find Z_1^*:

$$Z_1^* = 40 - 30i$$

Now calculate Γ:

$$\begin{aligned} \Gamma &= \frac{(20-10i)-(40-30i)}{(20-10i)+(40-30i)} \\ &= \frac{-20+20i}{60-40i} \end{aligned}$$

Rationalize:

$$\begin{aligned} \Gamma &= \frac{(-20+20i)(60+40i)}{(60-40i)(60+40i)} \\ &= \frac{-1200-800i+1200i+800i^2}{3600+1600} \\ &= \frac{-1200+400i-800}{5200} \\ &= \frac{-2000+400i}{5200} \\ &= -\frac{5}{13} + \frac{1}{13}i \end{aligned}$$

(c) For polar form:

$$|\Gamma| = \sqrt{\left(\frac{-5}{13}\right)^2 + \left(\frac{1}{13}\right)^2} = \sqrt{\frac{25+1}{169}} = \frac{\sqrt{26}}{13}$$

$$\arg(\Gamma) = 180-\tan^{-1}\left(\frac{1}{5}\right) = 180-11.31 = 168.69$$

$$\Gamma = \frac{\sqrt{26}}{13}(\cos 168.69 + i \sin 168.69)$$

8. **Mbarara Milk Processing - Cooling System**
At Mbarara milk cooling center, the temperature variation follows $T = 20 - 15e^{-0.1t}(\cos 0.5t + i \sin 0.5t)$°C, where t is time in minutes.
(a) Express the imaginary exponential in rectangular form at $t = \pi$ minutes.
(b) Find the real temperature at $t = \pi$ minutes.
Solution:
(a) At $t = \pi$ minutes:

$$\begin{aligned} e^{-0.1\pi}(\cos 0.5\pi + i\sin 0.5\pi) &= e^{-0.1\pi}(0+i) \\ &= e^{-0.314}i \\ &\approx 0.731i \end{aligned}$$

(b) The real temperature:

$$\begin{aligned} T(\pi) &= 20 - 15e^{-0.1\pi}(\cos 0.5\pi) \\ &= 20 - 15(0.731)(0) \\ &= 20\text{C} \end{aligned}$$

9. **Namanve Industrial Park - Motor Control**
An electric motor at Namanve Industrial Park has a transfer function with poles at $z_1 = -2+3i$ and $z_2 = -2-3i$.
   (a) Verify that z_1 and z_2 are complex conjugates.
   (b) Find $z_1 \cdot z_2$.
   (c) Calculate $|z_1|$ and interpret its significance for motor stability.

   Solution:
   (a) If $z_1 = -2+3i$, then:
   $$\overline{z_1} = -2-3i = z_2$$
   Yes, they are complex conjugates.
   (b) Product of conjugates:
   $$\begin{aligned} z_1 \cdot z_2 &= (-2+3i)(-2-3i) \\ &= 4+6i-6i-9i^2 \\ &= 4+9 \\ &= 13 \end{aligned}$$
   (c) Modulus:
   $$|z_1| = \sqrt{(-2)^2+3^2} = \sqrt{4+9} = \sqrt{13} \approx 3.606$$
   Since $|z_1| > 1$ and the real part is negative, the system is stable but oscillatory.

10. **Tororo Cement Factory - Vibration Analysis**
A grinding machine at Tororo Cement produces vibrations modeled by $z = 4(\cos 120 + i \sin 120)$. Use De Moivre's theorem to find z^5.

    Solution:
    Using De Moivre's theorem: $(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$
    $$\begin{aligned} z^5 &= 4^5(\cos(5\times 120) + i\sin(5\times 120)) \\ &= 1024(\cos 600 + i\sin 600) \\ &= 1024(\cos 240 + i\sin 240) \quad (\text{since } 600 = 240+360) \\ &= 1024\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \\ &= -512 - 512\sqrt{3}i \end{aligned}$$

11. **Uganda Railways - Track Alignment**
A proposed railway line from Kampala to Malaba has waypoints at positions $A = 5+12i$ km and $B = 9+20i$ km relative to Kampala.
   (a) Find the distance between points A and B.
   (b) Determine the midpoint of segment AB.
   (c) Find the direction from A to B (argument of $B-A$).

   Solution:
   (a) Distance:
   $$\begin{aligned} |B-A| &= |(9+20i)-(5+12i)| \\ &= |4+8i| \\ &= \sqrt{4^2+8^2} \\ &= \sqrt{16+64} \\ &= \sqrt{80} = 4\sqrt{5} \approx 8.94 \text{ km} \end{aligned}$$
   (b) Midpoint:
   $$M = \frac{A+B}{2} = \frac{(5+12i)+(9+20i)}{2} = \frac{14+32i}{2} = 7+16i$$
   (c) Direction:
   $$\begin{aligned} \arg(B-A) &= \arg(4+8i) \\ &= \tan^{-1}\left(\frac{8}{4}\right) \\ &= \tan^{-1}(2) \\ &= 63.43 \end{aligned}$$
   The railway line runs $63.43°$ North of East from A to B.

12. **Masindi Port - Wave Motion**
Water waves at Masindi Port on Lake Albert are represented by $w = 3e^{i(\pi t/4 - \pi/6)}$, where t is time in seconds.
   (a) Express w in rectangular form at $t = 2$ seconds.
   (b) Find the amplitude and phase of the wave.

   Solution:
   (a) At $t = 2$ seconds:
   $$\begin{aligned} w(2) &= 3e^{i(\pi\cdot 2/4 - \pi/6)} \\ &= 3e^{i(\pi/2 - \pi/6)} \\ &= 3e^{i\pi/3} \\ &= 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \\ &= 3\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ &= \frac{3}{2} + \frac{3\sqrt{3}}{2}i \\ &= 1.5 + 2.598i \end{aligned}$$
   (b) Amplitude: $|w| = 3$ meters
   Initial phase: $\phi = -\frac{\pi}{6}$ radians or -30

13. **Nile Breweries - Fermentation Tank**
A fermentation tank at Nile Breweries has a control system with characteristic equation $z^3 = -8i$. Find all three roots in both polar and rectangular forms.
Solution:
Express $-8i$ in polar form:

$$-8i = 8(\cos 270 + i \sin 270)$$

Using De Moivre's theorem:

$$z = \sqrt[3]{8}\left[\cos\left(\frac{270+360k}{3}\right) + i\sin\left(\frac{270+360k}{3}\right)\right]$$

where $k = 0, 1, 2$.

$$z = 2\left[\cos\left(\frac{270+360k}{3}\right) + i\sin\left(\frac{270+360k}{3}\right)\right]$$

For $k = 0$:

$$z_1 = 2(\cos 90 + i \sin 90) = 2i$$

For $k = 1$:

$$z_2 = 2(\cos 210 + i\sin 210) = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\sqrt{3} - i$$

For $k = 2$:

$$z_3 = 2(\cos 330 + i\sin 330) = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \sqrt{3} - i$$

14. **Soroti Solar Farm - Power Factor Correction**
Soroti Solar Farm has an inverter output of $V = 230 + 50i$ volts and current $I = 10 - 5i$ amperes.
(a) Calculate the apparent power $S = VI^*$.
(b) Find the power factor, which is $\cos(\arg(S))$.
(c) Determine the reactive power (imaginary part of S).
Solution:
(a) First find I^*:

$$I^* = 10 + 5i$$

Apparent power:

$$\begin{aligned} S &= (230 + 50i)(10 + 5i) \\ &= 2300 + 1150i + 500i + 250i^2 \\ &= 2300 + 1650i - 250 \\ &= 2050 + 1650i \text{ VA} \end{aligned}$$

(b) Find $\arg(S)$:

$$\arg(S) = \tan^{-1}\left(\frac{1650}{2050}\right) = \tan^{-1}(0.8049) = 38.82$$

Power factor:

$$\text{PF} = \cos(38.82) = 0.779 \text{ or } 77.9\%$$

(c) Reactive power:

$$Q = \text{Im}(S) = 1650 \text{ VAR}$$

15. **Gulu University - Antenna Design**
An antenna at Gulu University transmits at frequency corresponding to z where $z^4 = 81$. Find all possible values of z in rectangular form.
Solution:
Express 81 in polar form:

$$81 = 81(\cos 0 + i \sin 0)$$

Using De Moivre's theorem:

$$z = \sqrt[4]{81}\left[\cos\left(\frac{0+360k}{4}\right) + i\sin\left(\frac{0+360k}{4}\right)\right]$$

where $k = 0, 1, 2, 3$.

$$z = 3\left[\cos(90k) + i\sin(90k)\right]$$

For $k = 0$: $z_1 = 3(\cos 0 + i \sin 0) = 3$
For $k = 1$: $z_2 = 3(\cos 90 + i \sin 90) = 3i$
For $k = 2$: $z_3 = 3(\cos 180 + i \sin 180) = -3$
For $k = 3$: $z_4 = 3(\cos 270 + i \sin 270) = -3i$

16. **Kilembe Mines - Oscillating Equipment**
Equipment at Kilembe Mines vibrates with displacement $d = 10e^{-0.2t+i4t}$ mm, where t is time in seconds.
(a) Separate into real and imaginary parts.
(b) Find the displacement at $t = \frac{\pi}{8}$ seconds.
(c) Determine the damping factor and oscillation frequency.
Solution:
(a) Rewrite as:

$$\begin{aligned} d &= 10e^{-0.2t} \cdot e^{i4t} \\ &= 10e^{-0.2t}(\cos 4t + i \sin 4t) \\ &= 10e^{-0.2t}\cos 4t + 10ie^{-0.2t}\sin 4t \end{aligned}$$

Real part: $\text{Re}(d) = 10e^{-0.2t}\cos 4t$ mm
Imaginary part: $\text{Im}(d) = 10e^{-0.2t}\sin 4t$ mm

(b) At $t = \frac{\pi}{8}$:

$$d\left(\frac{\pi}{8}\right) = 10e^{-0.2\cdot\pi/8}\left(\cos\frac{4\pi}{8} + i\sin\frac{4\pi}{8}\right)$$
$$= 10e^{-0.0785}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$
$$= 10(0.924)(0+i)$$
$$= 9.24i \text{ mm}$$

(c) Damping factor: $\delta = 0.2\ \text{s}^{-1}$
Oscillation frequency: $\omega = 4$ rad/s, or $f = \frac{4}{2\pi} = 0.637$ Hz

17. **Jinja Bridge - Structural Analysis**
The Jinja Nile Bridge experiences forces that can be represented as complex numbers. If force $F_1 = 50(\cos 30 + i\sin 30)$ kN acts at one point and $F_2 = 40(\cos 150 + i\sin 150)$ kN at another:
(a) Find the resultant force $F_R = F_1 + F_2$ in rectangular form.
(b) Calculate $|F_R|$ and $\arg(F_R)$.
Solution:
(a) Convert to rectangular form:

$$F_1 = 50\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 25\sqrt{3} + 25i \text{ kN}$$

$$F_2 = 40\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -20\sqrt{3} + 20i \text{ kN}$$

Resultant:

$$F_R = (25\sqrt{3} + 25i) + (-20\sqrt{3} + 20i)$$
$$= 5\sqrt{3} + 45i$$
$$= 8.66 + 45i \text{ kN}$$

(b) Magnitude:

$$|F_R| = \sqrt{(5\sqrt{3})^2 + 45^2} = \sqrt{75 + 2025} = \sqrt{2100} = 45.83 \text{ kN}$$

Argument:

$$\arg(F_R) = \tan^{-1}\left(\frac{45}{5\sqrt{3}}\right) = \tan^{-1}(5.196) = 79.11$$

18. **Mbale Arabica Coffee - Roasting Temperature**
The roasting temperature at Mbale coffee factory varies according to $T = 200 + 50\text{cis}(0.4t)°\text{C}$, where $\text{cis}(\theta) = \cos\theta + i\sin\theta$ and t is time in minutes.
(a) Find the maximum and minimum real temperatures.
(b) At what times does the maximum temperature occur in the first 20 minutes?
Solution:
(a) Since $T = 200 + 50(\cos 0.4t + i\sin 0.4t)$:
Real temperature: $T_{\text{real}} = 200 + 50\cos 0.4t$
Maximum occurs when $\cos 0.4t = 1$:

$$T_{\max} = 200 + 50(1) = 250\text{C}$$

Minimum occurs when $\cos 0.4t = -1$:

$$T_{\min} = 200 + 50(-1) = 150\text{C}$$

(b) Maximum occurs when $\cos 0.4t = 1$:

$$0.4t = 2\pi n, \quad n = 0, 1, 2, \ldots$$

$$t = 5\pi n$$

For first 20 minutes: - $n = 0$: $t = 0$ minutes - $n = 1$: $t = 5\pi \approx 15.71$ minutes

19. **Fort Portal Tea Factory - Quality Control**
The moisture content of tea leaves at different processing stages is modeled by the locus $|z - (4 + 3i)| = 5$, where z represents position in a 2D drying chamber.
(a) Describe this locus geometrically.
(b) Find three points on this locus.
(c) Determine if the point $z = 8 + 6i$ lies on this locus.
Solution:
(a) The locus $|z - (4 + 3i)| = 5$ represents a circle with:
- Center: $(4, 3)$
- Radius: 5 units

(b) Three points on the circle:
When $z = 4 + 8i$:

$$|z-(4+3i)| = |4+8i-4-3i| = |5i| = 5$$

When $z = 9 + 3i$:

$$|z-(4+3i)| = |9+3i-4-3i| = |5| = 5$$

When $z = 4 - 2i$:

$$|z-(4+3i)| = |4-2i-4-3i| = |-5i| = 5$$

(c) Check if $z = 8 + 6i$ is on the locus:

$$|z - (4 + 3i)| = |(8 + 6i) - (4 + 3i)|$$
$$= |4 + 3i|$$
$$= \sqrt{16 + 9}$$
$$= \sqrt{25} = 5$$

Yes, the point $z = 8 + 6i$ lies on this locus.

20. **Murchison Falls - Hydroelectric Potential**
Engineers studying the Murchison Falls hydroelectric potential model water pressure as $P = (60 + 80i) + (20 - 40i)e^{i\pi/3}$ kPa.
   (a) Evaluate P in rectangular form.
   (b) Find $|P|$ and interpret it as the total pressure magnitude.

Solution:

(a) First evaluate $e^{i\pi/3}$:

$$e^{i\pi/3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

Now:

$$\begin{aligned}(20 - 40i)e^{i\pi/3} &= (20 - 40i)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\\ &= 10 + 10\sqrt{3}i - 20i - 20\sqrt{3}i^2\\ &= 10 + 10\sqrt{3}i - 20i + 20\sqrt{3}\\ &= (10 + 20\sqrt{3}) + (10\sqrt{3} - 20)i\\ &= 44.64 - 2.68i\end{aligned}$$

Therefore:

$$\begin{aligned}P &= (60 + 80i) + (44.64 - 2.68i)\\ &= 104.64 + 77.32i \text{ kPa}\end{aligned}$$

(b) Pressure magnitude:

$$\begin{aligned}|P| &= \sqrt{104.64^2 + 77.32^2}\\ &= \sqrt{10949.53 + 5978.38}\\ &= \sqrt{16927.91}\\ &= 130.11 \text{ kPa}\end{aligned}$$

This represents the total pressure magnitude at that point in the water system.

21. **Kampala City - Building Resonance**
A tall building in Kampala experiences wind-induced vibrations with natural frequency given by solutions to $z^2 + 4z + 13 = 0$.
   (a) Solve for z using the quadratic formula.
   (b) Express the solutions in polar form.
   (c) Determine the damped natural frequency (imaginary part) and damping ratio.

Solution:

(a) Using the quadratic formula:

$$\begin{aligned}z &= \frac{-4 \pm \sqrt{16 - 52}}{2}\\ &= \frac{-4 \pm \sqrt{-36}}{2}\\ &= \frac{-4 \pm 6i}{2}\\ &= -2 \pm 3i\end{aligned}$$

So $z_1 = -2 + 3i$ and $z_2 = -2 - 3i$.

(b) For polar form of $z_1 = -2 + 3i$:
Modulus: $|z_1| = \sqrt{4 + 9} = \sqrt{13}$
Argument: $\arg(z_1) = 180 - \tan^{-1}\left(\frac{3}{2}\right) = 180 - 56.31 = 123.69$

$$z_1 = \sqrt{13}(\cos 123.69 + i\sin 123.69)$$

Similarly, $z_2 = \sqrt{13}(\cos 236.31 + i\sin 236.31)$

(c) Damped natural frequency: $\omega_d = 3$ rad/s
The damping ratio can be found from:

$$\zeta = \frac{2}{\sqrt{13}} = 0.555 \text{ or } 55.5\%$$

The building is underdamped and will oscillate.

Key Learning Points

- Complex numbers effectively model periodic and oscillatory phenomena in electrical, mechanical, and natural systems.
- Polar and exponential forms simplify calculations involving rotations and periodic functions.
- De Moivre's theorem is powerful for finding roots and powers of complex numbers.
- Complex conjugates naturally appear in physical applications involving impedance and power calculations.
- The modulus represents magnitudes (distances, amplitudes, power levels) while arguments represent phases or directions.

Chapter 15

DIFFERENTIAL EQUATIONS

Introduction

Differential equations are mathematical equations that relate a function with its derivatives. They are fundamental tools in mathematics and science, describing how quantities change over time or space. In this chapter, we explore methods of solving differential equations and their applications to real-world problems in Uganda and East Africa, including population dynamics, economic modeling, agricultural planning, and healthcare systems.

A differential equation contains derivatives of one or more dependent variables with respect to one or more independent variables. For example, if y is a function of x, denoted $y = f(x)$, then an equation involving x, y, and derivatives of y such as $\frac{dy}{dx}$ or $\frac{d^2y}{dx^2}$ is called a differential equation.

The order of a differential equation is determined by the highest derivative present in the equation. A first-order differential equation contains only the first derivative $\frac{dy}{dx}$, while a second-order differential equation contains the second derivative $\frac{d^2y}{dx^2}$.

15.1 Solving Differential Equations Using Various Techniques

15.1.1 Formation of Differential Equations

Differential equations arise naturally when we describe rates of change in physical, biological, or economic systems. To form a differential equation from a given situation, we identify the variables involved and express the rate of change of one variable with respect to another.

Consider a general situation where a quantity y depends on another quantity x. If we know how the rate of change of y with respect to x depends on y and x themselves, we can write a differential equation.

For instance, if the rate of change of a population P with respect to time t is proportional to the current population, we write:

$$\frac{dP}{dt} = kP$$

where k is a constant of proportionality.

Example 26.1: Formation from a Physical Law

Newton's law of cooling states that the rate of change of temperature of an object is proportional to the difference between its temperature and the ambient temperature. If $T(t)$ is the temperature of the object at time t and T_a is the ambient temperature, form the differential equation.

Solution:

The rate of change of temperature is $\frac{dT}{dt}$. According to Newton's law:

$$\frac{dT}{dt} = -k(T - T_a)$$

where $k > 0$ is a constant. The negative sign indicates that if $T > T_a$, the temperature decreases, and if $T < T_a$, the temperature increases.

15.1.2 Relationship Between Linear Equations and Differential Equations

Linear equations and differential equations share structural similarities. A linear equation in x has the form:

$$ax + b = 0$$

where a and b are constants. Similarly, a first-order linear differential equation has the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x)$ and $Q(x)$ are functions of x. Both involve the unknown quantity linearly (raised to the first power and not multiplied by itself).

The key difference is that linear equations involve the unknown variable itself, while linear differential equations involve both the unknown function and its derivatives.

15.1.3 Classification of First-Order Differential Equations

First-order differential equations can be classified into several types based on their form. Understanding the type helps us choose the appropriate solution method.

Separable Differential Equations

A differential equation is separable if it can be written in the form:

$$\frac{dy}{dx} = f(x)g(y)$$

or equivalently:

$$h(y)dy = k(x)dx$$

where the variables can be separated on opposite sides of the equation.

Example: $\frac{dy}{dx} = xy$ is separable because we can write it as $\frac{1}{y}dy = xdx$.

Linear Differential Equations

A first-order linear differential equation has the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

This type is linear in y and $\frac{dy}{dx}$.

Example: $\frac{dy}{dx} + 2xy = x$ is a linear differential equation with $P(x) = 2x$ and $Q(x) = x$.

Homogeneous Differential Equations

A differential equation of the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

is called homogeneous. These equations can be solved using the substitution $v = \frac{y}{x}$, which gives $y = vx$.

Example: $\frac{dy}{dx} = \frac{x+y}{x}$ can be written as $\frac{dy}{dx} = 1 + \frac{y}{x}$, making it homogeneous.

Exact Differential Equations

A differential equation of the form:

$$M(x, y)dx + N(x, y)dy = 0$$

is exact if:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Example: $(2xy + 3)dx + (x^2 + 4y)dy = 0$ is exact if the partial derivative condition is satisfied.

15.1.4 Solution Techniques for First-Order Differential Equations

Separation of Variables

This method is used for separable differential equations. The steps are:

1. Rearrange the equation to separate variables: $h(y)dy = k(x)dx$

2. Integrate both sides: $\int h(y)dy = \int k(x)dx$

3. Solve for y if possible, including the constant of integration

Example 26.2: Coffee Cooling in Kampala

A cup of coffee at temperature $80C$ is placed in a room in Kampala where the ambient temperature is $25C$. The rate of cooling is proportional to the temperature difference. If the coffee cools to $60C$ in 5 minutes, find the temperature after 10 minutes.

Solution:

Let $T(t)$ be the temperature at time t minutes. By Newton's law of cooling:

$$\frac{dT}{dt} = -k(T - 25)$$

This is separable. Rearranging:

$$\frac{dT}{T - 25} = -k\,dt$$

Integrating both sides:

$$\int \frac{dT}{T - 25} = \int -k\,dt$$

$$\ln|T - 25| = -kt + C$$

$$T - 25 = Ae^{-kt}$$

where $A = e^C$

$$T = 25 + Ae^{-kt}$$

Using initial condition $T(0) = 80$:

$$80 = 25 + A$$

$$A = 55$$

So $T = 25 + 55e^{-kt}$

Using $T(5) = 60$:

$$60 = 25 + 55e^{-5k}$$

$$35 = 55e^{-5k}$$

$$e^{-5k} = \frac{35}{55} = \frac{7}{11}$$

$$-5k = \ln\frac{7}{11}$$

$$k = -\frac{1}{5}\ln\frac{7}{11} = \frac{1}{5}\ln\frac{11}{7}$$

At $t = 10$ minutes:

$$T(10) = 25 + 55e^{-10k} = 25 + 55\left(e^{-5k}\right)^2$$

$$T(10) = 25 + 55\left(\frac{7}{11}\right)^2 = 25 + 55 \times \frac{49}{121}$$

$$T(10) = 25 + \frac{2695}{121} = 25 + 22.27 = 47.27C$$

The coffee temperature after 10 minutes is approximately $47.3C$.

Example 26.3: Maize Grain Drying in Eastern Uganda

In a grain storage facility in Mbale, maize grains lose moisture at a rate proportional to the current moisture content. If grains with initial moisture content of 25% dry to 18% in 3 days, find:

(a) The moisture content after 7 days

(b) When the moisture content reaches 13%

Solution:

Let $M(t)$ be the moisture content (as a percentage) at time t days. Then:

$$\frac{dM}{dt} = -kM$$

where $k > 0$ is a constant. Separating variables:

$$\frac{dM}{M} = -k\,dt$$

Integrating:

$$\ln|M| = -kt + C$$

$$M = Ae^{-kt}$$

where $A = e^C$

Using $M(0) = 25$:

$$25 = A$$

So $M = 25e^{-kt}$

Using $M(3) = 18$:

$$18 = 25e^{-3k}$$
$$e^{-3k} = \frac{18}{25} = 0.72$$
$$-3k = \ln(0.72)$$
$$k = -\frac{1}{3}\ln(0.72) = 0.1096$$

(a) At $t = 7$ days:

$$M(7) = 25e^{-7(0.1096)} = 25e^{-0.7672} = 25(0.4642) = 11.6\%$$

(b) When $M = 13\%$:

$$13 = 25e^{-0.1096t}$$
$$e^{-0.1096t} = 0.52$$
$$-0.1096t = \ln(0.52) = -0.6539$$
$$t = \frac{0.6539}{0.1096} = 5.97 \text{ days}$$

The moisture content reaches 13% after approximately 6 days.

