SURDS

Well-Explained Mathematics Notes

Secondary School Mathematics • O-Level

What you will learn

- Meaning and examples of surds
- Perfect squares and simplifying square roots
- Adding, subtracting, multiplying and dividing surds
- Rationalising denominators
- Surds involving brackets and equations
- Common mistakes, revision exercises and answers

1. Meaning of surds

A **surd** is an irrational root that cannot be written exactly as a whole number or as a fraction of two integers. In school mathematics, surds commonly involve square roots, cube roots or other roots.

> **Worked example**
> $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and $\sqrt[3]{7}$ are surds. Their exact values cannot be written as terminating or recurring decimals.

Some roots are not surds because they simplify to rational numbers. For example, $\sqrt{9} = 3$, $\sqrt{(1/4)} = 1/2$ and $\sqrt[3]{8} = 2$.

Important vocabulary

• Radical sign: the symbol $\sqrt{}$.

• Radicand: the number or expression under the root sign. In $\sqrt{12}$, the radicand is 12.

• Index: tells you which root to take. In $\sqrt[3]{8}$, the index is 3. A square root has an understood index of 2.

• Exact form: $\sqrt{2}$ is exact; 1.414 is only an approximation.

Why do we use surds?

Surds let us keep answers exact instead of rounding them too early. For example, the diagonal of a square of side 1 unit is $\sqrt{2}$ units, not exactly 1.414 units.

2. Perfect squares and square roots

A perfect square is the result of multiplying an integer by itself.

Number	Square	Square root
1	$1 \times 1 = 1$	$\sqrt{1} = 1$
2	$2 \times 2 = 4$	$\sqrt{4} = 2$
3	$3 \times 3 = 9$	$\sqrt{9} = 3$
4	$4 \times 4 = 16$	$\sqrt{16} = 4$
5	$5 \times 5 = 25$	$\sqrt{25} = 5$
6	$6 \times 6 = 36$	$\sqrt{36} = 6$
7	$7 \times 7 = 49$	$\sqrt{49} = 7$
8	$8 \times 8 = 64$	$\sqrt{64} = 8$
9	$9 \times 9 = 81$	$\sqrt{81} = 9$
10	$10 \times 10 = 100$	$\sqrt{100} = 10$
11	$11 \times 11 = 121$	$\sqrt{121} = 11$
12	$12 \times 12 = 144$	$\sqrt{144} = 12$
13	$13 \times 13 = 169$	$\sqrt{169} = 13$
14	$14 \times 14 = 196$	$\sqrt{196} = 14$
15	$15 \times 15 = 225$	$\sqrt{225} = 15$

To simplify a square root, look for a perfect-square factor inside the radicand. Useful squares include 4, 9, 16, 25, 36, 49, 64, 81, 100, 121 and 144.

3. Simplifying surds

The key rule is: $\sqrt{(a \times b)} = \sqrt{a} \times \sqrt{b}$ when a and b are non-negative. Choose a factor that is a perfect square.

Worked example
Simplify $\sqrt{12}$.
Step 1: Write $12 = 4 \times 3$.
Step 2: $\sqrt{12} = \sqrt{(4 \times 3)} = \sqrt{4} \times \sqrt{3}$.
Step 3: $\sqrt{4} = 2$, so $\sqrt{\mathbf{12 = 2\sqrt{3}}}$.

Worked example
Simplify $\sqrt{45}$.
$45 = 9 \times 5$, so $\sqrt{45} = \sqrt{9} \times \sqrt{5} = \mathbf{3\sqrt{5}}$.

Worked example
Simplify $\sqrt{72}$.
$72 = 36 \times 2$, so $\sqrt{72} = \sqrt{36} \times \sqrt{2} = \mathbf{6\sqrt{2}}$.

Worked example
Simplify $\sqrt{200}$.
$200 = 100 \times 2$, so $\sqrt{200} = 10\sqrt{2}$.