Integrating Factor Method

This method is used for first-order linear differential equations of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

The steps are:

1. Identify $P(x)$ and $Q(x)$
2. Calculate the integrating factor: $I(x) = e^{\int P(x)dx}$
3. Multiply the entire equation by $I(x)$
4. The left side becomes $\frac{d}{dx}[I(x)y]$
5. Integrate: $I(x)y = \int I(x)Q(x)dx$
6. Solve for y

Example 26.4: Fertilizer Concentration in Lake Victoria

The concentration $C(t)$ of fertilizer runoff in a section of Lake Victoria (in mg/L) changes according to:

$$\frac{dC}{dt} + 0.2C = 5$$

where t is time in days. If the initial concentration is 10 mg/L, find the concentration after 10 days.

Solution:

This is a first-order linear differential equation with $P(t) = 0.2$ and $Q(t) = 5$.

The integrating factor is:

$$I(t) = e^{\int 0.2dt} = e^{0.2t}$$

Multiplying the equation by $e^{0.2t}$:

$$e^{0.2t}\frac{dC}{dt} + 0.2e^{0.2t}C = 5e^{0.2t}$$

The left side is $\frac{d}{dt}[e^{0.2t}C]$:

$$\frac{d}{dt}[e^{0.2t}C] = 5e^{0.2t}$$

Integrating both sides:

$$e^{0.2t}C = \int 5e^{0.2t}dt = \frac{5}{0.2}e^{0.2t} + K = 25e^{0.2t} + K$$

$$C = 25 + Ke^{-0.2t}$$

Using $C(0) = 10$:

$$10 = 25 + K$$
$$K = -15$$

Therefore:

$$C(t) = 25 - 15e^{-0.2t}$$

At $t = 10$ days:

$$C(10) = 25 - 15e^{-2} = 25 - 15(0.1353) = 25 - 2.03 = 22.97 \text{ mg/L}$$

The concentration after 10 days is approximately 23 mg/L.

Example 26.5: Salt Solution Mixing

A tank in a beverage processing plant in Jinja contains 100 liters of water. Brine containing 2 kg of salt per liter flows into the tank at 3 liters per minute. The mixture, kept uniform by stirring, flows out at the same rate. Find the amount of salt in the tank after 20 minutes.

Solution:

Let $S(t)$ be the amount of salt (in kg) at time t minutes. Salt enters at rate $3 \times 2 = 6$ kg/min. Salt leaves at rate $\frac{S}{100} \times 3 = \frac{3S}{100}$ kg/min.

$$\frac{dS}{dt} = 6 - \frac{3S}{100}$$

$$\frac{dS}{dt} + \frac{3S}{100} = 6$$

This is linear with $P(t) = \frac{3}{100} = 0.03$ and $Q(t) = 6$.

Integrating factor: $I(t) = e^{0.03t}$

Multiplying through:

$$\frac{d}{dt}[e^{0.03t} S] = 6e^{0.03t}$$

Integrating:

$$e^{0.03t} S = \frac{6}{0.03} e^{0.03t} + C = 200e^{0.03t} + C$$

$$S = 200 + Ce^{-0.03t}$$

Initially $S(0) = 0$:

$$0 = 200 + C$$

$$C = -200$$

Therefore:

$$S(t) = 200(1 - e^{-0.03t})$$

At $t = 20$ minutes:

$$S(20) = 200(1-e^{-0.6}) = 200(1-0.5488) = 200(0.4512) = 90.24 \text{ kg}$$

After 20 minutes, there are approximately 90.2 kg of salt in the tank.

Substitution Methods for Homogeneous Equations

For homogeneous differential equations of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$, we use the substitution $v = \frac{y}{x}$, which gives $y = vx$.

Differentiating: $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting into the original equation and simplifying leads to a separable equation in v and x.

Example 26.6: Road Construction Gradient

A road being constructed in Rwenzori region has its gradient at any point (x, y) given by:

$$\frac{dy}{dx} = \frac{2x + y}{x}$$

Find the equation of the road if it passes through the point $(1, 3)$.

Solution:

Rewrite the equation:

$$\frac{dy}{dx} = 2 + \frac{y}{x}$$

This is homogeneous. Let $v = \frac{y}{x}$, so $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting:

$$v + x\frac{dv}{dx} = 2 + v$$

$$x\frac{dv}{dx} = 2$$

$$\frac{dv}{dx} = \frac{2}{x}$$

This is separable:

$$dv = \frac{2}{x}dx$$

Integrating:

$$v = 2\ln|x| + C$$

Substituting back $v = \frac{y}{x}$:

$$\frac{y}{x} = 2\ln|x| + C$$

$$y = 2x\ln|x| + Cx$$

Using the condition $(1, 3)$:

$$3 = 2(1)\ln(1) + C(1) = 0 + C$$

$$C = 3$$

Therefore, the equation of the road is:

$$y = 2x\ln|x| + 3x$$

15.1.5 General and Particular Solutions

The general solution of a differential equation contains arbitrary constants. When we specify initial conditions or boundary conditions, we obtain a particular solution.

For a first-order differential equation, the general solution contains one arbitrary constant. An initial condition of the form $y(x_0) = y_0$ allows us to find this constant and obtain the particular solution.

15.2 Modeling and Analyzing Real-World Situations

Differential equations are powerful tools for modeling phenomena in various fields. In this section, we explore applications relevant to Uganda and East Africa.

15.2.1 Population Growth Models

Exponential Growth Model

The simplest population model assumes that the rate of population growth is proportional to the current population:

$$\frac{dP}{dt} = kP$$

where P is population, t is time, and k is the growth rate constant.

Solution: $P(t) = P_0e^{kt}$ where P_0 is the initial population.

Example 26.7: Population Growth in Kampala Metropolitan Area

The population of Kampala metropolitan area was approximately 3.5 million in 2020 and grew to 3.8 million in 2022. Assuming exponential growth:

(a) Find the growth model

(b) Predict the population in 2030

(c) Determine when the population will reach 5 million

Solution:

(a) Let $P(t)$ be the population (in millions) where t is years after 2020.

$$\frac{dP}{dt} = kP$$

Solution: $P(t) = P_0e^{kt}$

Using $P(0) = 3.5$:

$$P(t) = 3.5e^{kt}$$

Using $P(2) = 3.8$:

$$3.8 = 3.5e^{2k}$$
$$e^{2k} = \frac{3.8}{3.5} = 1.0857$$
$$2k = \ln(1.0857) = 0.0822$$
$$k = 0.0411$$

Growth model: $P(t) = 3.5e^{0.0411t}$ million

(b) In 2030, $t = 10$:

$$P(10) = 3.5e^{0.411} = 3.5(1.508) = 5.28 \text{ million}$$

(c) When $P = 5$:

$$5 = 3.5e^{0.0411t}$$
$$e^{0.0411t} = \frac{5}{3.5} = 1.4286$$

$$0.0411t = \ln(1.4286) = 0.3567$$

$$t = \frac{0.3567}{0.0411} = 8.68 \text{ years}$$

The population reaches 5 million in late 2028 (approximately 8.7 years after 2020).

Logistic Growth Model

In reality, populations face resource constraints. The logistic model accounts for a carrying capacity M:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

This equation shows that growth rate decreases as P approaches M.

Example 26.8: Fish Population in Lake Albert

A fish species in Lake Albert has a carrying capacity of 50,000 individuals. The population grows according to the logistic model with growth rate $k = 0.4$ per year. If the current population is 10,000, find:

(a) The differential equation

(b) The population after 5 years (using numerical estimation)

(c) When the population reaches 40,000

Solution:

(a) The differential equation is:

$$\frac{dP}{dt} = 0.4P\left(1 - \frac{P}{50000}\right)$$

$$\frac{dP}{dt} = 0.4P - \frac{0.4P^2}{50000}$$

$$\frac{dP}{dt} = 0.4P - 0.000008P^2$$

(b) This equation can be solved by separation of variables, but the integration is complex. Using partial fractions:

$$\frac{dP}{P(50000 - P)} = \frac{0.4}{50000}dt$$

After integration and applying initial condition $P(0) = 10000$:

$$P(t) = \frac{50000}{1 + 4e^{-0.4t}}$$

At $t = 5$ years:

$$P(5) = \frac{50000}{1 + 4e^{-2}} = \frac{50000}{1 + 4(0.1353)} = \frac{50000}{1.5412} = 32,441$$

The population after 5 years is approximately 32,400 fish.

(c) When $P = 40000$:

$$40000 = \frac{50000}{1 + 4e^{-0.4t}}$$

$$1 + 4e^{-0.4t} = \frac{50000}{40000} = 1.25$$

$$4e^{-0.4t} = 0.25$$

$$e^{-0.4t} = 0.0625$$

$$-0.4t = \ln(0.0625) = -2.773$$

$$t = 6.93 \text{ years}$$

The population reaches 40,000 fish in approximately 7 years.

15.2.2 Economic Models: Price and Commodity Dynamics

Price Adjustment Model

The rate of change of price can be related to excess demand:

$$\frac{dP}{dt} = k(D - S)$$

where P is price, D is demand, S is supply, and k is an adjustment coefficient.

Example 26.9: Coffee Price Dynamics in Uganda

The price of coffee beans in Uganda adjusts according to supply and demand. Suppose demand is $D = 100 - 2P$ thousand tons and supply is $S = 20 + 3P$ thousand tons, where P is price in thousands of shillings per ton. The price adjusts at a rate:

$$\frac{dP}{dt} = 0.1(D - S)$$

If the current price is 12,000 shillings per ton, find:

(a) The differential equation in terms of P

(b) The equilibrium price

(c) The price after 10 time units

Solution:

(a) Substituting D and S:

$$\frac{dP}{dt} = 0.1[(100 - 2P) - (20 + 3P)]$$

$$\frac{dP}{dt} = 0.1[80 - 5P]$$

$$\frac{dP}{dt} = 8 - 0.5P$$

(b) At equilibrium, $\frac{dP}{dt} = 0$:

$$8 - 0.5P = 0$$

$$P = 16$$

Equilibrium price is 16,000 shillings per ton.

(c) Solving the differential equation:

$$\frac{dP}{dt} + 0.5P = 8$$

This is linear. Integrating factor: $I = e^{0.5t}$

$$\frac{d}{dt}[e^{0.5t}P] = 8e^{0.5t}$$

$$e^{0.5t}P = 16e^{0.5t} + C$$

$$P = 16 + Ce^{-0.5t}$$

Using $P(0) = 12$:

$$12 = 16 + C$$

$$C = -4$$

Therefore: $P(t) = 16 - 4e^{-0.5t}$

At $t = 10$:

$$P(10) = 16 - 4e^{-5} = 16 - 4(0.0067) = 16 - 0.027 = 15.97$$

After 10 time units, the price is approximately 15,970 shillings per ton, very close to equilibrium.

15.2.3 Temperature Models

Newton's Law of Cooling

As seen earlier, temperature changes according to:

$$\frac{dT}{dt} = -k(T - T_a)$$

Example 26.10: Forensic Application - Time of Death Estimation

A body was discovered in a building in Entebbe. The ambient temperature was $22C$. When discovered at 8:00 AM, the body temperature was $30C$. One hour later, the temperature was $27C$. Assuming normal body temperature is $37C$, estimate the time of death.

Solution:

Using Newton's law: $T(t) = T_a + (T_0 - T_a)e^{-kt}$ where T_0 is initial temperature.

Let t be hours after death. At discovery ($t = t_1$): $T = 30C$

$$30 = 22 + (37 - 22)e^{-kt_1}$$

$$8 = 15e^{-kt_1}$$

$$e^{-kt_1} = \frac{8}{15}$$

... (1)

One hour later ($t = t_1 + 1$): $T = 27C$

$$27 = 22 + 15e^{-k(t_1+1)}$$

$$5 = 15e^{-kt_1}e^{-k}$$

Using equation (1):

$$5 = 15 \times \frac{8}{15} \times e^{-k}$$
$$5 = 8e^{-k}$$
$$e^{-k} = \frac{5}{8}$$
$$k = \ln \frac{8}{5} = 0.4700$$

From equation (1):

$$e^{-kt_1} = \frac{8}{15}$$
$$-kt_1 = \ln \frac{8}{15} = -0.6286$$
$$t_1 = \frac{0.6286}{0.4700} = 1.34 \text{ hours}$$

Since the body was discovered at 8:00 AM, death occurred approximately 1.34 hours earlier, at about 6:40 AM.

15.2.4 Drug Absorption and Medical Applications

Drug Concentration in Bloodstream

The rate of change of drug concentration can be modeled considering absorption and elimination:

$$\frac{dC}{dt} = A - kC$$

where A is absorption rate and k is elimination rate constant.

Example 26.11: Antimalarial Drug Treatment

A patient at Mulago Hospital receives an antimalarial drug. The drug is absorbed at a constant rate of 20 mg/hour and eliminated from the bloodstream at a rate proportional to the concentration, with elimination constant $k = 0.15$ per hour. Find:

(a) The differential equation for drug concentration $C(t)$

(b) The concentration after 8 hours if initial concentration is zero

(c) The steady-state concentration

Solution:

(a) The differential equation is:

$$\frac{dC}{dt} = 20 - 0.15C$$

(b) Rearranging to standard linear form:

$$\frac{dC}{dt} + 0.15C = 20$$

Integrating factor: $I = e^{0.15t}$

$$\frac{d}{dt}[e^{0.15t}C] = 20e^{0.15t}$$

$$e^{0.15t}C = \frac{20}{0.15}e^{0.15t} + K = 133.33e^{0.15t} + K$$

$$C = 133.33 + Ke^{-0.15t}$$

Using $C(0) = 0$:

$$0 = 133.33 + K$$

$$K = -133.33$$

Therefore: $C(t) = 133.33(1 - e^{-0.15t})$

At $t = 8$ hours:

$$C(8) = 133.33(1-e^{-1.2}) = 133.33(1-0.3012) = 133.33(0.6988) = 93$$

(c) At steady state, $\frac{dC}{dt} = 0$:

$$20 - 0.15C = 0$$
$$C = \frac{20}{0.15} = 133.33 \text{ mg}$$

The steady-state concentration is approximately 133 mg.

15.2.5 Agricultural Applications

Example 26.12: Pesticide Decay in Soil

In a cotton farm in Kasese, pesticide concentration in soil decreases at a rate proportional to the amount present. After application, the concentration is 500 mg/kg. After 30 days, it drops to 200 mg/kg. Agricultural guidelines require the concentration to be below 50 mg/kg before replanting. How long after application can replanting occur?

Solution:

Let $C(t)$ be concentration in mg/kg at time t days.

$$\frac{dC}{dt} = -kC$$

Solution: $C(t) = C_0 e^{-kt}$

Using $C(0) = 500$:

$$C(t) = 500e^{-kt}$$

Using $C(30) = 200$:

$$200 = 500e^{-30k}$$

$$e^{-30k} = 0.4$$

$$-30k = \ln(0.4) = -0.9163$$

$$k = 0.0305$$

When $C = 50$:

$$50 = 500e^{-0.0305t}$$

$$e^{-0.0305t} = 0.1$$

$$-0.0305t = \ln(0.1) = -2.3026$$

$$t = 75.5 \text{ days}$$

Replanting can occur approximately 76 days after pesticide application.

15.2.6 Mixing Problems

These involve substances flowing into and out of containers, changing concentration over time.

Example 26.13: Water Treatment Plant

A water treatment facility in Gulu has a tank containing 1000 liters of water with dissolved chlorine. Pure water flows in at 20 liters/minute, and the well-mixed solution flows out at the same rate. If the initial chlorine concentration is 50 mg/L, find:

(a) The amount of chlorine at time t

(b) When the concentration drops to 10 mg/L

Solution:

Let $A(t)$ be the amount of chlorine (in mg) at time t minutes.

Chlorine in: 0 mg/min (pure water)

Chlorine out: $\frac{A}{1000} \times 20 = \frac{A}{50}$ mg/min

$$\frac{dA}{dt} = 0 - \frac{A}{50} = -\frac{A}{50}$$

This is separable:

$$\frac{dA}{A} = -\frac{dt}{50}$$

$$\ln|A| = -\frac{t}{50} + C$$

$$A = Ke^{-t/50}$$

Initially, $A(0) = 50 \times 1000 = 50000$ mg:

$$50000 = K$$

Therefore: $A(t) = 50000e^{-t/50}$

Concentration $C(t) = \frac{A(t)}{1000} = 50e^{-t/50}$ mg/L

When $C = 10$ mg/L:

$$10 = 50e^{-t/50}$$

$$e^{-t/50} = 0.2$$

$$-\frac{t}{50} = \ln(0.2) = -1.609$$

$$t = 80.5 \text{ minutes}$$

The concentration drops to 10 mg/L after approximately 80.5 minutes.

15.3 Summary

In this chapter, we have explored differential equations and their applications to real-world problems in Uganda and East Africa. Key points include:

1. Differential equations relate functions with their derivatives and describe rates of change.

2. First-order differential equations can be classified as separable, linear, homogeneous, or exact.

3. Solution techniques include separation of variables, integrating factor method, and substitution methods.

4. Applications span population dynamics, economics, temperature modeling, medical treatments, agriculture, and environmental science.

5. Mathematical modeling using differential equations provides insights for prediction and decision-making in various sectors.

15.4 Practice Exercises

Exercise 26.1: Solve the differential equation $\frac{dy}{dx} = 3x^2y$ given that $y = 2$ when $x = 0$.

Exercise 26.2: A savings account in a Ugandan bank earns continuous compound interest at 8% per year. If 5 million shillings is deposited, find the balance after 10 years using the differential equation $\frac{dA}{dt} = 0.08A$.

Exercise 26.3: Solve the linear differential equation $\frac{dy}{dx} + 2y = 4x$ with initial condition $y(0) = 1$.

Exercise 26.4: The population of a bird species in Bwindi Impenetrable Forest follows the logistic model with carrying capacity 5000 and growth rate 0.3 per year. If the current population is 1000, write the differential equation and estimate the population after 5 years.

Exercise 26.5: A factory in Jinja discharges waste into a river. The pollution level $P(t)$ decreases according to $\frac{dP}{dt} = -0.25P + 10$ where t is in days. Find the steady-state pollution level.

Exercise 26.6: Tea cooling from $95C$ to $75C$ in 5 minutes in a room at $20C$. Using Newton's law of cooling, find when the tea reaches $50C$.

Exercise 26.7: Solve the homogeneous equation $x\frac{dy}{dx} = y + x$ with $y(1) = 2$.

Exercise 26.8: A tank contains 200 liters of brine with 30 kg of dissolved salt. Pure water enters at 4 L/min and the mixture leaves at 4 L/min. Find the amount of salt after 30 minutes.

Exercise 26.9: The rate of inflation in Uganda is modeled by $\frac{dI}{dt} = 0.05(10 - I)$ where I is the inflation rate in percent. If current inflation is 6%, find the inflation rate after 2 years.

Exercise 26.10: A crop pest population grows according to $\frac{dP}{dt} = 0.4P(1 - \frac{P}{10000})$ with initial population 1000. Find when the population reaches 8000.

15.5 Scenario-Based Questions on Differential Equations:

1. **Coffee Production in Mbale District**
   A coffee plantation in Mbale district observes that the rate of coffee berry maturation is proportional to the difference between the maximum possible mature berries (5000 kg) and the current amount of mature berries. If initially there are 500 kg of mature berries and after 2 weeks there are 1500 kg, find:
   (a) The differential equation modeling this situation
   (b) The amount of mature berries after 4 weeks

   Solution:
   (a) Let $M(t)$ be the mass of mature berries at time t weeks.

   $$\frac{dM}{dt} = k(5000 - M)$$

   (b) This is a separable differential equation:

   $$\frac{dM}{5000 - M} = k\,dt$$

   Integrating both sides:

   $$-\ln|5000 - M| = kt + C$$

   $$\ln|5000 - M| = -kt - C$$

   $$5000 - M = Ae^{-kt} \quad \text{where } A = e^{-C}$$

   $$M(t) = 5000 - Ae^{-kt}$$

   Using initial condition $M(0) = 500$:

   $$500 = 5000 - A \implies A = 4500$$

$$M(t) = 5000 - 4500e^{-kt}$$

Using $M(2) = 1500$:

$$1500 = 5000 - 4500e^{-2k}$$

$$e^{-2k} = \frac{7}{9}$$

$$k = -\frac{1}{2}\ln\left(\frac{7}{9}\right) \approx 0.1252$$

Therefore, $M(4) = 5000 - 4500e^{-4(0.1252)} = 5000 - 4500(0.6049) \approx 2278$ kg

2. **Malaria Spread in Gulu District**

The rate of increase of malaria cases in a Gulu district hospital is proportional to the product of infected people and susceptible people. If $I(t)$ represents infected people and $S(t)$ represents susceptible people, with $I + S = 10000$ (total population), and initially 100 people are infected:

(a) Form the differential equation

(b) Solve for $I(t)$ if after 1 week, 500 people are infected

Solution:

(a) $\frac{dI}{dt} = kI(10000 - I)$

(b) This is a separable equation:

$$\frac{dI}{I(10000 - I)} = k\,dt$$

Using partial fractions:

$$\frac{1}{I(10000 - I)} = \frac{1}{10000}\left(\frac{1}{I} + \frac{1}{10000 - I}\right)$$

Integrating:

$$\frac{1}{10000}\ln\left|\frac{I}{10000 - I}\right| = kt + C$$

With $I(0) = 100$:

$$C = \frac{1}{10000}\ln\left(\frac{1}{99}\right)$$

$$\ln\left|\frac{I}{10000 - I}\right| = 10000kt + \ln\left(\frac{1}{99}\right)$$

Using $I(1) = 500$:

$$\ln\left(\frac{500}{9500}\right) = 10000k + \ln\left(\frac{1}{99}\right)$$

$$k \approx 0.000269$$

$$\frac{I}{10000 - I} = \frac{1}{99}e^{10000kt}$$

$$I(t) = \frac{10000}{99e^{-10000kt} + 1}$$

3. **Temperature of Luwombo Dish**

A traditional Luwombo dish is removed from a charcoal stove at 95°C and placed in a room at 25°C. After 10 minutes, its temperature is 70°C. Using Newton's law of cooling, find:

(a) The differential equation governing the temperature

(b) The temperature after 20 minutes

(c) When the dish reaches 40°C

Solution:

(a) Let $T(t)$ be temperature at time t minutes.

$$\frac{dT}{dt} = k(T - 25)$$

(b) Solving by separation of variables:

$$\frac{dT}{T - 25} = k\,dt$$

$$\ln|T - 25| = kt + C$$

$$T = 25 + Ae^{kt}$$

With $T(0) = 95$: $A = 70$

$$T = 25 + 70e^{kt}$$

With $T(10) = 70$:

$$70 = 25 + 70e^{10k}$$

$$e^{10k} = \frac{9}{14}$$

$$k = \frac{1}{10}\ln\left(\frac{9}{14}\right) \approx -0.0442$$

Therefore:

$$T(20) = 25+70e^{20(-0.0442)} \approx 25+28.93 = 53.93\text{C}$$

(c) For $T = 40$:

$$40 = 25 + 70e^{-0.0442t}$$

$$t = \frac{\ln(15/70)}{-0.0442} \approx 35.6 \text{ minutes}$$

4. **Lake Victoria Water Level**

The rate of change of water level in Lake Victoria is modeled by $\frac{dh}{dt} = 0.5 - 0.02h$, where h is the height in meters above a reference point and t is time in months. If the initial water level is 15 m:

(a) Solve the differential equation

(b) Find the equilibrium water level

(c) Determine the water level after 6 months

Solution:

(a) This is a linear differential equation:

$$\frac{dh}{dt} + 0.02h = 0.5$$

Separating variables:

$$\frac{dh}{0.5 - 0.02h} = dt$$

$$\frac{dh}{25 - h} = 0.02\,dt$$

$$-\ln|25 - h| = 0.02t + C$$

$$h = 25 - Ae^{-0.02t}$$

With $h(0) = 15$: $A = 10$

$$h(t) = 25 - 10e^{-0.02t}$$

(b) Equilibrium occurs when $\frac{dh}{dt} = 0$:

$$0.5 - 0.02h = 0 \implies h = 25 \text{ m}$$

(c) $h(6) = 25 - 10e^{-0.12} \approx 25 - 8.87 = 16.13$ m

5. **Matooke Market Price Dynamics**

The price $P(t)$ (in UGX per bunch) of matooke in Nakasero market changes according to $\frac{dP}{dt} = 0.1P(50 - P)$, where t is in weeks. If the initial price is UGX 5,000 per bunch:

(a) Solve for $P(t)$

(b) What is the long-term equilibrium price?

Solution:

(a) This is separable (note: P is in thousands of UGX):

$$\frac{dP}{P(50 - P)} = 0.1\,dt$$

Using partial fractions:

$$\frac{1}{P(50 - P)} = \frac{1}{50}\left(\frac{1}{P} + \frac{1}{50 - P}\right)$$

$$\frac{1}{50}\ln\left|\frac{P}{50 - P}\right| = 0.1t + C$$

With $P(0) = 5$:

$$C = \frac{1}{50}\ln\left(\frac{1}{9}\right)$$

$$\ln\left|\frac{P}{50 - P}\right| = 5t + \ln\left(\frac{1}{9}\right)$$

$$\frac{P}{50 - P} = \frac{1}{9}e^{5t}$$

$$P(t) = \frac{50}{9e^{-5t} + 1}$$

(b) As $t \to \infty$, $P \to 50$ (thousands), so equilibrium price is UGX 50,000 per bunch.

6. **Kampala Population Growth**

Kampala's population grows according to $\frac{dP}{dt} = 0.035P$, where P is in millions and t is in years. If the population in 2020 was 1.5 million:

(a) Find $P(t)$ with 2020 as $t = 0$

(b) Predict the population in 2030

(c) When will the population double?

Solution:

(a) This is separable:

$$\frac{dP}{P} = 0.035\,dt$$

$$\ln|P| = 0.035t + C$$

$$P = Ae^{0.035t}$$

With $P(0) = 1.5$: $A = 1.5$

$$P(t) = 1.5e^{0.035t}$$

(b) For 2030, $t = 10$:

$$P(10) = 1.5e^{0.35} \approx 1.5(1.419) = 2.13 \text{ million}$$

(c) For doubling, $P = 3$:

$$3 = 1.5e^{0.035t}$$

$$t = \frac{\ln 2}{0.035} \approx 19.8 \text{ years (around 2040)}$$

7. **Radioactive Decay in Medical Imaging (Mulago Hospital)**

A radioactive isotope used for medical imaging at Mulago Hospital decays according to $\frac{dN}{dt} = -0.693N$, where N is the amount in grams and t is in hours. If initially there are 100 grams:

(a) Find $N(t)$

(b) Find the half-life

(c) How much remains after 3 hours?

Solution:

(a) Separating variables:

$$\frac{dN}{N} = -0.693\,dt$$

$$\ln|N| = -0.693t + C$$

$$N = Ae^{-0.693t}$$

With $N(0) = 100$: $A = 100$

$$N(t) = 100e^{-0.693t}$$

(b) Half-life occurs when $N = 50$:

$$50 = 100e^{-0.693t}$$

$$e^{-0.693t} = 0.5$$

$$t = 1 \text{ hour}$$

(c) $N(3) = 100e^{-0.693(3)} = 100e^{-2.079} \approx 12.5$ grams

8. **Cassava Fermentation Process**
In a cassava processing facility in Lira, the fermentation process follows $\frac{dB}{dt} = kB(M - B)$, where B is the amount of bacteria (in millions) and $M = 100$ is the maximum capacity. If $B(0) = 5$ and $B(2) = 20$ (time in days):
   (a) Determine k
   (b) Find when $B = 80$

Solution:
(a) Separating variables:

$$\frac{dB}{B(100 - B)} = k\,dt$$

Using partial fractions:

$$\frac{1}{100}\ln\left|\frac{B}{100 - B}\right| = kt + C$$

With $B(0) = 5$:

$$C = \frac{1}{100}\ln\left(\frac{1}{19}\right)$$

With $B(2) = 20$:

$$\frac{1}{100}\ln\left(\frac{20}{80}\right) = 2k + \frac{1}{100}\ln\left(\frac{1}{19}\right)$$

$$\ln\left(\frac{1}{4}\right) = 200k + \ln\left(\frac{1}{19}\right)$$

$$k = \frac{\ln(19/4)}{200} \approx 0.00782$$

(b) For $B = 80$:

$$\frac{1}{100}\ln\left(\frac{80}{20}\right) = 0.00782t + \frac{1}{100}\ln\left(\frac{1}{19}\right)$$

$$t \approx 5.53 \text{ days}$$

9. **Solar Panel Heating in Soroti**
A solar water heater in Soroti heats water according to $\frac{dT}{dt} = 0.08(60 - T) + 0.12S$, where T is water temperature (°C), S is solar intensity (constant at 50), and t is in hours. If initial temperature is 20°C:
   (a) Simplify and solve the differential equation
   (b) Find the equilibrium temperature

Solution:
(a) Simplifying:

$$\frac{dT}{dt} = 0.08(60-T)+6 = 4.8-0.08T+6 = 10.8-0.08T$$

$$\frac{dT}{dt} + 0.08T = 10.8$$

This is linear:

$$\frac{dT}{10.8 - 0.08T} = dt$$

$$\frac{dT}{135 - T} = 0.08\,dt$$

$$-\ln|135 - T| = 0.08t + C$$

$$T = 135 - Ae^{-0.08t}$$

With $T(0) = 20$: $A = 115$

$$T(t) = 135 - 115e^{-0.08t}$$

(b) Equilibrium: $\lim_{t\to\infty} T(t) = 135$C

10. **Fish Population in Lake Albert**
The fish population F (in thousands) in Lake Albert follows $\frac{dF}{dt} = 0.05F - 0.0001F^2 - H$, where $H = 2$ (harvesting rate in thousands per month). If $F(0) = 300$:
   (a) Find the equilibrium populations
   (b) Is the population sustainable?

Solution:
(a) At equilibrium, $\frac{dF}{dt} = 0$:

$$0.05F - 0.0001F^2 - 2 = 0$$

$$-0.0001F^2 + 0.05F - 2 = 0$$

$$F^2 - 500F + 20000 = 0$$

Using quadratic formula:

$$F = \frac{500 \pm \sqrt{250000 - 80000}}{2} = \frac{500 \pm \sqrt{170000}}{2}$$

$$F = \frac{500 \pm 412.31}{2}$$

$$F_1 \approx 456.2, \quad F_2 \approx 43.8$$

(b) The equation $\frac{dF}{dt} = -0.0001F^2 + 0.05F - 2$ represents a parabola opening downward. - For $F < 43.8$ or $F > 456.2$: $\frac{dF}{dt} < 0$ (decreasing) - For $43.8 < F < 456.2$: $\frac{dF}{dt} > 0$ (increasing)
Since $F(0) = 300$ is between the equilibria, the population will stabilize at $F = 456.2$ thousand (sustainable).

11. **Tuberculosis Treatment Model**
The number of TB patients $P(t)$ receiving treatment at a clinic in Mbarara follows $\frac{dP}{dt} + 0.15P = 25$, where t is in months and there are initially 50 patients:

(a) Solve for $P(t)$

(b) Find the steady-state number of patients

Solution:

(a) This is a first-order linear equation. Integrating factor: $\mu = e^{0.15t}$

Multiplying through:

$$e^{0.15t}\frac{dP}{dt} + 0.15e^{0.15t}P = 25e^{0.15t}$$

$$\frac{d}{dt}(Pe^{0.15t}) = 25e^{0.15t}$$

Integrating:

$$Pe^{0.15t} = \frac{25e^{0.15t}}{0.15} + C = \frac{500}{3}e^{0.15t} + C$$

$$P = \frac{500}{3} + Ce^{-0.15t}$$

With $P(0) = 50$: $C = 50 - \frac{500}{3} = -\frac{350}{3}$

$$P(t) = \frac{500}{3} - \frac{350}{3}e^{-0.15t}$$

(b) Steady state: $\lim_{t\to\infty} P(t) = \frac{500}{3} \approx 167$ patients

12. **Honey Production Rate**

A bee colony in Kapchorwa produces honey at a rate $\frac{dH}{dt} = 2t + 3H$, where H is honey in kg and t is in weeks. If $H(0) = 5$ kg:

(a) Solve the differential equation

(b) Find $H(4)$

Solution:

(a) This is linear: $\frac{dH}{dt} - 3H = 2t$

Integrating factor: $\mu = e^{-3t}$

$$e^{-3t}\frac{dH}{dt} - 3e^{-3t}H = 2te^{-3t}$$

$$\frac{d}{dt}(He^{-3t}) = 2te^{-3t}$$

Integrating by parts (let $u = 2t$, $dv = e^{-3t}dt$):

$$He^{-3t} = -\frac{2t}{3}e^{-3t} - \frac{2}{9}e^{-3t} + C$$

$$H = -\frac{2t}{3} - \frac{2}{9} + Ce^{3t}$$

With $H(0) = 5$:

$$5 = -\frac{2}{9} + C \implies C = \frac{47}{9}$$

$$H(t) = -\frac{2t}{3} - \frac{2}{9} + \frac{47}{9}e^{3t}$$

(b) $H(4) = -\frac{8}{3} - \frac{2}{9} + \frac{47}{9}e^{12} \approx 852,730$ kg

13. **Drug Concentration in Blood**

A patient at Nsambya Hospital receives a drug intravenously. The concentration $C(t)$ (mg/L) in blood follows $\frac{dC}{dt} = 5 - 0.2C$, where t is in hours and initially $C = 0$:

(a) Solve for $C(t)$

(b) Find the maximum concentration

(c) Find $C(5)$

Solution:

(a) Separating variables:

$$\frac{dC}{5 - 0.2C} = dt$$

$$\frac{dC}{25 - C} = 0.2\,dt$$

$$-\ln|25 - C| = 0.2t + K$$

$$C = 25 - Ae^{-0.2t}$$

With $C(0) = 0$: $A = 25$

$$C(t) = 25(1 - e^{-0.2t})$$

(b) Maximum concentration: $\lim_{t\to\infty} C(t) = 25$ mg/L

(c) $C(5) = 25(1 - e^{-1}) \approx 25(0.632) = 15.8$ mg/L

14. **Soil Erosion in Kabale Terraces**

The rate of soil loss S (tons/hectare) from a terrace in Kabale follows $\frac{dS}{dt} = 0.3\sqrt{S}$, where t is in months. If initially $S = 4$ tons:

(a) Solve for $S(t)$

(b) Find $S(12)$

Solution:

(a) Separating variables:

$$\frac{dS}{\sqrt{S}} = 0.3\,dt$$

$$S^{-1/2}dS = 0.3\,dt$$

$$2\sqrt{S} = 0.3t + C$$

With $S(0) = 4$: $C = 2\sqrt{4} = 4$

$$2\sqrt{S} = 0.3t + 4$$

$$\sqrt{S} = 0.15t + 2$$

$$S(t) = (0.15t + 2)^2$$

(b) $S(12) = (0.15(12) + 2)^2 = (1.8 + 2)^2 = 14.44$ tons/hectare

15. **Simsim Oil Extraction**

The rate of oil extraction from simsim seeds is $\frac{dV}{dt} = k(V_{\max} - V)$, where V is volume in liters and $V_{\max} = 50$ L. If $V(0) = 5$ L and $V(3) = 20$ L (time in hours):

(a) Find k

(b) When is $V = 40$ L?

Solution:

(a) Separating:

$$\frac{dV}{50 - V} = k\,dt$$

$$-\ln|50 - V| = kt + C$$

$$V = 50 - Ae^{-kt}$$

With $V(0) = 5$: $A = 45$

$$V = 50 - 45e^{-kt}$$

With $V(3) = 20$:

$$20 = 50 - 45e^{-3k}$$

$$e^{-3k} = \frac{2}{3}$$

$$k = -\frac{1}{3}\ln\left(\frac{2}{3}\right) \approx 0.135$$

(b) For $V = 40$:

$$40 = 50 - 45e^{-0.135t}$$

$$e^{-0.135t} = \frac{2}{9}$$

$$t = \frac{-\ln(2/9)}{0.135} \approx 11.15 \text{ hours}$$

16. **Electric Current in Rural Electrification**

In an RL circuit for a rural electrification project, the current $I(t)$ follows $L\frac{dI}{dt} + RI = V$, where $L = 5$ H, $R = 20\,\Omega$, $V = 100$ V, and $I(0) = 0$:

(a) Solve for $I(t)$

(b) Find the steady-state current

Solution:

(a) The equation becomes:

$$5\frac{dI}{dt} + 20I = 100$$

$$\frac{dI}{dt} + 4I = 20$$

Integrating factor: $\mu = e^{4t}$

$$\frac{d}{dt}(Ie^{4t}) = 20e^{4t}$$

$$Ie^{4t} = 5e^{4t} + C$$

$$I = 5 + Ce^{-4t}$$

With $I(0) = 0$: $C = -5$

$$I(t) = 5(1 - e^{-4t})$$

(b) Steady state: $\lim_{t\to\infty} I(t) = 5$ A

17. **Brewery Fermentation in Nile Breweries**

The alcohol concentration $A(t)$ during fermentation follows $\frac{dA}{dt} = 0.4A(12 - A)$, where t is in days. If $A(0) = 1\%$:

(a) Solve for $A(t)$

(b) Find when $A = 10\%$

Solution:

(a) Separating:

$$\frac{dA}{A(12 - A)} = 0.4\,dt$$

Partial fractions:

$$\frac{1}{12}\left(\frac{1}{A} + \frac{1}{12 - A}\right)dA = 0.4\,dt$$

$$\frac{1}{12}\ln\left|\frac{A}{12 - A}\right| = 0.4t + C$$

With $A(0) = 1$:

$$C = \frac{1}{12}\ln\left(\frac{1}{11}\right)$$

$$\ln\left|\frac{A}{12 - A}\right| = 4.8t + \ln\left(\frac{1}{11}\right)$$

$$\frac{A}{12 - A} = \frac{1}{11}e^{4.8t}$$

$$A = \frac{12}{11e^{-4.8t} + 1}$$

(b) For $A = 10$:

$$\frac{10}{2} = \frac{1}{11}e^{4.8t}$$

$$t = \frac{\ln 55}{4.8} \approx 0.835 \text{ days}$$

18. **Groundwater Contamination in Borehole**

A borehole in Arua experiences contamination where pollutant concentration $P(t)$ (mg/L) changes according to $\frac{dP}{dt} = 3 - 0.5P$, with $P(0) = 10$ mg/L and t in weeks:

(a) Solve for $P(t)$

(b) Find the safe concentration level (steady state)

(c) When does $P = 7$ mg/L?

Solution:

(a) Linear equation:

$$\frac{dP}{3 - 0.5P} = dt$$

$$\frac{dP}{6 - P} = 0.5\,dt$$

$$-\ln|6 - P| = 0.5t + C$$

$$P = 6 - Ae^{-0.5t}$$

With $P(0) = 10$: $A = -4$

$$P(t) = 6 + 4e^{-0.5t}$$

(b) Safe level (steady state): $\lim_{t\to\infty} P(t) = 6$ mg/L

(c) For $P = 7$:

$$7 = 6 + 4e^{-0.5t}$$

$$e^{-0.5t} = 0.25$$

$$t = \frac{\ln 4}{0.5} = 2.77 \text{ weeks}$$

19. **Banana Disease Spread in Rakai**

The rate of banana wilt disease spread is $\frac{dI}{dt} = 0.03I(500-I)$, where I is infected plants and t is in weeks. If $I(0) = 10$:

(a) Solve for $I(t)$

(b) Predict $I(10)$

(c) Find the inflection point

Solution:

(a) Separating:

$$\frac{dI}{I(500-I)} = 0.03\,dt$$

$$\frac{1}{500}\ln\left|\frac{I}{500-I}\right| = 0.03t + C$$

With $I(0) = 10$:

$$C = \frac{1}{500}\ln\left(\frac{1}{49}\right)$$

$$\ln\left|\frac{I}{500-I}\right| = 15t + \ln\left(\frac{1}{49}\right)$$

$$I = \frac{500}{49e^{-15t}+1}$$

(b) $I(10) = \frac{500}{49e^{-150}+1} \approx 500$ plants (essentially all infected)

(c) Inflection point at $I = 250$ (half of carrying capacity):

$$\frac{500}{49e^{-15t}+1} = 250$$

$$t = \frac{\ln 49}{15} \approx 0.26 \text{ weeks}$$

20. **Mobile Money Transaction Growth**

The daily mobile money transactions $M(t)$ (in billions UGX) in Uganda follow $\frac{dM}{dt} = 0.15M + 2$, where t is in months from January 2024, and $M(0) = 50$ billion:

(a) Solve for $M(t)$

(b) Project $M(12)$ for December 2024

Solution:

(a) Linear equation: $\frac{dM}{dt} - 0.15M = 2$

Integrating factor: $\mu = e^{-0.15t}$

$$\frac{d}{dt}(Me^{-0.15t}) = 2e^{-0.15t}$$

$$Me^{-0.15t} = -\frac{2}{0.15}e^{-0.15t} + C = -\frac{40}{3}e^{-0.15t} + C$$

$$M = -\frac{40}{3} + Ce^{0.15t}$$

With $M(0) = 50$:

$$C = 50 + \frac{40}{3} = \frac{190}{3}$$

$$M(t) = -\frac{40}{3} + \frac{190}{3}e^{0.15t}$$

(b) $M(12) = -\frac{40}{3} + \frac{190}{3}e^{1.8} \approx -13.33 + 381.56 = 368.23$ billion UGX

Chapter 16

SAMPLE PAPERS WITH SOLUTIONS

16.1 Uneb Sample Paper 1 2026

P425/1
PRINCIPAL MATHEMATICS
Paper 1
2026
3 hours

UGANDA NATIONAL EXAMINATIONS BOARD

Uganda Advanced Certificate of Education

PRINCIPAL MATHEMATICS

Paper 1

3 hours

INSTRUCTIONS TO CANDIDATES

*This examination paper consists of **two** sections; A and B. It has **five** examination items.*

*Section **A** consists of **one compulsory** item.*

*Section **B** has two parts, **I** and **II** each with two optional items. Respond to only **two** items from Section B, choosing **one** item from **each** of the parts, I and II.*

*Respond to **three** items **in all**.*

*Any additional item(s) responded to will **not** be scored.*

Begin each item on a fresh page.

***All** necessary working **must** be shown clearly.*

Graph paper is provided.

Silent, non-programmable scientific calculators and mathematical tables with a list of formulae may be used.

© 2026 Uganda National Examinations Board **Turn Over**

SECTION A

The item in this section is compulsory.

Item 1

Joseph is a final year civil engineering student who has been offered an opportunity to do industrial training in a construction company. He has joined at a time when the company has been contracted to design and construct a road connecting three trading centres; A, B and C that are on coordinates $(2, 3)$, $(-2, 1)$ and $(4, 7)$ respectively. The road should include a traffic-lights control tower to be installed at a point such that it is equidistant from all the three trading centres. The contractor has given the company two road design proposals; either to construct a circular road or a triangular road, depending on which one is of shorter length and the company has assigned this work to Joseph who needs assistance.

Task:

(a) Determine whether it is possible for the company to construct a triangular road through the three trading centres.
(b) Generate the equation Joseph can use to model the circular road design.
(c) Recommend, based on calculations, the appropriate road design the company should construct and the point where the traffic lights control tower shall be installed on that design.

SECTION B

Answer only two items from this section, choosing one item from each of the parts I and II.

PART I

Item 2

A cooperative society needs to distribute resources to farmers fairly, schedule activities efficiently and plan prize systems transparently using mathematical reasoning.

To balance this, 23 times the number of sacks given to each farmer minus the square of the number of sacks of maize given to each farmer, should be at least 120.

The organisers of the festival intend to have 7 helpers whom they also plan to reward with at most 4 prizes.

The prizes for the 4th, 3rd, 2nd, and 1st positions follow the terms of the expansion of $(3 + 2x)^8$ in ascending powers of x respectively, where x is \$1 (one dollar).

The organisers need to determine the number of ways to reward the prizes while following fair distribution standards.

The organisers plan to invite 45 guests to the festival with each subsequent guest arriving exactly after a 5-minute interval from the time of arrival of the previous guest and the first guest must arrive at 8:00 A.M.

The chief guest will be the last to arrive and 10 minutes after, the festival will be officially opened. The organisers need to determine the time for the official opening.

Task:

(a) Compute the range within which the number of sacks given to each farmer should lie.
(b) Calculate the
  (i) possible number of ways to reward the prize to the helpers.
  (ii) prize for each of the four positions.
(c) Determine the time for the official opening of the festival.

Item 3

A company that deals in fertilizers has been supplying coffee farmers with a fertilizer mixture to improve on their yields.

When using the mixture, the total quantity of coffee obtained in the four consecutive seasons was modeled as; Total yield (kg) = $(Ax + By)^3$, where A and B are quantity ratios of the amount of fertilizer (x) and water (y) respectively.

Each container of the mixture contains 6 kg of the fertilizer and 1 litre of water. The terms of the expansion of the Total yield represent the yields per season and the yields were 1728 kg and 216 kg of coffee in the first and last season respectively.

Over the past few years, the company has noticed that farmers are losing interest in the fertilizer mixture because of lack of expiry date information and limited branches of the company to serve farmers.

According to the company, the percentage effectiveness, P, of the fertilizer over a time period, t, in months, is modeled by the equation;

$$\log P = \log P_0 - C\log(t-4),$$

where C is the concentration constant in a given batch of fertilizers and P_0 is the initial percentage effectiveness.

The company is planning to manufacture a new batch of fertilizers on the 1st September, 2026 with a 95% effectiveness and a concentration constant of 0.2.

According to the company chemist, the fertilizer is considered fit for use when its percentage effectiveness doesn't go below 47.5% and this helps to determine the expiry date.

The company also wishes to set up three different sales outlets for its products whose locations are related by the equation;

$$Z^3 - 5Z^2 - 17Z - 13 = 0$$

where Z is the coordinate of any point on the site map.

However, information about the yields for the 2nd and 3rd seasons, actual expiry date and exact location of the sales outlets is missing, and the company manager needs it.

Task:

(a) Determine the season that gave the highest yields when the farmer used the mixture.
(b) Recommend, based on calculations, the actual expiry date to be indicated on the new label of the batch that will be manufactured on the 1st September 2026.
(c) Generate the coordinates of the sales outlets.

PART II

Item 4

A land owner is preparing to use a wire mesh to fence a rectangular piece of land that has an area of 39,200 m^2, with one side of the land along an existing straight wall.

Due to the existing wall, fencing is required on the other three sides only and a space for a 12 m wide gate must be left along the fence, on the side opposite the existing wall.