When there is a number outside the root

Worked example
Simplify $3\sqrt{20}$.
$\sqrt{20} = \sqrt{(4 \times 5)} = 2\sqrt{5}$. Therefore $3\sqrt{20} = 3 \times 2\sqrt{5} = \mathbf{6\sqrt{5}}$.

Cube roots

For cube roots, take out factors that are perfect cubes: 8, 27, 64, 125, 216, and so on.

Worked example
Simplify $\sqrt[3]{54}$.
$54 = 27 \times 2$, so $\sqrt[3]{54} = \sqrt[3]{27} \times \sqrt[3]{2} = \mathbf{3\sqrt[3]{2}}$.

4. Like and unlike surds

Like surds have the same simplified root part. For example, $2\sqrt{3}$ and $7\sqrt{3}$ are like surds. The expressions $\sqrt{2}$ and $\sqrt{3}$ are unlike surds.

Before deciding whether surds are like, simplify them first. For example, $\sqrt{12}$ and $\sqrt{27}$ simplify to $2\sqrt{3}$ and $3\sqrt{3}$, so they are like surds.

5. Addition and subtraction

You can add or subtract like surds by adding or subtracting their coefficients. The root part stays the same.

Worked example
$2\sqrt{5} + 6\sqrt{5} = (2 + 6)\sqrt{5} = \mathbf{8\sqrt{5}}$.

Worked example
$9\sqrt{7} - 4\sqrt{7} = (9 - 4)\sqrt{7} = \mathbf{5\sqrt{7}}$.

Worked example
Simplify $\sqrt{12} + \sqrt{27}$.
$\sqrt{12} = 2\sqrt{3}$ and $\sqrt{27} = 3\sqrt{3}$.
So $2\sqrt{3} + 3\sqrt{3} = \mathbf{5\sqrt{3}}$.

Worked example
Simplify $\sqrt{50} - \sqrt{8}$.
$\sqrt{50} = 5\sqrt{2}$ and $\sqrt{8} = 2\sqrt{2}$.
So $5\sqrt{2} - 2\sqrt{2} = \mathbf{3\sqrt{2}}$.

Remember: $\sqrt{2} + \sqrt{3}$ cannot be combined into $\sqrt{5}$. Unlike surds cannot be added by joining the numbers under the roots.

6. Multiplication of surds

Multiply the numbers outside the roots together and multiply the numbers inside the roots together.

Worked example
$\sqrt{3} \times \sqrt{5} = \sqrt{15}$.

Worked example
$2\sqrt{3} \times 4\sqrt{6} = (2 \times 4)\sqrt{(3 \times 6)} = 8\sqrt{18} = \mathbf{24\sqrt{2}}$, because $\sqrt{18} = 3\sqrt{2}$.

Worked example
$\sqrt{8} \times \sqrt{2} = \sqrt{16} = \mathbf{4}$.

Multiplying brackets

Use the distributive law: multiply each term in the first bracket by each term in the second bracket.

Worked example
$(\sqrt{2} + 3)(\sqrt{2} + 4)$
$= 2 + 4\sqrt{2} + 3\sqrt{2} + 12$
$= \mathbf{14 + 7\sqrt{2}}$.

Worked example
$(\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{2})$
$= 5 - 2 = \mathbf{3}$.

The identity $(a + b)(a - b) = a^2 - b^2$ is especially useful with surds.

7. Division of surds

For simple division, divide the numbers under the roots or separate the roots where appropriate.

Worked example
$\sqrt{18} \div \sqrt{2} = \sqrt{(18 \div 2)} = \sqrt{9} = \mathbf{3}$.

Worked example
$6\sqrt{15} \div 2\sqrt{3} = (6 \div 2)\sqrt{(15 \div 3)} = \mathbf{3\sqrt{5}}$.

A denominator containing a surd is often rationalised so that no root remains in the denominator.

8. Rationalising a denominator

Rationalising means changing a fraction into an equivalent form with a rational (non-surd) denominator.

Case A: One square root in the denominator

Worked example
Simplify $5/\sqrt{3}$.