According to the manager of the company that is going to fence the land, the rate at which the fence is installed after t hours is modeled by;

$$\frac{dL}{dt} = 12 - 0.1t; \qquad 0 \le t \le 12,$$

where t is the time in hours and L is the length fenced in a day.

The land is mostly flat, but has a depression to be filled with 200 m^3 of soil.

The land also has a small ant-hill in the centre whose shape is modeled by the equation;

$$y = 12 - 3x^2,$$

where x is the horizontal distance (in metres) from the centre of the ant-hill and y is the height of the ant-hill (in metres) above the flat field.

The ant-hill must be removed to ensure that the land is fully levelled and the soil will be used to fill the depression, however, the land owner is not sure whether that soil will completely fill the depression.

The land owner needs to determine the most efficient way to fence the land.

Task:

a) (i) Calculate the minimum length of wire mesh the land owner must buy to completely fence the land.
(ii) Compute the number of days required to install the wire mesh.
b) (i) Sketch the cross-section of the ant-hill.
(ii) Determine whether the depression will completely be filled.

Item 5

Senior six (S6) students are making items to exhibit during their annual competitions. The girls will compete in making moulds using clay while the boys will compete in packing bottles into a box using the shortest time possible.

Jalia had made a conical shaped mould of radius 10 cm and height 100 cm but she realised that many other girls had made the same shape. She would like to reshape it into a cylindrical mould with the biggest volume possible and needs assistance to determine that volume.

Margaret designed a spherical mould but she has realised that its radius is 9.8 cm yet she intended to mould one of 10 cm.

In the guidelines, only moulds whose error in volume does not exceed 8% will be approved for the competitions and Margaret is not sure whether her mould will be approved.

Moses has been practicing packing 100 bottles in a box similar to that used in the competitions and over time he has realised that he packs at a rate proportional to the square root of the bottles remaining.

In his most recent practice, he packed 19 bottles in 1 minute and according to the records, the winner in this category used 13 minutes last year.

Task:

(a) Determine the volume of the biggest cylinder Jalia can mould from her piece.
(b) Establish whether Margaret's mould will be approved for the competition.
(c) Evaluate whether Moses is likely to win the competition basing on the previous year's record.

END

SCORING GUIDE

In response to the tasks, the candidate is expected to: correctly identify the relevant concepts to manipulate in response to the task(s) in the scenario, select and apply correct method(s) to correctly manipulate the concepts identified, and correctly interpret accurate outputs from correctly identified and manipulated concepts to justify/inform decision.

SECTION A

ITEM 1

Task(a)

Proving that points/trading centres $A(2,3)$, $B(-2,1)$, and $C(4,7)$ form a triangle.

For line/road AB,

$$m_1 = \frac{1-3}{-2-2} = \frac{1}{2}$$

For line/road AC,

$$m_2 = \frac{7-3}{4-2} = 2$$

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{2 - \frac{1}{2}}{1 + 2 \times \frac{1}{2}}\right|$$

$$\theta = \tan^{-1}\left(\frac{3}{4}\right) = 36.87^\circ$$

Since the lines AB and AC meet at an angle of $\mathbf{36.87}^\circ$, then trading centres A, B and C form a triangle, and it is possible to construct the triangular road network.

Task(b)

Circle through points A, B, C.

General equation: $x^2 + y^2 + 2gx + 2fy + c = 0$

For $A(2,3)$:
$4 + 9 + 4g + 6f + c = 0 \Rightarrow 4g + 6f + c = -13 \quad (i)$

For $B(-2,1)$:
$4 + 1 - 4g + 2f + c = 0 \Rightarrow -4g + 2f + c = -5 \quad (ii)$

For $C(4,7)$:
$16+49+8g+14f+c = 0 \Rightarrow 8g+14f+c = -65 \quad (iii)$

$(iii) - (ii)$: $12g + 12f = -60 \Rightarrow g + f = -5 \quad (iv)$

$(iv) - (ii)$ [eliminating c]: Using the system,
$2g + f = -2 \Rightarrow 3 + f = -5 \Rightarrow f = -8,\ g = 3$

Substituting in (ii):
$4(3)+6(-8)+c = -13 \Rightarrow 12-48+c = -13 \Rightarrow c = 23$

Equation of the circular road: $x^2 + y^2 + 6x - 16y + 23 = 0$

Equations to model the triangular road network:

For road AB, $m_1 = \frac{1}{2} \Rightarrow \frac{y-3}{x-2} = \frac{1}{2} \Rightarrow y = \frac{1}{2}x + 2$

For road AC, $m_2 = 2 \Rightarrow \frac{y-3}{x-2} = 2 \Rightarrow y = 2x - 1$

For road BC,
$m_3 = \frac{7-1}{4-(-2)} = 1 \Rightarrow \frac{y-7}{x-4} = 1 \Rightarrow y = x + 3$

Task(c)

Lengths of roads.

For the circular road, $r = \sqrt{50}$

Length of the road = circumference
$= 2\pi\sqrt{50} = 44.4288$

For the triangular road:

$|AB| = \sqrt{20} = 2\sqrt{5}, \quad |AC| = \sqrt{20} = 2\sqrt{5}, \quad |BC| = \sqrt{72} = 6\sqrt{2}$

Total length of the road
$= \sqrt{20} + \sqrt{20} + \sqrt{72} = 17.4296$

The triangular road network should be constructed since it has shorter length.

Control tower for traffic lights should be installed at $(-3, 8)$.

ITEM 2

Task(a)

Let x be the number of sacks of maize.

$23x - x^2 \geq 120 \Rightarrow x^2 - 23x + 120 \leq 0 \Rightarrow (x-15)(x-8) \leq 0$

Critical values: $x = 8,\ x = 15$

	$x < 8$	$8 < x < 15$	$x > 15$
$(x-15)(x-8)$	+	−	+

Range: $8 \leq x \leq 15$

The number of sacks of maize given to each farmer should be $8 \leq x \leq 15$.

Task(b)

Combinations, ${}^nC_r$: $n = 7$, for $r \leq 4$

No. of ways $= {}^7C_4 + {}^7C_3 + {}^7C_2 + {}^7C_1 + {}^7C_0 = 35+35+21+7+1 = 99$ ways

$$(3+2x)^8 = 3^8 + 8(2x)(3)^7 + \frac{8\times 7}{2!}(2x)^2(3)^6 + \frac{8\times 7\times 6}{3!}(2x)^3(3)^5 + \cdots$$

$$= 6561 + 21870x + 29160x^2 + 19440x^3 + \cdots$$

Prizes:

- 4th position, prize amount = \$6561
- 3rd position, prize amount = \$21870
- 2nd position, prize amount = \$29160
- 1st position, prize amount = \$19440

Task(c)

Arithmetic progression & time.

Let the number of visitors be n, so $n = 45$.

The intervals in which the visitors arrive form an AP with first term $a = 0$ and common difference $d = 5$.

$T_n = a + (n-1)d$

Total time taken for last guest to arrive:
$T_{45} = 0 + 44 \times 5 = 220$ minutes $= 3$ hours 40 minutes.

Time when the chief guest (last guest) arrives
$= 8{:}00 + 3{:}40 = 11{:}40$ AM

Time for official opening of the festival $= 11{:}40 + 0{:}10 = 11{:}50$ AM

ITEM 3

Task(a)

Yield calculation.

$$(Ax + By)^3 = A^3x^3 + 3A^2x^2By + 3AxB^2y^2 + B^3y^3$$

With $x = 6,\ y = 1$:

$$(6A + B)^3 = 216A^3 + 108A^2B + 18AB^2 + B^3$$

1st season: $1728 = 216A^3 \Rightarrow A = 2$

Last season: $216 = B^3 \Rightarrow B = 6$

2nd season yield $= 108(2)^2(6) = 2592$ kg

3rd season yield $= 18(2)(6)^2 = 1296$ kg

Therefore, the 2nd season gives the highest yield when the farmer uses the fertilizer mixture.

Task(b)

Logarithm & expiry date.

$$\log P = \log P_0 - C\log(t-4) = \log\frac{P_0}{(t-4)^C} \Rightarrow P = \frac{P_0}{(t-4)^C}$$

$$t = \left(\frac{P_0}{P}\right)^{1/C} + 4$$

With $C = 0.2,\ P_0 = 95,\ P = 47.5$:

$$t = \left(\frac{95}{47.5}\right)^{1/0.2} + 4 = 36 \text{ months} = 3 \text{ years}$$

Manufacture date: 1st September, 2026
Expiry date: 1st September, 2029

Task(c)

$z^3-5z^2+17z-13=0, \qquad P(z)=z^3-5z^2+17z-13$

When $z=1$, $P(1)=0 \Rightarrow (z-1)$ is a factor.

By long division:
$z^3-5z^2+17z-13=(z-1)(z^2-4z+13)$

Solving $z^2-4z+13=0$:

$$z=\frac{4\pm\sqrt{16-52}}{2}=\frac{4\pm\sqrt{-36}}{2}=2\pm 3i$$

Therefore $z=1,\ 2\pm 3i$.

The coordinates of the sales outlets are $(1,0)$**,** $(2,3)$**,** $(2,-3)$**.**

ITEM 4

Task(a(i))

Fencing & optimization.

Let x and y be the length and width of the rectangular piece of land to be fenced.

Area: $xy=39200$

Perimeter: $P=2y+x-12$

From $xy=39200$, $y=\dfrac{39200}{x}$

$$P=\frac{78400}{x}+x-12$$

$$\frac{dP}{dx}=-\frac{78400}{x^2}+1$$

For minimum length, $\dfrac{dP}{dx}=0$:

$$-\frac{78400}{x^2}+1=0 \ \Rightarrow\ x=280 \text{ m}$$

$$y=\frac{39200}{280}=140 \text{ m}$$

Required length of the fence
$=2y+x-12=280+280-12$

The required length of the fence $=548$ **m**

Task(a(ii))

$$\frac{dL}{dt}=12-0.1t$$

$$\int_0^L dL=\int_0^{12}(12-0.1t)\,dt$$

$$L=\left[12t-\frac{0.1}{2}t^2\right]_0^{12}=144-7.2=136.8 \text{ m}$$

$$\text{Time taken}=\frac{548}{136.8}=4.0058 \text{ days} \approx 5 \text{ days.}$$

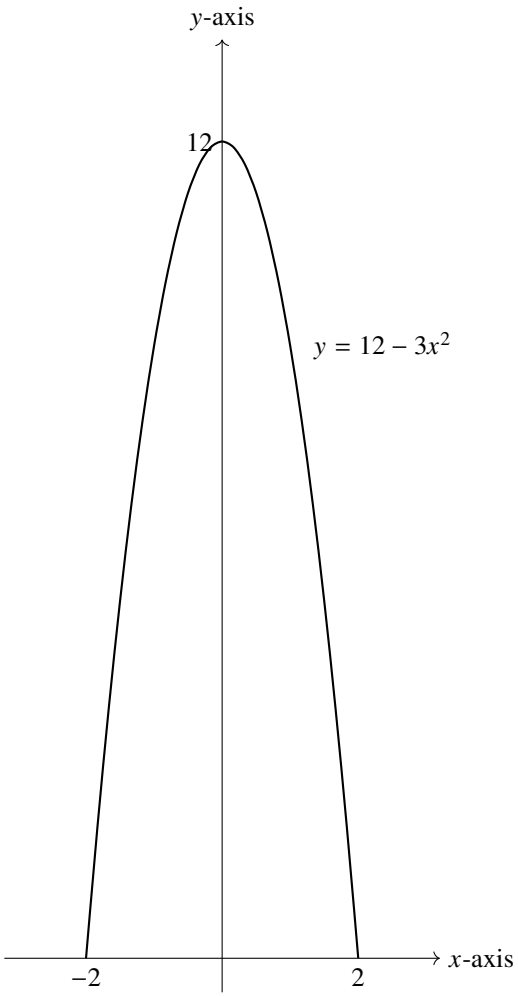
It will take five days to fence the land.

Task(b(i))

$y=12-3x^2, \qquad \dfrac{dy}{dx}=-6x, \qquad \dfrac{d^2y}{dx^2}=-6$ (a maximum)

For maximum, $-6x=0 \Rightarrow x=0$, giving $y=12$, so $(0,12)$ is the maximum point.

When $y=0$: $x=\pm 2$, so $(2,0)$ and $(-2,0)$ are the intercepts.



Task(b(ii))

$$\text{Volume of the Ant-hill}=\int_0^{12}\pi x^2\,dy$$

From $y = 12 - 3x^2$, $x^2 = \dfrac{12-y}{3}$

$$V = \int_0^{12} \pi\left(\frac{12-y}{3}\right)dy = \int_0^{12}\left(\frac{12\pi}{3} - \frac{\pi y}{3}\right)dy$$

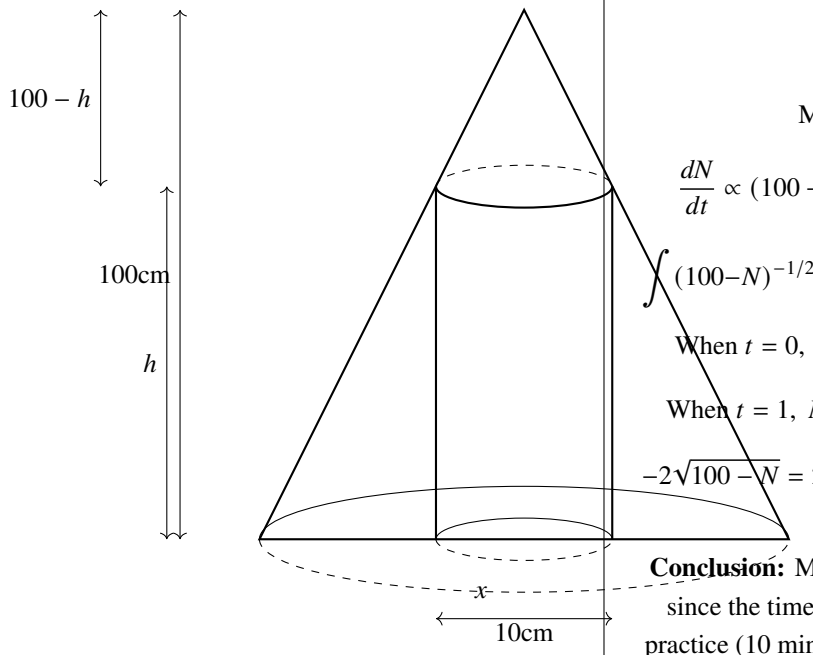
$$V = \left[4\pi y - \frac{\pi y^2}{6}\right]_0^{12} = (4\pi\times 12) - \frac{144\pi}{6} = 48\pi - 24\pi = 24\pi$$

$$V = 75.40 \text{ m}^3$$

The soil obtained by clearing the ant-hill will not be enough to fill the depression since it is < 200 m^3.

ITEM 5

Task(a)



By similar triangles:

$$\frac{100}{100-h} = \frac{10}{x} \Rightarrow 100x = 1000 - 10h \Rightarrow h = 100 - 10x$$

Volume:

$$V = \pi r^2 h = \pi x^2(100 - 10x) = \pi(100x^2 - 10x^3)$$

$$\frac{dV}{dx} = \pi(200x - 30x^2)$$

For max/min:

$$\pi(200x - 30x^2) = 0 \Rightarrow (20 - 3x) = 0 \Rightarrow x = \frac{20}{3} = 6\tfrac{2}{3} \text{ cm}$$

$$V_{max} = \pi\left(\frac{20}{3}\right)^2\left(100 - 10\times\frac{20}{3}\right) = \frac{400\pi}{9}\left(\frac{300-200}{3}\right) = \frac{4000\pi}{27} = 465.4211 \text{ cm}^3$$

The volume of the biggest cylinder Jalia can mould from her conical shaped piece is 465.4211 cm^3.

Task(b)

Error in $r = 10 - 9.8 = 0.2$

$$\%\text{ error} = \frac{0.2}{10}\times 100 = 2\%, \qquad \delta r = \frac{2r}{100} = 0.02r$$

$$V = \frac{4}{3}\pi r^3, \qquad \frac{dV}{dr} = 4\pi r^2, \qquad \delta V = \frac{dV}{dr}\cdot\delta r = 4\pi r^2(0.02r)$$

$$\frac{\delta V}{V}\times 100 = \frac{4\pi r^2(0.02r)}{\frac{4}{3}\pi r^3}\times 100 = 6\%$$

Margaret's mould will be approved for the competition since the error in the volume of her mould is 6% and therefore does not exceed the 8% error required for approval.

Task(c)

Moses – bottle problem.

$$\frac{dN}{dt} \propto (100-N)^{1/2} \Rightarrow \frac{dN}{dt} = k(100-N)^{1/2}$$

$$\int (100-N)^{-1/2}\,dN = \int k\,dt \Rightarrow -2(100-N)^{1/2} = kt + C$$

When $t = 0$, $N = 0$: $-2\sqrt{100} = C \Rightarrow C = -20$

When $t = 1$, $N = 19$: $-2\sqrt{81} = k - 20 \Rightarrow k = 2$

$$-2\sqrt{100-N} = 2t - 20 \Rightarrow \text{when } N = 100: \; -2\sqrt{0} = 2t - 20 \Rightarrow t = 10$$

Conclusion: Moses is likely to win the competition since the time he takes to pack the bottles during practice (10 minutes) is less than the time the winner in that category took last year (13 minutes).

Alternative: Let x represent the number of bottles remaining at time t.

$$\frac{dx}{dt} \propto \sqrt{x} \Rightarrow \frac{dx}{dt} = k\sqrt{x} \Rightarrow \int x^{-1/2}dx = \int k\,dt \Rightarrow 2\sqrt{x} = kt + C$$

At $t = 0$, $x = 100$: $2\sqrt{100} = C \Rightarrow C = 20$

At $t = 1$, $x = 81$: $2\sqrt{81} = k + 20 \Rightarrow k = -2$

$$2\sqrt{x} = -2t + 20 \Rightarrow \text{when } x = 0: \; 0 = -2t + 20 \Rightarrow t = 10 \text{ minutes}$$

Conclusion: Moses is likely to win the competition since the time he takes to pack the bottles during practice is less than the time the winner in that category took last year.

16.2 set 2
16.3 set 3
16.4 set 4
16.5 set 5
16.6 set 6
16.7 set 7
16.8 set 8
16.9 set 9
16.10 set 10
16.11 set 11
16.12 set 12
16.13 set 13
16.14 set 14
16.15 set 15

Chapter 17

PRACTICE PAPERS WITHOUT SOLUTIONS

17.1 set 1

P425/1 2 hours 30 minutes

Section A

The item in this section is compulsory.

Item 1

A certain City Council proposes the installation of a rotating surveillance camera above an elliptical garden in a public park. The garden is modelled by

$$9x^2 + 25y^2 - 54x - 100y - 44 = 0.$$

It is required to analyse the geometry and structural feasibility of the design. Two poles are positioned at the foci of the ellipse and serve as base supports from which two main support beams extend upward, meeting at a common mounting point above ground level. The first support beam is the line through $(5, 2, 6)$ and $(1, 2, 18)$. The second support beam is given by $\mathbf{r} = (1, 2, 6) + t(4, 0, 12)$. The beams intersect at the mounting point and form an angle that ensures structural stability. A rotating surveillance camera is installed at the intersection point. Its vertical displacement during rotation is modelled by $f(\theta) = \sqrt{3}\cos\theta + \sin\theta$, where θ is the angle of rotation.

For approval by the City Council, the design must satisfy: the support beams must intersect at a single mounting point; the mounting point must be exactly 12 m above ground; the maximum height of the camera must be less than 20 m and always above ground; and the angle between beams must be between 30° and 40°.

Task:

Determine whether the proposed installation is structurally sound and suitable for approval.

Section B

Answer only two items from this section, choosing one item from each of the parts I and II.

Part I

Item 2

At an S.6 Leavers' Party, the Organising Committee prepares a celebration combining decoration, entertainment, and a sound system to create a memorable experience. The decorations consist of 3 red balloons, 3 green balloons, 3 black balloons, and 3 orange balloons, where balloons of the same colour are identical. To create visually appealing patterns for photographs and stage decoration, the Committee considers different structured arrangements of the balloons under specific design conditions: Case 1: the three red balloons are grouped together as a single block. Case 2: both the three red balloons and the three green balloons are each grouped together as separate blocks. They need to find

out which arrangement yields the higher number of designs.

Inside the hall, a digital music system controls sound quality during performances. Stability is analysed using two response functions $f(a) = 32a + 59049$ and $g(a) = 59050(3^a)$, where engineers find the tuning parameter a by solving $f(a) = g(a)$ to ensure stable audio output is beyond 8.

At the same time, in a budgeting exercise, the Organising Committee is planning expenditures for decorations, catering, sound systems, and venue arrangements, where x represents one unit of budget allocation for a standard service package covering essential party requirements. The total projected expenditure is modelled by

$$2x^4 + 3x^3 - 4200x^2 + 6000x + 900000$$

and the standard service package cost structure is given by $x^2 + 50x + 200$. They need to determine any excess in the planning process, which should be less than one million to avoid misuse of the allocated funds.

For sound quality optimisation, the audio system also relies on impedance matching in the speaker circuit. Three components of the system are represented by complex numbers: $Z_1 = 5+12i$, $Z_2 = 2-i$, $Z_3 = 1+2i$. The overall impedance of the system is obtained by multiplying the first two components and then dividing by the third; they must evaluate its modulus to ensure it exceeds 9 Ω, confirming clear and stable sound output throughout the celebration.

Task:

Design and analyse the S.6 Leavers' Party system by solving stability, arrangement, budgeting, and impedance problems.

Item 3

A technology innovation hub is preparing a financial and operational assessment for a national exhibition, and management is required to make four key approval decisions.

The marketing department runs a customer-acquisition campaign in which new subscribers triple each week. In the first week, the sequence of subscribers is $6, 18, 54, \ldots$. Management will approve expansion only if the cumulative number of subscribers exceeds 100,000,000, achieved within a reasonable number of weeks.

A telecommunications system models signal efficiency by $E(x) = \sqrt[3]{1+4x}$, where x is a small performance improvement. They must expand this to approximate $E(x)$ up to the term in x^2, estimate $E(0.2)$, and determine system stability given that the efficiency increase must not exceed 25%.

The telecommunications unit is testing a digital signal represented by a complex number Z. During calibration, engineers find that the signal and its complex conjugate satisfy: the product of Z and Z^*, plus eight times the imaginary unit multiplied by Z, equals $9360 + 480i$. The system will only be approved for deployment if a valid complex signal exists and the product of the magnitudes of the two signals obtained is less than 10000.

The hub plans to select a team of 12 male workers from 15 eligible members. The selection process will only be approved if the number of possible selections does not exceed 800, ensuring that the administrative workload remains manageable.

Task:

Provide a clear final recommendation stating whether the Innovation Hub should approve all four systems or reject any of them, supported by your results.

Part II

Item 4

Wildlife authority officials of a certain region are assessing a forested area that is at risk of, or has been affected by, a bush-fire. The spatial extent of the affected forest land (in square kilometres) is modelled by the curve $y = x^2 - 10x$ and the straight line $y = -2x + 20$, where all distances are measured in kilometres. The sketch of these boundaries and the corresponding area of the forest are essential for planning fire-fighting operations, as they guide the strategic positioning of fire-fighting teams around the forest to ensure effective containment and suppression of the fire and prevention of further spread, which must be less than 350 km^2.

In addition, a bush-fire is spreading through a dry forest

region with a total area of 288 km^2. The rate at which the dry forest becomes burnt is assumed to be proportional to the product of the area already burnt and the area that remains unburnt. At 9:00 a.m., 2.8 km^2 of the forest had already been burnt, and by 9:30 a.m. this had increased to 18.4 km^2. Fire-extinguishing experts arrived at 10:00 a.m. From that time onward, the fire is considered to be under control only if the area already burnt is less than 100 km^2 of the total forest area.

Task:

Help the wildlife authority officials to:

(a) generate a sketch to be presented to the fire-extinguishing experts;
(b) decide whether the area should be prioritised for emergency response;
(c) determine whether the fire is already under control or still escalating based on the available data.

Item 5

A humanitarian water distribution project is being implemented in a drought-affected region to ensure continuous access to clean water. Engineers construct a rectangular tank inside a right circular conical protective shelter with a base radius of 9 m and a height of 21 m to house critical water storage infrastructure. Inside the shelter, a rectangular water tank with a square base is installed such that it fits perfectly within the cone. The tank rests on the circular base of the shelter, and its upper edges are constrained by the curved inner surface of the cone, ensuring maximum possible storage capacity without structural interference. The engineering team needs to determine the maximum storage capacity of the tank to establish whether it can meet the water demands of the affected community.

A water supply pump system is then used to fill the tank. The supply process is modelled by the differential equation

$$\frac{dV}{dt} = \frac{119622}{1200 - V}$$

where $V(t)$ is the volume of water in the tank (m^3) and t is the filling time (hours). Initially, the tank is empty. After operating the pump for 4 hours, engineers need to determine the amount of water delivered. The system will only be accepted if the calculated water volume is slightly greater than the maximum storage capacity of the tank. During the analysis, any mathematically invalid volume must be rejected.

Task:

Assist the project to decide whether the pump system is suitable, based on the storage capacity and the water supplied.

17.2 set 2

P425/1 2 hours 30 minutes

Section A

The item in this section is compulsory.

Item 1

A zone in one of the districts is facing three challenges: traffic congestion, poor mobile network at two trading centres, and frequent power cuts. The district has approved a zonal upgrade project to solve these problems.

A circular bypass road will be constructed around the congested centre. The path of the bypass is modelled by $x^2 + y^2 - 4x - 8y + 16 = 0$, where all distances are in kilometres. To reduce congestion, a new express road modelled by $y = x + 4$ will intersect the bypass at trading centres A and B.

To improve the network, a telecommunication mast is to be positioned at a point equidistant from the midpoint of AB and the centre of the circular bypass. For network clarity, the mast's construction point should lie on a linear path whose gradient is less than 1.5. A construction company has been contracted to tarmac both the circular road and the road from A to B; the cost of tarmacking is UGX 3 billion per kilometre, and the amount available for tarmacking is 48 billion.

At the top of a straight pole parallel to the telecommunication mast, passing through points $P(2, 2, 5)$ and $Q(4, 2, 7)$, a fixed solar panel will be installed to power the mast at night. The panel is to be fitted through three support points $C(8, 1, 1)$, $D(0, 2, -14)$ and $E(0, 0, -16)$. For sufficient solar energy to reach the panel, the acute angle between the pole and the panel should lie between 10° and 20°, and the point $(3, 3, -7)$ should lie on the panel.

Task:

Using mathematical justifications, help to:

(a) establish whether the network will be clear;
(b) establish whether the available amount of money is enough for tarmacking both roads;
(c) ascertain whether the telecommunication mast, the pole, and the solar panel meet the required specifications.

Section B

Answer only two items from this section, choosing one item from each of the parts I and II.

Part I

Item 2

A manufacturing company produces soap. Management is reviewing production costs, meeting customer demand, forming a quality committee, servicing machines, and monitoring product quality. The total cost $C(x)$ in UGX for producing x boxes of soap is modelled by $C(x) = Ex^2 + fx + g$. Company records show that 10 boxes cost UGX 320,000, 20 boxes cost UGX 720,000, and 30 boxes cost UGX 1,320,000. The revenue function is modelled by $R(x) = 40{,}000x$. A production run is considered profitable if the profit is greater than UGX 100,000 and the number of boxes produced is between 12 and 18.

To meet market demand, soap production was 200 boxes in the first month, 300 boxes in the second month, 450 boxes in the third month, continuing in the same pattern. Total production after 30 months must be at least 75,000,000 boxes to meet demand.

The company has 12 men and 10 women on its board. A quality assurance committee of 11 members must be formed, consisting of at least 8 men, with both genders represented, and the number of possible committees should not exceed 70,000.

Task:

As the company's mathematical consultant, use justification to help management:

(a) establish whether a single production run is profitable;
(b) ascertain whether the company can meet customer demand after 30 months;
(c) determine whether the quality assurance committee will be formed under an effective selection.

Item 3

A production plant processes timber in large volumes. Management needs to schedule machine maintenance and monitor product quality. The factory's monthly production is 10,000 m^3 in the first month, 15,000 m^3 in the second month, 20,000 m^3 in the third month, increasing by a constant amount each month. The levelling machine must be serviced when total cumulative production reaches 450,000 m^3. The machine operates safely only when its cutting speed V (rpm) satisfies $V^2 - 130V + 4000 \leq 0$, with $40 \leq V \leq 85$.

Timber quality is monitored using indices x and y satisfying $\log_3 x + \log_3 y = 3$ and $x + y = 12$. For good quality production, both x and y must lie between 2 and 12.

Task:

Use mathematical reasoning to:

(a) establish when the levelling machine will require servicing;
(b) ascertain the safety of the levelling machine based on its cutting speed;
(c) determine whether the timber produced is of good quality.

Part II

Item 4

A livestock farm is managed by a conservation company. The company must ensure continuous water supply for animals, construct efficient housing using limited fencing material, and maintain electrical safety standards.

Water for animals is stored in a tank and drains through an outlet at the base; at time t minutes, the depth of water is x metres. The rate at which the depth is decreasing is proportional to the square root of the depth. Monitoring started at 9:00 a.m. with a depth of 2 metres, and after 10 minutes the depth was 1.5 metres. The tank must be refilled when the depth reaches 1 metre.

Due to an increase in livestock, the company plans to fence part of a rectangular piece of land using UGX 1,200,000. The land already has a fence along one side. The shorter sides are fenced with wire costing shs 2000 per metre, while the longer side costs shs 3000 per metre. The fenced area will be suitable for the animals if it is greater than 29000 m^2.

The electric current supplied to the farm is modelled by $I(t) = \cos^2\left(\frac{t}{2}\right)$, t in seconds. For safety compliance, the mean value of the current over $0 \leq t \leq \frac{\pi}{2}$ must not exceed 0.9 Amperes.

Task:

Using mathematical justifications, help the company to:

(a) establish when the tank should be refilled;
(b) determine the suitability of the area to be fenced;
(c) establish the safety of the current supplied to the farm.

Item 5

A company intends to establish a forest reserve on a minimum of 10 km^2 of land. Part of the land available is bounded above by the curve $y = x(6 - x)$ and below by the line $y = 2x$, in square kilometres. For irrigation purposes, a cuboidal tank whose ratio of length to width is 4 : 3 and total surface area 376 m^2 is to be used. The tank must have a capacity of at least 450 m^3 to be deemed suitable.

During construction, the angle that the longest diagonal of the tank makes with the horizontal was measured as 29° instead of the design value of 30°. If the error was modelled using the sine function, the percentage error must not exceed 4% to satisfy acceptable standards.

Task:

Help the company to establish:

(a) whether the area bounded is sufficient for the forest reserve;
(b) the suitability of the water tank for its purpose;
(c) whether the error in measuring the longest diagonal angle is within the acceptable range.

17.3 set 3

P425/1 2 hours 30 minutes

Section A

The item in this section is compulsory.

Item 1

An agricultural development board near an industrial zone is planning to establish a secure perimeter and central infrastructure network to connect four regional storage and processing depots. On the district's Cartesian planning grid, the locations of the four depots are mapped as coordinates $A(2, 6)$, $B(5, 10)$, $C(8, 6)$, and $D(5, -2)$. The project plan requires constructing straight direct boundary roads between adjacent depots to enclose the zone. Additionally, two main internal service pipelines must be laid diagonally across the zone to connect opposite depots (A to C, and B to D). To maximise efficiency, a central security and monitoring tower is to be built at the exact point where these two diagonal pathways intersect.

The structural engineering firm handling the project must mathematically verify the exact geometric shape of the enclosed area to qualify for specific regional infrastructure funding and determine the optimal coordinates for the monitoring tower.

Task:

(a) Verify whether the boundary roads connecting the four depots enclose a region that forms a regular shape, and determine the precise coordinate location where the central monitoring tower should be built.
(b) Calculate the total length of the internal service pipelines required to connect the opposite depots, and determine the equations of the lines representing these two diagonal pathways.

Section B

Answer only two items from this section, choosing one item from each of the parts I and II.

Part I

Item 2

A large-scale grain milling and processing factory near an industrial zone is conducting a comprehensive oper-

ational audit to maximise its financial efficiency and reduce overhead expenses. The plant manager needs to evaluate three critical sectors: raw grain procurement optimisation, automated hydration control systems, and wholesale product inventory market readiness.

The factory's recent supply ledger shows three separate bulk batch purchases of maize flour, millet flour, and sorghum grain (in units of 100 kg bags):

Purchase Batch	Maize Flour (Bags)	Millet Flour (Bags)	Sorghum Grain (Bags)
Batch A	2	1	2
Batch B	1	3	1
Batch C	3	1	1

The total expenditure for batches A, B, and C was 750,000, 1,000,000, and 800,000 respectively. Management will classify future raw-material acquisitions as cost-friendly if a prospective order of 5 bags of maize flour, 4 bags of millet flour, and 2 bags of sorghum grain stays below a total budget of 2,100,000.

To maintain precise product moisture levels, the factory uses an automated processing bay where a liquid conditioning compound is introduced. Sensor telemetry records show the daytime-shift fluid volume as $\sqrt{11x+75}$ thousand litres, and the night-time-shift volume as $\sqrt{x+25}$ thousand litres. The total volume over a full 24-hour cycle is exactly 20 thousand litres. The hydration system successfully protects the grains from spoilage without risking structural degradation only if the chemical variable x does not exceed 15 units.

Furthermore, the facility holds an inventory of 25 tons of processed flour. A batch is cleared for export only if 90 times the number of tons in storage is strictly greater than the square of the tonnage plus 1400.

Task:

(a) Formulate and solve the mathematical models representing the raw material procurement costs and the automated hydration system variables.
(b) Evaluate the factory's inventory status against the regulatory rule and provide definitive recommendations to help the plant manager optimise operational decisions.

Item 3

A commercial fruit processing factory near an industrial zone specialises in manufacturing concentrated juices. Automated extraction chambers operate under controlled conditions. As the internal chamber pressure factor x increases, the efficiency percentage E of the extraction juice output decreases according to

$$E = ax^3 + bx^2 + cx + 40$$

where a, b, c are constant operation parameters. Quality control logs recorded:

Pressure Factor (x)	Efficiency (E)
1	37
2	32
3	19

An external consultant recommends running the process at $x = 4$ to conserve power, claiming a viable efficiency is maintained.

Two heavy-duty extraction pumps, P and Q, have workloads depending on running time t minutes: Pump P: 32^{t-1}; Pump Q: $28(3^t) - 9$. Both pumps must achieve a perfect equilibrium of workloads, with the required time not exceeding 3 minutes.

The factory stores its finished product in a multi-level automated warehouse of 9 storage zones, with total capacity represented by the terms of the binomial expansion $N = (x+3)^8$ hundred crates, where x is the baseline clearance height (m). Expansion will be approved only if the storage capacity calculated using the first four terms of the expansion, with $x = 0.2$ m, shows the facility can accommodate at least 550,000 crates.

Task:

(a) Help the factory officials to get the system of equations from the efficiency log to determine the constants a, b, and c, and analyse whether the consultant's recommended operating pressure yields a viable efficiency.
(b) Determine the operational time required for the two pumps to achieve equilibrium, and evaluate whether the warehouse capacity meets the criteria needed to approve the factory's distribution expansion plans.

Part II

Item 4

An agro-environmental cooperative based in a certain city designs specialised concrete water basins to harvest clean rainwater for livestock during drought seasons. The internal chamber of a newly proposed basin model is engineered by rotating a two-dimensional region bounded by the horizontal line $y = 6$ and the curve $y = x^2 + 2$ about the line $y = 6$.

To secure a localised green energy grant, the cooperative must print a standard technical brief document. The layout specifications dictate that the central text area dedicated to the engineering metrics must be shaped as a composite block: a central rectangle flanked by two identical right-angled triangles on its left and right sides. This entire combined central text block must cover an exact area of 150 cm^2. The document layout requires a fixed margin of 4 cm on both left and right sides of the paper, and a fixed margin of 2 cm at both top and bottom.

The district print-office guidelines state that for the document to fit standard evaluation folders, its total combined horizontal width (including both inner text blocks and side margins) must be kept within a strict limit between 18 cm and 35 cm. The cooperative wants to optimise the layout by finding the precise external dimensions that minimise the total combined surface area of the paper sheet, while preserving the mandatory central text block space.

Task:

(a) Determine the optimal external dimensions of the paper sheet that minimise its total area while meeting the print-office layout limits.
(b) Calculate the exact volume capacity of the water basin to complete the technical brief.

Item 5

An organic juice refinery is upgrading its packaging line to handle increased production volumes. The plant manager plans to channel refined fruit concentrate into cylindrical canisters whose height is exactly four times its radius. The canister fabrication plant has a known manufacturing tolerance error in the radius of 0.5%. The batch of canisters will be rejected if the corresponding percentage error in the total volume exceeds 1.6%.

The automated induction nozzle records the instantaneous height (h) and volume (V) of fluid in the canister after an elapsed time t seconds:

$$h = \left(\frac{t^2}{4} + 2t\right)\text{cm}, \qquad V = \left(\frac{t^2 + 6t}{t + 2}\right)\text{cm}^3.$$

The nozzle will be approved only if the rate of change of fluid volume with respect to height at $t = 2$ seconds does not exceed 0.5 cm^2, to prevent fluid turbulence and splashing.

The weekly operational energy cost of running the filling system is modelled by

$$\frac{dc}{dt} = 30te^{-t/4}$$

where c is cumulative cost in thousands of Ugandan shillings and t is the processing duration in weeks. The facility will transition to an alternative solar hybrid grid if the total accumulated cost over the first 4 weeks exceeds 180,000/=.

Task:

(a) Analyse the geometric tolerance dimensions of the canisters and determine whether the filling nozzle configuration satisfies the non-spill criteria at the specified time interval.
(b) Compute and solve the cost differential equation to advise the finance department on whether the refinery needs to switch to the alternative solar hybrid grid based on the four-week budget limit.

17.4 set 4

P425/1 2 hours 30 minutes

Section A

The item in this section is compulsory.

Item 1

A certain school is carrying out renovations ahead of re-opening. The school is to mount flood lights at $C(1, -2, 8)$ on the administration block to improve lighting. The school security guard patrols at night along the straight corridor route from $A(2, 3, 1)$ to $B(5, 7, 4)$. For proper visibility during night patrols,

the flood lights should be at most 9 m from the patrol corridor; otherwise the school will find another location. (All coordinates in metres.)

The girls' reading shelter is getting a modern curved arched roof to prevent water logging. Viewed from the front, the cross-section forms a downward-opening vertical parabola modelled by $y = 4 - \frac{x^2}{8}$ (height in metres above the ground, x the horizontal distance from the centre of the shelter). Rainwater collects at a central overhead collection node at $Q(0, 6)$. A straight drainage down-pipe must run from Q, touch the curved roof surface, and drain into a concrete collection trench at ground level at $T(6, 0)$.

The school management is also planning to install a CCTV camera whose rotation is modelled such that its distance of coverage in every rotation is $D = 20(3\sin\theta + 4\cos\theta + 6)$ metres. The school wishes to monitor a distance in the range $30 < D < 200$ metres.

Task:

Help the school administration make valid decisions on the flood lights, drainage pipes, and CCTV camera.

Item 2

A large-scale poultry farmer is conducting a comprehensive audit of his farm records to optimise operational efficiency and cut unnecessary expenses. He needs to evaluate three core areas of his business: feed procurement costs, automated flock hydration safety, and wholesale egg inventory marketability. The farm's recent inventory ledger shows three bulk purchases of layers mash, maize bran, and protein concentrate (mass of feed ×100 kg):

Purchase	Layers Mash	Maize Bran	Protein Conc.
A	3	1	2
B	1	2	1
C	2	1	3

The total amount used for purchases A, B, C was 2,800,000, 1,800,000 and 2,600,000 respectively. The farmer will consider these purchases cost-friendly if it takes less than 5,500,000 to purchase 400 kg of layers mash, 400 kg of maize bran, and 300 kg of protein concentrate.

The farm uses an automated drinking system where a vitamin boost is dissolved into the flock's daily water supply. Sensor telemetry shows the daytime water consumption as $\sqrt{3x + 700}$ litres, and night consumption as $\sqrt{x + 600}$ litres, with total consumption over 24 hours exactly 70 litres. The vitamin boost will successfully optimise egg production if its mass x doesn't exceed 300 g.

The farm has 33 trays of eggs in storage. A batch can only be sold if 130 times the number of available trays is strictly greater than the square of the number of trays plus 3000.

Task:

Help the poultry farmer make decisions on his farm using knowledge of algebra.

Item 3

A businessman set up a large-scale baking factory facing some challenges that need to be addressed. His machines mix ingredients at a constant temperature; whenever the temperature increases, the volume of bread obtained decreases according to

$$V = ax^3 + bx^2 + cx + 30$$

where a, b, c are constant amounts of ingredients used. The table below shows the volume of bread at different temperatures:

Temp. ×10 (°C)	Volume of bread, V
1	28
2	22
3	18

A specialist advises mixing ingredients at 45°C, claiming this is the optimal temperature.

The bakery has two machines A and B whose workloads relate to operational time t minutes as: Machine A: 22^{t+2}; Machine B: $33(2^t) - 8$. Both machines must perfectly balance and operational time t should not exceed 5 minutes.

The factory has a storage facility of 11 rooms represented by the terms of the expansion $N = (x + 2)^{10}$ hundreds of boxes, where x is the height of the box used for storage and transportation. The businessman would like to expand his storage space, but this will be done only if, using the first four terms of the expansion

with boxes of height 0.25 m, the available space cannot store 300,000 boxes.

Task:

Help the businessman make the necessary decisions in his factory using your knowledge of algebra.

Part II

Item 4

A farmer practising large-scale bee-keeping is facing a problem of large amounts of honey produced on his farm. He has decided to further process the honey and add value to it, pumping filtered honey into cylindrical tins whose height is three times the radius. The tins on the market have a factory error in radius of 0.5%. The farmer will not accept these tins if their corresponding error in volume exceeds 2%.

The pump machine records the height (h) and volume (V) of honey after time t seconds as

$$h = \left(\frac{t^2}{5} + 3t\right)\text{cm}, \qquad V = \left(\frac{t^2+4t}{t+1}\right)\text{cm}^3.$$

He will use this machine only if the rate of change of volume with height at $t = 3$ seconds does not exceed 0.3 cm^2, to ensure no spilling or wasting of honey.

The power cost of the pump machine is modelled by $\frac{dp}{dt} = 20te^{-t/5}$, where p is the cost in thousands of shillings per week. The farmer should consider other power sources if the cost for running the machine over 5 weeks exceeds 140,000/=.

Task:

Use your mathematical skills to help the farmer make valid decisions based on his plans.

Item 5

Students in a certain school have been complaining about unsafe drinking water. The school administration has come up with measures to solve this problem. A welder is contracted to make a cuboidal tank whose height h metres relates to the diagonal angle θ as $h = \tan\theta$ metres. The welder made a tank with diagonal angle 50° instead of 45°. The administration will only accept this tank if the error in the height is less than 0.2 m.

They also plan to buy a machine to pump water into the tank daily. The available machine is rated $\frac{dV}{dt} = \frac{100t}{(t+1)(t+2)}$ m^3 per hour. To save power, the administration needs a machine that can fill the tank with at least 70 m^3 of water in 5 hours.

Once the tank is filled with the required amount of water, they plan to add a water treatment chemical that decomposes at a rate $e^t\frac{dm}{dt} + me^t = t^2$, where m is the mass in grams of the chemical remaining after t hours. Initially 100 grams is added daily, but the water can only remain pure and safe for drinking if after 6 hours the amount of chemical remaining is not less than 0.4 grams.

Task:

Using your knowledge of calculus, help the school administration make valid decisions.

17.5 set 5

P425/1 2 hours 30 minutes

Section A

The item in this section is compulsory.

Item 1

Opio is developing a radar system for a certain radio. For strength of signals, there is a cyclic formation that passes through $(1, 1)$ with radius $\frac{\sqrt{10}}{2}$ units. The centre of the cyclic formation lies on the line $y = 3x - 7$. The point $(4, 7)$ must be within the cyclic formation for stability of signals.

The angle θ at which the radar system will transmit signals is given by $4\sin 2\theta + 8\cot 2\theta - 5\csc 2\theta = 0$. For the system to transmit signals at a higher radiation, there must be at least two angles θ between 100° and 200°.

The frequency wave will form a plane passing through $(-3, 4, -7)$, $(6, -1, -3)$ and $(-4, 3, -2)$. A radiation wave follows a straight line given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 8 \\ 2 \\ 8 \end{pmatrix}.$$

The angle at which the frequency wave and radiation wave intersect should be at least 30° or else there will be a lot of interference.

Task:

Help Opio to:

(a) determine whether the signals will be stable;
(b) find out whether the system will transmit signals at higher or average radiation;
(c) find the point at which the frequency wave will intersect with the radiation wave;
(d) determine whether there will be a lot of interference or not.

Section B

Answer only two items from this section, choosing one item from each of the parts I and II.

Part I

Item 2

Kenny is an engineer designing a curved bridge project to be constructed by his company. According to the design, the cable sag is to satisfy $\frac{(x-2)(x+2)}{x} \leq 3$, where x is the number of wire points in the sag. Kenny must get the range in which the number of wire points is to be maintained.

The maximum number x of vehicles that can run on the bridge at a time is modelled by $\log(\sqrt{1+x}) + \frac{5}{2}\log(1-x) = \log\sqrt{1-x^2} + 2$. Kenny must write a report indicating the maximum number of vehicles to avoid overweighing.

For the small deflections that are to happen, the intensity x has been modelled by $(1-x)^{1/3}$, expanded to the first four terms. The intensity has to be maintained at $\frac{1}{9}$; Kenny wants to use this intensity to find the approximate deflection of $\sqrt[3]{9}$ to determine the strength of the adapter to be used.

Task:

Help Kenny to:

(a) get the range in which the number of wire points in the cable sag are to be maintained;
(b) determine the maximum number of vehicles that can run on the bridge at once;
(c) find the approximation of the deflection of $\sqrt[3]{9}$.

Item 3

A certain mathematics club is running a coding contest of eight contestants who are to participate in a math quiz. Three participants out of the eight are to be selected for first, second and third places; the manager wants to know the number of selections in which this can be done.

The competition runs in rounds where a contestant earns \$10 in the first round and the amount triples every round. A contestant should not exceed a sum of \$20,000 to maintain the budget; the manager needs the number of rounds a contestant should participate in to stay within budget.

Each participant is given an identity card (ID) following the code Z where $Z = (2+5i)^2 + 5\left(\frac{7+2i}{1-2i}\right) - i(4-6i)$. To avoid repetition, $|Z|$ must be greater than 15 units, and to maintain ID quality, $\arg Z$ must be between $\frac{\pi}{2}$ and $\frac{4}{5}\pi$.

The club plans to use total scores n modelled by $1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$. If this model holds, the competition will take less than five days, otherwise more than six days.

Task:

Help the manager to determine:

(a) the number of selections that can be made for the three places;
(b) the number of rounds a contestant should participate in so as to maintain the budget;
(c) whether another ID code has to be used;
(d) the number of days in which the competition will take place.

Part II

Item 4

A certain biochemical company has secured a rectangular piece of land which is fenced on one of its sides. The company is to fence the remaining sides using 2500 m of wire. The director wants to know the maximum area that the available wire can cover.