Multiply top and bottom by √3:
(5/√3) × (√3/√3) = **5√3/3**.

Worked example
Simplify 2/√5.
(2/√5) × (√5/√5) = **2√5/5**.

Case B: A number times a root in the denominator

Worked example
Simplify 3/(2√7).
Multiply top and bottom by √7:
3/(2√7) × √7/√7 = **3√7/14**.

Case C: A binomial denominator

Use the conjugate. The conjugate of a + √b is a – √b, and vice versa. Their product removes the surd because (a + √b)(a – √b) = a² – b.

Worked example
Rationalise 1/(3 + √2).
Multiply top and bottom by 3 – √2:
= (3 – √2)/(9 – 2) = **(3 – √2)/7**.

Worked example
Rationalise 4/(√5 – 1).
Multiply top and bottom by √5 + 1:
= 4(√5 + 1)/(5 – 1) = **√5 + 1**.

9. Surds and powers

A square root can be written as a fractional power: √a = a^(1/2). A cube root is ∛a = a^(1/3).

• (√a)² = a for a ≥ 0.

• √(a²) = |a| in general; if a is non-negative, √(a²) = a.

• √a × √a = a for a ≥ 0.

Worked example
Simplify (√7)² + √49 = 7 + 7 = **14**.

10. Solving simple equations with surds

Worked example
Solve x√3 = 12.
x = 12/√3 = 12√3/3 = **4√3**.

Worked example
Solve √x = 5.
Square both sides: x = 25. Check: √25 = 5, so **x = 25**.

Check answers in the original equation, because squaring both sides can sometimes introduce an answer that does not work.

11. Common mistakes to avoid

• Do not write $\sqrt{a} + \sqrt{b} = \sqrt{(a + b)}$. This is generally false.

• Simplify surds before adding or subtracting.

• When rationalising, multiply both numerator and denominator by the same expression.

• When multiplying surds, multiply the radicands and then simplify.

• Remember $\sqrt{25} = 5$, not ± 5. The equation $x^2 = 25$ has solutions $x = 5$ and $x = -5$.

• Do not leave a perfect-square factor inside a simplified square root.

12. Quick revision examples

Simplify $\sqrt{28}$ Answer: $2\sqrt{7}$
Simplify $\sqrt{98}$ Answer: $7\sqrt{2}$
Simplify $4\sqrt{3} + 2\sqrt{3}$ Answer: $6\sqrt{3}$
Simplify $5\sqrt{2} - \sqrt{8}$ Answer: $3\sqrt{2}$
Multiply $\sqrt{6} \times \sqrt{24}$ Answer: 12
Simplify $(2\sqrt{5})(3\sqrt{10})$ Answer: $30\sqrt{2}$
Rationalise $1/\sqrt{6}$ Answer: $\sqrt{6}/6$
Rationalise $2/(\sqrt{3} + 1)$ Answer: $\sqrt{3} - 1$
Simplify $\sqrt{75} + \sqrt{12}$ Answer: $7\sqrt{3}$
Simplify $\sqrt{48} \div \sqrt{3}$ Answer: 4

13. Practice exercises

Try these questions on your own before checking the answer key.

• 1. State whether $\sqrt{16}$, $\sqrt{7}$ and $\sqrt[3]{27}$ are surds.

• 2. Simplify $\sqrt{20}$.

• 3. Simplify $\sqrt{63}$.

• 4. Simplify $3\sqrt{12}$.

• 5. Simplify $\sqrt{18} + \sqrt{8}$.

- 6. Simplify $\sqrt{75} - \sqrt{27}$.
- 7. Multiply $\sqrt{6}$ by $\sqrt{24}$.
- 8. Simplify $(2\sqrt{3})(5\sqrt{6})$.
- 9. Simplify $\sqrt{45} \div \sqrt{5}$.
- 10. Rationalise $4/\sqrt{2}$.
- 11. Rationalise $3/(\sqrt{5} + 2)$.
- 12. Expand $(\sqrt{3} + 2)(\sqrt{3} - 2)$.
- 13. Solve $\sqrt{x} = 8$.
- 14. Simplify $2\sqrt{7} + 3\sqrt{7} - \sqrt{7}$.
- 15. Explain why $\sqrt{2} + \sqrt{5}$ cannot be simplified by adding the radicands.