The engineer contracted to put the fence has identified that two of the sides of the fence will be modelled by $x = \frac{3t-1}{t}$ and $y = \frac{t^2+4}{t}$. The director wants to

identify the point(s) at which the fence will be turned, to avoid encroaching on neighbouring land.

The company is manufacturing a drug that kills infectious bacteria. The rate at which the bacteria die is directly proportional to the number of bacteria present at that time. On the first day the drug was administered when there were 5500 bacteria, and by the fourth day there were 4000 bacteria. Administering the drug is meant to stop when the number of bacteria falls below 1375.

Task:

Help the company director to determine the:

(a) maximum area that the available wire will cover;
(b) point(s) at which the fence will be turned;
(c) day on which the patient should stop being administered with the drug.

Item 5

Musa is a director of a certain construction company that has been contracted to design and construct roads in a hilly area. In the plan, the first road will have a slope of $\cos 2x \sin 2x$ at the point $\left(\frac{\pi}{8}, \frac{1}{2}\right)$. This road must pass through the existing trading centre located at $\left(\frac{\pi}{24}, \frac{5}{4}\right)$. If the road does not pass through this centre, the design has to be adjusted.

The second road has an equation modelled by $y = \frac{3x+3}{x(3-x)}$. Before construction begins, Musa must give in the sketch of this road for record purposes.

Task:

Help Musa to:

(a) decide whether the first road is to be adjusted or not;
(b) make a sketch of the second road.

17.6 set 6

P425/1 2 hours 30 minutes

Section A

The item in this section is compulsory.

Item 1

An extension worker mapped a village demonstration farm on Cartesian coordinates in three dimensions. He installed a sprinkler at point $S(1, 3, 2)$ inside a greenhouse. The greenhouse roof is the plane $3x - y + 2z = 7$. A water pipe running along the line $\mathbf{r} = (2 + \mu)\mathbf{i} + (-1 + 2\mu)\mathbf{j} + (1 - 2\mu)\mathbf{k}$ enters the roof at point P. The extension worker wishes to move the sprinkler closer if the distance between S and P is more than 5 m.

Two boundaries of the sprinkler spray zone on the greenhouse floor have equations $y = 2x - 1$ and $3x + 4y = 8$. The extension worker wishes to change the nozzle if the angle between the two spray boundaries is less than $50°$.

During calibration, the sprinkler rotates through angle x satisfying $2\sin x + 5\cos x - 1 = 0$ $(0° \leq x \leq 180°)$. If the angle is less than $100°$, the sprinkler will be adjusted.

A faulty nozzle sprays water such that it hits the floor at a point $Q(x, y)$ whose distance from $A(2, 1)$ is twice its distance from $B(5, 3)$. The extension worker wishes to relocate a plant at $(7, 4)$ if water can't reach it.

Task:

Based on mathematical computations, advise the extension worker on the decisions to take.

Section B

Answer only two items from this section, choosing one item from each of the parts I and II.

Part I

Item 2

A college administration is planning activities for the Annual Academic Excellence and Innovation Week. To ensure fairness in allocating study materials, organising student awards, and scheduling the official opening ceremony, the administration applies mathematical reasoning.

To allocate revision booklets fairly, 25 times the number of booklets given to each student minus the square of that number should be at least 144. For fairness, each student must get at least 8 booklets.

The administration has selected 8 student leaders, but only 5 will receive certificates of excellence. The number of possible selections of certificate winners should not exceed 60, and each cash prize should not be more than 120 million.

The certificates for the 5th, 4th, 3rd, 2nd and 1st positions correspond respectively to the first five terms in ascending powers of x in the expansion of $(4 + 3x)^7$, where x represents shs. 5.

On the day of the ceremony, 50 invited guests are expected. The first guest arrives at 9:10 a.m., and each subsequent guest arrives t minutes after the previous one. The guest of honour is the last person to arrive, and 15 minutes later the ceremony will officially be declared open. The ceremony is to run for 3 hours 40 minutes and must end by 5:59 p.m.

Task:

Help the administration decide whether to carry on the planned format of booklet distribution and certificate awarding, and also advise on the arrival time interval (t) for the individual guests so that the function ends on time.

Item 3

A security team was tasked to design a stepped security pillar to be constructed in a sugar-cane plantation to monitor and fight fire outbreaks. The pillar provides a high viewpoint for surveillance and quick response.

The heights of the steps above the ground increase by the same vertical distance from one step to the next. The height of the 5th step is three times the 7th step. The total height of the first twelve steps is 21 m. The team wishes to redesign the pillar if the total height of the first five steps is less than 15 m.

The safety team wishes to reduce response time by placing two communication radios a distance of x tens of metres from the pillar. One radio shows a signal strength of $3\log_8 x$, and the other radio shows a signal strength of $6\log_x 4$. For the safety team to hear clearly without echo, the sum of the two signal strengths must be 8. They wish to adjust the radios if their distance from the pillar is less than 40 m.

The team wishes to position at least four solar powered lights. The main light is positioned at $-8 + 8\sqrt{3}\,i$ metres. The other three lights must be positioned at the cube roots of the main light position. They wish to relocate the lights if the sum of the moduli of the positions is less than 10 m.

Task:

Based on mathematical evidence, advise the security team on the decisions to make.

Part II

Item 4

A farmer in a certain district is establishing a beef cattle project on his land along a river. He is to construct a rectangular paddock along the river forming one side of the enclosure; only the remaining three sides require fencing. The farmer has 1120 m of fencing available.

Veterinary guidelines recommend that each bull requires 500 m^2 of grazing space. The farmer stocks the farm at a rate proportional to the square of the number of available bulls. Initially there are 50 bulls, and after 8 weeks there are 75 bulls. The farmer wishes to start selling bulls when the farm is stocked to full capacity.

To maintain the pasture throughout the year, an irrigation system is installed. The main irrigation pipe follows the path $y = \frac{x}{2}\sqrt{10 - x}$, where x is the horizontal distance (m) east of the water source and y is the distance (m) north of the water source. Engineers are to install inspection gadgets at points where the pipe changes direction, which must be 10 metres apart.

Task:

(a) Determine the maximum number of bulls that can be grazed according to the veterinary guideline.
(b) Advise the farmer on when to start the sales.
(c) Make a recommendation on the engineer's inspection points.

Item 5

A community irrigation project is constructing a decorative water feature and a storage system for a farm. The side profile of the decorative basin is modelled by $y = x(4 - x)^2$, $0 \le x \le 4$, in metres. The region enclosed by the curve and the x-axis is rotated about

the x-axis to form a solid basin whose volume must be at least 450 m^3, otherwise there is a need to redesign the decorative feature.

At the same project, workers use a spherical water pot of radius 2.5 m for processing the water. The pot is filled with water to a depth of 3.6 m. The water is then poured into a cylindrical container of base radius 6 m. For the project to be a success, the height of the cylindrical tank must not be greater than 0.5 m.

Task:

With mathematical justifications, analyse the project.

17.7 set 7

P425/1 2 hours 30 minutes

Section A

The item in this section is compulsory.

Item 1

A certain country is experiencing a crisis due to war. The war has affected mainly three towns at locations $T_1(7, 3, 6), T_2(1, 3, 9)$, and $T_3(5, -4, 3)$, all coordinates in kilometres. An emergency response team is to supply relief items to the affected people. The helicopter has to be stationed at location $H(10, 2, 16)$ km. For efficient supply, the shortest distance between H and the plane containing the three towns should be at least 8 km.

The team has also set up its own medical camp at position $P(7, 1)$ metres, and the casualties are in a circular health centre with boundaries at $A(5, 0)$, $B(0, 5)$, and $C(-3, 4)$, all in hundreds of metres. The casualties are moved from the health centre to the medical camp using a straight road which touches one of the entrances of the health centre. For safety, the journey should not exceed 1200 m.

From the medical camp, a helicopter picks up the casualties. The helicopter landing is modelled by $\cos 2\theta + \cos\theta = 0$. For safe landing, the acute angle of landing should not exceed 70°.

Task:

Using mathematical skills, help the response team to know whether their plans will be successful.

Section B

Answer only two items from this section, choosing one item from each of the parts I and II.

Part I

Item 2

A farmer is setting up a new solar pump to solve the problem of water on his farm. The pump uses a controller switch with components Z_1, Z_2, Z_3 whose pumping force while operating is $F = \frac{Z_1^2}{Z_2 Z_3}$ kN. For the safety of the pump, the size of the pumping force should be greater than 2 kN and its angle when switched on must be obtuse, where $Z_1 = 2 + 3i$, $Z_2 = 1 - 2i$, $Z_3 = 2 + i$.

The pump sends water to the tank every hour such that the amount pumped per hour is defined by $V = 100(1 + 20x)^n$ litres, where n is the number of operational hours and the terms of the expansion represent the volume pumped each hour. x is the pump code depending on the hours set; on his farm, water is pumped for 10 hours and the code for this time is $\frac{1}{20}$. He will only use this pump if in the first 4 hours it pumps not less than 15000 litres.

The pump sends water to the tanks using pipes of length l_1 and l_2, where $l_1 = 9w$ and $l_2 = 3w + 1$, w being the width of the pipes. The total length of these pipes is 108 m, but the required pressure can be realised only if the width is not more than 2.

Task:

Help the farmer to make a valid decision using mathematical skills.

Item 3

A telecom company is upgrading to a 5G network and running pre-launch stability checks on the following station components.

The engineering team is balancing the complex signal vectors coming from two independent transmitter sub-circuits. The first sub-circuit generates a complex signal value of $S_1 = \frac{2x + 4i}{2y + x}$, and the second sub-circuit generates $S_2 = \frac{x}{y - i}$. For optimal modulation stability, both complex signal vectors must be

equal. Furthermore, the total combined signal amplitude $A = \sqrt{x^2 + y^2}$ must be at least 2 to clear the receiver's noise floor.

The primary antenna mast undergoes a stress test where its allocated aggregate channel bandwidth scales exponentially every hour. In the first three hours, the allocated bandwidth blocks are 400 MHz, 800 MHz, and 1600 MHz respectively. The scaling sequence passes verification if the difference in spectrum capacity between the 7th and the 5th hour is at least 1900 MHz.

A secondary mast utilises a dual-phase local oscillator to generate carrier frequencies. The frequency profile (GHz) for Phase 1 is $F_1 = f^3 - 61$, and Phase 2 is $F_2 = 15f^2 - 74f + 59$. For stability, the frequencies of both phases must be equal, and the system must yield exactly one valid operational frequency between 1 GHz and 4 GHz.

Task:

Using your mathematical skills, help the telecom company to make valid recommendations.

Part II

Item 4

A fresh graduate in fisheries has decided to set up a fish pond to solve the problem of unemployment. He plans to construct a hemispherical pond of radius 5 m and fill it with water to full capacity. The volume of water in the pond is $V = \frac{1}{3}\pi h^2(3r - h)$ m^3, where r is the pond radius and h is the depth of water. Water evaporates at a constant rate of 4π m^3 per day, and he needs to top up the water whenever the depth is 2 m. But for the safety of the fish, the pond should be topped up at a time when the rate at which the depth is falling is not less than 0.2 metres per day.

The fish species feed at a rate modelled by $\frac{dM}{dt} + M = t$, where M is the mass of feeds (kg) remaining after t days. He plans to initially put 100 kg of feeds and add the same amount every two days. To ensure fish don't starve, the amount remaining after two days should not go below 10 kg.

The fish species increase at a rate proportional to the number of fish present at any time. He plans to initially place 200 fish, and after 1 month expects this number to double. He plans to start harvesting after 6 months; if at that time the population is more than 1300 fish, the quality of fish won't meet market standards.

Task:

Using mathematical skills, help the fresh graduate make valid decisions on his fish pond plans.

Item 5

An agriculture student doing industrial training on a certain farm has discovered some challenges and come up with the following recommendations.

The cows on the farm drink water from a cylindrical trough whose volume is $V = 9\pi h^3$ m^3. When the cows start drinking, the depth h of water after t hours is $h = \ln(10 - t)$ metres. The farm manager fills the trough to maximum capacity at 6:00 a.m., but the water has always not been enough for the cows for a full day. The student recommends topping up at noon, but this is only possible if at this time the volume of water is reducing at a rate not less than 10π m^3 per hour.

He also recommends adding a vitamin mixture to boost growth and digestion, at a rate $\frac{dm}{dt} + 80t = 17$, where m is the mass in grams of the mixture remaining after t hours. Initially 1 kg of mixture is added, and the booster can only have the required effect if after 5 hours the mass remaining is not less than 50 g.

He also discovered that the cows feed on grass provided at a rate proportional to the square root of the amount remaining. If given 900 kg in the morning, they can eat 500 kg in 4 hours; at this rate he thinks this amount is enough for the cows for at least 10 hours.

Task:

Help the agriculture student to make a valid report on his recommendations using your knowledge of calculus.

17.8 set 8

P425/1 2 hours 30 minutes

Section A

The item in this section is compulsory.

Item 1

A farmer who practices agriculture has been facing a challenge of lack of spacing for movements in his old banana plantation. According to new directives by the ministry of agriculture, the distance from where the first seed is planted to where the second seed is planted successively must be at least 10 metres.

In his new banana plantation, the farmer dug circular holes to start planting the seedlings; the first seedling was placed at the central point of the first circular hole with points $A(4,1)$, $B(1,4)$ and $C(6,5)$ lying on it. The second seedling was placed at the central point of the path described by the movement from a variable point $A(4,1)$ so that the movement to A and the movement to B are at right angles, where the pattern remained the same for the whole garden in a straight form.

To address spacing, the farmer decided to plant the next seedlings in a straight format with seedlings placed at respective positions $2\mathbf{i}+3\mathbf{k}-5\mathbf{j}$, $-3\mathbf{i}+4\mathbf{j}+\mathbf{k}$, $-2\mathbf{k}+7\mathbf{i}$. The farmer used a thread to check whether his seedlings align.

The farmer also put two supports on each banana plant in the old garden. The height of the banana plant was 6 metres (ignoring the leaves), and the two supports meet at a common point on the ground 4 metres away. The first support touches the top of the banana plant, and the second is at the stem 2 metres from the top to which the first support touches the plant.

Task:

(a) By calculations, help the farmer check whether he met the directives of the ministry of agriculture.
(b) Are the farmer's seedlings properly aligned? Provide evidence.
(c) Write the mathematical relationship that can determine the positioning of other points.
(d) What is the inclination between the two supports for effectiveness?

Section B

Answer only two items from this section, choosing one item from each of the parts I and II.

Part I

Item 2

During the construction of a highway, engineers faced a challenge of numerous anti-hills along the way; the first anti-hill appeared and was mathematically recorded as $y = (12 - 3x^2)$ metres, which had to be removed to have a straight horizontal road. The engineers wanted to know the peak height of the anti-hill from the ground and the horizontal distance x corresponding to the peak point.

Among the available group of engineers, 8 work in the monitoring department while 6 are directly involved in active work. For an anti-hill to be worked on, there must be at least 3 engineers from each department so that 8 signatories approve the work as complete.

After construction, rectangular pavers were used on the sides with length l and width three-quarters of the length. The dimensions satisfy the condition that twice the square of the length less four times the width plus ten is at least nine.

Task:

(a) Help the engineer determine the peak height of the anti-hill.
(b) How wide was the anti-hill; hence clearly sketch it.
(c) In how many ways can an anti-hill be worked?
(d) Which range of the length does not satisfy the above condition?

Item 3

During a mathematics seminar at a school, the first presenters were Jean, James and Joel respectively. They introduced themselves with their respective ages, which were consecutive such that the sum of their ages was 36 and the product of their ages was 1680.

Jean presented her work as the sketch of the roots of $Z^3+1=0$, while Joel presented about obtaining the exact value of $(6.75)^5$ following what he had previously expanded in $(2x+3y)^5$.

Task:

(a) Help the colleagues present the ratio of Jean's age to James' age.
(b) Clearly sketch the roots correctly as Jean did present them.
(c) Obtain the value that Joel obtained without using the calculator.

Item 4

A school owns a rectangular garden on the outside of the school for agricultural students to practise their projects. The garden consists of an existing wall, and the other remaining sides are to be fenced with a barbed wire mesh; the total length of the other sides to be fenced is 36 feet. The school intends to know the dimensions of the garden and the greatest area that the garden occupies.

For science students not offering agriculture, the girls chose to make conical pots as their project with radius 10 cm and height 50 cm using available clay, while the boys chose to make hollow hemispherical pots of radius 12 cm. While the conical pot was still wet, a girl cut out the largest cylindrical pot from the conical pot without wastage. After the project, a boy discovered that his pot had a radius of 11.95 cm relative to the required one, leading to variations in volume due to such moulding mistakes.

Task:

(a) Help the school establish the largest area occupied by the school garden.
(b) Verify whether the capacity of the cylindrical pot cut out is exactly 625π cm^3.
(c) Determine the percentage error that can occur while calculating the capacity of any liquid due to the reluctancy of a boy.

Item 5

The price fluctuations of maize (q) in kilograms in the market were recorded by an economist, and the change in quantity demanded due to change in price was recorded as $\frac{dp}{dq} = 120 - 0.2q$. No price is charged for no quantity demanded, and the economist is interested in determining the price (USD) of 300 kg of the commodity as well as the optimal output.

He also analysed the market value of coffee and noticed that at a nearby shop, the quantity of coffee sold varies at a rate directly proportional to the square of the cube root of the quantity remaining unsold. Initially, the shop had 1000 kg of coffee, and after two days the quantity sold had reached 271 kg.

Task:

(a) Help the economist to determine the quantity of maize (q) and the respective optimal price of maize as well as the price charged for 300 kg of maize.
(b) How long does it take for the coffee store to be emptied?
(c) Express the quantity of coffee sold in terms of time period, hence determine the quantity that had remained at the end of the 14th day.

17.9 set 9

P425/1 2 hours 30 minutes

Section A

The item in this section is compulsory.

Item 1

A city planning authority is optimising its urban transport infrastructure by mapping key service hubs using a three-dimensional coordinate system. Current hubs are located at $M(2,4,3)$ km, $N(8,6,1)$ km, and $P(4,10,2)$ km. Planners identified a site for a new logistics depot at $Q(14,8,4)$ km. The project director will only approve the depot if it lies on the straight supply line defined by hubs M and N.

For emergency rapid response, a fire station is proposed at $R(2,4,8)$ km. Planners must determine the shortest distance from the fire station to the plane containing hubs M, N and P; the fire service response will only be confirmed adequate if this distance is less than 10 km.

During expansion, engineers mapped three sensor nodes along the perimeter of a circular restricted access zone: $X(16,4)$ km, $Y(4,16)$ km, and $Z(-4,4)$ km. The city authority must determine whether a newly planned emergency access road passing through $S(4,15)$ km lies within this restricted zone.

Finally, the signal reach of a mobile weather station is defined by $S = (24\sin 2\theta + 7\cos 2\theta - 3)$ km. The weather bureau has a target of maximum signal reach of at least 20 km.

Task:

(a) Will the project director approve the new logistics depot location?
(b) Help the planners to check whether the fire service response distance is adequate.

(c) Does the emergency access road lie within the restricted security zone?
(d) Will the engineers achieve the required signal reach for the weather station?

Section B

Answer only two items from this section, choosing one item from each of the parts I and II.

Part I

Item 2

A manufacturing enterprise has embarked on a strategic upgrade to optimise its industrial liquid filtration and power control systems. During the initial phase, the company planned to increase the number of filtration units installed annually: 10 units in the first year, 20 units in the second, growing by the same constant ratio. The budget can support the expansion only if the total number of units installed during the first ten years does not exceed 12,000.

The procurement office purchased three types of electrical components from three vendors ('000 USD):

Vendor	Volt. Reg.	Sensors	Transf.	Total Cost
Alpha	2	1	3	45
Beta	3	2	1	35
Gamma	1	3	2	40

The finance department needs to determine the individual cost of each component and decide whether an allocated budget of 200,000 USD is sufficient to purchase an additional 5 voltage regulators, 4 sensors, and 6 transformers.

In the final phase, a complex automation system was implemented to manage control signals. The reference signal is $Z_0 = (2 + 3i)$, and the control signals are modelled by $Z^3 - 3Z^2 + 7Z - 5 = 0$. The system will remain stable if $|Z_0 - Z| < 4$ for each control signal.

Task:

(a) Help the planning team establish if the installation programme is financially sustainable.
(b) Advise the management team to take well-informed decisions as they undertake the remaining two projects.

Item 3

In a rural village, the village development committee agreed to collect funds monthly to install a community solar power grid: UGX 20,000 in the first month, increasing by UGX 5,000 each month for 20 months. The chair wants to determine whether the contributions will meet the budget target of UGX 50 million, given 400 participating households.

The local cooperative introduced a productivity bonus for solar technicians based on hours worked per week. The total hours x must satisfy: the square of the hours plus 4,000 must be less than or equal to 125 times the number of hours, to qualify for the bonus. A senior technician worked 55 hours in a certain week.

The maintenance crew consists of 6 junior and 5 senior technicians. The supervisor wants a daily maintenance team of 5 with an equal number of junior and senior technicians (or as close as possible), effective only if the total number of possible teams exceeds 100.

The security key x for a new encryption system must be at least 2 units for strength. During testing, the overall security rating combines $\log_x 9$ and $2\log_3 x$; the system functions correctly only when the total rating equals 3 units.

Task:

(a) Perform a financial feasibility study to determine if the budget target will be met.
(b) Advise on the effectiveness of the team selection process.
(c) Determine the eligibility of the senior technician for the performance bonus.
(d) Evaluate the technical viability of the new encryption system.

Part II

Item 4

A local cooperative is establishing an apiary project and intends to use 100 metres of protective netting to enclose a rectangular area for the bee hives. The project requires a minimum area of 500 m^2 for the hives and pollinator gardens to be effective.

The cooperative has designed a central honey-processing facility with a cylindrical storage vat, constructed such that the sum of its height and base radius is 15 metres. The facility requires a storage capacity of at least 400 m^3 to be viable.

The initial bee colony population is 2,000, changing according to $\frac{dp}{dt} = 0.05p - 150$, where p is the population after t years. For sustainability, the population must reach at least 4,000 within 10 years.

Task:

Help the cooperative establish whether:

(a) the available netting is sufficient to enclose the required area for the apiary;
(b) the proposed storage vat design can provide enough capacity to sustain the processing facility;
(c) the colony population will remain sustainable after 10 years.

Item 5

A local paint company's production profit function over time (t years) is modelled by $\frac{dp}{dt} = 0.04p(1 - 0.02)$. After the first year, the recorded profit was 500,000. Management forecasts that in five years the profit will reach UGX 2 million; the board requires an assessment of this growth target.

The company designs a container from a square sheet of reinforced plastic 60×60 cm, formed by cutting squares of side y cm from each corner and folding the sides upward. For efficient shipping, the company requires a container of maximum volume; only containers with volume between 15,000 cm^3 and 18,000 cm^3 meet durability standards.

A specialised firm produces small spherical metallic canisters, during whose manufacture a 1.5% measurement error occurred in the radius. Any canister with a volume error exceeding 5% will not be certified.

Task:

(a) Help the company know whether the projected profit growth target is achievable, and whether the container design meets the required volume standards for durability.
(b) Determine whether the spherical metallic canisters will be certified for distribution.

17.10 set 10

P425/1 2 hours 30 minutes

Section A

The item in this section is compulsory.

Item 1

A secondary school plans to construct a modern classroom block with a hidden roof, an underground rainwater harvesting tank, and a CCTV security system. The roof surfaces are represented by the planes $3x + 2y - 3z = 6$ and $4x - y + z = 8$. Rainwater from the roof flows into a central gutter formed by the line of intersection of the two planes, emptying into a collection box located at $B(1, -15, -11)$ m. For effective drainage, the collection box should lie on the line of intersection of the two roof planes, and the angle between the two roof surfaces must not exceed 70°.

The circular top of the underground water tank passes through $P(2, 3)$ m and $Q(8, 11)$ m, the endpoints of a diameter. The nearest perimeter wall is represented by $3x + 4y + 2 = 0$. For safe construction, the shortest distance from the centre of the tank to the wall must be at least 8 metres.

The maximum coverage distance D (m) of the CCTV camera is $D = 100\sin(\theta + 45°)\cos(\theta - 45°) + 120\cos 2\theta$. The school requires a camera whose maximum distance coverage exceeds 150 metres.

Task:

Use mathematical justifications to analyse the engineer's blueprint and provide a comprehensive evaluation report advising the school administration on whether the hidden roof, the water tank placement, and the CCTV camera specifications meet the operational and structural requirements.

Section B

Answer only two items from this section, choosing one item from each of the parts I and II.

Part I

Item 2

A certain company offers three-year renewable contracts for its employees. In the existing contract, each employee gets a basic pay of 600,000/= monthly multiplied by a positive performance factor X, calculated using the initial efficiency index $P = 3^x$ and the final efficiency index $Q = 3^{4-x}$. The system requires a total combined index value of 82.

The company wishes to come up with a new payment plan aimed at increasing the total amount an employee earns at the end of 36 months. Scheme A offers an initial monthly pay of 2,000,000/= with a guaranteed 1% compound growth rate applied at the end of every month for 36 months. Scheme B offers a fixed monthly pay of 1,800,000/= for 3 years with a large lump sum gratuity payout at the end of the 3 years.

The company's performance model states that the final gratuity $R(t)$ in millions after t years is $P(t) = g(t)Q(t) + R(t)$ where $P(t) = 2t^3 - 7t^2 + 13t + 5$ and $g(t) = t^2 - t$, evaluated at the end of the contract.

The company has 5 female and 6 male supervisors. In the new contract these supervisors should form daily teams of 4 supervisors with equal numbers of males and females. The company will consider having more supervisors if this arrangement does not yield more than 130 team days.

Task:

Help the company to make valid decisions using your knowledge of algebra.

Item 3

Old students of a certain school have fundraised money to construct a new academic block, housing four distinct rooms. They contracted an engineer with material purchase records from recent projects:

Project	Cement (100 bags)	Sand (×10 trucks)	Stones (×10 trucks)
A	26	5	3
B	46	9	5
C	12	4	2

The total amount spent on projects A, B, C was 128 million, 226 million and 70 million UGX respectively. The building committee allocated 100 million for the purchase of 1000 bags of cement, 80 trucks of sand and 50 trucks of stones.

An architect designed the block such that the floor areas of the four rooms correspond to the first four terms of the expansion of $A = \frac{200}{1-x}$ m^2, where x is the architectural reduction factor. The committee will adopt this design only if the total area of the four rooms is at most 600 square metres using a reduction factor of 0.8.

The school surveyor identified $Z_0 = (30 + 40i)$ m as the centre point for the new block. The proposed site sits between two existing classroom blocks whose coordinates Z are linked by $Z^2 = (119 + 120i)$ m^2. The committee will only approve the proposed site if it satisfies $|Z_0 - Z| \geq 35$ metres.

Task:

Using your knowledge of algebra, help the building committee analyse the recommendations and make valid decisions on the proposed new block.

Part II

Item 4

A certain boy dropped out of school due to lack of school fees and decided to apply the skill of making soap. He contracted a welder to manufacture containers with square cross-section for moulding bars of soap. The welder made a percentage error of 1% when measuring the side of the cross-section. The length of the bars is 10 times the side of the cross-section. Only bars of soap whose percentage error in volume does not exceed 4% are allowed on the market.

He has square sheets of hard paper measuring 50 cm by 50 cm from which he wishes to make the biggest boxes for packaging bars of soap. The box is made by cutting off equal squares from each corner and folding, to contain 10 bars of soap each of approximately 900 cm^3.

A market survey found that soap sells at a rate $\frac{dN}{dt} = \frac{4t^2 + 13t + 7}{(t+1)(t+2)}$, where N is hundreds of bars sold after t days. He will consider the business profitable if after every 7 days he can sell more than 2000 bars of soap.

Task:

Using your skills in calculus, help the young entrepreneur to make valid decisions.

Item 5

Suzan has discovered that there are many nursery schools in her locality and decided to start an ice cream business. She purchased moulds that make conical ice cream packs; however, the vertical angle θ is 32° instead of 30°. This will result in losses if the corresponding error in the radius of the cone exceeds 0.2 cm. The relationship between the radius and the vertical angle is $r = 9\tan\frac{\theta}{2}$.

She plans to use milk and sugar mixed according to $\frac{ds}{dm} = m\ln m$, where m is the amount of milk (litres) and s is the amount of sugar (grams), with 1 litre of milk requiring 30 g of sugar. She will only use this formula if her expected daily usage of 50 litres of milk requires less than 5 kg of sugar.

Suzan plans to use a mixture of chocolate and vanilla flavours that degrades at a rate proportional to the amount remaining. Specifically, 5% of the flavour mixture degrades in 3 hours. She will only consider this mixture if, after 12 hours, not more than 20% of the flavour mixture has degraded.

Task:

Help Suzan to make valid decisions on her new ice cream business plans using your calculus skills.

17.11 set 11

P425/1 2 hours 30 minutes

Section A

The item in this section is compulsory.

Item 1

A district water authority plans to establish a solar-powered water monitoring station within a circular water distribution zone located on level ground. Three boundary points of the zone were recorded as $A(1,2)$, $B(7,2)$, and $C(4,5)$, to be used to determine the equation of the circular zone and its area so that the monitoring station can be accurately positioned at the centre.

The monitoring equipment is mounted on a mast represented by $\mathbf{r} = (4,3,0) + \lambda(0,1,14)$, where the device is fixed at $\lambda = 1$. Its calibration includes a depression-angle control system governed by $2\sin 5\theta\cos 5\theta - 2\cos^2 5\theta - 2\sin^2 5\theta + 3 = 0$ for $0^\circ < \theta < 90^\circ$.

A maintenance officer stationed at $D(20,15,12)$ carries out routine inspections. For operational efficiency, the mast must be placed at the centre of the circular zone, and the area enclosed by the zone must not exceed 30 m^2. To minimise construction costs, the distance between the maintenance officer and the monitoring device must be less than 35 m, and the mast must be nearly vertical, making an angle between 87° and 90° with the horizontal ground.

Task:

Using mathematical justification, verify whether the proposed design satisfies all operational and structural requirements.

Section B

Answer only two items from this section, choosing one item from each of the parts I and II.

Part I

Item 2

A national infrastructure agency is evaluating three salary schemes for new technical posts over 4 months. The first job pays a monthly salary following an arithmetic progression starting at 1,000,000 UGX with a common difference of 75,000 UGX, plus a bonus following a geometric progression starting at 30,000 UGX and doubling each month.

The second job bases monthly pay on committee efficiency, where a committee of at least 9 staff is chosen from 12 members; each valid formation contributes 3,050 UGX to the compensation index.

The third job uses a monthly pay from a logarithmic performance indicator given by two times $\log_{10} 100$ plus $\log_{10} 10$ minus $\log_{10} 5$, scaled using a factor of 320,000 UGX.

Task:

Determine the best job to choose based on earnings after 4 months.

Item 3

A construction company is undertaking the construction of a 12-storey smart classroom tower. For cost balancing under engineering standards, the unit costs of materials are constrained by: five times the cost of cement plus four times the cost of sand plus three times the cost of gravel equals 530 million shillings; three times the cost of cement plus six times the cost of sand plus twice the cost of gravel equals 370 million shillings; seven times the cost of cement plus four times the cost of sand plus five times the cost of gravel equals 790 million shillings.

The company wishes to use 100 bags of cement, 15 trips of sand and 120 bags of gravel; the total budget is 15,000 million shillings.

The staircase design follows an arithmetic progression where the 4th floor requires 37 tiles and the 10th floor requires 91 tiles; determine whether the available 1000 tiles are enough for all 14 floors.

Engineers model the position of a steel beam tip using complex numbers. The initial position is $z_1 = 3+3i\sqrt{3}$. The beam undergoes a structural transformation due to load redistribution represented by multiplication with $2 + i$. A stabilising adjustment is then applied, $z_2 = 3 + 3i$.

Four engineers are assigned to supervise four critical floors; the number of possible inspection arrangements must be at least 15 distinct valid schedules.

Task:

Using mathematical modelling, verify whether the construction project is feasible in terms of cost, design, structural stability, and scheduling efficiency.

Part II

Item 4

An industrial manufacturing complex consists of two major production units: a metal fabrication section and a shoe manufacturing section. In the metal fabrication section, a worker is designing an open-top storage box from a rectangular sheet of metal measuring 4800 cm by 1300 cm. Equal squares of side x cm are cut from each of the four corners and the remaining edges are folded upwards to form the sides of the box, intended to store finished shoe products.

The rate of shoe production is given by $t^2e^{0.01t}$ shoes per hour, t hours after production begins. After 120 hours of continuous production, management intends to use the model to forecast total output for planning export contracts. Each shoe occupies a volume of 530 cm^3.

Task:

Help the complex find out whether the metal box has sufficient capacity to store all the shoes produced.

Item 5

A geological research team in a western region is investigating a mountain suspected to have dormant volcanic activity. The mountain profile is modelled by $y = -0.01x^2 + 100$, where y is the height (m) and x is the horizontal distance (m) from the central axis. A cross-sectional sketch is required to determine the mountain's topography and verify whether it has a single symmetrical peak suitable for magma accumulation.

The mountain is assumed to form a solid of revolution when the curve is rotated about the y-axis. The resulting volume represents the volcanic rock structure used to estimate magma storage capacity; volcanic activity is possible only if the volume of molten magma underground is at least 1.2×10^6 m^3.

The rock temperature varies with depth; the rate of change of temperature is proportional to the difference between the geothermal reference temperature of 900°C and the rock temperature. At the surface the temperature is 60°C; at depths of 200 m and 400 m the temperatures are 180°C and 420°C respectively. Volcanicity is possible when the temperature reaches 800°C or higher when the height is below 3000 m.

Task:

Determine whether:

(a) its symmetric central peak favours volcanic accumulation, from an accurate sketch;
(b) the volume of the volcanic rock structure is sufficient for volcanic activity;

(c) the region shows evidence of possible volcanic activity due to the temperature reaching 800°C.

17.12 set 12

P425/1 2 hours 30 minutes

Section A

The item in this section is compulsory.

Item 1

A farmers' association operates three mixed farms that supply food to a nearby trading centre. To improve resource management, the association monitors the use of nitrogen fertiliser (x), organic compost (y) and irrigation water pumping hours (z). Data from three production cycles produced the system of equations:

$$3x+2y-z=4, \qquad 2x+y+3z=9, \qquad x+3y-2z=-1.$$

The research team analyses seasonal crop yield variability, establishing that crop productivity is governed by the quadratic $3x^2-6x+2=0$, where α and β are its roots. Engineers proposed a mathematical model for a new automated nutrient dispenser whose roots are $\left(2\alpha+\frac{1}{\beta}\right)$ and $\left(2\alpha+\frac{1}{\alpha}\right)$.

Task:

(a) Using either the matrix method or the elimination method, solve the system of equations and determine the values of x, y and z.
(b) Without solving the quadratic yield equation, determine the value of (i) $\alpha+\beta$; (ii) $\alpha\beta$.
(c) Hence, form the new calibrated quadratic function equation for the automated dispenser.

Section B

Answer only two items from this section, choosing one item from each of the parts I and II.

Part I

Item 2

A homeowner is willing to roof his newly constructed house; the site engineer explained that the roof of the building should be in a triangular form ABC, with the following statements:

The ties and struts are to be strong if $\frac{\sin(A-B)}{\sin(A+B)}=\frac{a^2-b^2}{b^2}$.

The angle of inclination of ties to struts must be α defined by $7\cos 2\alpha+6\sin 2\alpha=5$ where $0^\circ\leq\alpha\leq 180^\circ$.

The length of sides a and b of the roof must make $x^4+6x^3+13x^2+ax+b$ a perfect square.

Task:

Help the homeowner to determine:

(a) if the ties and struts are strong;
(b) the angle of inclination of ties to struts;
(c) the length of sides a and b of the roof.

Item 3

Palm oil growing is one of the main farming activities performed in a certain district. During an agricultural expo ceremony, the government suggested land of 946400 km^2 along the shores of the lake to be used in the partial fractions of

$$\frac{11x+12}{(2x+3)(x+2)(x-3)}.$$

Resident farmers were told to use the sum of the positive partial fractions for growing palm oil, and the fraction with the negative integral sign will be used for construction of a house.

Task:

(a) Express the fraction given above into simple fractions. Taking $x=5$, obtain the real fractions for growing palm oil and construction.
(b) Determine the exact land in km^2 meant for growing palm oil and construction.
(c) Obtain the remaining land in km^2 and, with a reason, suggest any activity that can be done on it.

Part II

Item 4

A community has requested an engineering group to design a local water supply and drainage channel network near a village trading centre. The cross-section profile of a natural hill ridge valley is modelled by

$y = 6x - x^2$. A straight pipeline must be constructed along the valley floor following a path modelled by $y = x$.

Task:

As the leading hydraulic structural engineer on the project:

(a) Find the coordinate points on the grid map where the drainage pipeline intersects the parabolic ridge.
(b) Sketch the enclosed cross-section profile of the valley safety zone showing the boundaries clearly.
(c) Set up a structural definite integral expression and evaluate it to compute the exact regional area enclosed between the curve and the line.

Item 5

A group of students has been given the following set of questions to discuss during a seminar.

(a) Using small changes in differential calculus, show that $627.5^{1/4} \approx 5\frac{1}{200}$.
(b) If $x(t) = (t-2)^{3/2}$ and $y(t) = t^2$, prove that $\dfrac{d^2y}{dx^2} = \dfrac{4(t-4)}{9(t-2)^2}$.
(c) If $y(t) = \tan t$ and $x(t) = \ln(1+\sin t)$, show that $\dfrac{d^2y}{dx^2} = \sec^5 t\,(1+\sin t)^2(1+2\sin t)$.
(d) A curve $f(x)$ is given by parametric equations $x(\theta) = 2\cos\theta$ and $y(\theta) = \cos\theta - 2\sin 2\theta$, $0 \le \theta \le 2\pi$.
(e) The point P lies on $f(x)$ where $\theta = \dfrac{\pi}{4}$. Find the coordinates of point P.
(f) Show that the equation of the normal to $f(x)$ at point P is given by $4x + 2y = 5\sqrt{2}$.

17.13 set 13

P425/1 2 hours 30 minutes

Section A

The item in this section is compulsory.

Item 1

At a certain national park, rangers have established that zebras normally move along a straight path between two points $A(2,3,15)$ and $B(10,5,5)$, and the giraffes move along a straight path between two points $C(4,-2,-4)$ and $D(6,7,19)$. To save time for tourists, the rangers want to take tourists interested in seeing these two animals to the location P where these paths intersect. The rangers wish that the z-component of this location should not exceed 10 m for the tourists to be able to climb to this point.

These animals have been observed to meet at this location at times that satisfy $3\cos 2t - 2\sin t = 2$, where t can be converted into hours passed midday using the fact that for every 360° there are 24 hours. The rangers wish to be at the location at a time not later than 2:00 p.m. for safety reasons.

The rangers have also shown tourists three trees whose locations are $W(20,12)$, $X(16,-2)$ and $Y(14,15)$ near the camp, located at the boundary of a circular region that was safe for tourists at night. The tourists demanded to know whether the waterfall at point $Z(9,12)$ was within this safe zone.

Task:

Help the rangers obtain the information they need and make the right decisions.

Section B

Answer only two items from this section, choosing one item from each of the parts I and II.

Part I

Item 2

A neighbour is planning to start a fish farming business. He was told that fish grows well if the pond has the right measurements. The right length L, width W and height H (metres) for every n (in thousands) fingerlings, are given by $L \pm W\sqrt{H}$, obtained by dividing the number n of fingerlings by the magnitude of the square root of $14 + 6\sqrt{5}$. He plans to start with 4000 fingerlings, and wishes that the size of the pond does not exceed 20 m^3.

The profits P (millions) from this business depend on the number n (thousands) of fingerlings sold per harvest, equal to the ratio of the fingerlings increased by 2, all divided by the square of the fingerlings less five times the fingerlings and increased by 6. He wants the range of fingerlings that will give him at least one million shillings per harvest.

The cost (hundreds of thousands) for farming n (thou-

sands) fingerlings is given by dividing the number of fingerlings by the magnitude of the principal argument of the complex number $z = \frac{(2-i)(5+12i)}{(1+2i)^2}$. He wishes the cost does not exceed his planned budget of 300,000.

Task:

Help your neighbour make informed decisions before starting this business.

Item 3

An old woman rearing cows and goats wants to start rearing a new breed of sheep for wool. She recalls that on some day she sold a cow and three goats and bought one sheep, and had 2.4 million left. On another day, she sold three cows and one goat and bought one sheep, and had 4.4 million left. On the day she bought three sheep, she had sold a cow and a goat, and was left with 200,000 shillings. She wants to buy 5 sheep from the sale of two cows and 2 goats and remain with a balance of not less than 1 million.

The veterinarian advised that the number n of sheep per shelter, permutation the width of the shelter, must equal 30 times the number of sheep per shelter combination the length of the shelter, with length 5 m and width 4 m; she wants this number to exceed 7.

This breed multiplies geometrically at a common ratio of 2 sheep per year; starting with 10 sheep, she wants to know if in five years the number exceeds 300.

The price of sheep in the next five years will be three times the sum of the moduli of the square roots of the complex number $1 + i\sqrt{3}$, in hundreds of thousands; she wants to know if it exceeds 800,000.

Task:

Help the old woman make informed decisions.

Part II

Item 4

A company dealing in ceramics wants to design a new vessel whose edge is defined by the curve $y = 8 - x^3$. The vessel surface is formed by rotating the area cut off by this edge and the axes about the x-axis, then made hollow on the inside. The area enclosed should not exceed 13 sq units, and the volume of the vessel should not be less than 250 cubic units.

The company's earnings increase at a rate directly proportional to the product of the square of its earnings increased by twenty times the earnings, and twice the time less 1. At the start of manufacturing this vessel, total earnings will be 50 million, and after half a year the earnings will have doubled. The company wants to know the earnings after two years; if they do not exceed 150 million, it will venture into another product.

Task:

Help the company obtain the right information to direct its decisions.

Item 5

A roads authority wants to construct two straight roads. Road l_1 should be tangential to the river defined by $2x^2 + xy = 5$ at the point where $x = 1$, and road l_2 should be perpendicular to the first road at the point the first road touches the river. The authority also wishes to fund the bridge where road l_2 meets the river again, as long as the sum of the coordinates at this point exceeds 5. The authority also wishes to use the area enclosed by the river and road l_2 from $x = 0.2$ to $x = 1$ for a recreational centre if it exceeds 5 square units.

The rate at which a contracted company constructs a given length of road is inversely proportional to the length constructed reduced by ten, where time is in months. The company constructed 10 km in three months and 50 km in a year. The two roads have a total length of 100 km; the authority wants the time to be less than three years, or else another company shares the roads.

Task:

Help the authority to obtain the necessary information and make informed decisions.

17.14 set 14

P425/1 2 hours 30 minutes

Section A

The item in this section is compulsory.

Item 1

A telecommunications company is expanding its network coverage in a certain country by constructing additional communication towers and installing transmission equipment. Engineers use a three-dimensional coordinate system to map the locations of existing towers at $A(4,6,5)$ km, $B(10,8,3)$ km, and $C(6,12,1)$ km, with a proposed tower at $D(16,10,1)$ km. The manager will approve the installation only if the proposed tower lies on the straight transmission path determined by towers A and B.

For emergency operations, a new control station is proposed at $E(4,6,10)$ km. Engineers must determine the shortest distance from the control station to the plane containing towers A, B, and C; links are considered reliable only if that distance is less than 8 km.

Engineers also mapped three inspection points on the boundary of a circular security zone: $P(18,6)$ km, $Q(6,18)$ km, and $R(-6,6)$ km. They must determine whether a road passing through $T(6,-6)$ km lies within the secure zone.

The signal transmission distance from any tower is $D = (30\sin 2\theta + 40\cos 2\theta - 2)$ km. The engineers require a maximum transmission distance of at least 40 km to ensure nearby underserved towns are covered.

Task:

(a) Will the manager approve the new tower location?
(b) Determine whether the communication links will be reliable.
(c) Determine whether the road lies within the secure zone.
(d) Determine whether the required signal transmission distance will be achieved.

Section B

Answer only two items from this section, choosing one item from each of the parts I and II.

Part I

Item 2

In a certain school, parents agreed to contribute money every term in addition to school fees to buy a school bus: Shs 30,000 in the first term, increasing by Shs 10,000 each term for 15 terms. The head teacher wants to determine whether the contributions will be sufficient to meet the target budget of Shs 700 million, given a current enrolment of 500 students.

A weekly performance allowance requires that the total hours x worked by a teacher satisfy: the square of the hours plus 3600 is at most 136 times the number of hours. A teacher worked 56 hours in a certain week.

The prep committee consists of 5 male and 4 female teachers, forming daily teams of four with an equal number of males and females; the arrangement is effective only if the number of possible teams exceeds 50.

The school's encryption system's security key x is considered strong if x is at most 3 units. The overall security rating combines $\log_x 4$ and $2\log_2 x$; the system functions correctly only when the total security rating equals 4 units.

Task:

(a) Determine whether the parents' contributions will be sufficient for the purchase of the school bus.
(b) Determine whether the teacher qualifies for the bonus.
(c) Determine whether the prep committee's daily team arrangement will be effective.
(d) Determine whether the security key is strong.

Item 3

A regional development authority is working to improve water supply, energy production, and digital monitoring services for rural communities through international funding programmes. In the first phase, 24 household water tanks were installed in the first year, and 48 in the second, growing by the same constant ratio; the budget can support expansion only if the total number of tanks installed during the first nine years does not exceed 130,000.

Three types of equipment were purchased from three contractors (in hundreds of dollars):

Contractor	Water Pumps	Filtration Units	Solar Panels	Total Cost
A	2	1	3	86
B	3	2	1	91
C	1	4	2	104

The finance department must determine the cost of each item and decide whether the remaining budget of USD 44,000 is sufficient to purchase an additional 10 water pumps, 8 filtration units, and 12 solar control panels.

A digital monitoring system's reference point is $Z_0 = (5 + 12i)$ km, and the stations are modelled by $Z^3 - 8Z^2 + 37Z - 50 = 0$. Water pressure will be sufficient if the distance $|Z_0 - Z|$ from each station to the pumps is less than 20 km.

Task:

Using mathematical reasoning, advise the management team so that it can make informed decisions during the project.

17.15 set 15

P425/1 2 hours 30 minutes

Section A

The item in this section is compulsory.

Item 1

A technology company innovation mathematics club is designing a drone-navigation and signal-processing system for agricultural monitoring. Two drones were launched from a control station along different routes, and at one point they are supposed to meet. One drone took the route

$$\mathbf{r}_1 = \lambda \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix},$$

while the other took the route

$$\mathbf{r}_2 = \lambda \begin{pmatrix} 0 \\ -2 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.$$

The controller is not sure where the two drones will meet, nor the angle between the two routes.

The same company designed a rotating spotlight for a cultural gala. The brightness of the light at any instant depends on the angle of rotation; when the rotation angle is A (acute), the brightness after three rotations is $\sin 3A$. They must confirm this identity is mathematically correct and determine the value of A when $3\cos A - 4\sin A = 5$.