14. Answers to practice exercises

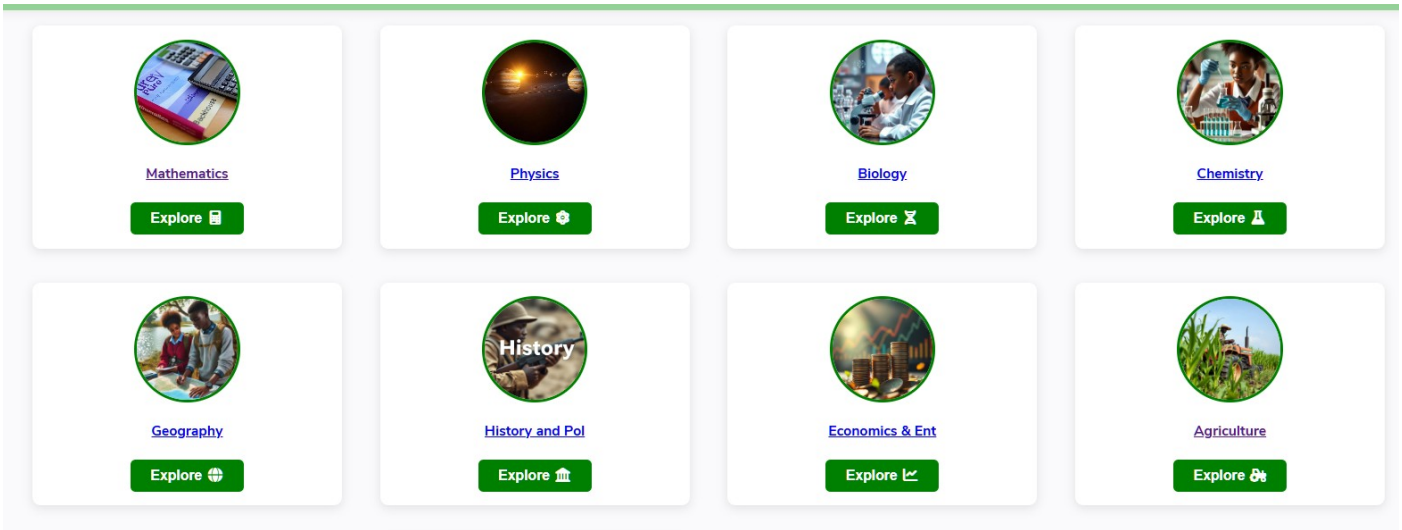
- 1. $\sqrt{16} = 4$ and $\sqrt[3]{27} = 3$ are not surds; $\sqrt{7}$ is a surd.
- 2. $2\sqrt{5}$.
- 3. $3\sqrt{7}$.
- 4. $6\sqrt{3}$.
- 5. $5\sqrt{2}$.
- 6. $2\sqrt{3}$.
- 7. 12.
- 8. $30\sqrt{2}$.
- 9. 3.
- 10. $2\sqrt{2}$.
- 11. $3\sqrt{5} - 6$.
- 12. -1.
- 13. $x = 64$.
- 14. $4\sqrt{7}$.
- 15. They are unlike surds; the radicands cannot simply be added.

15. Exam tips

- Show each step clearly; marks are often awarded for correct working.
- Look for the largest perfect-square factor when simplifying.
- For addition and subtraction, simplify each surd first and then collect like terms.
- For rationalising a binomial, use its conjugate and apply difference of two squares.
- Keep answers exact unless the question specifically asks for a decimal approximation.

End of notes • Revise the worked examples, then attempt the practice questions without looking at the answers.

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