At the centre of a certain city, a circular recreational park is constructed such that its boundary is $x^2 + y^2 - 8x + 6y - 11 = 0$. An elliptical racing track for testing autonomous vehicles has equation $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

Task:

(a) Help the controller out of the challenge.
(b) As a student of pure mathematics, help the company clearly verify that the two equations are equal and find the value of A.
(c) Help the company determine the centre and radius of the circular park, and the eccentricity of the ellipse.

Section B

Answer only two items from this section, choosing one item from each of the parts I and II.

Part I

Item 2

A company wants to install two signal boosters to improve communication between fishing boats and the mainland along a lake shore. One emergency signal beacon was located at the complex point $z = x + iy$, satisfying $|z - (3 + 2i)| = 5$, meaning the beacon remained at a constant distance from the relay station located at $3 + 2i$.

Technicians in the same company were investigating the performance of an automated glaze-mixing and delivery system used during porcelain tile production. The rate at which a liquid glaze moved through one section of the pipe network was modelled by

$$G(t) = \frac{5t + 9}{(t - 2)(t^2 + 3t + 4)},$$

where t is time in minutes. They needed to express the function in partial fractions of the form

$$\frac{5t + 9}{(t - 2)(t^2 + 3t + 4)} = \frac{A}{t - 2} + \frac{Bt + C}{t^2 + 3t + 4}.$$

While calibrating the machine, they discovered that

three operating parameters x, y and z satisfy:

$2x+y-z = 7, \qquad x-3y+2z = -4, \qquad 3x+2y+z = 12.$

Task:

(a) Interpret geometrically the locus represented by $|z-(3+2i)| = 5$, determine its centre and radius, hence sketch the locus.
(b) Express $G(t)$ into partial fractions and hence determine the value of $G(1)$.
(c) Solve the simultaneous equations to determine x, y and z.

Item 3

A technology company designs a data storage system in which the amount of data processed in the nth cycle is modelled by $f(x) = x^3 + ax^2 + bx + 2$ such that when $f(x)$ is divided by $x-1$ the remainder is 4, and when divided by $x+2$ the remainder is also 4.

The total amount of data processed after n cycles follows the sequence $5, 10, 20, \ldots$, and they need a sum greater than 500,000.

To estimate future storage requirements, engineers use the expansion of $\sqrt{\dfrac{1+x}{1-x}}$ in ascending powers of x up to the term in x^2, and use the expansion of $(1+x)^{1/2}$ up to the term in x^3 to find the value of $\sqrt{1.08}$ to 4 decimal places.

Task:

(a) Find the values of a and b.
(b) Help the company find the smallest number of terms of the geometric progression.
(c) Help the engineers expand what is before them and use the information appropriately.

Part II

Item 4

At a carpentry workshop, a decorative wooden panel is being designed for the reception area of a hotel. The edge of the panel is shaped using a computer-controlled cutting machine following the model $y = x^3-x^2-5x+6$. The designer wants to identify the exact point where the curve changes direction, to smoothen sharp bends and ensure structural stability.

The same carpenter also deals in metal fabrication and was instructed to design an arch frame for an advertising board at a roadside sign-post workshop. The arc is modelled by $y = x^2 + 4x + 3$, and the fabricators need a clear sketch of the curve to identify key features before fabrication begins.

Task:

(a) Help the designers to find the coordinates of the turning points and distinguish its nature.
(b) Help the fabricator to sketch the above curve, clearly showing all important features.

Item 5

In a certain village, an agricultural extension officer carried out research to investigate the effect of a certain chemical on a virus infection in crops. The rate at which the virus population is destroyed by the chemical is directly proportional to the population of the virus at that time. Initially the population was p_0, and at t months later it was found to be p. The officer wanted to know how long it will take for only 5% of the original population to remain.

The officer was also tasked to find the particular solution of the differential equation $x\dfrac{dy}{dx} + 3y = 5x^2$ given $y(1) = 3$.

Task:

(a) Form a differential equation connecting p and t.
(b) Given that the virus population reduced to one third of the initial population in four months, solve the equation in (a).
(c) Find how long it will take for only 5% of the original population to remain.
(d) Solve the differential equation $x\dfrac{dy}{dx} + 3y = 5x^2$ with $y(1) = 3$.

17.16 set 16

P425/1 2 hours 30 minutes

Section A

The item in this section is compulsory.

Item 1

A certain road passes through points $(3, -2)$ and $(2, 1)$. A second road passes through a point $(6, -7)$ but meets

the first road at 90° at a certain junction. Engineers are planning to construct a third road to obey the equation $5x + 12y - 18 = 0$, but the junction is a very busy trading centre and it must be at least 2 km from this road for the safety of the people doing business there.

Due to increased crime, engineers installed three cameras at points $A(3, 2, -6)$, $B(7, 8, 4)$ and $C(13, 17, 19)$, but are unsure whether these three cameras are in the same line. The pole on which camera C was installed is supported by a strong wire to the ground, making an angle θ to the horizontal where θ is acute and $6\csc^2\theta + 7\cot\theta = 9$.

Before the trading centre, there is a roundabout satisfying $x^2 + y^2 + 2x - 4y - 11 = 0$. A straight road touches this roundabout and moves to a second point at $(3, 5)$. A rider plans to ride his motorcycle from that point to the roundabout through this straight road, but the fuel he has can only cover 5 km. (Hint: assume 1 unit = 1.4 km.)

Task:

(a) Determine whether the trading centre is at a safe location from the third road for the people doing business there.
(b) Establish whether the cameras are in one straight line, and determine the angle the wire supporting the pole on which camera C was installed makes with the horizontal.
(c) Recommend, based on calculations, if the rider should rely on the fuel he has for the journey to the roundabout.

Section B

Answer only two items from this section, choosing one item from each of the parts I and II.

Part I

Item 2

Three students made purchases from the same shop on the same day, all items of the same quality. Annet bought 5 books, 9 sets and 7 compo pens. Betty bought 8 books, 1 set and 4 compo pens. Cissy bought 6 books, 2 sets and 8 compo pens. Annet used a total of sh 65,500. Betty and Cissy spent a total of sh 90,500. Cissy spent sh 18,500 less than Annet.

Another student needs 7 books, 4 sets and 6 compo pens but was given only sh 27,000 for the shopping, and is not sure whether this money will be enough.

The profit made by the shop owner after selling books is given by squaring the number of books sold, adding 300,000, and subtracting seven hundred times the number of books sold. The smallest profit was made on a public holiday, but the owner created strategies to increase sales such that he could get at most sh 180,000 as profit on the following Sunday, though he was not certain of the number of books to sell that Sunday.

The shop has two rooms. The first room is 4 metres by 2 metres, while the second room has a square base whose length is 1 metre more than the length of the diagonal of the first room. The area of the second room is in the form $a + b\sqrt{c}$, and this room is locked with a three-digit pin obtained by arranging the roots of $x^3 - 9x^2 + 23x - 15 = 0$ from the biggest to the smallest.

Task:

(a) Check whether the second student has enough money for the items he needs.
(b) Compute the profit made on the public holiday and the range of the number of books to be sold on the following Sunday.
(c) Determine the base area of the second room and the three-digit pin used to lock this room.

Item 3

A group of seven students (excluding prefects) is needed from 6 boys and 5 girls available to represent the rest in a workshop; the girls must be more than the boys. These students will move with teachers and prefects, whose number is given by $(9 + 2i)^2(3 - i)$.

They will also take craft materials to be sold, priced in a certain progression: the cheapest craft is sh 3,000, the second sh 6,000, the third sh 12,000. A buyer is interested in buying the ninth craft material and needs to know the price.

The actual number of members in the workshop who never came from the sending school is the coefficient of x^7 in the expansion of $\left(3x^2 + \frac{4}{x}\right)^8$.

During the meeting, the students were told to go for

lunch at 1:15 p.m. and expected to return after time t (hours), where $0 < t < 1$ and $\log_3 t - \log_t 9 = 1$.

Task:

(a) Determine the number of ways the group needed can be selected, and the number of teachers and prefects to move with the students for the workshop.
(b) Help find the price of the craft material of interest, and the number of members in the workshop who never came from the sending school.
(c) Generate the time at which the students are expected to come back from lunch.

Part II

Item 4

On a certain fundraising ceremony, a decorator was hired to decorate the venue. The venue was to have disco lights obeying the equation $y = \dfrac{x^2 - 2x - 8}{x^2 - 1}$.

There were p VIP seats and q ordinary seats, where $\dfrac{257x - 1392}{x^2 + x - 12} = \dfrac{p}{x - 3} + \dfrac{q}{x + 4}$. Each VIP seat was hired at sh 50,000 and each ordinary seat at sh 20,000; all seats were hired out.

The tent in which the ordinary seats were placed obeys $y = 8 + 2x - x^2$, where x is the distance from its centre.

Task:

(a) Sketch the region to be decorated by lights.
(b) Determine the total amount of money received from the hired seats.
(c) Calculate the area covered by the tent hosting ordinary seats with the ground.

17.17 set 17

P425/1 2 hours 30 minutes

Section A

The item in this section is compulsory.

Item 1

In a certain trading centre, cargo vehicles move following a road pattern described by the curve $2x^2 + 4x - 1 = 0$, whose roots are α and β. When the mayor of the municipality visited for inspection, he realised there was congestion and decided to construct a new road of the same nature, with roots $\dfrac{1 - \alpha}{\beta}$ and $\dfrac{1 - \beta}{\alpha}$.

The revenue department advised the mayor that to construct this new road easily, the tax levied on traders should be increased by an approximate factor of $\sqrt[3]{24}$, corrected to three significant figures, obtained by expanding $(1 + x)^{1/3}$. To get a favourable value that does not exploit traders, the expansion should be to the term in x^3 and the value $x = \frac{1}{9}$ substituted.

After analysing the length of this new road, the engineering team found it to be

$$\left(\frac{2\sqrt{2}}{3 - \sqrt{2}}\right) \div \left(\frac{4 + \sqrt{2}}{3 + \sqrt{2}}\right) \text{ km.}$$

The mayor tasked them to express it as a linear expression of the form $a + b\sqrt{2}$.

The mayor has decided to form a committee of four members among whom is the chairperson of the market, choosing from a group of four engineers and five revenue specialists, such that the committee can have one or two engineers.

To protect the account where the revenue money is placed, he has decided to hide the last value of the password in the solution of the equation $\sqrt{x - 5} + \sqrt{x - 5} = 0$.

Task:

Help the mayor to formulate the report for proper documentation.

Section B

Answer only two items from this section, choosing one item from each of the parts I and II.

Part I

Item 2

At a certain school, the security team was tasked to install cameras in the library to monitor noise makers. The technical team discovered that the sitting plan of the library was designed such that the columns follow the plane $2x + y - z = 5$, the rows follow the plane $x + 3y + 2z = 5$, and the book shelf follows the plane $x - y + z = 1$. They came up with a plan of installing a 3-dimensional camera at the centre of the three planes.

The estimated installation cost was UGX 18 million, which looked expensive. However, it was suggested that the school should invest UGX 1.5 million at the beginning of every month in a savings SACCO offering a compound interest of 14% per month for eight months without withdrawals, so that the interest earned can be used to fix the cameras.

The security team also identified that a group of 8 members should be selected from 15 non-teaching staff to boost security, and chairs whose area is the product of the solutions of $9^m - 12(3^m) + 27 = 0$ should be brought to minimise congestion in the library.

The library lights, which should emit a light of frequency $\frac{1+\sqrt{3}\,i}{1-2i}$ Hz so as not to interfere with the new camera, need their modulus and principal argument determined before turning off.

Task:

Help the security team to design the logical report to be presented to the head teacher.

Item 3

The electrical grid at a certain school has two grid lines: one for non-lighting electrical components and one for the lighting system. The head teacher discovered that the lighting system is mismanaged, leading to unnecessary losses. After a careful study, he found that the teaching faculty is at coordinate $(2, 9)$, the girls' dormitory at $(5, 0)$, and the boys' dormitory at $(-1, 8)$, and are in circular form. He needs to determine the central point where the switch can be placed to control the three buildings, and the cost of the wire connecting each building from the central switch.

To eliminate trespass, the head teacher plans to construct a pathway of pavers, R_1, passing through $(2, 2)$ and $(3, 3)$, connecting the teaching faculty to the dining hall. A pathway of pavers, R_2, passing through $(2, 0)$ and $(-1, -1)$, connects the girls' dormitory and the canteen. These two pathways meet, and he plans to put a school monument at this point.

To make security tight, cameras should be placed at point $A(2, \sqrt{3}, -3)$ on the girls' dormitory and $B(1, 0, -1)$ on the teaching faculty block, following a Cartesian equation joining the two points. For a wide field of view, the cameras should be positioned at angles satisfying $12\cos\theta + 5\sin\theta = 2$ within $0° \leq \theta \leq 360°$.

A parabolic lamp placed at $(at^2, 2at)$, producing a parallel beam of light, is tangentially sending a light ray to the head teacher's office. He wants to reposition the lamp at the same point but perpendicular to the current path. (Hint: cost of 1 unit of wire is UGX 20,000; equation of a parabola is $y^2 = 4ax$.)

Task:

Help the head teacher to address the concerns.

Part II

Item 4

A trader's daily cost of producing x batches of produce is defined by $\frac{2x^2+5x+7}{(x+1)(x+1)^2}$, which her accountant explains can be broken down into simpler parts for better analysis.

Motivated by her success, the trader has started a juice business on the side, using conical disposable cups of height h cm and radius r cm related by $r = \frac{h}{2}$. While filling the cups, juice is pumped at a rate of 10π cm^{3}s^{-1}; she wants to analyse the rate at which the height of the juice rises when the height is 6 cm.

She would also like to construct a restaurant on a piece of land enclosed by the curve $y = (2+x)^2$ from $x = 0$ to $x = 1$. She estimates the area using the trapezium rule with 6 ordinates (correct to 4 decimal places), and later performs integration to get the actual value, needing to know the percentage deviation of the estimate from the actual area.

Task:

Use the knowledge of calculus to break down the trader's concerns following logical steps.

Item 5

A chemical factory dealing in the production of pesticides works with a concentrate x and concentrate y to produce a final product. The two concentrates are related by $y = \frac{\sin x}{x}$. A new chemical engineer confirmed that the concentrates satisfy the model

$x\frac{d^2y}{dx^2}+2\frac{dy}{dx}+xy=0$; the manager was not convinced by this submission.

The factory packs its product in a bottle container of volume 10 cm^3; the company auditor advised that they are making losses due to the high cost of packaging material. The manager tasked the engineer to design a cylindrical container of the same volume, using the radius and height that give the least possible surface area, and to show how the surface area can be modelled as $A=2\pi r^2+\frac{20}{r}$.

The manager wants to build a warehouse on company land bounded by $y=2x^2+7x+3$ and $y=9+4x-x^2$, and needs to know the shape and size of the land bounded by the two functions.

Task:

Help the engineer to satisfy the demands of the manager.

Item 6

A farmer has built a new fish pond and needs to understand several mathematical concepts to manage its efficiency. An expert said the fish growth pattern is modelled by $p(t)=e^{2t}\cos t$ or $\frac{d^2p}{dt^2}-4\frac{dp}{dt}+5p=0$; the farmer did not accept that fish growth could model both functions.

The quantity of food, x, to be given to the fish satisfies $Q(x)=x^3-2x^2-4x+10=0$. The exact quantity of food that does not waste food (a root) lies between $x=3$ kg and $x=4$ kg, and can be obtained using the Newton-Raphson method correct to 4 decimal places.

The volume of the water reservoir to maintain the ponds should be generated by rotating the curve $y=kx^2$ about the y-axis, so that when the surface depth of the central reservoir is h metres, the surface area A is proportional to h and the volume is proportional to h^2. Since the pond is located in a dry place, the rate of loss of water due to evaporation is λA m^2 per day; a differential equation relating the height h and time t should be formed. With $\lambda=\frac{1}{2}$, the farmer asked how many days it will take for the depth of water in the reservoir to decrease from 20 m to 2 m.

Task:

Help the farmer to address his concerns step by step based on the information given by the expert.

17.18 set 18

P425/1 2 hours 30 minutes

Section A

The item in this section is compulsory.

Item 1

An extension worker mapped a village demonstration farm using Cartesian coordinates in three dimensions. He installed a sprinkler at the point $S(1,3,2)$ inside a greenhouse. The greenhouse roof is the plane $3x-y+2z=7$. A water pipe, running along the line $\mathbf{r}=(2+\mu)\mathbf{i}+(-1+2\mu)\mathbf{j}+(1-2\mu)\mathbf{k}$, enters the greenhouse through the roof at point P. The extension worker wishes to move the sprinkler closer if the distance between S and P is more than 5 m.

Two boundaries of the sprinkler spray zone on the greenhouse floor have equations $y=2x-1$ and $3x+4y=8$. The extension worker wishes to change the nozzle if the angle between the two spray boundaries is less than $50°$.

During calibration, the sprinkler rotates through an angle x satisfying $2\sin x+5\cos x-1=0, 0°\le x\le 180°$. If the angle is less than $100°$, the sprinkler will be adjusted.

A faulty nozzle sprays water such that it hits the floor at a point $Q(x,y)$ whose distance from the point $A(2,1)$ is twice its distance from the point $B(5,3)$. The extension worker wishes to relocate a plant at $(7,4)$ if water cannot reach it.

Task:

Based on mathematical computations, advise the extension worker on the decisions to take.

Section B

Answer only two items from this section, choosing one item from each of the parts I and II.

Part I

Item 2

A college administration is planning activities for the Annual Academic Excellence and Innovation Week. To ensure fairness in allocating study materials, organising students' awards, and scheduling the official opening ceremony, the administration applies mathematical reasoning.

To allocate revision booklets fairly, 25 times the number of booklets given to each student minus the square of the number of booklets given to each student should be at least 144. For fairness, each student must get at least 8 booklets.

The administration has selected 8 student leaders, but only 5 of them will receive certificates of excellence. For their plan to continue, the number of possible selections of certificate winners should not exceed 60, and each cash prize should not be more than Shs 120 million.

The certificates for the 5th, 4th, 3rd, 2nd, and 1st positions correspond respectively to the first five terms in ascending powers of x in the expansion of $(4 + 3x)^7$, where x represents Shs 5.

On the day of the ceremony, 50 invited guests are expected. The first guest arrives at 9:10 a.m., and each subsequent guest arrives t minutes after the previous one. The guest of honour is the last person to arrive, and 15 minutes later the ceremony will officially be declared open. The ceremony is to run for 3 hours 40 minutes and must end by 5:59 p.m.

Task:

Help the administration decide whether to carry on with the planned format of booklet distribution and certificate awarding. Also advise on the arrival-time interval t for the individual guests so that the function ends on time.

Item 3

A security team was tasked to design a strong security pillar to be constructed in a sugar-cane plantation to monitor and fight fire outbreaks. The pillar will provide a high viewpoint for surveillance and quick response.

The heights of the steps above the ground increase by the same vertical distance from one step to the next. The height of the 5th step is three times that of the 7th step. The total height of the first twelve steps is 21 m. The team wishes to redesign the pillar if the total height of the first five steps is less than 15 m.

The security team wishes to reduce response time by placing two communication radios a distance of x tens of metres from the pillar. One radio shows a signal strength of $3\log_8 x$, and the other radio shows a signal strength of $6\log_x 4$. For the safety team to hear clearly without echo, the sum of the two signal strengths must be 8. The team wishes to adjust the radios if their distance from the pillar is less than 40 m.

The team wishes to position at least four solar-powered lights. The main light is positioned at $-8 + 8\sqrt{3}\,i$ metres on the plantation grid. The other three lights must be positioned at the cube roots of the main light position. The team wishes to relocate the lights if the sum of the moduli of the positions of these lights is less than 10 m.

Task:

Based on mathematical evidence, advise the security team on the decisions to make.

Part II

Item 4

A farmer in a certain district is establishing a beef-cattle project on his land along a river. He is to construct a rectangular paddock along the river, which forms one side of the enclosure; only the remaining three sides require fencing. The farmer has 1120 m of fencing available.

Veterinary guidelines recommend that each bull requires 500 m^2 of grazing space. The farmer stocks the farm at a rate proportional to the square of the number of available bulls. Initially, there are 50 bulls on the farm, and after 8 weeks there are 75 bulls. The farmer wishes to start selling bulls when the farm is stocked to full capacity.

To maintain the pasture throughout the year, an irrigation system is installed. The main irrigation pipe follows the path $y = \frac{x}{2}\sqrt{10 - x}$, where x represents the horizontal distance, in metres, east of the water source and y is the distance, in metres, north of the water source. Engineers are to install inspection gadgets at

points where the pipe changes direction; these points must be 10 metres apart.

Task:

(a) Determine the maximum number of bulls that can be grazed according to the veterinary guideline.
(b) Advise the farmer on when to start the sales.
(c) Make a recommendation on the engineers' inspection points.

Item 5

A community irrigation project is constructing a decorative water feature and a storage system for a farm. The side profile of the decorative basin is modelled by $y = x(4 - x)^2$, $0 \leq x \leq 4$, where the units are metres. The region enclosed by the curve and the x-axis is rotated about the x-axis to form a solid basin whose volume must be at least 450 m^3; otherwise, the decorative feature must be redesigned.

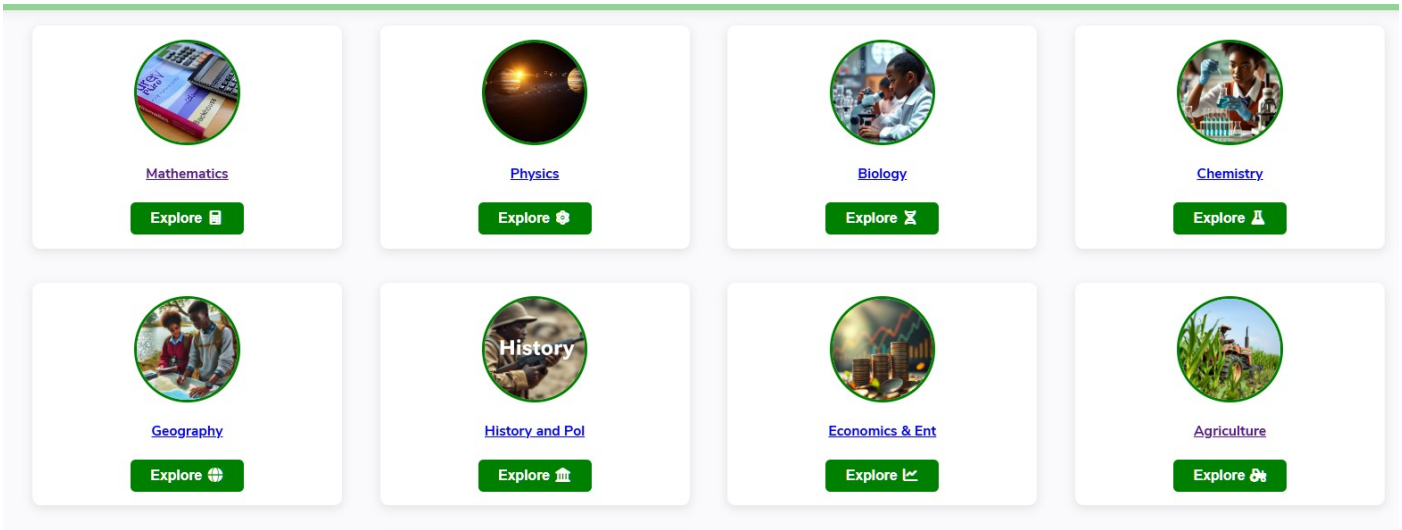
At the same project, workers use a spherical water pot of radius 2.5 m for processing the water. The pot is filled with water to a depth of 3.6 m. The water is then poured into a cylindrical container of base radius 6 m. For the project to be successful, the height of water in the cylindrical tank must not be greater than 0.5 m.

Task:

With mathematical justifications, analyse the project.

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